3. Fuzzy Investment Analysis Using Capital Budgeting and Dynamic Programming Techniques

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In an uncertain economic decision environment, an expert's knowledge about discounting cash flows consists of a lot of vagueness instead of randomness. Cash amounts and interest rates are usually estimated by using educated guesses based on expected values or other statistical techniques to obtain them. Fuzzy numbers can capture the difficulties in estimating these parameters. In this chapter, the formulas for the analysis of fuzzy present value, fuzzy equivalent uniform annual value, fuzzy future value, fuzzy benefit-cost ratio, and fuzzy payback period are developed and given some numeric examples. Then the examined cash flows are expanded to geometric and trigonometric cash flows and using these cash flows fuzzy present value, fuzzy future value, and fuzzy annual value formulas are developed for both discrete compounding and continuous compounding. The fuzzy dynamic programming is applied to the situation where each investment in the set has the following characteristics: the amount to be invested has several possible values, and the rate of return varies with the amount invested. Each sum that may be invested represents a distinct level of investment, and the investment therefore has multiple levels. A fuzzy present worth based dynamic programming approach is used. A numeric example for a multilevel investment with fuzzy geometric cash flows is given. A computer software named FUZDYN is developed for various problems such as alternatives having different lives, different uniform cash flows, and different ranking methods.

3.1 Introduction

The purpose of this chapter is to develop the fuzzy capital budgeting techniques and a fuzzy dynamic programming method for multilevel investments. The analysis of fuzzy future value, fuzzy present value, fuzzy rate of return, fuzzy benefit/cost ratio, fuzzy payback period, fuzzy equivalent uniform annual value are examined for the case of discrete compounding and continuous compounding.

To deal with vagueness of human thought, Zadeh [3.37] first introduced the fuzzy set theory, which was based on the rationality of uncertainty due to imprecision or vagueness. A major contribution of fuzzy set theory is its capability of representing vague knowledge. The theory also allows mathematical operators and programming to apply to the fuzzy domain.

A fuzzy number is a normal and convex fuzzy set with membership function $\mu_A(x)$ which both satisfies normality: $\mu_A(x)$=1, for at least one $x \in R$ and convexity: $\mu_A(x') \geq \mu_A(x_1) \Lambda \mu_A(x_2)$, where $\mu_A(x) \in [0,1]$ and $\forall x' \in [x_1, x_2]$. '$\Lambda$' stands for the minimization operator.

Quite often in finance future cash amounts and interest rates are estimated. One usually employs educated guesses, based on expected values or other statistical techniques, to obtain future cash flows and interest rates. Statements like *approximately between \$ 12,000 and \$ 16,000* or *approximately between 10% and 15%* must be translated into an exact amount, such as *\$ 14,000* or *12.5%* respectively. Appropriate fuzzy numbers can be used to capture the vagueness of those statements.

A tilde will be placed above a symbol if the symbol represents a fuzzy set. Therefore, $\tilde{P}, \tilde{F}, \tilde{G}, \tilde{A}, \tilde{i}, \tilde{r}$ are all fuzzy sets. The membership functions for these fuzzy sets will be denoted by $\mu(x\left|\tilde{P}\right.), \mu(x\left|\tilde{F}\right.), \mu(x\left|\tilde{G}\right.)$, etc. A fuzzy number is a special fuzzy subset of the real numbers. The extended operations of fuzzy numbers are given in the appendix. A triangular fuzzy number (TFN) is shown in Fig. 3.1 The membership function of a TFN ($\tilde{M}$)defined by

$$\mu(x\left|\tilde{M}\right.) = (m_1, f_1(y\left|\tilde{M}\right.)/m_2, m_2/f_2(y\left|\tilde{M}\right.), m_3) \tag{3.1}$$

where $m_1 \prec m_2 \prec m_3$, $f_1(y\left|\tilde{M}\right.)$ is a continuous monotone increasing function of y for $0 \leq y \leq 1$ with $f_1(0\left|\tilde{M}\right.) = m_1$ and $f_1(1\left|\tilde{M}\right.) = m_2$ and $f_2(y\left|\tilde{M}\right.)$ is a continuous monotone decreasing function of y for $0 \leq y \leq 1$ with $f_2(1\left|\tilde{M}\right.) = m_2$ and $f_2(0\left|\tilde{M}\right.) = m_3$. $\mu(x\left|\tilde{M}\right.)$ is denoted simply as $(m_1/m_2, m_2/m_3)$.

A flat fuzzy number (FFN) is shown in Fig. 3.2 The membership function of a FFN, $\tilde{V}$ is defined by

$$\mu(x\left|\tilde{V}\right.) = (m_1, f_1(y\left|\tilde{V}\right.)/m_2, m_3/f_2(y\left|\tilde{V}\right.), m_4) \tag{3.2}$$

where $m_1 \prec m_2 \prec m_3 \prec m_4$, $f_1(y\left|\tilde{V}\right.)$ is a continuous monotone increasing function of y for $0 \leq y \leq 1$ with $f_1(0\left|\tilde{V}\right.) = m_1$ and $f_1(1\left|\tilde{V}\right.) = m_2$ and $f_2(y\left|\tilde{V}\right.)$ is a continuous monotone decreasing function of y for $0 \leq y \leq 1$ with $f_2(1\left|\tilde{V}\right.) = m_3$ and $f_2(0\left|\tilde{V}\right.) = m_4$. $\mu(y\left|\tilde{V}\right.)$ is denoted simply as $(m_1/m_2, m_3/m_4)$.

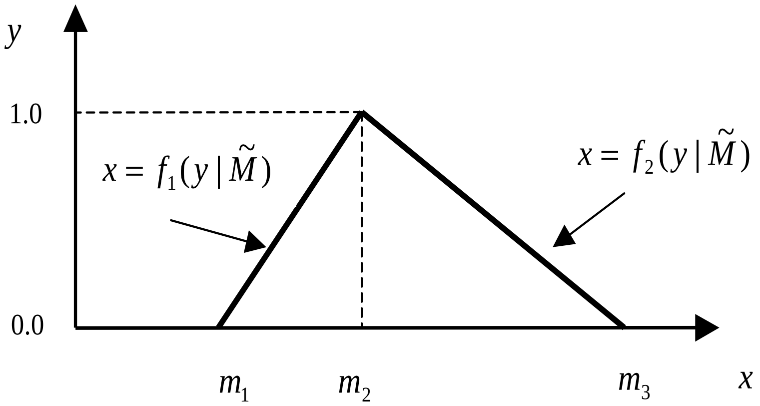

Fig. 3.1. A triangular fuzzy number, $\tilde{M}$

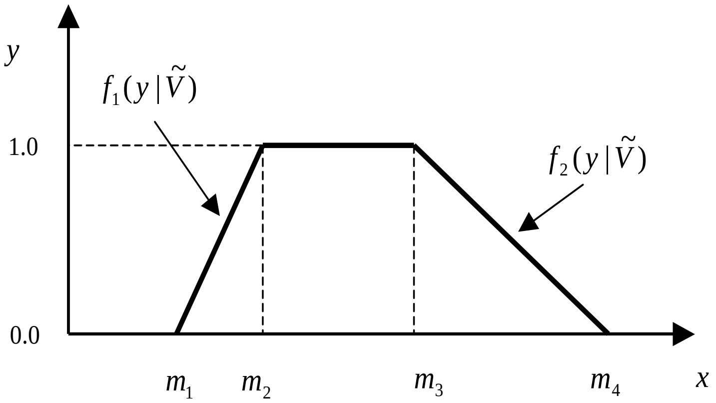

Fig. 3.2. A flat fuzzy number, $\tilde{V}$

The fuzzy sets $\tilde{P}, \tilde{F}, \tilde{G}, \tilde{A}, \tilde{i}, \tilde{r}$ are usually fuzzy numbers but n will be discrete positive fuzzy subset of the real numbers [3.5]. The membership function $\mu(x \mid \tilde{n})$ is defined by a collection of positive integers n_i, $1 \leq i \leq K$, where

$$\mu(x \mid \tilde{n}) = \begin{cases} \mu(n_i \mid \tilde{n}) = \lambda_i, & 0 \leq \lambda_i \leq 1 \\ 0 \quad , & \text{otherwise} \end{cases} \tag{3.3}$$

Karsak [3.26] develops some measures of liquidity risk supplementing fuzzy discounted cash flow analysis. Iwamura and Liu [3.16] develop chance constrained integer programming models for capital budgeting in fuzzy environments. Boussabaine and Elhag [3.4] examine the possible application of the fuzzy set theory to the cash flow analysis in construction projects. Dimitrovski and Matos [3.9] present an approach to including nonstatistical uncertainties in utility economic analysis by modelling uncertain variables with fuzzy numbers. Kuchta [3.28] proposes fuzzy equivalents of all the classical capital budgeting methods.

3.2 Fuzzy Present Value (PV) Method

The present-value method of alternative evaluation is very popular because future expenditures or receipts are transformed into equivalent dollars now. That is, all of the future cash flows associated with an alternative are converted into present dollars. If the alternatives have different lives, the alternatives must be compared over the same number of years.

Chiu and Park [3.8] propose a present value formulation of a fuzzy cash flow. The result of the present value is also a fuzzy number with nonlinear membership function. The present value can be approximated by a TFN. Chiu and Park [3.8]'s formulation is

$$\begin{aligned} P\tilde{V} = \; & \Big[\sum_{t=0}^{n}\Big(\frac{\max(P_t^{l(y)},0)}{\prod_{t'=0}^{t}(1+r_{t'}^{r(y)})} + \frac{\min(P_t^{l(y)},0)}{\prod_{t'=0}^{t}(1+r_{t'}^{l(y)})}\Big), \\ & \sum_{t=0}^{n}\Big(\frac{\max(P_t^{r(y)},0)}{\prod_{t'=0}^{t}(1+r_{t'}^{l(y)})} + \frac{\min(P_t^{r(y)},0)}{\prod_{t'=0}^{t}(1+r_{t'}^{r(y)})}\Big)\Big] \end{aligned} \tag{3.4}$$

where$P_t^{l(y)}$: the left representation of the cash at time t, $P_t^{r(y)}$: the right representation of the cash at time t, $r_t^{l(y)}$: the left representation of the interest rate at time t, $r_t^{r(y)}$: the right representation of the interest rate at time t.

Buckley's [3.5] membership function for $\tilde{P}_n$,

$$\mu(x\,\Big|\tilde{P}_n) = (p_{n1}, f_{n1}(y\,\Big|\tilde{P}_n)/p_{n2}, p_{n2}/f_{n2}(y\,\Big|\tilde{P}_n), p_{n3}) \tag{3.5}$$

is determined by

$$f_{ni}(y|\tilde{P}_n) = f_i(y|\tilde{F})(1+f_k(y|\tilde{r}))^{-n} \tag{3.6}$$

for $i = 1,2$ where $k = i$ for negative $\tilde{F}$ and $k = 3 - i$ for positive $\tilde{F}$. Ward [3.34] gives the fuzzy present value function as

$$P\tilde{V} = (1+r)^{-n}(a,b,c,d) \tag{3.7}$$

where (a, b, c, d) is a trapezoidal fuzzy number.

3.3 Fuzzy Capitalized Value Method

A specialized type of cash flow series is perpetuity, a uniform series of cash flows that continues indefinitely. An infinite cash flow series may be appropriate for such very long-term investment projects as bridges, highways, forest harvesting, or the establishment of endowment funds where the estimated life is 50 years or more.

