Chapter 2

Geometry of Linear Programming

The intent of this chapter is to provide a geometric interpretation of linear programming problems. To conceive fundamental concepts and validity of different algorithms encountered in optimization, convexity theory is considered the key of this subject. The last section is on the graphical method of solving linear programming problems.

2.1 Geometric Interpretation

Let $\mathbb{R}^n$ denote the n-dimensional vector space (Euclidean) defined over the field of reals. Suppose $X, Y \in \mathbb{R}^n$. For $X = (x_1, x_2, \ldots, x_n)^T$ and $Y = (y_1, y_2, \ldots, y_n)^T$ we define the distance between X and Y as

$$|X - Y| = \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2\right]^{1/2}.$$

Neighbourhood. Let X_0 be a point in $\mathbb{R}^n$. Then δ-neighbourhood of X_0, denoted by $N_\delta(X_0)$ is defined as the set of points satisfying

$$N_\delta(X_0) = \{X \in \mathbb{R}^n : |X - X_0| < \delta,\ \delta > 0\}.$$

But

$$N_\delta(X_0) \setminus X_0 = \{X \in \mathbb{R}^n : 0 < |X - X_0| < \delta\}$$

will be termed the deleted neighbourhood of X_0.

In $\mathbb{R}^2$, $N_\delta(X_0)$ is a circle without circumference, and in $\mathbb{R}^3$, $N_\delta(X_0)$ is sphere without boundary, and for $\mathbb{R}$, an open interval on the real line. For $n > 3$, figures are hypothetical.

Let $S \subset \mathbb{R}^n$. We give few elementary definitions.

Boundary point. A point X_0 is called a boundary point of S if each deleted neighbourhood of X_0 intersects S and its compliment S^c.

Interior point. A point $X_0 \in S$ is said to be an interior point of S, if there exists a neighbourhood of X_0 which is contained in S.

Open set. A set S is said to be open if for each $X \in S$ there exists a neighbourhood of X which is contained in S.

For example, $S = \{X \in \mathbb{R}^n : |X - X_0| < 2\}$ is an open set. The well known results: (i) A set is open $\Leftrightarrow$ it contains all its interior points, and (ii) The union of any number of open sets is an open set, are left as exercises for the reader.

Close set. A set S is closed if its compliment S^c is open.

For example, $S = \{X \in \mathbb{R}^n : |X - X_0| \leq 3\}$ is a closed set. Again a useful result arises: intersection of any number of closed sets is closed.

A set S in $\mathbb{R}^n$ is bounded if there exists a constant $M > 0$ such that $|X| \leq M$ for all X in S.

Definition 1. A line joining X_1 and X_2 in $\mathbb{R}^n$ is a set of points given by the linear combination

$$L = \{X \in \mathbb{R}^n : X = \alpha_1 X_1 + \alpha_2 X_2,\ \alpha_1 + \alpha_2 = 1\}.$$

Obviously,

$$L^+ = \{X : X = \alpha_1 X_1 + \alpha_2 X_2,\ \alpha_1 + \alpha_2 = 1\ , \alpha_2 \geq 0\}$$

is a half-line originating from X_1 in the direction of X_2 as, for α_2=0, $X = X_1$ and α_2=1, $X = X_2$.

Similarly,

$$L^- = \{X : X = \alpha_1 X_1 + \alpha_2 X_2,\ \alpha_1 + \alpha_2 = 1\ , \alpha_1 \geq 0\}$$

is a half-line emanating from X_2 in the direction of X_1 as, for α_1=0, $X = X_2$ and α_1=1, $X = X_1$.

Definition 2. A point $X \in \mathbb{R}^n$ is called a convex linear combination (clc) of two points X_1 and X_2, if it can be expressed as

$$X = \alpha_1 X_1 + \alpha_2 X_2,\ \alpha_1, \alpha_2 \geq 0,\ \alpha_1 + \alpha_2 = 1.$$

Geometrically, speaking convex linear combination of any points X_1 and X_2 is a line segment joining X_1 and X_2.

For example, let $X_1 = (1, 2)$ and $X_2 = (3, 7)$. Then,

$$X = \left(\frac{1}{3}\right)(1,2) + \left(\frac{2}{3}\right)(3,7) = \left(\frac{1}{3}, \frac{2}{3}\right) + \left(2, \frac{14}{3}\right) = \left(\frac{7}{3}, \frac{16}{3}\right)$$

is a point lying on the line joining X_1 and X_2.

Convex set. A set S is said to be convex if clc of any two points of S belongs to S, i.e.,

$$X = \alpha_1 X_1 + \alpha_2 X_2 \in S, \;\; \alpha_1 + \alpha_2 = 1, \;\; \alpha_1, \alpha_2 \geq 0 \;\; \forall \; X_1, X_2 \in S.$$

Geometrically, this definition may be interpreted as the line segment joining every pair of points X_1, X_2 of S lies entirely in S. For more illustration, see Fig. 2.1.

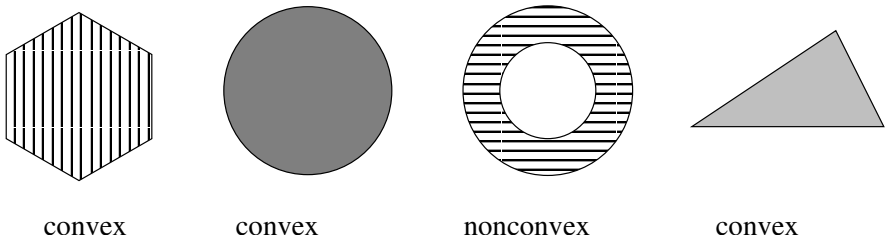


Figure 2.1

By convention empty set is convex. Every singleton set is convex. A straight line is a convex set, and a plane in $\mathbb{R}^3$ is also a convex set. Convex sets have many pleasant properties that give strong mathematical back ground to the optimization theory.

Theorem 1. Intersection of two convex sets is a convex set.

Proof. Let S_1 and S_2 be two convex sets. We have to show that $S_1 \cap S_2$ is a convex set. If this intersection is empty or singleton there is nothing to prove.

Let X_1 and X_2 be two arbitrary points in $S_1 \cap S_2$. Then $X_1, X_2 \in S_1$ and $X_1, X_2 \in S_2$. Since S_1 and S_2 are convex, we have

$$\alpha_1 X_1 + \alpha_2 X_2 \in S_1 \text{ and } \alpha_1 X_1 + \alpha_2 X_2 \in S_2, \;\; \alpha_1, \alpha_2 \geq 0, \;\; \alpha_1 + \alpha_2 = 1.$$

Thus, $\alpha_1 X_1 + \alpha_2 X_2 \in S_1 \cap S_2$, and hence $S_1 \cap S_2$ is convex.

Remarks. 1. Moreover, it can be shown that intersection of any number of convex sets is a convex set, see Problem 3.

