

61

The Divergence, Rotation and Laplacian

...Stokes was a very important formative influence on subsequent generations of Cambridge men, including Maxwell. With Green, who in turn had influenced him, Stokes followed the work of the French, especially Lagrange, Laplace, Fourier, Poisson and Cauchy. This is seen most clearly in his theoretical studies in optics and hydrodynamics; but it should also be noted that Stokes, even as an undergraduate, experimented incessantly. Yet his interests and investigations extended beyond physics, for his knowledge of chemistry and botany was extensive, and often his work in optics drew him into those fields. (Parkinson)

Appointed professor of mathematics at the Ecole Polytechnique in 1809 Ampère held posts there until 1828. Ampère and Cauchy shared the teaching of analysis and mechanics and there was a great contrast between the two with Cauchy's rigorous analysis teaching leading to great mathematical progress but found extremely difficult by students who greatly preferred Ampère's more conventional approach to analysis and mechanics. (O'Connor and Robertson)

61.1 Introduction

We saw previously that the gradient of a function of several variables is a practically useful *differential operator*. In this chapter, we introduce some other useful operators, including the *divergence*, *rotation* and the *Laplacian*, together with the gradient play a fundamental role in mathematical modeling in science and engineering. We first define the operators in $\mathbb{R}^2$ and

then in $\mathbb{R}^3$, noting that the rotation takes somewhat different forms in $\mathbb{R}^2$ and $\mathbb{R}^3$.



Fig. 61.1. Napoleon to Laplace (1749–1827): “You have written this huge book on the system of the world without once mentioning the Author of the Universe”. Laplace to Napoleon: “Sire, I had no need of this hypothesis”

61.2 The Case of $\mathbb{R}^2$

We recall that the *gradient* of a function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, denoted $\text{grad } u$ or ∇u , is the vector-valued function formed by the first order partial derivatives of u , i.e.

$$\text{grad } u = \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right).$$

The *divergence* of a vector function $u = (u_1, u_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, denoted $\text{div } u$ or $\nabla \cdot u$, is the scalar function defined by

$$\text{div } u = \nabla \cdot u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}.$$

Formally, we have

$$\nabla \cdot u = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) \cdot (u_1, u_2)$$

where we may think of $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)$ “as a vector” and let the dot indicate a “scalar product”. This idea applies to all the formulas below involving ∇ combined with the operators $\cdot$ and $\times$.

The *rotation* of a vector function $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, denoted by $\text{rot } u$ or $\nabla \times u$, is the scalar function

$$\text{rot } u = \nabla \times u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) \times (u_1, u_2).$$

If $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a scalar function, then $\text{rot } u = \nabla \times u$ is defined as the vector function

$$\text{rot } u = \nabla \times u = \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1} \right).$$

The different appearances of $\text{rot } u = \nabla \times u$, with u a scalar or $u = (u_1, u_2)$ a vector function will be explained when we pass to $\mathbb{R}^3$ below. For now, it may be helpful to recall the different appearances of $a \times b$ with $a, b \in \mathbb{R}^2$ or $a, b \in \mathbb{R}^3$.

The following identities follow directly from the definitions for any function u :

$$\begin{aligned} \nabla \cdot (\nabla \times u) &= \text{div } (\text{rot } u) = 0, & (u : \mathbb{R}^2 \rightarrow \mathbb{R}^2) \\ \nabla \times (\nabla u) &= \text{rot } (\text{grad } u) = 0, & (u : \mathbb{R}^2 \rightarrow \mathbb{R}). \end{aligned} \quad (61.1)$$

Finally, the *Laplacian* Δu of a function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$\Delta u = \nabla \cdot (\nabla u) = \text{div } (\text{grad } u) = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2},$$

where $\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_i} \right)$.

61.3 The Laplacian in Polar Coordinates

In polar coordinates $x = (x_1, x_2) = (r \cos(\theta), r \sin(\theta))$ with $r \geq 0$ and $0 \leq \theta < 2\pi$, the Laplacian takes the form

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}. \quad (61.2)$$

This follows by a routine computation using that the Jacobian of the mapping $x = (r \cos(\theta), r \sin(\theta))$, in the notation (54.9) is given by

$$\frac{d(x_1, x_2)}{d(r, \theta)} = \begin{pmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{pmatrix},$$

so that

$$\frac{d(r, \theta)}{d(x_1, x_2)} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta)/r & \cos(\theta)/r \end{pmatrix},$$

and thus by the Chain rule

$$\frac{\partial}{\partial x_1} = \cos(\theta) \frac{\partial}{\partial r} - \frac{\sin(\theta)}{r} \frac{\partial}{\partial \theta} \quad \text{and} \quad \frac{\partial}{\partial x_2} = \sin(\theta) \frac{\partial}{\partial r} + \frac{\cos(\theta)}{r} \frac{\partial}{\partial \theta}.$$