In the nonfuzzy case, if a present value P is deposited into a fund at interest rate r per period so that a payment of size A may be withdrawn each and every period forever, then the following relation holds between P, A, and r:

$$P = \frac{A}{r} \tag{3.8}$$

In the fuzzy case, lets assume all the parameters as triangular fuzzy numbers: $\tilde{P} = (p_1, p_2, p_3)$ or $\tilde{P} = (((p_2 - p_1)y + p_1), (p_2 - p_3)y + p_3)$ and $\tilde{A} = (a_1, a_2, a_3)$ or $\tilde{A} = (((a_2 - a_1)y + a_1), (a_2 - a_3)y + a_3)$ and $\tilde{r} = (r_1, r_2, r_3)$ or $\tilde{r} = ((r_2 - r_1)y + r_1, (r_2 - r_3)y + r_3)$, where y is the membership degree of a certain point of A and r axis. If $\tilde{A}$ and $\tilde{r}$ are both positive,

$$\tilde{P} = \tilde{A} \emptyset \tilde{r} = (a_1/r_3, a_2/r_2, a_3/r_1) \tag{3.9}$$

or

$$\tilde{P} = (((a_2-a_1)y+a_1)/((r_2-r_3)y+r_3), ((a_2-a_3)y+a_3)/((r_2-r_1)y+r_1)) \tag{3.10}$$

If $\tilde{A}$ is negative and $\tilde{r}$ is positive,

$$\tilde{P} = \tilde{A} \emptyset \tilde{r} = (a_1/r_1, a_2/r_2, a_3/r_3) \tag{3.11}$$

or

$$\tilde{P} = (((a_2-a_1)y+a_1)/((r_2-r_1)y+r_1), ((a_2-a_3)y+a_3)/((r_2-r_3)y+r_3)) \tag{3.12}$$

Now, let $\tilde{A}$ be an expense every nth period forever, with the first expense occurring at n. For example, an expense of $(\$5,000, \$7,000, \$9,000)$ every third year forever, with the first expense occurring at t=3. In this case, the fuzzy effective rate $\tilde{e}$ may be used as in the following:

$$f_i(y|\,\tilde{e}) = (1 + (1/m) f_i(y|\,\tilde{r}'))^m - 1 \tag{3.13}$$

where $i = 1, 2; f_1(y|\,\tilde{e})$:a continuous monotone increasing function of y; $f_2(y|\,\tilde{e})$:a continuous monotone decreasing function of y; m: the number of compounding per period; $\tilde{r}'$: the fuzzy nominal interest rate per period. The membership function of $\tilde{e}$ may be given as

$$\mu(x|\,\tilde{e}) = (e_1, f_1(y|\,\tilde{e})/e_2, e_2/f_2(y|\,\tilde{e}), e_3) \tag{3.14}$$

If $\tilde{A}$ and $f_i(y|\,\tilde{e})$ are both positive,

$$\tilde{P} = \tilde{A}\emptyset\tilde{e} = [((a_2 - a_1)y + a_1)/f_2(y|\,\tilde{e}), ((a_2 - a_3)y + a_3)/f_1(y|\,\tilde{e})] \quad (3.15)$$

If $\tilde{A}$ is negative and $f_i(y|\,\tilde{e})$ is positive,

$$\tilde{P} = \tilde{A}\emptyset\tilde{e} = [((a_2 - a_1)y + a_1)/f_1(y|\,\tilde{e}), ((a_2 - a_3)y + a_3/f_2(y|\,\tilde{e}))] \quad (3.16)$$

$(a_2 - a_1)y + a_1$and $(a_2 - a_3)y + a_3$ can be symbolized as $f_1(y|\,\tilde{a})$ and $f_2(y|\,\tilde{a})$ respectively.

3.4 Fuzzy Future Value Method

The future value (FV) of an investment alternative can be determined using the relationship

$$FV(r) = \sum_{t=0}^{n} P_t(1 + r)^{n-t} \quad (3.17)$$

where *FV(r)* is defined as the future value of the investment using a minimum attractive rate of return *(MARR)* of *r%.* The future value method is equivalent to the present value method and the annual value method.

Chiu and Park's [3.8] formulation for the fuzzy future value has the same logic of fuzzy present value formulation:

$$\{\sum_{t=0}^{n-1}[\max(P_t^{l(y)},0)\prod_{t'=t+1}^{n}(1+r_{t'}^{l(y)}) + \min(P_t^{l(y)},0)\prod_{t'=t+1}^{n}(1+r_{t'}^{r(y)})] + P_n^{l(y)},$$

$$\sum_{t=0}^{n-1}[\max(P_t^{r(y)},0)\prod_{t'=t+1}^{n}(1+r_{t'}^{r(y)}) + \min(P_t^{r(y)},0)\prod_{t'=t+1}^{n}(1+r_{t'}^{l(y)})] + P_n^{r(y)}\}$$

$$(3.18)$$

Buckley's [3.5] membership function $\mu(x|\,\tilde{F})$ is determined by

$$f_i(y \mid \tilde{F}_n) = f_i(y \mid \tilde{P})(1 + f_i(y \mid \tilde{r}))^n \quad (3.19)$$

For the uniform cash flow series, $\mu(x|\,\tilde{F})$ is determined by

$$f_{ni}(y \mid \tilde{F}) = f_i(y \mid \tilde{A})\beta(n, f_i(y|\,\tilde{r})) \quad (3.20)$$

where i =1,2 and $\beta(n,r) = (((1 + r)^n - 1)/r)$ and $\tilde{A} \succ 0$ and $\tilde{r} \succ 0$.

3.5 Fuzzy Benefit/Cost Ratio Method

The benefit/cost ratio (*BCR)* is often used to assess the value of a municipal project in relation to its cost; it is defined as

$$BCR = \frac{B - D}{C} \tag{3.21}$$

where B represents the equivalent value of the benefits associated with the project, D represents the equivalent value of the disbenefits, and C represents the project's net cost. A BCR greater than 1.0 indicates that the project evaluated is economically advantageous. In BCR analysis, costs are not preceded by a minus sign.

When only one alternative must be selected from two or more mutually exclusive (stand-alone) alternatives, a multiple alternative evaluation is required. In this case, it is necessary to conduct an analysis on the incremental benefits and costs. While calculating $\Delta B_{2-1}/\Delta C_{2-1}$ ratio, the costs and benefits of the alternative with higher first cost are subtracted from the costs and benefits of the alternative with smaller first cost. Suppose that there are two mutually exclusive alternatives. In this case, for the incremental BCR analysis ignoring disbenefits the following ratios must be used:

$$\Delta B_{2-1}/\Delta C_{2-1} = \Delta PVB_{2-1}/\Delta PVC_{2-1} \tag{3.22}$$

where PVB: present value of benefits, PVC: present value of costs.
If $\Delta B_{2-1}/\Delta C_{2-1} \geq 1.0$, the alternative 2 is preferred.

In the case of fuzziness, first, it will be assumed that the largest possible value of Alternative 1 for the cash in year t is less than the least possible value of Alternative 2 for the cash in year t. The fuzzy incremental BCR is

$$\Delta\tilde{B}\Big/\Delta\tilde{C} = \left(\frac{\sum_{t=0}^{n}(B_{2t}^{l(y)} - B_{1t}^{r(y)})(1+r^{r(y)})^{-t}}{\sum_{t=0}^{n}(C_{2t}^{r(y)} - C_{1t}^{l(y)})(1+r^{l(y)})^{-t}}, \frac{\sum_{t=0}^{n}(B_{2t}^{r(y)} - B_{1t}^{l(y)})(1+r^{l(y)})^{-t}}{\sum_{t=0}^{n}(C_{2t}^{l(y)} - C_{1t}^{r(y)})(1+r^{r(y)})^{-t}}\right) \tag{3.23}$$

If $\Delta\tilde{B}\Big/\Delta\tilde{C}$ is equal or greater than (1, 1, 1), Alternative 2 is preferred.

In the case of a regular annuity, the fuzzy $\tilde{B}\Big/\tilde{C}$ ratio of a single investment alternative is

$$\tilde{B}/\tilde{C} = \left(\frac{A^{l(y)}\gamma(n, r^{r(y)})}{C^{r(y)}}, \frac{A^{r(y)}\gamma(n, r^{l(y)})}{C^{l(y)}}\right) \tag{3.24}$$

where $\tilde{C}$ is the first cost and $\tilde{A}$ is the net annual benefit, and
$\gamma(n, r) = (((1+r)^n - 1)/(1+r)^n r)$.

The $\Delta\tilde{B}/\Delta\tilde{C}$ ratio in the case of a regular annuity is

$$\varDelta\tilde{B}/\varDelta\tilde{C} = (\frac{(A_2^{l(y)} - A_1^{r(y)})\gamma(n, r^{r(y)})}{C_2^{r(y)} - C_1^{l(y)}}, \frac{(A_2^{r(y)} - A_1^{l(y)})\gamma(n, r^{l(y)})}{C_2^{l(y)} - C_1^{r(y)}}) \quad (3.25)$$

3.6 Fuzzy Equivalent Uniform Annual Value (EUAV) Method

The *EUAV* means that all incomes and disbursements (irregular and uniform) must be converted into an equivalent uniform annual amount, which is the same each period. The major advantage of this method over all the other methods is that it does not require making the comparison over the least common multiple of years when the alternatives have different lives [3.3]. The general equation for this method is

$$EUAV = A = NPV\gamma^{-1}(n, r) = NPV[\frac{(1+r)^n r}{(1+r)^n - 1}] \quad (3.26)$$

where *NPV* is the net present value. In the case of fuzziness, $N\tilde{P}V$ will be calculated and then the fuzzy $EU\tilde{A}V$ ($\tilde{A_n}$) will be found. The membership function $\mu(x \mid \tilde{A}_n)$ for $\tilde{A}_n$ is determined by

$$f_{ni}(y | \tilde{A}_n) = f_i(y | N\tilde{P}V)\gamma^{-1}(n, f_i(y \mid \tilde{r})) \quad (3.27)$$

and *TFN(y)* for fuzzy *EUAV* is

$$\tilde{A}_n(y) = (\frac{NPV^{l(y)}}{\gamma(n, r^{l(y)})}, \frac{NPV^{r(y)}}{\gamma(n, r^{r(y)})}) \quad (3.28)$$

3.7 Fuzzy Payback Period (FPP) Method

The payback period method involves the determination of the length of time required to recover the initial cost of investment based on a zero interest rate ignoring the time value of money or a certain interest rate recognizing the time value of money. Let C_{j0} denote the initial cost of investment alternative j, and R_{jt} denote the net revenue received from investment j during period t. Assuming no other negative net cash flows occur, the smallest value of m_j ignoring the time value of money such that

$$\sum_{t=1}^{m_j} R_{jt} \geq C_{j0} \quad (3.29)$$

or the smallest value of m_j recognizing the time value of money such that

$$\sum_{t=1}^{m_j} R_{jt}(1+r)^{-t} \geq C_{j0} \quad (3.30)$$

defines the payback period for the investment j. The investment alternative having the smallest payback period is the preferred alternative. In the case of fuzziness, the smallest value of m_j ignoring the time value of money such that

$$(\sum_{t=1}^{m_j} r_{1jt}, \sum_{t=1}^{m_j} r_{2jt}, \sum_{t=1}^{m_j} r_{3jt}) \geq (C_{1j0}, C_{2jo}, C_{3j0}) \tag{3.31}$$

and the smallest value of m_j recognizing the time value of money such that

$$\begin{aligned}(\sum_{t=1}^{m_j} \frac{R_{jt}^{l(y)}}{(1+r^{r(y)})^t}, \sum_{t=1}^{m_j} \frac{R_{jt}^{r(y)}}{(1+r^{l(y)})^t}) \geq \\ ((C_{2j0} - C_{1j0})y + C_{1j0}, (C_{2j0} - C_{3j0})y + C_{3j0})\end{aligned} \tag{3.32}$$

defines the payback period for investment j, where r_{kjt} : the kth parameter of a triangular fuzzy R_{jt}; C_{kj0}: the kth parameter of a triangular fuzzy C_{j0}; $R_{jt}^{l(y)}$: the left representation of a triangular fuzzy R_{jt}; $R_{jt}^{r(y)}$: the right representation of a triangular fuzzy R_{jt}. If it is assumed that the discount rate changes from one period to another, $(1+r^{r(y)})^t$ and $(1+r^{l(y)})^t$ will be $\prod_{t'=1}^{t}(1+r_{t'}^{r(y)})$ and $\prod_{t'=1}^{t}(1+r_{t'}^{l(y)})$ respectively.