2. The union of two or more convex sets may not be convex. As an example the sets $S_1 = \{(x_1, 0) : x_1 \in \mathbb{R}\}$ and $S_2 = \{(0, x_2) : x_2 \in \mathbb{R}\}$ are convex in $x_1 - x_2$ plane, but their union

$$S_1 \cup S_2 = \{(x_1, 0), (0, x_2) : x_1, x_2 \in \mathbb{R}\}$$

is not convex, since $(2, 0), (0, 2) \in S_1 \cup S_2$, but their clc,

$$\left(\frac{1}{2}\right)(2, 0) + \left(\frac{1}{2}\right)(0, 2) = (1, 1) \notin S_1 \cup S_2.$$

Hyperplanes and Half-spaces. A plane in $\mathbb{R}^3$ is termed a hyperplane. The equation of hyperplane in $\mathbb{R}^3$ is the set of points $(x_1, x_2, x_3)^T$ satisfying

$$a_1x_1 + a_2x_2 + a_3x_3 = \beta.$$

Extending the above idea, a hyperplane in $\mathbb{R}^n$ is the set of points $(x_1, x_2, \ldots, x_n)^T$ satisfying the linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \beta$$

or $a^TX = \beta$, where $a = (a_1, a_2, \ldots, a_n)^T$. Thus, a hyperplane in $\mathbb{R}^n$ is the set

$$H = \{X \in \mathbb{R}^n : a^TX = \beta\}. \tag{2.1}$$

A hyperplane separates the whole space into two closed half-spaces

$$H_L = \{X \in \mathbb{R}^n : a^TX \leq \beta\}, \quad H_U = \{X \in \mathbb{R}^n : a^TX \geq \beta\}.$$

Removing H results in two disjoint open half-spaces

$$H'_L = \{X \in \mathbb{R}^n : a^TX < \beta\}, \quad H'_U = \{X \in \mathbb{R}^n : a^TX > \beta\}.$$

From (2.1), it is clear that the defining vector a of hyperplane H is orthogonal to H. Since, for any two vectors X_1 and $X_2 \in H$

$$a^T(X_1 - X_2) = a^TX_1 - a^TX_2 = \beta - \beta = 0.$$

Moreover, for each vector $X \in H$ and $W \in H'_L$,

$$a^T(W - X) = a^TW - a^TX < \beta - \beta = 0.$$

This shows that the normal vector a makes an obtuse angle with any vector that points from the hyperplane toward the interior of H_L. In other words, a is directed toward the exterior of H_L. Fig. 2.2 illustrates the geometry.

$$\left\{X \in \mathbb{R}^n : a^T X = \beta\right\}$$

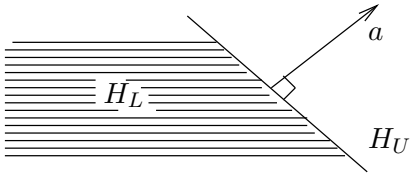


Figure 2.2

Theorem 2. A hyperplane in $\mathbb{R}^n$ is a closed convex set.

Proof. The hyperplane in $\mathbb{R}^n$ is the set $S = \{X \in \mathbb{R}^n : a^T X = \alpha\}$. We prove that S is closed convex set. First, we show that S is closed. To do this we prove S^c is open, where

$$S^c = \left\{X \in \mathbb{R}^n : a^T X < \alpha\right\} \cup \left\{X \in \mathbb{R}^n : a^T X > \alpha\right\} = S_1 \cup S_2.$$

Let $X_0 \in S^c$. Then $X_0 \notin S$. This implies

$$a^T X_0 < \alpha \quad \text{or} \quad a^T X_0 > \alpha.$$

Suppose $a^T X_0 < \alpha$. Let $a^T X_0 = \beta < \alpha$. Define

$$N_\delta(X_0) = \left\{X \in \mathbb{R}^n : |X - X_0| < \delta,\ \delta = \frac{\alpha - \beta}{|a|}\right\}. \tag{2.2}$$

If $X_1 \in N_\delta(X_0)$, then in view of (2.2),

$$\begin{aligned} a^T X_1 - a^T X_0 &\le \left|a^T X_1 - a^T X_0\right| \\ &= \left|a^T (X_1 - X_0)\right| \\ &= \left|a^T\right| |X_1 - X_0| \\ &< \alpha - \beta. \end{aligned}$$

But $a^T X_0 = \beta$. This implies $a^T X_1 < \alpha$ and hence $X_1 \in S_1$. Since X_1 is arbitrary, we conclude that $N_\delta(X_0) \subset S_1$. This implies S_1 is open.

Similarly, it can be shown that $S_2 = \{X : a^T X_0 > \alpha\}$ is open. Now, $S^c = S_1 \cup S_2$ is open (being union of open sets) which proves that S is closed.

Let $X_1, X_2 \in S$. Then $a^T X_1 = \alpha$ and $a^T X_2 = \alpha$, and consider

$$X = \beta_1 X_1 + \beta_2 X_2,\ \beta_1, \beta_2 \ge 0,\ \beta_1 + \beta_2 = 1$$

and operating a^T, note that

$$a^T X = \beta_1 a^T X_1 + \beta_2 a^T X_2 = \beta_1\beta + \beta_2\beta = \beta(\beta_1 + \beta_2) = \beta.$$

Thus, $X \in S$ and hence S is convex.

Theorem 3. A half-space $S = \{X \in \mathbb{R}^n : a^T X \leq \alpha\}$ is a closed convex set.

Proof. Let $S = \{X \in \mathbb{R}^n : a^T X \leq \alpha\}$. Suppose $X_0 \in S^c$. Then $a^T X_0 > \alpha$. Now, $a^T X_0 = \beta > \alpha$. Consider the neighbourhood $N_\delta(X_0)$ defined by

$$N_\delta(X_0) = \left\{ X \in \mathbb{R}^n : |X - X_0| < \delta,\ \delta = \frac{\beta - \alpha}{|a|} \right\}.$$

Let X_1 be an arbitrary point in $N_\delta(X_0)$. Then

$$a^T X_0 - a^T X_1 \leq \left| a^T X_1 - a^T X_0 \right| = \left| a^T \right| |X_1 - X_0| < \beta - \alpha$$

Since $a^T X_0 = \beta$, we have

$$-a^T X_1 < -\alpha \Rightarrow a^T X_1 > \alpha \Rightarrow X_1 \in S^c \Rightarrow N_\delta(X_0) \subset S^c.$$

This implies S^c is open and hence S is closed.

Take $X_1, X_2 \in S$. Hence $a^T X_1 \leq \alpha$, $a^T X_2 \leq \alpha$. For

$$X = \alpha_1 X_1 + \alpha_2 X_2,\ \alpha_1, \alpha_2 \geq 0,\ \alpha_1 + \alpha_2 = 1.$$

Note that

$$\begin{aligned} a^T X &= a^T (\alpha_1 X_1 + \alpha_2 X_2) \\ &= \alpha_1 a^T X_1 + \alpha_2 a^T X_2 \\ &\leq \alpha_1 \alpha + \alpha_2 \alpha \\ &= \alpha(\alpha_1 + \alpha_2) = \alpha. \end{aligned}$$

This implies $X \in S$, and hence S is convex.

Polyhedral set. A set formed by the intersection of finite number of closed half-spaces is termed polyhedron or polyhedral.

If the intersection is nonempty and bounded, it is called a polytope. For a linear programme in standard form, we have m hyperplanes

$$H_i = \left\{ X \in \mathbb{R}^n : a_i^T X = b_i,\ X \geq 0,\ b_i \geq 0, \quad i = 1, 2, \ldots, m \right\},$$

where $a_i^T = (a_{i1}, a_{i2}, \ldots, a_{in})$ is the ith row of the constraint matrix A and b_i is the ith element of the right-hand vector b.

Moreover, for a linear program in standard form, the hyperplanes $H = \{X \in \mathbb{R}^n : C^T X = \beta,\ \beta \in \mathbb{R}\}$ depict the contours of the linear objective function, and cost vector C^T becomes the normal of its contour hyperplanes.

Set of feasible solutions. The set of all feasible solutions forms a feasible region, generally denoted by P_F, and this is the intersection of hyperplanes H_i, i=1,2,...,m and the first octant of $\mathbb{R}^n$.

Note that each hyperplane is intersection of two closed half-spaces H_L and H_U, and the first octant of $\mathbb{R}^n$ is the intersection of n closed half-spaces $\{x_i \in \mathbb{R} : x_i \geq 0\}$. Hence the feasible region is a polyhedral set, and is given by

$$P_F = \{X \in \mathbb{R}^n : AX \leq b,\ X \geq 0,\ b \geq 0\}.$$

When P_F is not empty, the linear programme is said to be consistent. For a consistent linear programme with a feasible solution $X^* \in P_F$, if $C^T X^*$ attains the minimum or maximum value of the objective function $C^T X$ over the feasible region P_F, then we say X^* is an optimal solution to the linear programme.