61.4 Some Basic Examples

The function $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $u(x) = \frac{1}{2}(x_1, x_2)$, satisfies

$$\nabla \cdot u(x) = 1.$$

The function $v : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $v(x) = \frac{1}{2}(-x_2, x_1)$, satisfies

$$\nabla \times v(x) = 1.$$

The function $w : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $w(x) = \frac{1}{4}(x_1^2 + x_2^2)$, satisfies

$$\Delta w = 1.$$

We plot these basic examples in Fig. 61.2

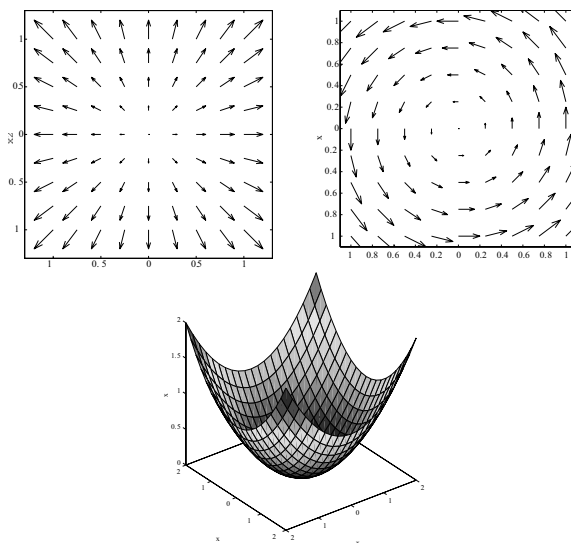


Fig. 61.2. Basic examples satisfying $\nabla \cdot u = 1$, $\nabla \times v = 1$ and $\Delta w = 1$, respectively

We see that $u(x)$ “explodes”, $v(x)$ “rotates” and $w(x)$ is a “hump”.

61.5 The Laplacian Under Rigid Coordinate Transformations

It follows from the form of the Laplacian in polar coordinates, that the Laplacian is invariant under rotations and translations in $\mathbb{R}^2$, i.e. so-called

rigid transformations of the form

$$\begin{aligned}\tilde{x}_1 &= \cos(\alpha)x_1 + \sin(\alpha)x_2 + a_1 \\ \tilde{x}_2 &= -\sin(\alpha)x_1 + \cos(\alpha)x_2 + a_2,\end{aligned}$$

where (x_1, x_2) are the old coordinates and $(\tilde{x}_1, \tilde{x}_2)$ the new ones. In other words, the Laplacian takes exactly the same form in the two coordinate systems:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \frac{\partial^2 u}{\partial \tilde{x}_1^2} + \frac{\partial^2 u}{\partial \tilde{x}_2^2}.$$

This fact is reflected in the observation that the Laplace operator typically occurs in *isotropic* models that have the same properties in all directions.

61.6 The Case of $\mathbb{R}^3$

The *gradient* of a function $u : \mathbb{R}^3 \rightarrow \mathbb{R}$, denoted $\text{grad } u$ or ∇u , is the vector-valued function formed by the set of first order partial derivatives of u , i.e.

$$\text{grad } u = \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \right).$$

For a vector function $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, the divergence $\text{div } u$ is a scalar function defined by

$$\text{div } u = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i},$$

and $\text{rot } u$ is the vector function

$$\text{rot } u = \nabla \times u = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right).$$

We now explain the relation of the operator of rotation $\nabla \times$ in $\mathbb{R}^3$ to the operator of rotation $\nabla \times$ in $\mathbb{R}^2$ introduced above. Consider first a function $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of the form $u = (u_1, u_2, 0)$ with u_1 and u_2 being independent of x_3 so that effectively $u_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $u_i = u_i(x_1, x_2)$ for $i = 1, 2$. We have

$$\nabla \times u = \left(0, 0, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) = (0, 0, \nabla \times (u_1, u_2)).$$

Secondly, if $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ has the form $u = (0, 0, u_3)$ with u_3 independent of x_3 , so that effectively $u_3 : \mathbb{R}^2 \rightarrow \mathbb{R}$, then

$$\nabla \times u = \left(\frac{\partial u_3}{\partial x_2}, -\frac{\partial u_3}{\partial x_1}, 0 \right) = (\nabla \times u_3, 0).$$