3.8 Ranking Fuzzy Numbers

It is now necessary to use a ranking method to rank the TFNs such as Chiu and Park's [3.8], Chang's [3.6] method, Dubois and Prade's [3.10] method, Jain's [3.17] method, Kaufmann and Gupta's [3.27] method, Yager's [3.36] method. These methods may give different ranking results and most methods are tedious in graphic manipulation requiring complex mathematical calculation. In the following, three of the methods, which do not require graphical representations, are given.

Kaufmann and Gupta [3.27] suggest three criteria for ranking TFNs with parameters (a,b,c). The dominance sequence is determined according to priority of:

1. Comparing the ordinary number (a+2b+c)/4
2. Comparing the mode, (the corresponding most promise value), b, of each TFN.
3. Comparing the range, c-a, of each TFN.

The preference of projects is determined by the amount of their ordinary numbers. The project with the larger ordinary number is preferred. If the ordinary numbers are equal, the project with the larger corresponding most promising value is preferred. If projects have the same ordinary number and most promising value, the project with the larger range is preferred.

Liou and Wang [3.32] propose the total integral value method with an index of optimism $\omega \in [0,1]$. Let $\tilde{A}$ be a fuzzy number with left membership function $f_{\tilde{A}}^{L}$ and right membership function $f_{\tilde{A}}^{R}$. Then the total integral value is defined as:

$$E_\omega\left(\tilde{A}\right) = \omega E_R\left(\tilde{A}\right) + (1-\omega)\, E_L\left(\tilde{A}\right) \tag{3.33}$$

where

$$E_R\left(\tilde{A}\right) = \int_{\alpha}^{\beta} x f_{\tilde{A}}^{R}(x)\, dx \tag{3.34}$$

$$E_L\left(\tilde{A}\right) = \int_{\gamma}^{\delta} x f_{\tilde{A}}^{L}(x)\, dx \tag{3.35}$$

where $-\infty \le \alpha \le \beta \le \gamma \le \delta \ge +\infty$ and a trapezoidal fuzzy number is denoted by $(\alpha, \beta, \gamma, \delta)$. For a triangular fuzzy number,$\tilde{A} = (a, b, c)$,

$$E_\omega\left(\tilde{A}\right) = \frac{1}{2}\left[\omega\,(a+b) + (1-\omega)\,(b+c)\right] \tag{3.36}$$

and for a trapezoidal fuzzy number, $\tilde{B} = (\alpha, \beta, \gamma, \delta)$,

$$E_\omega\left(\tilde{B}\right) = \frac{1}{2}\left[\omega\,(\gamma+\delta) + (1-\omega)\,(\alpha+\beta)\right] \tag{3.37}$$

Chiu and Park's [3.8] weighted method for ranking TFNs with parameters (a, b, c) is formulated as

$$((a+b+c)/3) + wb$$

where w is a value determined by the nature and the magnitude of the most promising value. The preference of projects is determined by the magnitude of this sum.

The computer software developed by the authors, FUZDYN, has the ability to use many ranking methods that are tedious in graphic manipulation requiring complex mathematical calculation. To select the ranking method required by the decision maker, the following form in Fig. 3.3 is used:

3.9 Fuzzy Internal Rate of Return (IRR) Method

The *IRR* method is referred to in the economic analysis literature as the discounted cash flow rate of return, internal rate of return, and the true rate of return. The internal rate of return on an investment is defined as the rate

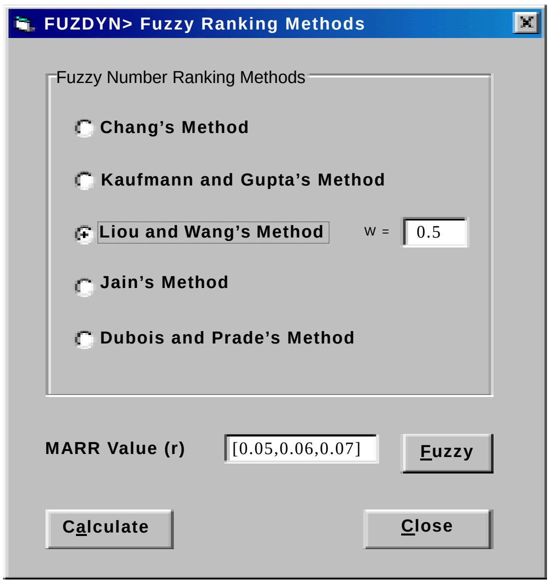

Fig. 3.3. The form of fuzzy ranking methods

of interest earned on the unrecovered balance of an investment. Letting r^* denote the rate of return, the equation for obtaining r^* is

$$\sum_{t=1}^{n} P_t(1+r^*)^{-t} - FC = 0 \tag{3.38}$$

where P_t is the net cash flow at the end of period t.

Assume the cash flow $\tilde{F} = \tilde{F}_0, \tilde{F}_1, ..., \tilde{F}_N$ is fuzzy. $\tilde{F}_n$ is a negative fuzzy number and the other $\tilde{F}_i$ may be positive or negative fuzzy numbers. The fuzzy $IRR(\tilde{F}, n)$ is a fuzzy interest rate $\tilde{r}$ that makes the present value of all future cash amounts equal to the initial cash outlay. Therefore, the fuzzy number $\tilde{r}$ satisfies

$$\sum_{i=1}^{n} PV_{k(i)}(\tilde{F}_i, \tilde{r}) = -\tilde{F}_0 \tag{3.39}$$

where $\sum$ is fuzzy addition, $k(i)$=1 if $\tilde{F}_i$ is negative and $k(i)$=2 if $\tilde{F}_i$ is positive.

Buckley [3.5] shows that such simple fuzzy cash flows may not have a fuzzy IRR and concludes that the IRR technique does not extend to fuzzy cash flows. Ward [3.34] considers Eq. (3.38) and explains that such a procedure can not be applied for the fuzzy case because the right hand side of Eq. (3.38) is fuzzy, 0 is crisp, and an equality is impossible.

3.10 An Expansion to Geometric and Trigonometric Cash Flows

When the value of a given cash flow differs from the value of the previous cash flow by a constant percentage, $g\%$, then the series is referred to as a *geometric series*. If the value of a given cash flow differs from the value of the previous cash flow by a sinusoidal wave or a cosinusoidal wave, then the series is referred to as a *trigonometric series* .

3.10.1 Geometric Series–Fuzzy Cash Flows in Discrete Compounding

The present value of a crisp geometric series is given by

$$P = \sum_{n=1}^{N} F_1(1+g)^{n-1}(1+i)^{-n} = \frac{F_1}{1+g}\sum_{n=1}^{N}(\frac{1+g}{1+i})^n \tag{3.40}$$

where F_1 is the first cash at the end of the first year. When this sum is made, the following present value equation is obtained:

$$P = \begin{cases} F_1[\frac{1-(1+g)^N(1+i)^{-N}}{i-g}], & i \neq g \\ \frac{NF_1}{1+i} \quad , & i = g \end{cases} \tag{3.41}$$

and the future value is

$$F = \begin{cases} F_1[\frac{(1+i)^N-(1+g)^N}{i-g}], & i \neq g \\ NF_1(1+i)^{N-1} \quad , & i = g \end{cases} \tag{3.42}$$

In the case of fuzziness, the parameters used in Eq.(3.40) will be assumed to be fuzzy numbers, except project life. Let $\gamma(i,g,N) = [\frac{1-(1+g)^N(1+i)^{-N}}{i-g}], i \neq g$. As it is in Fig. 3.1 and Fig. 3.2, when k=1, the left side representation will be depicted and when k=2, the right side representation will be depicted. In this case, for $i \neq g$

$$f_{Nk}(y \mid \tilde{P}_N) = f_k(y \mid \tilde{F}_1)\gamma(f_{3-k}(y|\tilde{i}), f_{3-k}(y \mid \tilde{g}), N) \tag{3.43}$$

In Eq.(3.43), the least possible value is calculated for $k = 1$ and $y = 0$; the largest possible value is calculated for $k = 2$ and $y = 0$; the most promising value is calculated for k= 1 or $k = 2$ and $y = 1$.

To calculate the future value of a fuzzy geometric cash flow, let $\zeta(i,g,N) = [\frac{(1+i)^N-(1+g)^N}{i-g}], i \neq g$. Then the fuzzy future value is

$$f_{Nk}(y|\tilde{F}_N) = f_k(y \mid \tilde{F}_1)\zeta(f_k(y \mid \tilde{i}), f_k(y \mid \tilde{g}), N) \tag{3.44}$$

In Eq. (3.44), the least possible value is calculated for $k = 1$ and $y = 0$; the largest possible value is calculated for $k = 2$ and $y = 0$; the most promising value is calculated for $k = 1$ or $k = 2$ and $y = 1$. This is also valid for the formulas developed at the rest of the paper.

The fuzzy uniform equivalent annual value can be calculated by using Eq. (3.45):

$$f_{Nk}(y \mid \tilde{A}) = f_k(y \mid \tilde{P}_N)\vartheta(f_k(y \mid \tilde{i}), N) \tag{3.45}$$

where $\vartheta(i,N) = [\frac{(1+i)^N i}{(1+i)^N - 1}]$ and $f(y \mid \tilde{P}_N)$ is the fuzzy present value of the fuzzy geometric cash flows.

3.10.2 Geometric Series–Fuzzy Cash Flows in Continuous Compounding

In the case of crisp sets, the present and future values of discrete payments are given by Eq.(3.46) and Eq.(3.47) respectively:

$$P = \begin{cases} F_1[\frac{1-e^{(g-r)N}}{e^r - e^g}], & r \neq g \\ \frac{NF_1}{e^r}, & g = e^r - 1 \end{cases} \tag{3.46}$$

$$F = \begin{cases} F_1[\frac{e^{rN} - e^{gN}}{e^r - e^g}], & r \neq g \\ NF_1 e^{r(N-1)}, & g = e^r - 1 \end{cases} \tag{3.47}$$

and the present and future values of continuous payments are given by Eq.(3.48) and Eq.(3.49) respectively:

$$P = \begin{cases} F_1[\frac{1-e^{N(g-r)}}{r-g}], & r \neq g \\ \frac{NF_1}{1+r}, & r = g \end{cases} \tag{3.48}$$

$$F = \begin{cases} F_1[\frac{e^{rN} - e^{gN}}{r-g}], & r \neq g \\ \frac{NF_1 e^{rN}}{1+r}, & r = g \end{cases} \tag{3.49}$$

The fuzzy present and future values of the fuzzy geometric discrete cash flows in continuous compounding can be given as in Eq.(3.50) and Eq.(3.51) respectively:

$$f_{Nk}(y \mid \tilde{P}_N) = f_k(y \mid \tilde{F}_1)\beta(f_{3-k}(y \mid \tilde{r}), f_{3-k}(y \vert \tilde{g}), N) \tag{3.50}$$

$$f_{Nk}(y \mid \tilde{F}) = f_k(y \mid \tilde{F}_1)\tau(f_k(y \mid \tilde{r}), f_k(y \mid \tilde{g}), N) \tag{3.51}$$

where $\beta(r, g, N) = \frac{1-e^{(g-r)N}}{e^r - e^g}, r \neq g$ for present value and $\tau(r, g, N) = \frac{e^{rN} - e^{gN}}{e^r - e^g}, r \neq g$ for future value.