Moreover, we say a linear programme has a bounded feasible region, if there exists a positive constant M such that for every $X \in P_F$, we have $|X| \leq M$. On the other hand for minimization problem, if there exists a constant K such that $C^T X \geq K$ for all $X \in P_F$, then we say linear programme is bounded below. Similarly, we can define bounded linear programme for maximization problem.

Remarks. 1. In this context, it is worth mentioning that a linear programme with bounded feasible region is bounded, but the converse may not be true, i.e., a bounded LPP need not to have a bounded feasible region, see Problem 2.

2. In $\mathbb{R}^3$, a polytope has prism like shape.

Converting equalities to inequalities. To study the geometric properties of an LPP we consider the LPP in the form, where the constraints are of the type $\leq$ or $\geq$, i.e.,

$$\begin{aligned} \text{opt} \quad & z = C^T X \\ \text{s.t.} \quad & g_i(X) \leq \text{ or } \geq 0,\ i = 1, 2, \ldots, m \\ & X \geq 0. \end{aligned}$$

In case there is equality constraint like $x_1 + x_2 - 2x_3 = 5$, we can write this as (equivalently): $x_1 + x_2 - 2x_3 \leq 5$ and $x_1 + x_2 - 2x_3 \geq 5$. This tells us that m equality constraints will give rise to $2m$ inequality constraints. However, we can reduce the system with m equality constraints to an equivalent system which has $m+1$ inequality constraints.

Example 1. Show that the system having m equality constraints

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1, 2, \ldots, m$$

is equivalent to the system with $m + 1$ inequality constraints.

To motivate the idea, note that $x = 1$ is equivalent to the combination $x \leq 1$ and $x \geq 1$. As we can check graphically, the equations $x = 1$ and $y = 2$ are equivalent to the combinations $x \leq 1$, $y \leq 2$, $x + y \geq 3$. The other way to write equivalent system is $x \geq 1$, $y \geq 2$, $x + y \leq 3$. This idea can further be generalized to m equations. Consider the system of m equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1, 2, \ldots, m.$$

This system has the equivalent form:

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad \text{and} \quad \sum_{j=1}^{n} a_{ij}x_j \geq b_i$$

or

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad \text{and} \quad \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right) x_j \geq \sum_{i=1}^{m} b_i.$$

If we look at the second combination, then the above system is equivalent to

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i, \quad \text{and} \quad \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right) x_j \leq \sum_{i=1}^{m} b_i.$$

Consider the constraints and nonnegative restrictions of an LPP in its standard form

$$\begin{aligned} x_1 + x_2 + s_1 &= 1 \\ -x_1 + 2x_2 + s_2 &= 1 \\ x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

Although it has four variables, the feasible reason P_F can be represented as a two dimensional graph. Write the basic variables s_1 and s_2 in terms of nonbasic variables, see Problem 28, and use the conditions $s_1 \geq 0,\ s_2 \geq 0$ to have

$$\begin{aligned} &x_1 + x_2 \leq 1 \\ &-x_1 + 2x_2 \leq 1 \\ &x_1,\ x_2 \geq 0 \end{aligned}$$

and is shown in Fig. 2.3.

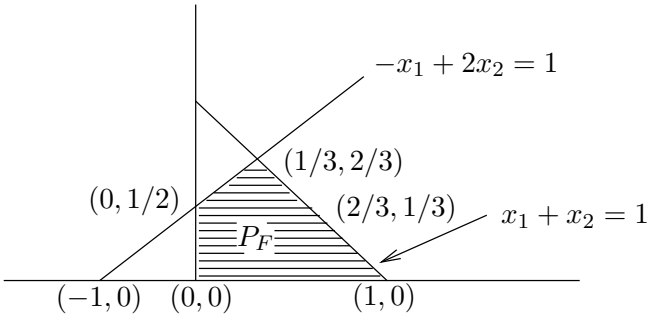


Figure 2.3

Remark. If a linear programming problem in its standard has n variables and $n-2$ nonredundant constraints, then the LPP has a two-dimensional representation. Why?, see Problem 28.

Theorem 4. The set of all feasible solutions of an LPP (feasible region P_F) is a closed convex set.

Proof. By definition $P_F = \{X : AX = b, X \geq 0\}$. Let X_1 and X_2 be two points of P_F. This means that $AX_1 = b,\ AX_2 = b,\ X_1 \geq 0,\ X_2 \geq 0$. Consider

$$Z = \alpha X_1 + (1-\alpha)X_2,\ 0 \leq \alpha \leq 1.$$

Clearly, $Z \geq 0$ and $AZ = \alpha AX_1 + (1-\alpha)AX_2 = \alpha b + (1-\alpha)b = b$. Thus, $Z \in P_F$, and hence P_F is convex.

Remark. Note that in above theorem b may be negative, i.e., the LPP may not be in standard form. The only thing is that we have equality constraint. If b is negative, we multiply by minus in the constraint.

Alternative proof. Each constraint $a_i^T X = b_i,\ i = 1, 2, \ldots m$ is closed (being a hyperplane), and hence intersection of these m hyperplanes ($AX = b$) is closed. Further, each nonnegative restriction $x_i \geq 0$

is a closed (being closed half-spaces) is closed, and hence their intersection $X \geq 0$ is closed. Again, intersection $P_F = \{AX = b, X \geq 0\}$ is closed. This concludes that P_F is a closed convex set.

Clc's of two or more points. Convex linear combination of two points gives a line segment. The studies on different regions need the clc of more than two points. This motivates the idea of extending the concept.

Definition 3. The point X is called a clc of m points $X_1, X_2, \ldots, X_m$ in $\mathbb{R}^n$, if there exist scalars $\alpha_i,\ i = 1, 2, \ldots, m$ such that

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_m X_m,\ \alpha_i \geq 0,\ \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1.$$

Remark. This definition includes clc of two points also. Henceforth, whenever we talk about clc, it means clc of two or more points.

Theorem 5. A set S is convex $\Leftrightarrow$ every clc of points in S belongs to S.

Proof. ($\Leftarrow$) Given that every clc of points in S belongs to S includes the assertion that every clc of two points belongs to S. Hence S is convex.

($\Rightarrow$) Suppose S is convex. we prove the result by induction. S is convex $\Rightarrow$ clc of every two points in S belongs to S. Hence the theorem is true for clc of two points. Assume that theorem is true for clc of n points. we must show that it is true for $n + 1$ points. Consider

$$X = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_n X_n + \beta_{n+1} X_{n+1}$$

such that

$$\beta_1 + \beta_2 + \cdots + \beta_{n+1} = 1,\ \beta_i \geq 0.$$

If $\beta_{n+1} = 0$, then X, being clc of n points belongs to S (by assumption). If $\beta_{n+1} = 1$, then $\beta_1 = \beta_2 = \cdots = \beta_n = 0$ and $X = 1.X_{n+1}$, the theorem is trivially true. Assume, $\beta_{n+1} \neq 0$ or 1, i.e., $0 < \beta_{n+1} < 1$. Now, $\beta_1 + \beta_2 + \cdots + \beta_n \neq 0$.

$$X = \frac{\beta_1 + \beta_2 + \cdots + \beta_n}{\beta_1 + \beta_2 + \cdots + \beta_n} (\beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_n X_n) + \beta_{n+1} X_{n+1}$$

or

$$X = (\beta_1 + \beta_2 + \cdots + \beta_n)(\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n) + \beta_{n+1} X_{n+1},$$

where, $\alpha_i = \beta_i/(\beta_1 + \beta_2 + \cdots + \beta_n),\ i = 1, 2, \ldots, n$. Clearly, $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Hence, by assumption $\alpha_1 X_1 + \alpha_2 X_2 + \cdots +$

$\alpha_n X_n = Y$ (say) belongs to S. Again,

$$X = (\beta_1 + \beta_2 + \cdots + \beta_n)Y + \beta_{n+1}X_{n+1}$$

such that

$$\beta_1 + \beta_2 + \cdots + \beta_n \geq 0,\ \beta_{n+1} \geq 0,\ \sum_{i=1}^{n+1} \beta_i = 1.$$

Thus, X is the clc of two points and hence belongs to S.

