3

Lattice Boltzmann Method

3.1 Introduction

The lattice Boltzmann method is a discrete computational method based upon the lattice gas automata - a simplified, fictitious molecular model. It consists of three basic tasks: lattice Boltzmann equation, lattice pattern and local equilibrium distribution function. The former two are standard, which is the same for fluid flows. The latter determine what flow equations are solved by the lattice Boltzmann model, which is often derived for certain flow equations such as the equations for shallow water flows. These tasks are described in this chapter.

3.2 Lattice Boltzmann Equation

The lattice Boltzmann method is originally evolved from the LGA, i.e. the equation for the LGA is replaced with the lattice Boltzmann equation. As mentioned in the preface, the LBE can effectively be viewed as a special discrete form of the continuum Boltzmann equation [10], leading it to be self-explanatory in statistical physics. It is generally valid for fluid flows such as shallow water flows. According to the origin of the LBM, the lattice Boltzmann equation consists of two steps: a streaming step and a collision step. In the streaming step, the particles move to the neighbouring lattice points which is governed by

$$f_\alpha(\mathbf{x} + \mathbf{e}_\alpha \Delta t, t + \Delta t) = f'_\alpha(\mathbf{x}, t) + \frac{\Delta t}{N_\alpha e^2} e_{\alpha i} F_i(\mathbf{x}, t), \tag{3.1}$$

where f_α is the distribution function of particles; f_α is the value of f'_α after the streaming; $e = \Delta x / \Delta t$; Δx is the lattice size; Δt is the time step; F_i is the component of the force in i direction; $\mathbf{e}_\alpha$ is the velocity vector of a particle in the α link and N_α is a constant, which is decided by the lattice pattern as

$$N_\alpha = \frac{1}{e^2} \sum_\alpha e_{\alpha i} e_{\alpha i}. \tag{3.2}$$

In the collision step, the arriving particles at the points interact one another and change their velocity directions according to scattering rules, which is expressed as

$$f'_\alpha(\mathbf{x}, t) = f_\alpha(\mathbf{x}, t) + \Omega_\alpha[f(\mathbf{x}, t)], \tag{3.3}$$

in which Ω_α is the collision operator which controls the speed of change in f_α during collision.

Theoretically, Ω_α is generally a matrix, which is decided by the microscopic dynamics. Higuera and Jiménez [11] first introduced an idea to linearize the collision operator around its local equilibrium state. This greatly simplifies the collision operator. Based on this idea, Ω_α can be expanded about its equilibrium value [36],

$$\Omega_\alpha(f) = \Omega_\alpha(f^{eq}) + \frac{\partial \Omega_\alpha(f^{eq})}{\partial f_\beta}(f_\beta - f_\beta^{eq}) + O[(f_\beta - f_\beta^{eq})^2], \tag{3.4}$$

where f_α^{eq} is the local equilibrium distribution function.

The solution process of the lattice Boltzmann equation is characterized by $f_\beta \to f_\beta^{eq}$, implying $\Omega_\alpha(f^{eq}) \approx 0$. After the higher-order terms in Eq. (3.4) are neglected, we obtain a linearized collision operator,

$$\Omega_\alpha(f) \approx \frac{\partial \Omega_\alpha(f^{eq})}{\partial f_\beta}(f_\beta - f_\beta^{eq}). \tag{3.5}$$

If assuming the local particle distribution relaxes to an equilibrium state at a single rate τ [12, 13],

$$\frac{\partial \Omega_\alpha(f^{eq})}{\partial f_\beta} = -\frac{1}{\tau}\delta_{\alpha\beta}, \tag{3.6}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta function,

$$\delta_{\alpha\beta} = \begin{cases} 0, & \alpha \neq \beta, \\ 1, & \alpha = \beta, \end{cases} \tag{3.7}$$

we can write Eq. (3.5) as

$$\Omega_\alpha(f) = -\frac{1}{\tau}\delta_{\alpha\beta}(f_\beta - f_\beta^{eq}), \tag{3.8}$$

resulting in the lattice BGK collision operator [14],

$$\Omega_\alpha(f) = -\frac{1}{\tau}(f_\alpha - f_\alpha^{eq}), \tag{3.9}$$

and τ is called as the single relaxation time. This makes the lattice Boltzmann equation extremely simple and efficient; hence it is widely used in a lattice

Boltzmann model for fluid flows. With reference to Eq. (3.9), the streaming and collision steps are usually combined into the following lattice Boltzmann equation,

$$f_\alpha(\mathbf{x} + \mathbf{e}_\alpha \Delta t, t + \Delta t) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tau}(f_\alpha - f_\alpha^{eq}) + \frac{\Delta t}{N_\alpha e^2} e_{\alpha i} F_i, \tag{3.10}$$

which is the most popular form of the lattice Boltzmann equation in use today.