The fuzzy present and future values of the fuzzy geometric continuous cash flows in continuous compounding can be given as in Eq.(3.52) and Eq.(3.53) respectively:

$$f_{Nk}(y|\tilde{P}_N) = f_k(y|\tilde{F}_1)\eta(f_{3-k}(y \mid \tilde{r}), f_{3-k}(y \mid \tilde{g}), N) \tag{3.52}$$

$$f_{Nk}(y \mid \tilde{F}_N) = f_k(y \mid \tilde{F}_1)\upsilon(f_k(y \mid \tilde{r}), f_k(y \mid \tilde{g}), N) \tag{3.53}$$

where $\eta(r, g, N) = \frac{1-e^{(g-r)N}}{r-g}$, $\upsilon(r, g, N) = \frac{e^{rN} - e^{gN}}{r-g}, r \neq g$

3.10.3 Trigonometric Series–Fuzzy Continuous Cash Flows

In Fig. 3.4, the function of the semi-sinusoidal wave cash flows is depicted. This function, $h(t)$, is given by Eq.(3.54) in the crisp case:

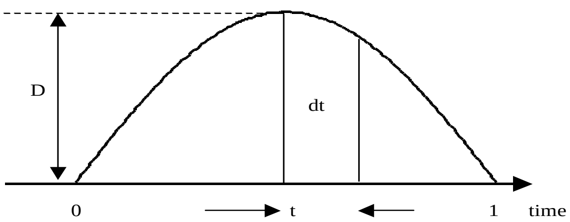

Fig. 3.4. Semi sinusoidal wave cash flow function

$$h(t) = \begin{cases} D\sin(\pi t), & 0 \leq t \leq 1 \\ 0 \quad , & \text{otherwise} \end{cases} \tag{3.54}$$

The future value of a semi-sinusoidal cash flow for T=1 and g is defined by Eq. (3.55) :

$$V(g, 1) = D\int_0^1 e^{r(1-t)}\sin(\pi t)\,dt = D[\frac{\pi(2+g)}{r^2+\pi^2}] \tag{3.55}$$

Fig. 3.5 shows the function of a cosinusoidal wave cash flow. This function, $h(t)$, is given by Eq.(3.56):

$$h(t) = \begin{cases} D(\cos(2\pi t) + 1), & 0 \leq t \leq 1 \\ 0 \quad , & \text{otherwise} \end{cases} \tag{3.56}$$

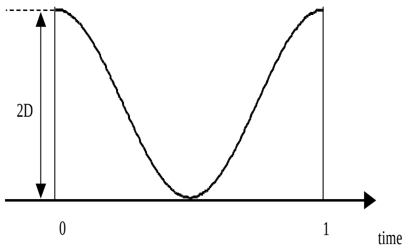

Fig. 3.5. Cosinusoidal wave cash flow function

The future value of a cosinusoidal cash flow for T=1 and g is defined as

$$V(g,1) = D\int_0^1 e^{r(1-t)}(\cos(2\pi\, t)+1)dt = D[\frac{gr}{r^2+4\pi^2}+\frac{g}{r}] \qquad (3.57)$$

Let the parameters in Eq. (3.55), r and g, be fuzzy numbers. The future value of the semi-sinusoidal cash flows as in Fig. 3.6 is given by

$$f_{Nk}(y \mid \tilde{F}_N) = f_k(y \mid \tilde{D})\phi(f_{3-k}(y \mid \tilde{r}), f_k(y \mid \tilde{g}))\varphi(f_k(y \mid \tilde{r}), N) \qquad (3.58)$$

where $\phi(r,g) = \pi(2+g)/(r^2+\pi^2)$, $\varphi(r,N) = (e^{rN}-1)/(e^r-1)$.

The present value of the semi-sinusoidal cash flows is given by Eq. (3.59):

$$f_{Nk}(y \mid \tilde{P}_N) = f_k(y \mid \tilde{D})\phi(f_{3-k}(y \mid \tilde{r}), f_k(y \mid \tilde{g}))\psi(f_{3-k}(y \mid \tilde{r}), N) \qquad (3.59)$$

where $\psi(r,N) = (e^{rN}-1)/((e^r-1)e^{rN})$.

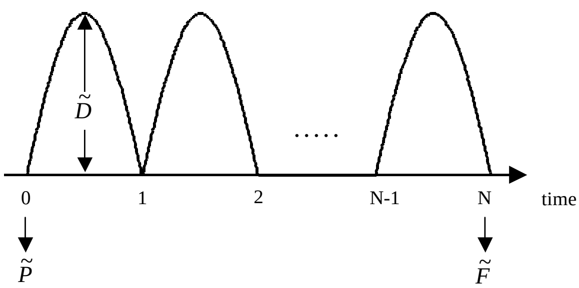

Fig. 3.6. Fuzzy sinusoidal cash flow diagram

3.10.4 Numeric Example I

The continuous profit function of a firm producing ice cream during a year is similar to semi-sinusoidal wave cash flows whose g is around 4%. The maximum level in \$ of the ice-cream sales is between the end of June and the beginning of July. The profit amount obtained on this point is around

\$120,000. The firm manager uses a minimum attractive rate of return of around 10%, compounded continuously and he wants to know the present worth of the 10-year profit and the possibility of having a present worth of \$1,500,000.

'Around \$120,000' can be represented by a TFN, (\$100,000;\$120,000;\$130,000). 'Around 10%' can be represented by a TFN, (9%;10%;12%). 'Around 4%' can be represented by a TFN, (3%;4%;6%)

$f_2(y|\tilde{r}) = 0.12 - 0.02y$ $\quad$ $f_1(y|\tilde{r}) = 0.09 + 0.01y$

$f_1(y|\tilde{D}) = 100,000 + 20,000y$ $\quad$ $f_2(y|\tilde{D}) = 130,000 - 10,000y$

$f_{10,1}(y|\tilde{P_{10}}) = f_1(y|\tilde{D})\Phi(f_2(y|\tilde{r}), f_1(y|\tilde{g}))\Psi(f_2(y|\tilde{r}), 10)$

$f_{10,2}(y|\tilde{P_{10}}) = f_2(y|\tilde{D})\Phi(f_1(y|\tilde{r}), f_2(y|\tilde{g}))\Psi(f_1(y|\tilde{r}), 10)$

$f_1(y|\tilde{g}) = 0.03 + 0.01y$ $\quad$ $f_2(y|\tilde{g}) = 0.06 - 0.02y$

$$f_{10,1}(y|\tilde{P}_{10}) = (100,000 + 20,000y) \times \left[\frac{\pi(2.03 + 0.01y)}{[(0.12 - 0.02y)^2 + \pi^2]}\right]\left[\frac{e^{(0.12-0.02y)10} - 1}{e^{0.12-0.02y} - 1}\right]\frac{1}{e^{(0.12-0.02y)10}}$$

$$f_{10,2}(y|\tilde{P}_{10}) = (130,000 - 10,000y) \times \left[\frac{\pi(2.06 - 0.02y)}{[(0.09 + 0.01y)^2 + \pi^2]}\right]\left[\frac{e^{(0.09+0.01y)10} - 1}{e^{0.09+0.01y} - 1}\right]\frac{1}{e^{(0.09+0.01y)10}}$$

For $y = 1$, the most possible value is $f_{10,1}(y|\tilde{P}_{10}) = f_{10,2}(y|\tilde{P}_{10}) = \$467,870.9$.
For $y = 0$, the smallest possible value is $f_{10,1}(y|\tilde{P}_{10}) = \$353,647.1$.
For $y = 0$, the largest possible value is $f_{10,2}(y|\tilde{P}_{10}) = \$536,712.8$.

It seems to be impossible to have a present worth of \$1,500,000.

The present and future values of the fuzzy cosinusoidal cash flows as in Fig. 3.7 can be given by Eq. (3.60) and Eq. (3.61) respectively:

$$f_{Nk}(y \mid \tilde{P}_N) = f_k(y \mid \tilde{D})\xi(f_{3-k}(y \mid \tilde{r}), f_k(y \mid \tilde{g}))\Psi(f_{3-k}(y \mid \tilde{r}), N) \qquad (3.60)$$

where $\xi(r, g) = [\frac{gr}{r^2+4\pi^2} + \frac{g}{r}]$ and the fuzzy future value is

$$f_{Nk}(y \mid \tilde{F}_N) = f_k(y \mid \tilde{D})\xi(f_{3-k}(y \mid \tilde{r}), f_k(y \mid \tilde{g}))\varphi(f_k(y \mid \tilde{r}), N) \qquad (3.61)$$

3.10.5 Numeric Example II

The continuous cash flows of a firm is similar to cosinusoidal cash flows. The maximum level of the cash flows during a year is around \$780,000. The fuzzy nominal cost of capital is around 8% per year. The fuzzy geometric growth rate of the cash flows is around 4% per year. Let us compute the future worth of a 10 year working period.

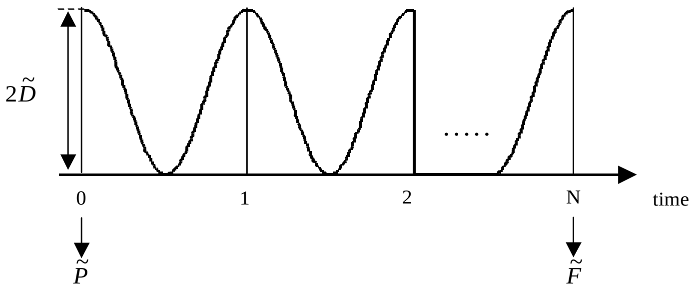

Fig. 3.7. Fuzzy sinusoidal cash flow diagram

Let us define
$\tilde{D} = (\$300,000; \$390,000; \$420,000)$,$f_1(y|\tilde{D}) = 350,000 + 40,000y$,
$f_2(y|\tilde{D}) = 420,000 - 30,000y$, $\tilde{r} = (6\%; 8\%; 10\%)$, $f_1(y|\tilde{r}) = 0.06 + 0.02y$,
$f_2(y|\tilde{r}) = 0.10 - 0.02y$, $\tilde{g} = (3\%; 4\%; 5\%)$, $f_1(y|\tilde{g}) = 0.03 + 0.01y$,
$f_2(y|\tilde{g}) = 0.05 - 0.01y$

$$f_{10,1}(y|\tilde{F}_{10}) = (350,000 + 40,000y) \times \left[\frac{(0.03 + 0.01y)(0.10 - 0.02y)}{(0.10 - 0.02y)^2 + 4\pi^2} + \frac{0.03 + 0.01y}{0.10 - 0.02y}\right]\left[\frac{e^{(0.06+0.02y)10} - 1}{e^{0.06+0.02y} - 1}\right]$$

$$f_{10,2}(y|\tilde{F}_{10}) = (420,000 - 30,000y) \times \left[\frac{(0.05 - 0.01y)(0.06 + 0.02y)}{(0.06 + 0.02y)^2 + 4\pi^2} + \frac{0.05 - 0.01y}{0.06 + 0.02y}\right]\left[\frac{e^{(0.10-0.02y)10} - 1}{e^{0.10-0.02y} - 1}\right]$$

For $y = 1$, the most possible value is $f_{10,1}(y|\tilde{F}_{10}) = \$2,869,823.5$.
For $y = 0$, the smallest possible value is $f_{10,1}(y|\tilde{F}_{10}) = \$1,396,331.5$.
For $y = 0$, the largest possible value is $f_{10,2}(y|\tilde{F}_{10}) = \$5,718,818.9$.