Convex hull. Let S be a nonempty set. Then convex hull of S, denoted by $[S]$ is defined as all clc's of points of S,

$$[S] = \{X \in \mathbb{R}^n : X \text{ is clc of points in } S\}.$$

Remarks. 1. By convention $[\emptyset] = \{0\}$.

2. The above discussion reveals that the convex hull of finite number of points $X_1, X_2, \ldots, X_m$ is the convex combination of the m points. This is the convex set having at most m vertices. Here, at most means some points may be interior points. Moreover, convex hull generated in this way is a closed convex set.

3. The convex hull of m points is given a special name as convex polyhedron.

Theorem 6. Let S be a nonempty set. Then the convex hull $[S]$ is the smallest convex set containing S.

Proof. Let $X, Y \in [S]$. Then

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n,\ \alpha_i \geq 0,\ \sum_{i=1}^{n} \alpha_i = 1,$$

$$Y = \beta_1 Y_1 + \beta_2 Y_2 + \cdots + \beta_m Y_m,\ \beta_j \geq 0,\ \sum_{j=1}^{m} \beta_j = 1.$$

Consider the linear combination $\alpha X + \beta Y,\ \alpha, \beta \geq 0,\ \alpha + \beta = 1$, and note that

$$\begin{aligned} \alpha X + \beta Y &= \alpha(\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n) \\ &\quad + \beta(\beta_1 Y_1 + \beta_2 Y_2 + \cdots + \beta_m Y_m) \\ &= (\alpha\alpha_1)X_1 + \cdots + (\alpha\alpha_n)X_n + (\beta\beta_1)Y_1 + \cdots + (\beta\beta_m)Y_m. \end{aligned}$$

Now, each $\alpha\alpha_i \geq 0$, $\beta\beta_j \geq 0$ and

$$\sum_{i=1}^{n}(\alpha\alpha_i) + \sum_{j=1}^{m}(\beta\beta_j) = \alpha\sum_{i=1}^{n}\alpha_i + \beta\sum_{j=1}^{m}\beta_j = \alpha + \beta = 1,$$

i.e., $\alpha X + \beta Y$ is a clc of points $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$. This implies that $\alpha X + \beta Y \in [S]$ and hence $[S]$ is convex.

Clearly, it contains S because each $X \in S$ can be written as $X = 1.X + 0.Y$, i.e., clc of itself. To prove that $[S]$ is the smallest convex set containing S, we show that if there exists another convex set T containing S, then $[S] \subset T$.

Suppose T is a convex set which contains S. Take any element $X \in [S]$. Then

$$X = \alpha_1 X_1 + \cdots + \alpha_n X_n, \;\; \alpha_i \geq 0, \;\; \sum_{i=1}^{n}\alpha_i = 1 \;\; \forall\, X_1, X_2, \ldots, X_n \in S.$$

Since $S \subset T$, it follows that $X_1, X_2, \ldots, X_n \in T$ and, moreover convexity of T ensures that

$$\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n = X \in T.$$

Hence $[S] \subset T$.

Remark. If S is convex, then $S = [S]$.

For convex set $S \subset \mathbb{R}^n$, a key geometric figure is due to the following separation theorem. The proof is beyond the scope of the book.

Theorem 7 (Separation theorem). Let $S \subset R^n$ and X be a boundary point of S. Then there is a hyperplane H containing X with S contained either in lower half-plane or upper half-plane.

Based on Theorem 7 we can define a supporting hyperplane H to be the hyperplane such that (i) the intersection of H and S is nonempty; (ii) lower half-plane contains S, see Fig. 2.4.

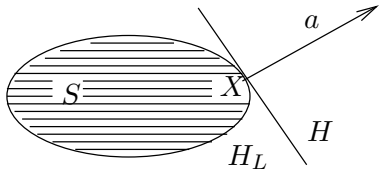


Figure 2.4

One very important fact to point out here is that the intersection set of the polyhedral set and the supporting hyperplane with negative cost vector C^T as its normal provides optimal solution of an LPP. This is the key idea of solving linear programming problems by the graphical method.

To verify this fact, let us take

$$\min x_0 = -x_1 - 2x_2$$

as the objective function for the LPP whose feasible region is shown in Fig. 2.3. Note that $-x_1-2x_2 = -80$ is the hyperplane passing through $(0, 40)$ and the vector $-C^T = (1, 2)$ is normal to this plane. This is a supporting hyperplane passing through $(0, 40)$, since $H_L = \{(x_1, x_2) : x_1 + 2x_2 \leq 80\}$ contains P_F and is satisfied by the points $(20, 20)$ and $(30, 0)$.

However, the hyperplane passing through $(20, 20)$ which is normal to $-C^T = (1, 2)$ is given by $-x_1 - 2x_2 = -60$. This is not a supporting hyperplane as point $(0, 40)$ is not in $\{(x_1, x_2) : x_1 + 2x_2 \leq 60\}$. Similarly, it can be shown that hyperplane at $(3, 0)$ which is normal to $-C^T$ is also not a supporting hyperplane. This implies that $x_1 = 0,\ x_2 = 40$ is the optimal solution.

2.2 Vertices and Basic Feasible Solutions

Definition 4. A point X of a convex set S is said to be an vertex (extreme point) of S if X is not a clc of any other two distinct points of S, i.e., X can not be expressed as

$$X = \alpha_1 X_1 + \alpha_2 X_2,\ \alpha_1, \alpha_2 > 0,\ \alpha_1 + \alpha_2 = 1.$$

In other words, a vertex is a point that does not lie strictly within the line segment connecting two other points of the convex set.

From the pictures of convex polyhedron sets, especially in lower dimensional spaces it is clear to see the vertices of a convex polyhedron, Analyze the set P_F as depicted in Fig. 2.3. Further, we note the following observations:

(a) $A(0, 0)$, $B(1, 0)$, $C(1/3, 2/3)$ and $D(0, 1/2)$ are vertices and moreover, these are boundary points also. But every boundary point need not be a vertex. In Fig. 2.3, as E is boundary point of the

feasible region P_F but not vertex, since it can be written as clc of distinct points of the set P_F as

$$\left(\frac{2}{3}, \frac{1}{3}\right) = \left(\frac{1}{2}\right)(1,0) + \left(\frac{1}{2}\right)\left(\frac{1}{3}, \frac{2}{3}\right).$$

(b) All boundary points of $\{(x, y) : x^2 + y^2 \leq 9\}$ are vertices. Hence, vertices of a bounded closed set may be infinite. However, in an LPP, if P_F is bounded and closed, then it contains finite number of vertices, see Theorem 9.

(c) Needless to mention whenever we talk about vertex of a set S, it is implied that S is convex.

(d) If S is unbounded, then it may not have a vertex, e.g., , $S = \mathbb{R}^2$.

(e) If S is not closed, then it may not have vertex, e.g., , $S = \{(x, y) : 0 < x < 1, 2 < y < 3\}$ has no vertex.

Remark. Here extreme points are in reference to convex sets. However, extreme points of a function will be defined in Chapter 13.

To characterize the vertices of a feasible region $P_F = \{X \in \mathbb{R}^n : AX = b,\ X \geq 0\}$ of a given LPP in standard form, we may assume A is an $m \times n$ matrix with $m < n$ and also denote the jth column of the coefficient matrix A by A_j, $j = 1, 2, \ldots, n$. Then, for each $X = (x_1, x_2, \ldots, x_n)^T \in P_F$, we have

$$x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = b.$$

Therefore, A_j is the column of A corresponding to the jth component x_j of X.

Theorem 8. A point X of feasible region P_F is a vertex of $P_F \Leftrightarrow$ the columns of A corresponding to the positive components of X are linearly independent.