3.3 Lattice Pattern

Lattice pattern in the lattice Boltzmann method has two functions: representing grid points and determining particles' motions. The former plays a similar role to that in the traditional numerical methods. The latter defines a microscopic model for molecular dynamics. In addition, the constant N_α in Eq. (3.10) is determined by the lattice pattern.

In 2D situations, there are generally two types of lattice patterns: square lattice and hexagonal lattice suggested in the literature. Their examples are shown in Figs. 3.1 and 3.2, respectively. Depending to the number of particle speed at lattice node, the square lattice can have 4-speed, 5-speed, 8-speed or 9-speed models, and the hexagonal lattice can have 6-speed and 7-speed models. Not all of these models have sufficient lattice symmetry which is a dominant requirement for recovery of the correct flow equations [7]. Theoretical analysis and numerical studies indicate that both 9-speed square lattice (see Fig. 3.1) and 7-speed hexagonal lattice (see Fig. 3.2) have such property and satisfactory performance in numerical simulations. They are widely used in the lattice Boltzmann methods. However, the recent study has shown that the 9-speed square lattice usually gives more accurate results than that based on hexagonal lattice [37]. Furthermore, the use of the square lattice provides an easy way to implement different boundary conditions [17], e.g. only by using the square lattice, can the force term associated with a gradient and boundary conditions be accurately and easily determined. Hence the 9-speed square lattice is preferred in use today and it is applied in this book throughout (see Appendix A for a lattice Boltzmann model on the 7-speed hexagonal lattice).

On the 9-speed square lattice shown in Fig. 3.1, each particle moves one lattice unit at its velocity only along the eight links indicated with 1 - 8, in which 0 indicates the rest particle with zero speed. The velocity vector of particles is defined by

$$\mathbf{e}_\alpha = \begin{cases} (0,0), & \alpha = 0, \\ e\left[\cos\frac{(\alpha-1)\pi}{4}, \sin\frac{(\alpha-1)\pi}{4}\right], & \alpha = 1,3,5,7, \\ \sqrt{2}e\left[\cos\frac{(\alpha-1)\pi}{4}, \sin\frac{(\alpha-1)\pi}{4}\right], & \alpha = 2,4,6,8. \end{cases} \tag{3.11}$$

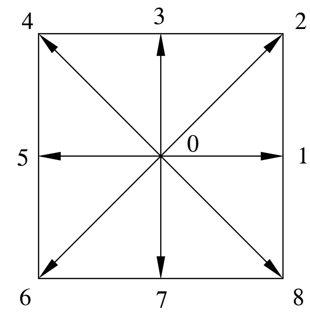

Fig. 3.1. 9-speed square lattice.

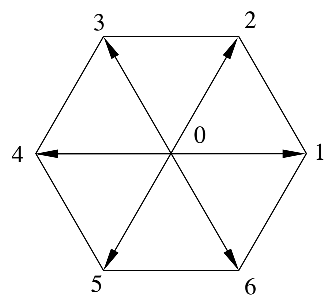

Fig. 3.2. 7-speed hexagonal lattice.

It is easy to show that the 9-speed square lattice has the following basic features,

$$\sum_\alpha e_{\alpha i} = \sum_\alpha e_{\alpha i} e_{\alpha j} e_{\alpha k} = 0, \tag{3.12}$$

$$\sum_\alpha e_{\alpha i} e_{\alpha j} = 6e^2 \delta_{ij}, \tag{3.13}$$

$$\sum_\alpha e_{\alpha i} e_{\alpha j} e_{\alpha k} e_{\alpha l} = 4e^4 (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - 6e^4 \Delta_{ijkl}. \tag{3.14}$$

where $\Delta_{ijkl} = 1$ if $i = j = k = l$, otherwise $\Delta_{ijkl} = 0$.