3.11 Dynamic Programming for Multilevel Investment Analysis

Dynamic programming is a technique that can be used to solve many optimization problems. In most applications, dynamic programming obtains solutions by working backward from the end of a problem toward the beginning, thus breaking up a large, unwieldy problem into a series of smaller, more tractable problems. The characteristics of dynamic programming applications are [3.35]

- The problem can be divided into stages with a decision required at each stage.

– Each stage has a number of states associated with it.
– The decision chosen at any stage describes how the state at the current stage is transformed into the state at the next stage.
– Given the current state, the optimal decision for each of the remaining stages must not depend on previously reached states or previously chosen decisions.
– If the states for the problem have been classified into one of T stages, there must be a recursion that relates the cost or reward earned during stages t, $t+1,\ldots,T$ to the cost or reward earned from stages $t+1$, $t+2,\ldots,T$.

The dynamic programming recursion can often be written in the following form. For a min problem with fixed output:

$$f_t\left(i\right) = \min\{\ (\text{cost during stage t}) + f_{t+1}\left(\text{new state at stage t} + 1\right)\ \} \tag{3.62}$$

and for a max problem with fixed input, it is

$$f_t\left(i\right) = \max\{\ (\text{benefits during state t}) + f_{t+1}\left(\text{new state at stage t} + 1\right)\ \} \tag{3.63}$$

or for a max problem neither input nor output fixed, it is

$$f_t\left(i\right) = \max\left\{('\text{benefits} - \text{costs}'\ \text{during state t}) + f_{t+1}\left(\text{new state at stage t} + 1\right)\right\} \tag{3.64}$$

where the minimum in Eq. (3.62) or maximum in Eq. (3.63) and Eq. (3.64) is over all decisions that are allowable, or feasible, when the state at stage t is i. In Eq. (3.62), $f_t\left(i\right)$ is the minimum cost and in Eq. (3.63) the maximum benefit incurred from stage t to the end of the problem, given that at stage t the state is i.

In deterministic dynamic programming, a specification of the current state and current decision is enough to tell us with certainty the new state and costs during the current stage. In many practical problems, these factors may not be known with certainty, even if the current state and decision are known. When we use dynamic programming to solve problems in which the current period's cost or the next period's state is random, we call these problems *probabilistic dynamic programming problems (PDPs).* In a *PDP*, the decision-maker's goal is usually to minimize expected cost incurred or to maximize expected reward earned over a given time horizon.

Many *PDPs* can be solved using recursions of the following forms.
For min problems:

$$f_t\left(i\right) = \min_a\Big[(\text{ expected cost during stage t}|\,\text{i, a}\ ) + \sum_j p\left(j|\,i,a,t\right) f_{t+1}\left(j\right)\Big] \tag{3.65}$$

and for max problems:

$$f_t(i) = \max_a \Big[(\text{expected reward during stage } \ t|\,\mathrm{i,a}) + \sum_j p\,(j|\, i, a, t) f_{t+1}\,(j) \Big] \tag{3.66}$$

where

i : the state at the beginning of stage t.

a: all actions that are feasible when the state at the beginning of stage t is i.

$p\,(j|\,i, a, t)$:the probability that the next period's state will be j, given that the current state is i and action a is chosen.

In the formulations above, we assume that benefits and costs received during later years are weighted the same as benefits and costs received during earlier years. But later benefits and costs should be weighted less than earlier benefits and costs. We can incorporate the time value of money into the dynamic programming recursion in the following way. For a max problem with neither input nor output fixed,

$$f_t(i) = \max \{ ('\text{benefits} - \text{costs}' \text{ during state t}) + \tfrac{1}{(1+r)} f_{t+1}\,(\text{new state at stage t} + 1) \} \tag{3.67}$$

where r is the time value of money.

Many capital budgeting problems allow of a *dynamic* formulation. There may actually be several decision points, but even if this is not so, if the decision problem can be divided up into *stages* then a discrete dynamic expression is possible. Many problems allow of either static or dynamic expression. The choice of form would be up to the problem solver. Characteristically, a dynamic economizing model allocates scarce resources between alternative uses between initial and terminal times. In the case of equal-life multilevel investments, each investment in the set has the following characteristic: the amount to be invested has several *possible* values, and the rate of return varies with the amount invested. Each sum that may be invested represents a distinct *level* of investment, and the investment therefore has multiple levels. Examples of multilevel investments may be the purchase of labor-saving equipment where several types of equipment are available and each type has a unique cost. The level of investment in labor-saving equipment depends on the type of equipment selected. Another example is the construction and rental of an office building, where the owner-builder has a choice concerning the number of stories the building is to contain [3.29].

3.11.1 Fuzzy Dynamic Programming: Literature Review

Fuzzy dynamic programming has found many applications to real-world problems: Health care, flexible manufacturing systems, integrated regional development, transportation networks and transportation of hazardous waste, chemical engineering, power and energy systems, water resource systems.

Li and Lai [3.31] develop a new fuzzy dynamic programming approach to solve hybrid multi-objective multistage decision-making problems. They present a methodology of fuzzy evaluation and fuzzy optimization for hybrid multi-objective systems, in which the qualitative and quantitative objectives are synthetically considered. Esogbue [3.11] presents the essential elements of fuzzy dynamic programming and computational aspects as well as various key real world applications. Fu and Wang [3.13] establish a model in the framework of fuzzy project network by team approach under the consideration of uncertain resource demand and the budget limit. The model is transformed into a classical linear program formula and its results show that the cause-effect relations of insufficient resources or over due of the project is identified for better management. Lai and Li [3.30] develop a new approach using dynamic programming to solve the multiple-objective resource allocation problem. There are two key issues being addressed in the approach. The first one is to develop a methodology of fuzzy evaluation and fuzzy optimization for multiple-objective systems. The second one is to design a dynamic optimization algorithm by incorporating the method of fuzzy evaluation and fuzzy optimization with the conventional dynamic programming technique. Esogbue [3.12] considers both time and space complexity problems associated with the fuzzy dynamic programming model. Kahraman *et al.* [3.23] use fuzzy dynamic programmingto combine equal-life multilevel investments. Huang *et al.*[3.14] develop a fuzzy dynamic programmingapproach to solve the direct load control problem of the air conditioner loads. Kacprzyk and Esogbue [3.18] survey major developments and applications of fuzzy dynamic programming which is advocated as a promising attempt at making dynamic programming models more realistic by a relaxation of often artificial assumptions of precision as to the constraints, goals, states and their transitions, termination time, etc. Chin [3.7] proposes a new approach using fuzzy dynamic programmingto decide the optimal location and size of compensation shunt capacitors for distribution systems with harmonic distortion. The problem is formulated as a fuzzy dynamic programming of the minimization of real power loss and capacitor cost under the constraints of voltage limits and total harmonic distortion. Hussein and Abo-Sinna [3.15] propose a new approach using fuzzy dynamic programmingto solve the multiple criteria resource allocation problems. They conclude that solutions obtained by the approach are always efficient; hence an “optimal” compromise solution can be introduced. Berenji [3.2] develops a new algorithm called Fuzzy Q-Learning, which extends Watkin’s Q-Learning method. It is used for decision processes in which the goals and/or the constraints, but not necessarily the system under control, are fuzzy in nature. He shows that fuzzy Q-Learning provides an alternative solution simpler than the Bellman-Zadeh’s[3.1] fuzzy dynamic programming approach.

3.11.2 Crisp Dynamic Programming for Multilevel Investments

The solution of a dynamic programming problem of multilevel investments consists of the following steps:

1. Devise all possible investments that encompass plans A and B alone, applying an upper limit of \$$L$ to the amount invested. Compound the corresponding annual dividends. Let Q denote the amount of capital to be allocated to the combination of plans A and B, where Q can range from \$ X to \$ kX where k=1, 2, 3… Although both plans A and B fall within our purview in this step, it is understood that Q can be allocated to A alone or to B alone.
2. Identify the most lucrative combination of Plans A and B corresponding to every possible value of Q.
3. Devise all possible investments that encompass plans A, B, and C, and identify the most lucrative one.

Now let's consider the selection among multilevel investments when crisp cash flows are known. In other words, let's deal with the problem from capital budgeting viewpoint.

Newnan [3.33] shows that independent proposals competing for funding should be picked according to their IRR values- monotonically from highest to lowest. Ranking on present-worth values (computed at a specified MARR) may not give the same results. Given a specified minimum attractive rate of return(MARR) value, Newnan [3.33] suggests that proposals be ranked on the basis of

$$\text{Ranking ratio} = \frac{\text{Proposal PW(MARR)}}{\text{Proposal first cost}} \tag{3.68}$$

where PW is the present worth of a proposal. The larger ratio indicates the better proposal.

Now assume that cash flows for l independent proposals that have passed a screening based on a MARR of $r\%$ are given in Table 3.1 and we have a \$$L$ capital limitation. The problem is which combination of proposals should be funded. The solution consists of the following steps:

1. Devise all possible investments that encompass proposals 1 and 2 alone, applying an upper limit of \$$L$ to the amount invested. Compute the present worth of each proposal in the possible combinations using the discounted cash flow techniques. \$$L$ can be allocated to proposal 1 alone or to proposal 2 alone or to any other combination.
2. Identify the most lucrative combination of proposals 1 and 2 corresponding to every possible value of \$$L$, using the ranking ratio in Eq.(3.68).
3. Devise all possible investments that encompass proposals 1, 2, and 3, and identify the most lucrative one as in step 2.

Table 3.1. Cash flows for l independent proposals

End-of-period-cash-flow, $						
Proposal	Investment,$	Period 1	Period 2	Period 3	...	Period n
1	$X	CF_{11}^{1}	CF_{12}^{1}	CF_{13}^{1}	...	CF_{1n}^{1}
	$2X	CF_{11}^{2}	CF_{12}^{2}	CF_{13}^{2}	...	CF_{1n}^{2}
	$3X	CF_{11}^{3}	CF_{12}^{3}	CF_{13}^{3}	...	CF_{1n}^{3}
	...	...	...	...	...	...
	$$k$X	CF_{11}^{k}	CF_{12}^{k}	CF_{13}^{k}	...	CF_{1n}^{k}
2	$X	CF_{21}^{1}	CF_{22}^{1}	CF_{23}^{1}	...	CF_{2n}^{1}
	$2X	CF_{21}^{2}	CF_{22}^{2}	CF_{23}^{2}	...	CF_{2n}^{2}
	$3X	CF_{21}^{3}	CF_{22}^{3}	CF_{23}^{3}	...	CF_{2n}^{3}
	...	...	...	...	...	...
	$$k$X	CF_{21}^{k}	CF_{22}^{k}	CF_{23}^{k}	...	CF_{2n}^{k}
...	...	...	...	...	...	...
l	$X	CF_{l1}^{1}	CF_{l2}^{1}	CF_{l3}^{1}	...	CF_{ln}^{1}
	$2X	CF_{l1}^{2}	CF_{l2}^{2}	CF_{l3}^{2}	...	CF_{ln}^{2}
	$3X	CF_{l1}^{3}	CF_{l2}^{3}	CF_{l3}^{3}	...	CF_{ln}^{3}
...	...	...	...	...	...	...
	$$k$X	CF_{l1}^{k}	CF_{l2}^{k}	CF_{l3}^{k}	...	CF_{ln}^{k}

4. Continue increasing the number of proposals in the combination until the number is l and identify the most lucrative combination.