Proof. Without loss of generality, we may assume that the components of X are zero except for the first p components, namely

$$X = \left[\frac{\overline{X}}{0}\right], \quad \overline{X} = (x_1, x_2, \ldots, x_p)^T > 0.$$

We also denote the first p columns of matrix A by $\overline{A}$. Hence $AX = \overline{A}\ \overline{X} = b$.

($\Rightarrow$) Suppose that the columns of $\overline{A}$ are not linearly independent, then there exists nonzero vector $\overline{w}$ (at least one of the p components is nonzero) such that $\overline{A}\overline{w} = 0$. Now, define

$$Y = \overline{X} + \delta\overline{w} \quad \text{and} \quad Z = \overline{X} - \delta\overline{w}$$

For sufficiently small $\delta > 0$, we note that $Y, Z \geq 0$ and

$$\overline{A}Y = \overline{A}Z = \overline{A}\ \overline{X} = b.$$

We further define

$$Y_1 = \left[\frac{\overline{Y}}{0}\right] \quad \text{and} \quad Z_1 = \left[\frac{\overline{Z}}{0}\right]$$

Note that $Y_1, Z_1 \in P_F$ and $X = (1/2)Y_1 + (1/2)Z_1$. In other words X is not an vertex of P_F.

($\Leftarrow$) Suppose that X is not vertex of P_F, then $X = \alpha Y_1 + (1 - \alpha)Z_1,\ \alpha \geq 0$ for some distinct $Y_1, Z_1 \in P_F$. Since $Y_1, Z_1 \geq 0$ and $0 \leq \alpha \leq 1$, the last $n - p$ components of Y_1 must be zero, as

$$0 = \alpha y_j + (1 - \alpha)z_j,\ \ j = p + 1, p + 2, \ldots, n.$$

Consequently, we have a nonzero vector $w = X - Y_1$ $(X \neq Y_1)$ such that

$$\overline{A}w = Aw = AX - AY_1 = b - b = 0.$$

This shows that columns of $\overline{A}$ are linearly dependent.

Consider an LPP in standard, $AX = b$, suppose we have, $n =$ number of unknowns; $m =$ number of equations. Assume $m < n$ (otherwise the problem is over-specified). However, after introducing the slack and surplus variables, generally this assumption remains valid. Let $r(A)$ and $r(A, b)$ be the ranks of matrix A and augmented matrix (A, b), respectively.

(i) $r(A) = r(A, b)$ guarantees the consistency, i.e., $AX = b$ has at least one solution.

(ii) $r(A) \neq r(A, b)$ the system is inconsistent, i.e., $AX = b$ has no solution. For example

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 4x_1 + 2x_2 - x_3 &= 5 \\ 9x_1 + 5x_2 - x_3 &= 11 \end{aligned}$$

has no solution and hence inconsistent.

For consistent systems we have difficulty when $r(A) = r(A, b) < m =$ number of equations. It means that all m rows are not linearly independent. Hence, any row may be written as linear combination of other rows. We consider this row (constraint) as redundant. For example

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ 2x_1 + x_2 - x_3 &= 3 \\ 5x_1 + x_2 &= 10 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

In this example $r(A) = r(A, b) = 2$ which is less than number of equations. The third constraint is the sum of the first constraint and two times of the second constraint. Hence, the third constraint is redundant. Actually, if the system is consistent, then the rows in echelon from of the matrix which contain pivot elements are to be considered. All nonpivot rows are redundant.

Another type of redundancy happens when $r(A) = r(A, b)$, but some of the constraint does not contribute any thing to find the optimal solution. Geometrically, a constraint is redundant if its removal leaves the feasible solution space unchanged. However, in this case $r(A) = r(A, b) = m$, the number of equations. Such type of cases we shall deal in Chapter 3. The following simple example illustrates the fact.

$$\begin{aligned} x_1 + x_2 + s_1 &= 1 \\ x_1 + 2x_2 + s_2 &= 2 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

Here $r(A) = r(A, b) = 2$. The vertices of the feasible region are $(0, 0)$, $(1, 0)$ and $(0, 1)$. The second constraint is redundant as it contributes only $(0, 1)$ which is already given by the first constraint.

It is advisable to delete redundant constraints, if any, before an LPP is solved to find its optimal solution, otherwise computational difficulty may arise.

Basic solutions. Let $AX = b$ be a system of m simultaneous linear equations in n unknowns ($m < n$) with $r(A) = m$. This means that there exist m linearly independent column vectors. In this case group these linearly independent column vectors to form a basis B and leave the remaining $n - m$ columns as nonbasis N. In other words, we can rearrange $A = [B|N]$.

We can also rearrange the components of any *solution* vector X in the corresponding order, namely

$$X = \left[\frac{X_B}{X_N}\right]$$

For a component in X_B, its corresponding column is in the basis B. Similarly, components in X_N correspond to a nonbasis matrix N.

If all $n - m$ variables X_N which are not associated with columns of B are equated to zero, then the solution of the resulting system $BX_B = b$ is called a basic solution of $AX = b$. Out of n columns, m columns can be selected in $n!/m!(n - m)!$ ways. The m variables (left after putting $n - m$ variables equal to zero) are called the basic variables and remaining $n - m$ variables as nonbasic variables. The matrix corresponding to basic variables is termed the basis matrix.

Basic feasible solution. A basic solution of the system $AX = b$, $X \geq 0$ is called a basic feasible solution.

Nondegenerate BFS. If all the m basic variables in a BFS are positive than it is called nondegenerate basic feasible solution.

The following result is a direct consequences of Theorem 8.

Corollary 1. A point $X \in P_F$ is an vertex of $P_F \Leftrightarrow X$ is a basic feasible solution corresponding to some basis B.

Proof. By Theorem 8, we have

$$\begin{aligned} X \in P_F \text{ is vertex} \Leftrightarrow\ & \text{columns } A_i \text{ for } x_i > 0\ (i = 1 \text{ to } m) \text{ are LI} \\ \Leftrightarrow\ & B = [A_1, \ldots, A_m] \text{ is a nonsingular matrix of } X \\ \Leftrightarrow\ & X \text{ is a basic feasible solution.} \end{aligned}$$

Degenerate BFS. A BFS which is not nondegenerate is called degenerate basic feasible solution, i.e., at least one of the basic variable is at zero level in the BFS.

Remarks. 1. Corollary 1 reveals that there exists a one-one correspondence between the set of basic feasible solutions and set of vertices of P_F only in the absence of degeneracy. Actually, in case of degeneracy, a vertex may correspond to many degenerate basic feasible solutions. Examples 3 will make the remark more clear and justified.

2. When we select m variables out of n variables to define a basic solution it is must that the matrix B formed with the coefficients of

m variables must be nonsingular, otherwise we may get no solution or infinity of solutions (not basic), see Problem 12, Problem set 2.

3. Every basic solution of the system $AX = b$ is a basic feasible solution. Why?

Example 2. Without sketching P_F find the vertices for the system

$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ 2x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0. \end{aligned}$$

First, write the system in standard form

$$\begin{aligned} -x_1 + x_2 + s_1 &= 1 \\ 2x_1 + x_2 + s_2 &= 2. \end{aligned}$$

Here $n = 4, m = 2 \Rightarrow 4!/2!2! = 6$. Hence the system has at most 6 basic solutions. To find all the basic solutions we take any of the two variables as basic variables from the set $\{x_1, x_2, s_1, s_2\}$ to have

$$\left(\frac{1}{3}, \frac{4}{3}, 0, 0\right), \quad (1, 0, 2, 0), \quad (-1, 0, 0, 4),$$

$$(0, 2, -1, 0), \quad (0, 1, 0, 1), \quad (0, 0, 1, 2).$$

Thus, 4 are basic feasible solutions. The solution set is nondegenerate and hence there exists one-one correspondence between BFS and vertices, i.e., the feasible region P_F has 4 vertices.

Note that $(-1, 0, 0, 4)$ and $(0, 2, -1, 0)$ are basic solutions but not feasible.