We conclude that $\nabla \times u$ for $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\nabla \times u$ for $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, may be viewed as special cases of $\nabla \times u$ for $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

The *Laplacian* Δu of a function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$\Delta u = \nabla \cdot (\nabla u) = \operatorname{div} (\operatorname{grad} u) = \sum_{i=1}^3 \frac{\partial^2 u}{\partial x_i^2}.$$

By direct computation we verify the following identities:

$$\begin{aligned} \nabla \cdot (\nabla \times u) &= 0, \\ \nabla \times (\nabla u) &= 0, \\ \nabla \times (\nabla \times u) &= -\Delta u + \nabla(\nabla \cdot u). \end{aligned} \tag{61.3}$$

61.7 Basic Examples, Again

The function $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $u(x) = \frac{1}{3}x$, satisfies

$$\nabla \cdot u(x) = 1.$$

The function $v : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $v(x) = \frac{1}{2}(-x_2, x_1, 0)$, satisfies

$$\nabla \times v(x) = (0, 0, 1).$$

The function $w : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $w(x) = \frac{1}{6}\|x\|^2$, satisfies

$$\Delta w = 1.$$

We plot these basic examples in Fig. 61.3

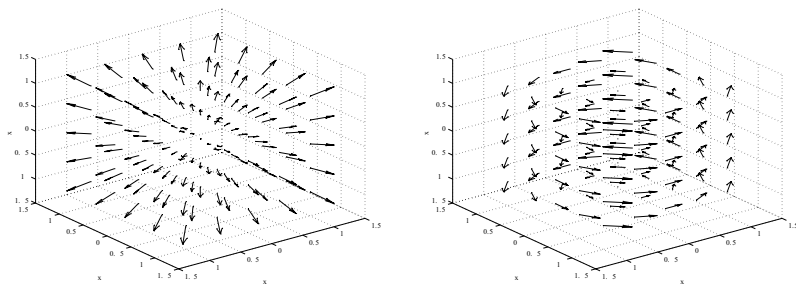


Fig. 61.3. Basic examples in $\mathbb{R}^3$ satisfying $\nabla \cdot u = 1$, $\nabla \times v = 1$

We see again that $u(x)$ “explodes”, $v(x)$ “rotates” along the x_3 axis while the “hump” $w(x)$ is difficult to visualize.

61.8 The Laplacian in Spherical Coordinates

In *spherical coordinates*.

$$x = (x_1, x_2, x_3) = (r \sin(\varphi) \cos(\theta), r \sin(\varphi) \sin(\theta), r \cos(\varphi)),$$

where $r \geq 0$, $0 \leq \theta < 2\pi$ and $0 \leq \varphi < \pi$, the Laplacian is given by

$$\Delta u = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 u}{\partial \varphi^2}. \quad (61.4)$$

The Laplacian is invariant under orthogonal coordinate transformations in $\mathbb{R}^3$.

Example 61.1. Consider the velocity field generated by rotation around a vector $\omega \in \mathbb{R}^3$ with angular speed $\|\omega\|$, that is the vector field

$$v(x) = \omega \times x.$$

We compute

$$\begin{aligned} \nabla \times v(x) &= \nabla \times (\omega_2 x_3 - \omega_3 x_2, \omega_3 x_1 - \omega_1 x_3, \omega_1 x_2 - \omega_2 x_1) \\ &= (2\omega_1, 2\omega_2, 2\omega_3) = 2\omega. \end{aligned}$$

We conclude that the rotation $\nabla \times v(x)$ of a velocity field $v(x)$ generated by a rotation according to a given vector ω is equal to 2ω . This motivates the name of the differential operator $\nabla \times$ as the “rotation”.

Example 61.2. A basic formula of electromagnetics expressing Ampère’s law states that the *magnetic field* H generated by a unit electrical current flowing through the x_3 -axis in the positive direction, is given by

$$H(x) = H(x_1, x_2, x_3) = \frac{1}{2\pi} \frac{(-x_2, x_1, 0)}{x_1^2 + x_2^2} \quad \text{for } x_1^2 + x_2^2 > 0. \quad (61.5)$$