In Table 3.1, CF_{lt}^{k} indicates the cash flow of proposal l in period t at kth level of investment.

3.11.3 Fuzzy Dynamic Programming for Multilevel Investments

Assume that we know the fuzzy cash flows of multilevel investments and we deal with the problem from capital budgeting viewpoint. Given a fuzzy specified (MARR) value, proposals can be ranked on the basis of

$$\text{Ranking ratio} = \frac{\text{Proposal fuzzy PW(MARR)}}{\text{Proposal fuzzy first cost}} \tag{3.69}$$

where PW is the present worth of a proposal. The larger ratio indicates the better proposal. Kahraman *et al.* [3.22] and Kahraman [3.19] use fuzzy present worth and fuzzy benefit/cost ratio analysis for the justification of manufacturing technologies and for public work projects.

Now assume that cash flows for l independent proposals that have passed a screening based on a MARR of $\tilde{r}$% are given in Table 3.2 and we have a $\$\tilde{L}$ capital limitation. In Table 3.2, $\tilde{CF}_{lt}^{k}$ indicates the fuzzy cash flow of proposal l in period t at kth level of investment. The problem is which combination of proposals should be funded. The solution consists of the following steps:

1. Devise all possible investments that encompass proposals 1 and 2 alone, applying an upper limit of $\$\tilde{L}$ to the fuzzy amount invested. Compute the fuzzy present worth of each proposal in the possible combinations using the fuzzy discounted cash flow techniques [3.20], [3.21]. $\$\tilde{L}$ can be allocated to proposal 1 alone or to proposal 2 alone or to any other combination.
2. Identify the most lucrative combination of proposals 1 and 2 corresponding to every possible value of $\$\tilde{L}$, using the ranking ratio in Eq. (3.69). Use a ranking method of fuzzy numbers to identify the most lucrative combination.
3. Devise all possible investments that encompass proposals 1, 2, and 3, and identify the most lucrative one as in step 2. Use a ranking method of fuzzy numbers to identify the most lucrative combination.
4. Continue increasing the number of proposals in the combination until the number is l and identify the most lucrative combination. Use a ranking method of fuzzy numbers to identify the most lucrative combination.

Table 3.2. Fuzzy cash flows for l independent proposals

End-of-period-cash-flow, $						
Proposal	Investment,$	Period 1	Period 2	Period 3	...	Period n
1	$\$\tilde{X}$	$C\tilde{F}_{11}^1$	$C\tilde{F}_{12}^1$	$C\tilde{F}_{13}^1$	...	$C\tilde{F}_{1n}^1$
	$\$2\tilde{X}$	$C\tilde{F}_{11}^2$	$C\tilde{F}_{12}^2$	$C\tilde{F}_{13}^2$	...	$C\tilde{F}_{1n}^2$
	$\$3\tilde{X}$	$C\tilde{F}_{11}^3$	$C\tilde{F}_{12}^3$	$C\tilde{F}_{13}^3$	...	$C\tilde{F}_{1n}^3$
	...	...	...	...	...	...
	$\$k\tilde{X}$	$C\tilde{F}_{11}^k$	$C\tilde{F}_{12}^k$	$C\tilde{F}_{13}^k$	...	$C\tilde{F}_{1n}^k$
2	$\$\tilde{X}$	$C\tilde{F}_{21}^1$	$C\tilde{F}_{22}^1$	$C\tilde{F}_{23}^1$	...	$C\tilde{F}_{2n}^1$
	$\$2\tilde{X}$	$C\tilde{F}_{21}^2$	$C\tilde{F}_{22}^2$	$C\tilde{F}_{23}^2$	...	$C\tilde{F}_{2n}^2$
	$\$3\tilde{X}$	$C\tilde{F}_{21}^3$	$C\tilde{F}_{22}^3$	$C\tilde{F}_{23}^3$	...	$C\tilde{F}_{2n}^3$
	...	...	...	...	...	...
	$\$k\tilde{X}$	$C\tilde{F}_{21}^k$	$C\tilde{F}_{22}^k$	$C\tilde{F}_{23}^k$	...	$C\tilde{F}_{2n}^k$
...	...	...	...	...	...	...
l	$\$\tilde{X}$	$C\tilde{F}_{l1}^1$	$C\tilde{F}_{l2}^1$	$C\tilde{F}_{l3}^1$	...	$C\tilde{F}_{ln}^1$
	$\$2\tilde{X}$	$C\tilde{F}_{l1}^2$	$C\tilde{F}_{l2}^2$	$C\tilde{F}_{l3}^2$	...	$C\tilde{F}_{ln}^2$
	$\$3\tilde{X}$	$C\tilde{F}_{l1}^3$	$C\tilde{F}_{l2}^3$	$C\tilde{F}_{l3}^3$	...	$C\tilde{F}_{ln}^3$
	...	...	...	...	...	...
	$\$k\tilde{X}$	$C\tilde{F}_{l1}^k$	$C\tilde{F}_{l2}^k$	$C\tilde{F}_{l3}^k$	...	$C\tilde{F}_{ln}^k$

3.11.4 A Numeric Example

A firm has $(15000, 21000, 27000) available for investment, and three investment proposals are under consideration. Each proposal has these features:

the amount that can be invested is a multiple of \$(5000, 7000, 9000); the investors receive annual unequal receipts; each proposal has a useful life of three years. Table 3.3 lists the annual geometric receipts corresponding to the various fuzzy levels of investment. Devise the most lucrative composite investment using fuzzy dynamic programming. The company-specified MARR value, $\tilde{r}$%, is (5%, 6%, 7%) per year.

Table 3.3. Fuzzy cash flows for three independent proposals (× \$1,000)

Proposal	Investment, \$	Year 1	Year 2	Year 3
1	\$(5, 7, 9)	(3, 4, 5)	(3.3, 4.4, 5.5)	(3.63, 4.84, 6.05)
	\$(10, 14, 18)	(5, 6, 7)	(5.6, 6.72, 7.84)	(6.272, 7.526, 8.78)
	\$(15, 21, 27)	(8, 9, 10)	(9.12, 10.26, 11.4)	(10.396, 11.696, 12.996)
2	\$(5, 7, 9)	(3, 4, 6)	(3.3, 4.4, 6.6)	(3.63, 4.84, 7.392)
	\$(10, 14, 18)	(4, 6, 7)	(4.48, 6.72, 7.84)	(5.017, 7.526, 8.78)
	\$(15, 21, 27)	(5, 9, 10)	(5.7, 10.26, 11.4)	(6.498, 11.696, 12.996)
3	\$(5, 7, 9)	(3, 3, 4)	(3.3, 3.3, 4.4)	(3.630, 3.63, 4.84)
	\$(10, 14, 18)	(5, 7, 7)	(5.6, 7.84, 7.84)	(6.272, 7.526, 7.526)
	\$(15, 21, 27)	(8, 9, 12)	(9.12, 10.26, 13.68)	(10.396, 11.696, 15.595)

In FUZDYN, the project definition is as in Fig. 3.8.

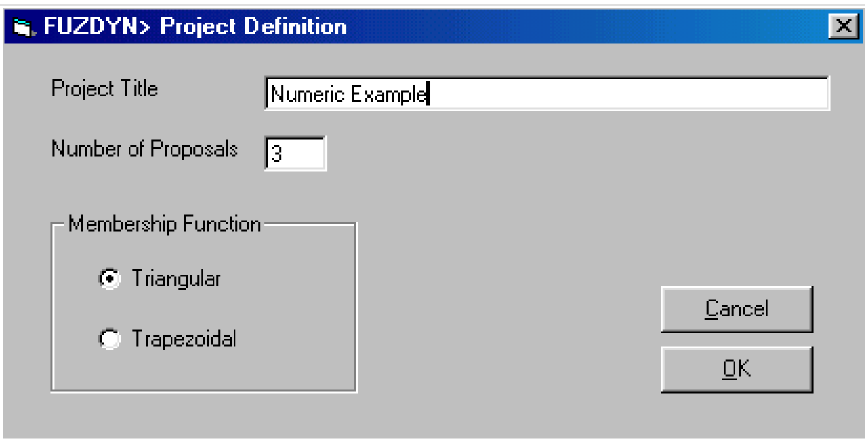

Fig. 3.8. Project definition

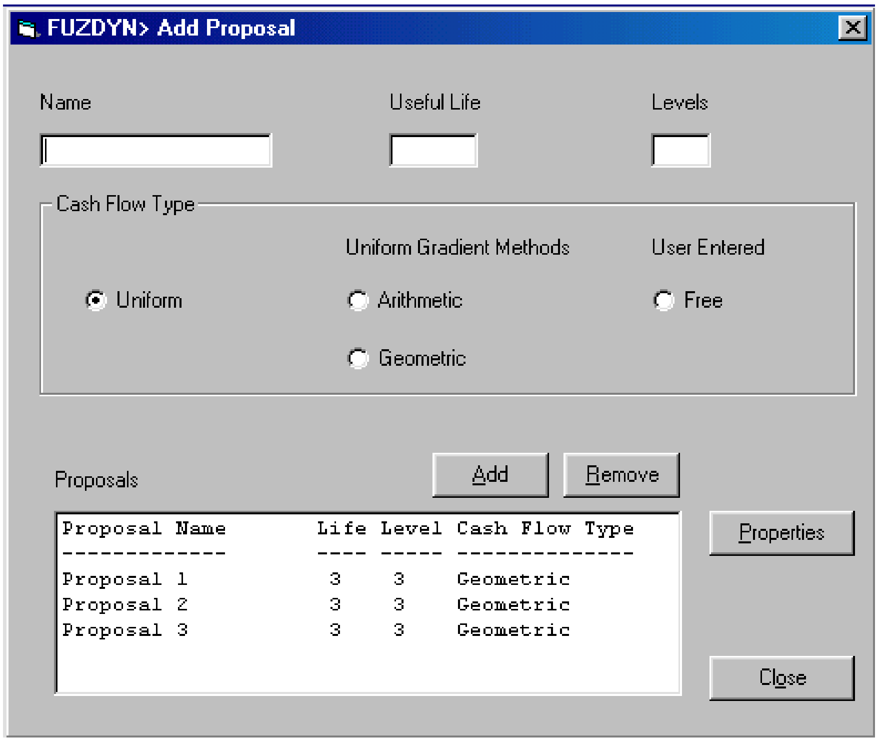

Fig. 3.9. The form of parameter input for proposals

As it can be seen from Table 3.3, the geometric growth rates (g) for the annual receipts at the investment levels are 10%, 12%, and 14% respectively and they are given as crisp rates in the problem and $f_1\left(y|\,\tilde{r}\right) = 0.05 + 0.01y$, $f_2\left(y|\,\tilde{r}\right) = 0.07 - 0.01y$, $\gamma\left(f_{3-k}\left(y|\,\tilde{r}\right), g, n\right)$, k=1,2.

In FUZDYN, data input for proposals is shown in Fig. 3.9. In Fig. 3.10 the data regarding fuzzy investment cost, fuzzy growth rate, and the benefit of the first year are entered and in Fig. 3.11, it is shown how a fuzzy number is entered.