Example 3. The system $AX = b$ is given by

$$\begin{aligned} x_1 + x_2 - 8x_3 + 3x_4 &= 2 \\ -x_1 + x_2 + x_3 - 3x_4 &= 2. \end{aligned}$$

Determine the following:

(i) a nonbasic feasible solution;

(ii) a basic solution with x_1, x_4 as basic variables;

(iii) a vertex which corresponds to two different basic feasible solutions.

(iv) all nondegenerate basic feasible solutions

Here $X = (x_1, x_2, x_3, x_4)^T$, $b = (2, 2)^T$ and the coefficient matrix A and the augmented matrix (A, b) are written as

$$A = \begin{bmatrix} 1 & 1 & -8 & 3 \\ -1 & 1 & 1 & -3 \end{bmatrix}; \quad (A, b) = \begin{bmatrix} 1 & 1 & -8 & 3 & 2 \\ -1 & 1 & 1 & -3 & 2 \end{bmatrix}$$

We make the following observations:

(i) Since $r(A) = r(A, b) = 2 <$ number of unknowns, the system is consistent and has infinity of solutions with two degrees of freedom. The row reduced echelon form of (A, b) is

$$\begin{bmatrix} 1 & 0 & -9/2 & 3 & 0 \\ 0 & 1 & -7/2 & 0 & 2 \end{bmatrix}$$

This gives

$$\begin{aligned} x_1 &= \frac{9}{2}x_3 - 3x_4 \\ x_2 &= \frac{7}{2}x_3 + 2 \end{aligned} \tag{2.3}$$

Let us assign $x_3 = x_4 = 2$, i.e., nonzero values to x_3 and x_4 to have $x_1 = 3$ and $x_2 = 9$. Thus one of the feasible solution is $(3, 9, 2, 2)$ but not a basic feasible solution, since at least two variables must be at zero level. Moreover, $(-3, 2, 0, 1)$ is a nonbasic feasible solution of the system.

(ii) If x_1 and x_4 are chosen basic variables, then the system becomes

$$\begin{aligned} x_1 + 3x_4 &= 2 \\ -x_1 - 3x_4 &= 2 \end{aligned}$$

The system has no solution with x_1, x_4 as basic variables.

(iii) Further, for x_1, x_2 as basic variables,

$(0, 2, 0, 0)$ is a BFS with basis matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

while, for x_2, x_4 as basic variables,

$$(0,2,0,0) \text{ is a BFS with basis matrix } \begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix}$$

Both BFS given by $(0,2,0,0)$ seem to be same but these are different as their basis matrices are different. However, both BFS correspond to the same vertex $(0,2)$. The vertex $(0,2)$ can be identified by writing the equivalent form of LPP as two-dimensional graph in $x_1 - x_2$ plane.

(iv) All nondegenerate basic feasible solutions of the system are

$$(-18/7, 0, 4/7, 0), \ (0, 0, -4/7, -6/7).$$

Note that these vectors are BFS even though some entries are negative, because all variables are unrestricted.

Remarks. 1. Note that system (2.3) can be written as

$$1x_1 + 0x_2 - \tfrac{9}{2}x_3 + 3x_4 = 0$$
$$0x_1 + 1x_2 - \tfrac{7}{2}x_3 + 0x_4 = 2$$

This is canonical form of constraint equations. Further, note that

$$\begin{bmatrix} -8 \\ 1 \end{bmatrix} = -\frac{9}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{7}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 3 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This extracts a good inference that if x_1 and x_2 are basic variables and are pivoted, then coefficients of x_3 and x_4 are coordinate vectors of A_3 and A_4 column vectors of A with respect to the basic vectors A_1 and A_2. This phenomenon will be used in simplex method to be discussed in next chapter.

2. All basic solutions can also be obtained by pivoting at any two variables and assigning zero value to the remaining variables. This can be done in 6 ways and hence will give at most six different basic solutions.

3. For the existence of all basic solutions it is necessary that column vectors of the coefficient matrix A must be linearly independent. It is also possible that after keeping requisite number of variables at zero level the remaining system may have infinity of solutions, see Problem 12(c). This happens when at least two columns are not linearly independent. In Problem 12, A_2 and A_5 are not linearly independent. We do not term these infinity solutions as basic solutions because for a basic solution the coefficient matrix formed with the coefficients of basic variables (in order) must be nonsingular.

Theorem 9. The set of all feasible solutions P_F of an LPP has finite number of vertices.

Proof. Let P_F be the set of all feasible solutions of the LPP, where constraints are written in standard form, $AX = b, b \geq 0$. If A is of full rank m, i.e., each $m \times m$ submatrix of A is nonsingular, then the system has

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

basic solutions. Further, we add the condition $X \geq 0$, then we can say $AX = b$ has at most $n!/m!(n-m)!$ basic feasible solutions. As there exists one to one correspondence between BFS set and set of all vertices, provided nondegeneracy persists in the problem, we conclude that set of vertices may have at most $n!/m!(n-m)!$ elements and hence number of vertices is finite.

In case of degeneracy more than one BFS may correspond to the same vertex. Hence, in this situation number of vertices will be less than $n!/m!(n-m)!$, and again the vertices are finite.

Note that two basic feasible solutions (vertices) are adjacent, if they use $m-1$ basic variables in common to form basis. For example, in Figure 2.3, it is easy to verify that $(0, 1/2)$ is adjacent to $(0, 0)$ but not adjacent to $(1, 0)$ since $(0, 1/2)$ takes x_2 and s_1 as basic variables, while $(0, 0)$ takes s_1 and s_2 and $(1, 0)$ takes x_1 and s_2. Under the nondegeneracy assumption, since each of the $n-m$ nonbasic variables could replace one current basic variable in a given basic feasible solution, we know that every BFS (vertex) has $n-m$ neighbours. Actually, each BFS can be reached by increasing the value of one nonbasic from zero to positive and decreasing the value of one basic variable from positive to zero. This is the basic concept of pivoting in simplex method to be discussed in next chapter.

Suppose that the feasible region P_F is bounded, in other words it

is a polytope as shown in Fig. 2.5.

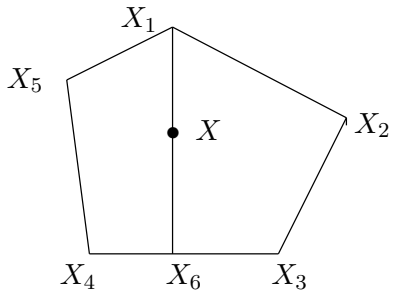


Figure 2.5

From this figure, it is easy to observe that each point of P_F can be represented as a convex combination of finite number of vertices of P_F, in particular X is the clc of the vertices $X_1,\ X_3,\ X_4$.

This idea of convex resolution can be verified for a general polyhedron (may be unbounded) with the help of following definition:

Definition 5. An extremal direction of a polyhedron set is a nonzero vector $d \in \mathbb{R}^n$ such that for each $X_0 \in P_F$, the ray

$$\{X \in \mathbb{R}^n : X = X_0 + \alpha d,\ \alpha \geq 0\}$$

is contained in P_F.

Remark. From the definition of feasible region, we see that a nonzero vector $d \in \mathbb{R}^n$ is an extremal direction of $P_F \Leftrightarrow Ad = 0$ and $d \geq 0$. Also, P_F is unbounded $\Leftrightarrow$ P_F has an extremal direction.

Considering the vertices and extremal directions every point in P_F can be represented by the following useful result known as the resolution theorem.

Theorem 10 (Resolution theorem). Let $B = \{V_i \in \mathbb{R}^n : i \in \mathbb{Z}\}$ be the set of all vertices of P_F with a finite index set $\mathbb{Z}$. Then, for each $X \in P_F$, we have

$$X = \sum_{i \in \mathbb{Z}} \alpha_i V_i + d, \qquad \sum_{i \in \mathbb{Z}} \alpha_i = 1,\ \ \alpha_i \geq 0, \tag{2.4}$$

where d is either the zero vector or an extremal direction of P_F.