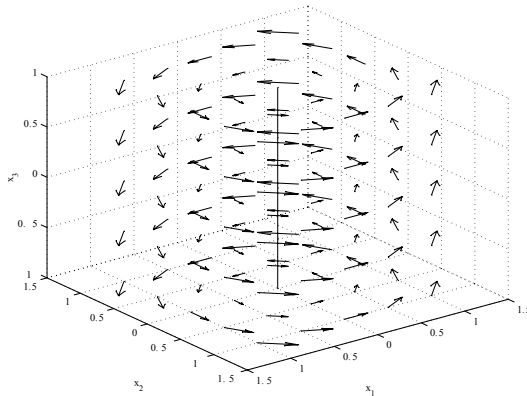


Fig. 61.4. The magnetic field around a current through the x_3 -axis

We compute

$$\nabla \times H(x) = \frac{1}{2\pi} \left(0, 0, \frac{\partial}{\partial x_1} \frac{x_1}{x_1^2 + x_2^2} - \frac{\partial}{\partial x_2} \frac{-x_2}{x_1^2 + x_2^2} \right) = 0 \text{ for } x_1^2 + x_2^2 > 0.$$

Thus $\nabla \times H(x) = 0$ for $x_1^2 + x_2^2 > 0$, which is just *Ampere's Law* $\nabla \times H = J$, where J is the current density, noting that $J(x) = 0$ for $x_1^2 + x_2^2 > 0$, i.e. outside the x_3 -axis. Ampere's Law is one of *Maxwell's equations*. Below we shall show how to interpret the equation $\nabla \times H(x) = J(x)$ for $x_1^2 + x_2^2 = 0$ and motivate the factor $\frac{1}{2\pi}$ in (61.5).

Chapter 61 Problems

61.1. Let $F = (5x_1 - 3x_1x_2 + x_3^2, \sin(x_1)\cos(x_1) + x_1, \sin(x_1)\exp(x_1x_2))$. With $x = (1, 2, 3)$, compute (a) $\nabla \cdot F$, (b) $\nabla \times F$, (c) $\nabla(\nabla \cdot F)$, (d) $\nabla \times (\nabla \times F)$.

61.2. Interpret the expression $(\nabla \times \nabla)u$ in a reasonable way and show that $(\nabla \times \nabla)u = 0$ for any u . Compare with $\nabla \times (\nabla \times u)$.

61.3. Show that for appropriate function u and v

1. $\nabla(uv) = (\nabla u)v + u(\nabla v)$,
2. $\nabla \cdot (uv) = (\nabla u) \cdot v + u(\nabla \cdot v)$,
3. $\nabla \times (uv) = (\nabla u) \times v + u(\nabla \times v)$,
4. $\nabla \cdot (u \times v) = v \cdot (\nabla \times u) - u \cdot (\nabla \times v)$,
5. $\nabla \times (u \times v) = (v \cdot \nabla)u - (\nabla \cdot u)v - (u \cdot \nabla)v + (\nabla \cdot v)u$,
6. $\nabla(u \cdot v) = (u \cdot \nabla)v + (v \cdot \nabla)u + u \times (\nabla \times v) + v \times (\nabla \times u)$.

61.4. Compute $\nabla(r \cdot F(r))$ where $r = \|x\|$.

61.5. Prove that the velocity field $v(x) = \omega \times x$, where $\omega \in \mathbb{R}^3$ is a given vector, satisfies $\nabla \cdot v(x) = 0$. Interpret the result in fluid mechanical terms.

61.6. Prove directly using the Chain rule that the Laplacian in $\mathbb{R}^2$ and $\mathbb{R}^3$ is invariant under rigid coordinate transformations.

61.7. Prove (61.3), (61.2) and (61.4).

61.8. Show that if $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, then $\nabla \times (\nabla \times u) = \text{rot}(\text{rot } u) = -\Delta u$.

61.9. Show that the function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $u(x) = c_1 \log(\|x\|) + c_2$ with c_1 and c_2 constants, is a solution of the Laplace equation $\Delta u(x) = 0$ in $\mathbb{R}^2$ for $x \neq 0$.

61.10. Prove that the function $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $u(x) = c_1 \|x\|^{-1} + c_2$, with c_1 and c_2 constants, is a solution of Laplace's equation $\Delta u(x) = 0$ in $\mathbb{R}^3$ for $x \neq 0$.

61.11. Show that the divergence is invariant under rigid coordinate transformations. Does the rotation have the same property?

All the effects of Nature are only the mathematical consequences of a small number of immutable laws. (Laplace)



<http://www.springer.com/978-3-540-00891-0>

Applied Mathematics: Body and Soul

Calculus in Several Dimensions

Eriksson, K.; Estep, D.; Johnson, C.

2004, XLIV, 426 p., Hardcover

ISBN: 978-3-540-00891-0