For the total investment of \$(15000, 21000, 27000) in proposals 1 and 2:

– Investment in proposal 1: \$ (15000, 21000, 27000) and proposal 2: \$ 0

We find $f_1\left(y|\,\tilde{F}_1\right) = 1000y + 8000$, $f_2\left(y|\,\tilde{F}_1\right) = 10000 - 1000y$.

For k= 1, $f_{3,1}\left(y|\,\tilde{P}\right) = (1000y + 8000)\left[\frac{(1.14)^3(1.07-0.01y)^{-3}-1}{0.07+0.01y}\right]$ and for $y = 0$, $f_{3,1}\left(y|\,\tilde{P}\right)$ =\$ 23,929 and for $y = 1$, $f_{3,1}\left(y|\,\tilde{P}\right)$ =\$ 27,442. For $k = 2$, $f_{3,2}\left(y|\,\tilde{P}\right) = (10000 - 1000y)\left[\frac{1-(1.14)^3(1.05+0.01y)^{-3}}{0.01y-0.09}\right]$ and for $y = 0$, $f_{3,2}\left(y|\,\tilde{P}\right)$ =\$ 31,090.

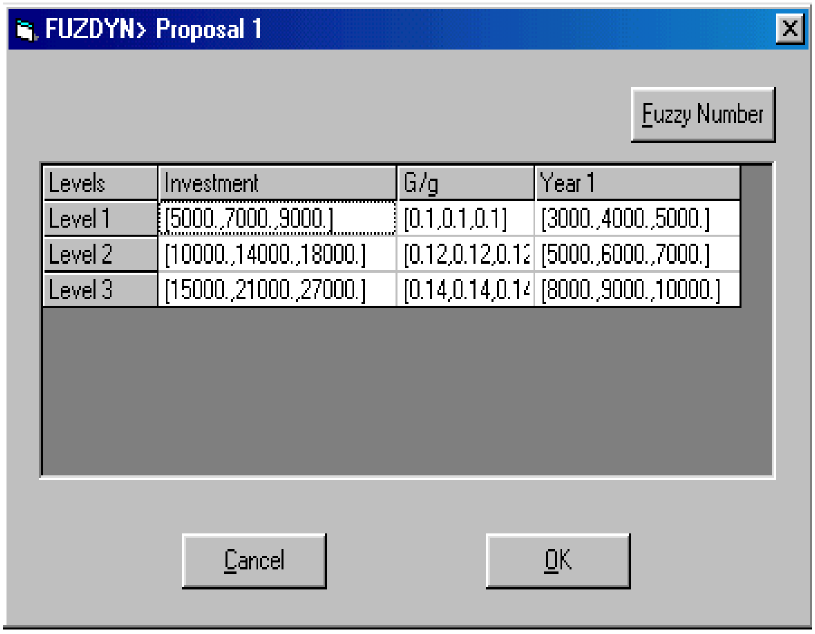

Fig. 3.10. The forms related to data input for proposals

Now we can calculate the net PW and the fuzzy ranking ratio:

$$
\begin{aligned}
NPW_1 &= \$\,(23,929; 27,442; 31,090) - \$\,(15,000; 21,000; 27,000) \\
&= \$\,(-3,071; +6,442; +16,090)
\end{aligned}
$$

$$
\begin{aligned}
\text{Ranking ratio} &= \frac{\text{Proposal fuzzy PW(MARR)}}{\text{Proposal fuzzy first cost}} \\
&= \frac{\$\,(-3,071; +6,442; +16,090)}{\$\,(15,000; 21,000; 27,000)} \\
&= (-0.114; +0,307; +1,073)
\end{aligned}
$$

– Investment in proposal 1: \$ (10000, 14000, 18000) and proposal 2: \$ (5000, 7000, 9000)

For proposal 1:

$f_1\left(y|\,\tilde{F}_1\right) = 1000y + 5000,\;\; f_2\left(y|\,\tilde{F}_1\right) = 7000 - 1000y.$

For $k = 1$, $f_{3,1}\left(y|\,\tilde{P}\right) = (1000y + 5000)\left[\frac{(1.12)^3(1.07-0.01y)^{-3}-1}{0.05+0.01y}\right]$ and for $y = 0$, $f_{3,1}\left(y|\,\tilde{P}\right)$ =\$ 14,684 and for $y = 1$, $f_{3,1}\left(y|\,\tilde{P}\right)$ =\$ 17,960.

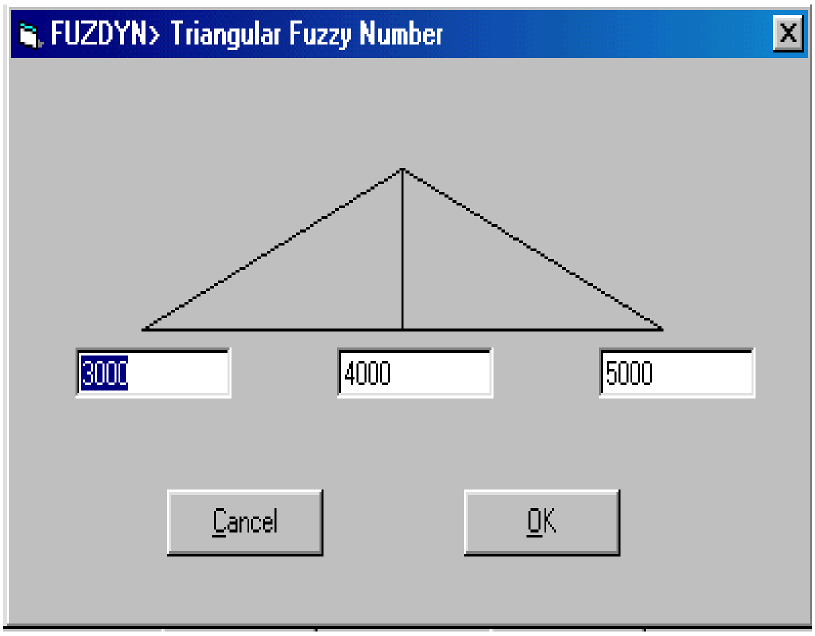

Fig. 3.11. The forms related to data input for proposals

For $k=2$, $f_{3,2}\left(y|\tilde{P}\right)=(7000-1000y)\left[\frac{1-(1.12)^3(1.05+0.01y)^{-3}}{0.01y-0.07}\right]$ and for $y=0$, $f_{3,2}\left(y|\tilde{P}\right)=$\$ 21,363

For proposal 2:

$f_1\left(y|\tilde{F}_1\right)=1000y+3000,\;\; f_2\left(y|\tilde{F}_1\right)=6000-2000y$

For $k=1$, $f_{3,1}\left(y|\tilde{P}\right)=(1000y+3000)\left[\frac{(1.10)^3(1.07-0.01y)^{-3}-1}{0.03+0.01y}\right]$ and for $y=0$, $f_{3,1}\left(y|\tilde{P}\right)=$\$ 8,649 and for $y=1$, $f_{3,1}\left(y|\tilde{P}\right)=$\$ 11,753.

For $k=2$, $f_{3,2}\left(y|\tilde{P}\right)=(6000-2000y)\left[\frac{1-(1.10)^3(1.05+0.01y)^{-3}}{0.01y-0.05}\right]$ and for $y=0$, $f_{3,2}\left(y|\tilde{P}\right)=$\$ 17,972.

Now we can calculate the net PW and the fuzzy ranking ratio:

$$\begin{aligned}PW_{1,2} &= PW_1+PW_2\\ &= \$\,(14,684;17,690;21,393)+\$\,(8,649;11,753;17,972)\\ &= \$\,(23,333;29,443;39,365)\end{aligned}$$

$$\begin{aligned}NPW_{1,2} &= \$\,(23,333;29,443;39,365)-\$\,(15,000;21,000;27,000)\\ &= \$\,(-3,667;+8,443;+24,365)\end{aligned}$$

$$\text{Ranking ratio} = \frac{\$\,(-3,667;+8,443;+24,365)}{\$\,(15,000;21,000;27,000)} = (-0.136;+0.402;+1.624)$$

– Investment in proposal 1: \$ (5000, 7000, 9000) and proposal 2: \$ (10000, 14000, 18000)

For proposal 1:
$f_1\left(y|\tilde{F}_1\right) = 1000y + 3000, \quad f_2\left(y|\tilde{F}_1\right) = 5000 - 1000y.$
For $k = 1$, $f_{3,1}\left(y|\tilde{P}\right) = (1000y + 3000)\left[\frac{(1.10)^3(1.07-0.01y)^{-3}-1}{0.03+0.01y}\right]$ and for $y = 0$, $f_{3,1}\left(y|\tilde{P}\right) =$\$ 8,649 and for $y = 1$, $f_{3,1}\left(y|\tilde{P}\right) =$\$ 11,753.
For $k = 2$, $f_{3,2}\left(y|\tilde{P}\right) = (5000 - 1000y)\left[\frac{1-(1.10)^3(1.05+0.01y)^{-3}}{0.01y-0.05}\right]$ and for $y = 0$, $f_{3,2}\left(y|\tilde{P}\right) =$\$ 14,977.

For proposal 2:
$f_1\left(y|\tilde{F}_1\right) = 2000y + 4000, \quad f_2\left(y|\tilde{F}_1\right) = 7000 - 1000y$
For $k = 1$, $f_{3,1}\left(y|\tilde{P}\right) = (2000y + 4000)\left[\frac{(1.12)^3(1.07-0.01y)^{-3}-1}{0.05+0.01y}\right]$ and for y= 0, $f_{3,1}\left(y|\tilde{P}\right) =$\$ 11,747 and for $y = 1$, $f_{3,1}\left(y|\tilde{P}\right) =$\$ 17,960.
For k =2, $f_{3,2}\left(y|\tilde{P}\right) = (7000 - 1000y)\left[\frac{1-(1.12)^3(1.05+0.01y)^{-3}}{0.01y-0.07}\right]$ and for $y = 0$, $f_{3,2}\left(y|\tilde{P}\right) =$\$ 21,363.

Now we can calculate the net PW and the fuzzy ranking ratio:

$$\begin{aligned} PW_{1,2} &= PW_1 + PW_2 \\ &= \$\,(8,649;11,753;14,977) + \$\,(11,747;17,960;21,363) \\ &= \$\,(20,396;29,713;36,340) \end{aligned}$$

$$\begin{aligned} NPW_{1,2} &= \$\,(20,396;29,713;36,340) - \$\,(15,000;21,000;27,000) \\ &= \$\,(-6,604;+8,713;+21,340) \end{aligned}$$

$$\text{Ranking ratio} = \frac{\$\,(-6,604;+8,713;+21,340)}{\$\,(15,000;21,000;27,000)} = (-0.025;+0.415;+1.423)$$

– Investment in proposal 1: \$ 0 and proposal 2: \$ (15000, 21000, 27000)

We find $f_1\left(y|\tilde{F}_1\right) = 4000y + 5000$, $f_2\left(y|\tilde{F}_1\right) = 10000 - 1000y$.
For k= 1, $f_{3,1}\left(y|\tilde{P}\right) = (4000y + 5000)\left[\frac{(1.14)^3(1.07-0.01y)^{-3}-1}{0.07+0.01y}\right]$ and for y= 0, $f_{3,1}\left(y|\tilde{P}\right)$ =$ 14,956 and for $y = 1$, $f_{3,1}\left(y|\tilde{P}\right)$ =$ 27,442.
For k =2, $f_{3,2}\left(y|\tilde{P}\right) = (10000 - 1000y)\left[\frac{1-(1.14)^3(1.05+0.01y)^{-3}}{0.01y-0.09}\right]$ and for $y = 0$, $f_{3,2}\left(y|\tilde{P}\right)$ =$ 31,090.
Now we can calculate the net PW and the fuzzy ranking ratio:

$$\begin{aligned} NPW_2 = &\ \$\,(14,956; 27,442; 31,090) - \$\,(15,000; 21,000; 27,000) \\ = &\ \$\,(-12,044; +6,442; +16,090) \end{aligned}$$

$$\begin{aligned} \text{Ranking ratio} &= \frac{\$\,(-12,044; +6,442; +16,090)}{\$\,(15,000; 21,000; 27,000)} \\ &= (-0.446; +0,307; +1,073) \end{aligned}$$

To select the most lucrative combination of an investment of $ (15,000; 21,000; 27,000), we will use Liou and Wang's [3.32] method. For a moderately optimistic decision-maker, $\omega = 0.5$.