Proof. To prove the theorem by the induction, we let p be the number of positive components of $X \in P_F$. When $p = 0$, $X = (0, 0, \ldots, 0)$ is obviously a vertex. Assume that the theorem holds

for $p = 0, 1, \ldots, k$ and X has $k+1$ positive components. If X is a vertex, then there is nothing to prove. If X is not a vertex, we let $X^T = (x_1, x_2, \ldots, x_{k+1}, 0, \ldots, 0) \in \mathbb{R}^n$ such that $(x_1, x_2, \ldots, x_{k+1}) > 0$ and $A = [\overline{A}|N]$, $\overline{A}$ is the matrix corresponding to positive components of X. Then, by Theorem 8, the columns of $\overline{A}$ are linearly independent, in other words there exists a nonzero vector $\overline{w} \in \mathbb{R}^{k+1}$ such that $\overline{A}\overline{w} = 0$. We define $w = (\overline{w}, 0, \ldots, 0) \in \mathbb{R}^n$, then $w \neq 0$ and $Aw = \overline{A}\overline{w} = 0$. There are three possibilities: $w \geq 0$, $w < 0$ and w has both positive and negative components.

For $w \geq 0$, consider $X(\theta) = X + \theta w$ and pick θ^* to be the largest value of θ such that $X^* = X(\theta^*)$ has at least one more zero component than X. Then follow the induction hypothesis to show that the theorem holds. Similarly, show that in the remaining two cases, the theorem still holds.

The direct consequences of the resolution theorem are:

Corollary 2. If P_F is a bounded feasible region (polytope), then each point $X \in P_F$ is a convex linear combination of its vertices.

Proof. Since P_F is bounded, by the remark following Definition 5 the extremal direction d is zero, and application of (2.4) ensures that X is a clc of vertices.

Corollary 3. If P_F is nonempty, then it has at least one vertex.

Example 4. Consider the set

$$P_F = \{(x_1, x_2) : x_1 + x_2 \leq 1,\ -x_1 + 2x_2 \leq 1,\ x_1, x_2 \geq 0\}.$$

Show that a point of P_F may be clc of different vertices.

Take the point $(1/3, 1/6) \in P_F$. Now

$$\left(\frac{1}{3}, \frac{1}{6}\right) = \frac{1}{3}(0,0) + \frac{1}{3}(1,0) + \frac{1}{3}\left(0, \frac{1}{2}\right) + 0\left(\frac{1}{3}, \frac{2}{3}\right)$$

or,

$$\left(\frac{1}{3}, \frac{1}{6}\right) = \frac{3}{8}(0,0) + \frac{5}{16}(1,0) + \frac{1}{16}\left(\frac{1}{3}, \frac{2}{3}\right) + \frac{1}{4}\left(0, \frac{1}{2}\right).$$

Thus, an additional information is gathered here that a point of P_F may have different clc's of its vertices.

2.3 Basic Theorem of Linear Programming

Theorem 11. The maximum of the objective function f(X) of an LPP occurs at least at one vertex of P_F, provided P_F is bounded.

Proof. Given that the LPP is a maximization problem. Suppose that maximum of $f(X)$ occurs at some point X_0 in the feasible region P_F. Thus,

$$f(X) \leq f(X_0) \ \forall \ X \in P_F.$$

We show that this $f(X_0)$ occurs at some vertex of P_F. Since P_F is bounded, closed and convex and the problem is LPP, it contains finite number of vertices $X_1, X_2, \ldots, X_n$. Hence,

$$f(X_i) \leq f(X_0), \ i = 1, 2, \ldots, n. \tag{2.5}$$

By Corollary 2, each $X_0 \in P_F$ can be written as clc of its vertices, i.e.,

$$X_0 = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n, \ \ \alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha_i = 1.$$

Using linearity of f, we have

$$f(X_0) = \alpha_1 f(X_1) + \alpha_2 f(X_2) + \cdots + \alpha_n f(X_n).$$

Let

$$f(X_k) = \max \{f(X_1), f(X_2), \ldots, f(X_n)\},$$

where $f(X_k)$ is one of the values of $f(X_1), f(X_2), \ldots, f(X_n)$. Then

$$f(X_0) \leq \alpha_1 f(X_k) + \alpha_2 f(X_k) + \cdots + \alpha_n f(X_k) = f(X_k). \tag{2.6}$$

Combining (2.5) and (2.6), we have $f(X_0) = f(X_k)$. This implies that the maximum value $f(X_0)$ is attained at the vertex X_k and hence the result.

The minimization case can be treated on parallel lines just by reversing the inequalities. Note that in the minimization case, we define $f(X_k) = \min\{f(X_1), f(X_2), \ldots, f(X_n)\}$.

Thus, we have proved that the optimum of an LPP occurs at some vertex of P_F, provided P_F is bounded.

Remark. Theorem 11 does not rule out the possibility of having an optimal solution at a point which is not vertex. It simply says among all optimal solutions to an LPP at least one of them is a vertex. The following theorem further strengthens Theorem 11.

Theorem 12. In an LPP, if the objective function $f(X)$ attains its maximum at an interior point of P_F, then f is constant, provided P_F is bounded.

Proof. Given that the problem is maximization, and let X_0 be an interior point of P_F, where maximum occurs, i.e.,

$$f(X) \leq f(X_0) \ \forall \ X \in P_F.$$

Assume contrary that $f(X)$ is not constant. Thus, we have $X_1 \in P_F$ such that

$$f(X_1) \neq f(X_0), \ \ f(X_1) < f(X_0).$$

Since P_F is nonempty bounded closed convex set, it follows that X_0 can be written as a clc of two points X_1 and X_2 of P_F

$$X_0 = \alpha X_1 + (1-\alpha)X_2, \ 0 < \alpha < 1.$$

Using linearity of f, we get

$$f(X_0) = \alpha f(X_1) + (1-\alpha)f(X_2) \Rightarrow f(X_0) < \alpha f(X_0) + (1-\alpha)f(X_2).$$

Thus $f(X_0) < f(X_2)$. This is a contradiction and hence the theorem.

2.4 Graphical Method

This method is convenient in case of two variables. By Theorem 11, the optimum value of the objective function occurs at one of the vertices of P_F. We exploit this result to find an optimal solution of any LPP. First, we sketch the feasible region and identify its vertices. Compute the value of objective function at each vertex, and take largest of these values to decide the optimal value of the objective function, and the vertex at which this largest value occurs is the optimal solution. For minimization problem we consider the smallest value.

Example 5. Solve the following LPP by the graphical method

$$\begin{aligned} \max \quad & z = x_1 + 5x_2 \\ \text{s.t.} \quad & -x_1 + 3x_2 \leq 10 \\ & x_1 + x_2 \leq 6 \\ & x_1 - x_2 \leq 2 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Rewrite each constraint in the forms:

$$-\frac{x_1}{10} + \frac{x_2}{10/3} \leq 1$$

$$\frac{x_1}{6} + \frac{x_2}{6} \leq 1$$

$$\frac{x_1}{2} - \frac{x_2}{2} \leq 1$$

Draw the each constraint first by treating as linear equation. Then use the inequality condition to decide the feasible region. The feasible region and vertices are shown in Fig. 2.6.

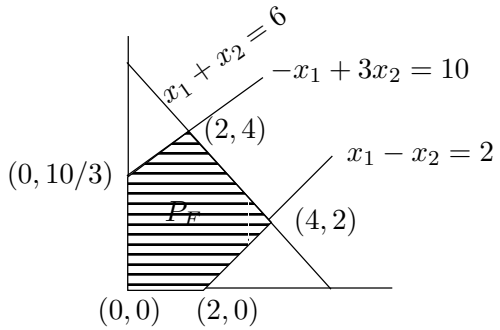


Figure 2.6

The vertices are $(0,0)$, $(2,0)$, $(4,2)$, $(2,4)$, $(0,10/3)$.

The values of the objective function is computed at these points are

$z = 0$	at	$(0,0)$
$z = 2$	at	$(2,0)$
$z = 14$	at	$(4,2)$
$z = 22$	at	$(2,4)$
$z = 50/3$	at	$(0,10/3)$

Obviously, the maximum occurs at vertex $(2,4)$ with maximum value 22. Hence,

$$\text{optimal solution: } x_1 = 2,\ x_2 = 4, \quad z = 22.$$

Example 6. A machine component requires a drill machine operation followed by welding and assembly into a larger subassembly. Two

versions of the component are produced: one for ordinary service and other for heavy-duty operation. A single unit of the ordinary design requires 10 min of drill machine time, 5 min of seam welding, and 15 min for assembly. The profit for each unit is $ 100. Each heavy-duty unit requires 5 min of screw machine time, 15 min for welding and 5 min for assembly. The profit for each unit is $ 150. The total capacity of the machine shop is 1500 min; that of the welding shop is 1000 min; that of assembly is 2000 min. What is the optimum mix between ordinary service and heavy-duty components to maximize the total profit?

Let x_1 and x_2 be number of ordinary service and heavy-duty components.

The LPP formulation is

$$\begin{aligned} \max \quad & z = 100x_1 + 150x_2 \\ \text{s.t.} \quad & 10x_1 + 5x_2 \leq 1500 \\ & 5x_1 + 15x_2 \leq 1000 \\ & 15x_1 + 5x_2 \leq 2000 \\ & x_1, x_2 \geq 0 \text{ and are integers.} \end{aligned}$$

Draw the feasible region by taking all constraints in the format as given in Example 5 and determine all the vertices. The vertices are $(0,0)$, $(400/3,0)$, $(125,25)$, $(0,200/3)$. The optimal solution exists at the vertex $x_1 = 125,\ x_2 = 25$ and the maximum value: $z = 16250$.

Problem Set 2

1. Which of the following sets are convex

(a) $\{(x_1,x_2) : x_1x_2 \leq 1\}$; (b) $\{(x_1,x_2) : x_1^2 + x_2^2 < 1\}$;

(c) $\{(x_1,x_2) : x_1^2 + x_2^2 \geq 3\}$; (d) $\{(x_1,x_2) : 4x_1 \geq x_2^2\}$;

(e) $\{(x_1,x_2) : 0 < x_1^2 + x_2^2 \leq 4\}$; (f) $\{(x_1,x_2) : |x_2| = 5\}$.

2. Prove that a linear program with bounded feasible region must be bounded, and give a counterexample to show that the converse need not be true.

3. Prove that arbitrary intersection of convex sets is convex.

4. Prove that the half-space $\{X \in \mathbb{R}^n : a^TX \geq \alpha\}$ is a closed convex set.

5. Show that the convex sets in $\mathbb{R}^n$ satisfy the following properties.

(a) If S is a convex set and β is a real number, the set

$$\beta S = \{\beta X : X \in S\}$$

is convex;

(b) If S_1 and S_2 are convex sets in $\mathbb{R}^n$, then the set

$$S_1 \pm S_2 = \{X_1 \pm X_2 : X_1 \in S_1,\ X_2 \in S_2\}$$

is convex.

6. A point X_v in S is a vertex of $S \Leftrightarrow S \setminus \{X_v\}$ is convex.

7. Write the system

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 - 4x_3 &= -5 \end{aligned}$$

into its equivalent system which contains only three inequality constraints.

8. Define the convex hull of a set S as

$$[S] = \{\cap_{i\in\Lambda} A_i : A_i \supset S \text{ and } A_i \text{ is convex}\}.$$

Show that this definition and the definition of convex hull in Section 2.1 are equivalent.

9. Using the definition of convex hull in Problem 8, show that $[S]$ is the smallest convex set containing S.

10. Find the convex hull of the following sets

(a) $\{(1,1),(1,2),(2,0),(0,-1)\}$; (b) $\{(x_1,x_2) : x_1^2 + x_2^2 > 3\}$;
(c) $\{(x_1,x_2) : x_1^2 + x_2^2 = 1\}$; (d) $\{(0,0),(1,0),(0,1)\}$.

11. Prove that convex linear combinations of finite number of points is a closed convex set.

Suggestion. For convexity, see Theorem 5.

12. Consider the following constraints of an LPP:

$$x_1 + 2x_2 + 2x_3 + x_4 + x_5 = 6$$
$$x_2 - 2x_3 + x_4 + x_5 = 3$$

Identify (a) all nondegenerate basic feasible solutions; (b) all degenerate basic infeasible solutions (c) infinity of solutions.

Suggestion. Here x_1, x_2, x_3, x_4, x_4 are unrestricted and hence every basic solution will be a basic feasible solution.

13. Use the resolution theorem to prove the following generalization of Theorem 11.
For a consistent LP in its standard form with a feasible region P_F, the maximum objective value of $z = C^T X$ over P_F is either unbounded or is achieved at least at one vertex of P_F.

14. Prove Theorems 11 and 12 for the minimization case.

15. Prove that if the optimal value of an LPP occurs at more than one vertex of P_F, then it also occurs at clc of these vertices.

16. Consider the above problem and mention whether the point other than vertices where optimal solution exists is a basic solution of the LPP.

17. Show that set of all optimal solutions of an LPP is closed convex set.

18. Consider the system $AX = b,\ X \geq 0,\ b \geq 0$ (with m equations and n unknowns). Let X be a basic feasible solution with $p < m$ components positive. How many different bases will correspond to X due to degeneracy in the system.

19. In view of problem 18, the BFS $(0, 2, 0, 0)$ of Example 3 has one more different basis. Find this basis.

20. Write a solution of the constraint equations in Example 3 which is neither basic nor feasible.

21. Let X_0 be an optimal solution of the LPP min $z = C^T X$, subject to $AX = b$ in standard form and let X^* be any optimal solution when C is replaced by C^*. Then prove that

$$(C^* - C)^T (X^* - X_0) \geq 0.$$

22. To make the graphical method work, prove that the intersection set of the feasible domain P_F and the supporting hyperplane whose normal is given by the negative cost vector $-C^T$ provides the optimal solution to a given linear programming problem.

23. Find the solution of following linear programming problems using the graphical method

(a)
$$\begin{aligned} \min \quad & z = -x_1 + 2x_2 \\ \text{s.t.} \quad & -x_1 + 3x_2 \leq 10 \\ & x_1 + x_2 \leq 6 \\ & x_1 - x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(b)
$$\begin{aligned} \max \quad & z = 3x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 - 2x_2 \leq -1 \\ & -x_1 + 2x_2 \geq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

24. Prove that a feasible region given by n variables and $n-2$ nonredundant equality constraints can be represented by two dimensional graph.

25. Let us consider the problem

$$\begin{aligned} \max \quad & z = 3x_1 - x_2 + x_3 + x_4 + x_5 - 10 \\ \text{s.t.} \quad & 3x_1 + x_2 + x_3 + x_4 + x_5 = 10 \\ & x_1 + x_3 + 2x_4 + x_5 = 15 \\ & 2x_1 + 3x_2 + x_3 + x_4 + 2x_5 = 12 \\ & x_1, x_2, x_3, x_4, x_4, x_5 \geq 0 \end{aligned}$$

Using Problem 24, write this as a two-dimensional LPP and then find its optimal solution by graphical method.

Suggestion. Do pivoting at x_3, x_4, x_5.

26. Consider the LPP:

$$\begin{aligned} \max \quad & z = x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 2 \\ & -x_1 + x_2 + x_4 = 4 \\ & x_2, x_3, x_4 \geq 0 \end{aligned}$$

(a) Determine all the basic feasible solutions of problem;

(b) Express the problem in two-dimensional plane;

(c) Show that the optimal solution obtained using (a) and (b) are same.

(d) Is there any basic infeasible solution? if, yes find it.

27. What difficulty arises if all the constraints are taken as strict inequalities?

28. Show that by properly choosing c_i's in objective function of an LPP, every vertex can be made optimal.

29. Let X be a basic solution of the system $AX = b$ having both positive and negative variables. How can X be reduced to a BFS.



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