Table 3.4. Identifying the most lucrative combination of $ (15,000; 21,000; 27,000) for the first stage

Ranking ratio, $\tilde{A}$	$E_\omega(\tilde{A}) = \frac{1}{2}\left[\omega\,(a+b) + (1-\omega)\,(b+c)\right]$
(−0.114; +0.307; +1.073)	0.393
(−0.136; +0.402; +1.624)	0.573*
(−0.025; +0.415; +1.423)	0.557
(−0.446; +0.307; +1.073)	0.310

As it can be seen from Table 3.4, the most lucrative combination is to invest $ (10,000; 14,000; 18,000) in proposal 1 and invest $ (5,000; 7,000; 9,000) in proposal 2.

For the total investment of $ (10000, 14000, 18000) in proposals 1 and 2:

– Investment in proposal 1: (10000, 14000, 18000) and proposal 2: $ 0

$$\begin{aligned} \text{Ranking ratio} &= \frac{\$\,(-3,316; +3,960; +11,363)}{\$\,(10,000; 14,000; 18,000)} \\ &= (-0.184; +0.283; +1.136) \end{aligned}$$

– Investment in proposal 1: $ (5000, 7000, 9000) and proposal 2: $ (5000, 7000, 9000)

$$\text{Ranking ratio} = \frac{\$\,(-702; +9,506; +22,949)}{\$\,(10,000; 14,000; 18,000)} = (-0.039; +0.679; +2.295)$$

– Investment in proposal 1: \$ 0 and proposal 2: \$ (10000, 14000, 18000)

$$\text{Ranking ratio} = \frac{\$\,(-6,253; +3,960; 11,363)}{\$\text{ (10000, 14000, 18000)}} = (-0.347; +0.283; +1.136)$$

To select the most lucrative combination of an investment of \$ (10,000; 14,000; 18,000), we will again use Liou and Wang's [3.32] method. For a moderately optimistic decision-maker, $\omega = 0.5$.

Table 3.5. Identifying the most lucrative combination of \$ (10,000; 14,000; 18,000) for the second stage

Ranking ratio, $\tilde{A}$	$E_\omega\left(\tilde{A}\right) = \frac{1}{2}\left[\omega\,(a+b) + (1-\omega)\,(b+c)\right]$
(−0.184; +0.282; +1.136)	0.379
(−0.039; +0.679; +2.295)	0.904*
(−0.347; +0.283; +1.136)	0.339

As it can be seen from Table 3.5, the most lucrative combination is to invest \$ (5,000; 7,000; 9,000) in proposal 1 and invest \$ (5,000; 7,000; 9,000) in proposal 2.

For the total investment of \$ (5000, 7000, 9000) in proposals 1 and 2:

– Investment in proposal 1: \$ (5000, 7000, 9000) and proposal 2: \$ 0

$$\text{Ranking ratio} = \frac{\$\,(-351; +4,753; +9,977)}{\$\text{ (5000, 7000, 9000)}} = (-0.039; +0.679; +1.995)$$

– Investment in proposal 1: \$ 0 and proposal 2: \$ (5000, 7000, 9000)

$$\text{Ranking ratio} = \frac{\$\,(-351; +4,753; +12,972)}{\$\,(5,000; 7,000; 9,000)} = (-0.039; +0.679; +2.594)$$

It is obvious that the most lucrative combination of an investment of \$ (5,000; 7,000; 9,000) is to invest \$ (5,000; 7,000; 9,000) in proposal 2.

Now we will devise all possible investments that encompass proposals 1, 2, and 3, and identify the most lucrative one.

– Investment in proposals 1+2: \$ (15000, 21000, 27000) and proposal 3: \$ 0

$$\text{Ranking ratio} = \frac{\$\,(-3,667;+8,443;+24,365)}{\$\,(15,000;21,000;27,000)}$$
$$= (-0.136;+0.402;+1.624)$$

– Investment in proposals 1+2: \$ (10000, 14000, 18000) and proposal 3: \$ (5000, 7000, 9000)

$$\text{Ranking ratio} = \frac{\$\,(-1,053;+11,321;+29,930)}{\$\,(15,000;21,000;27,000)}$$
$$= (-0.039;+0.539;+1.995)$$

– Investment in proposals 1+2: \$ (5000, 7000, 9000) and proposal 3: \$ (10000, 14000, 18000)

$$\text{Ranking ratio} = \frac{\$\,(-3,667;+11,707;+24,335)}{\$\,(15,000;21,000;27,000)}$$
$$= (-0.136;+0.557;+1.622)$$

– Investment in proposals 1+2: \$ 0 and proposal 3: \$ (15000, 21000, 27000)

$$\text{Ranking ratio} = \frac{\$\,(-3,071;+6,442;+10,308)}{\$\,(15,000;21,000;27,000)}$$
$$= (-0.114;+0.307;+0.687)$$

To select the most lucrative combination of an investment of \$ (15,000; 21,000; 27,000), we will again use Liou and Wang's [3.32] method. For a moderately optimistic decision-maker, $\omega = 0.5$.

Table 3.6. Identifying the most lucrative combination of \$ (15,000; 21,000; 27,000) for the last stage

Ranking ratio, $\tilde{A}$	$E_\omega(\tilde{A}) = \frac{1}{2}\left[\omega\,(a+b) + (1-\omega)\,(b+c)\right]$
$(-0.136;+0.402;+1.624)$	0.573
$(-0.039;+0.539;+1.995)$	0.759*
$(-0.136;+0.557;+1.622)$	0.650
$(-0.114;+0.307;+0.687)$	0.328

As it can be seen from Table 3.6, the most lucrative combination is to invest \$ (10,000; 14,000; 18,000) in proposal 1 and proposal 2 and invest \$ (5,000; 7,000; 9,000) in proposal 3. Then the final solution is to invest \$ (5000, 7000, 9000) in proposal 1 and \$ (5000, 7000, 9000) in proposal 2, and \$ (5,000; 7,000; 9,000) in proposal 3.

The final solution found by FUZDYN is given in Fig. 3.12:

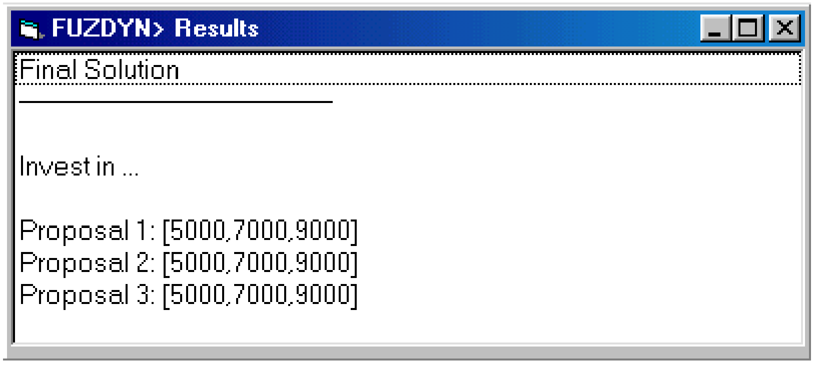

Fig. 3.12. Final solution

3.12 Conclusions

In this chapter, capital budgeting techniques in the case of fuzziness and discrete compounding have been studied. The cash flow profile of some investments projects may be geometric or trigonometric. For these kind of projects, the fuzzy present, future, and annual value formulas have been also developed under discrete and continuous compounding in this chapter. Fuzzy set theory is a powerful tool in the area of management when sufficient objective data has not been obtained. Appropriate fuzzy numbers can capture the vagueness of knowledge. The other financial subjects such as replacement analysis, income tax considerations; continuous compounding in the case of fuzziness can be also applied [3.24], [3.25]. Comparing projects with unequal lives has not been considered in this paper. This will also be a new area for a further study. Dynamic programming is a powerful optimization technique that is particularly applicable many complex problems requiring a sequence of interrelated decisions. In the paper, we presented a fuzzy dynamic programming application for the selection of independent multi level investments. This method should be used when imprecise or fuzzy input data or parameters exist. In multilevel mathematical programming, input data or parameters are often imprecise or fuzzy in a wide variety of hierarchical optimization problems such as defense problems, transportation network designs, economical analysis, financial control, energy planning, government regulation, equipment scheduling, organizational management, quality assurance, conflict resolution and so on. Developing methodologies and new concepts for solving fuzzy and possibilistic multilevel programming problems is a practical and interesting direction for future studies.

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Appendix

One of the most basic concepts of fuzzy set theory which can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. Let X be a cartesian product of universes $X = X_1 \ldots X_r$, and $\tilde{A}_1, ..., \tilde{A}_r$ be r fuzzy sets in X_1 ,. . . ,X$_r$, respectively. f is a mapping from X to a universe Y, $y = f(x_1$,. . . ,x$_r)$. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in Y by

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y\ ))|\, y = f(x_1, ..., x_r), (x_1, ..., x_r) \in X\} \tag{A.1}$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{(x_1,...,x_r)\in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1), ..., \mu_{\tilde{A}_r}(x_r)\}, if\ f^{-1}(y) \neq \emptyset \\ 0 \qquad\qquad\qquad\qquad\qquad , \quad \text{otherwise} \end{cases} \tag{A.2}$$

where f^{-1} is the inverse of f.

Assume $\tilde{P}$=(a, b, c) and $\tilde{Q}$=(d, e, f). $a,\ b,\ c,\ d,\ e,\ f$ are all positive numbers. With this notation and by the extension principle, some of the extended algebraic operations of triangular fuzzy numbers are expressed in the following.

Changing Sign

$$-(a, b, c) = (-c, -b, -a) \tag{A.3}$$

or

$$-(d, e, f) = (-f, -e, -d) \tag{A.4}$$

Addition

$$\tilde{P} \oplus \tilde{Q} = (a + d, b + e, c + f) \tag{A.5}$$

and

$$k \oplus (a, b, c) = (k + a, k + b, k + c) \tag{A.6}$$

or

$$k \oplus (d, e, f) = (k + d, k + e, k + f) \tag{A.7}$$

if k is an ordinary number (a constant).

Subtraction

$$\tilde{P} - \tilde{Q} = (a - f, b - e, c - d) \tag{A.8}$$

and

$$(a, b, c) - k = (a - k, b - k, c - k) \tag{A.9}$$

or

$$(d, e, f) - k = (d - k, e - k, f - k) \tag{A.10}$$

if k is an ordinary number.

Multiplication

$$\tilde{P} \otimes \tilde{Q} = (ad, be, cf) \tag{A.11}$$

and

$$k \otimes (a, b, c) = (ka, kb, kc) \tag{A.12}$$

or

$$k \otimes (d, e, f) = (kd, ke, kf) \tag{A.13}$$

if k is an ordinary number.

Division

$$\tilde{P} \oslash \tilde{Q} = (a/f, b/e, c/d) \tag{A.14}$$

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