Using Eq. (3.11) to evaluate the terms in Eq. (3.2), we have

$$N_\alpha = \frac{1}{e^2} \sum_\alpha e_{\alpha x} e_{\alpha x} = \frac{1}{e^2} \sum_\alpha e_{\alpha y} e_{\alpha y} = 6. \tag{3.15}$$

Substitution of the above equation into Eq. (3.10) leads to

$$f_\alpha(\mathbf{x} + \mathbf{e}_\alpha \Delta t, t + \Delta t) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tau}(f_\alpha - f_\alpha^{eq}) + \frac{\Delta t}{6e^2} e_{\alpha i} F_i, \tag{3.16}$$

which is the most common form of a lattice Boltzmann model used for simulating fluid flows.

3.4 Local Equilibrium Distribution Function

Determining a suitable local equilibrium function plays an essential role in the lattice Boltzmann method. It is this function that decides what flow equations are solved by means of the lattice Boltzmann equation (3.16). In order to apply the equation (3.16) for solution of the 2D shallow water equations (2.53) and (2.54), we derive a suitable local equilibrium function f_α^{eq} in this section.

According to the theory of the lattice gas automatalattice gas automata, an equilibrium function is the Maxwell-Boltzmann equilibrium distribution function, which is often expanded as a Taylor series in macroscopic velocity to its second order [10, 38]. However, the use of such equilibrium function in the lattice Boltzmann equation can recover the Navier-Stokes equation only. This severely limits the capability of the method to solve flow equations. Thus an alternative and powerful way is to assume that an equilibrium function can be expressed as a power series in macroscopic velocity [39], i.e.,

$$f_\alpha^{eq} = A_\alpha + B_\alpha e_{\alpha i} u_i + C_\alpha e_{\alpha i} e_{\alpha j} u_i u_j + D_\alpha u_i u_i. \tag{3.17}$$

This turns out to be a general approach, which is successfully used for solution of various flow problems [40, 41], demonstrating its accuracy and suitability. Hence it is used. Since the equilibrium function has the same symmetry as the lattice (see Fig. 3.1), there must be

$$A_1 = A_3 = A_5 = A_7 = \bar{A}, \qquad A_2 = A_4 = A_6 = A_8 = \tilde{A}, \tag{3.18}$$

and similar expressions for B_α, C_α and D_α. Accordingly, it is convenient to write Eq. (3.17) in the following form,

$$f_\alpha^{eq} = \begin{cases} A_0 + D_0 u_i u_i, & \alpha = 0, \\ \bar{A} + \bar{B} e_{\alpha i} u_i + \bar{C} e_{\alpha i} e_{\alpha j} u_i u_j + \bar{D} u_i u_i, & \alpha = 1,3,5,7, \\ \tilde{A} + \tilde{B} e_{\alpha i} u_i + \tilde{C} e_{\alpha i} e_{\alpha j} u_i u_j + \tilde{D} u_i u_i, & \alpha = 2,4,6,8. \end{cases} \tag{3.19}$$

The coefficients such as A_0, $\bar{A}$ and $\tilde{A}$ can be determined based on the constraints on the equilibrium distribution function, i.e. it must obey the conservation relations such as mass and momentum conservations. For the shallow water equations, the local equilibrium distribution function (3.19) must satisfy the following three conditions,

$$\sum_\alpha f_\alpha^{eq}(\mathbf{x}, t) = h(\mathbf{x}, t), \tag{3.20}$$

$$\sum_\alpha e_{\alpha i} f_\alpha^{eq}(\mathbf{x}, t) = h(\mathbf{x}, t) u_i(\mathbf{x}, t), \tag{3.21}$$

$$\sum_\alpha e_{\alpha i} e_{\alpha j} f_\alpha^{eq}(\mathbf{x}, t) = \frac{1}{2} g h^2(\mathbf{x}, t) \delta_{ij} + h(\mathbf{x}, t) u_i(\mathbf{x}, t) u_j(\mathbf{x}, t). \tag{3.22}$$

Once the local equilibrium function (3.17) is determined under the above constraints, the calculation of the lattice Boltzmann equation (3.16) leads to the solution of the 2D shallow water equations (2.53) and (2.54) (The proof is given in Section 3.6).

Substituting Eqs. (3.19) into Eq. (3.20) yields

$$\begin{aligned} & A_0 + D_0 u_i u_i + \\ & 4\bar{A} + \sum_{i=1,3,5,7} \bar{B} e_{\alpha i} u_i + \sum_{i=1,3,5,7} \bar{C} e_{\alpha i} e_{\alpha j} u_i u_j + 4\bar{D} u_i u_i + \\ & 4\tilde{A} + \sum_{i=2,4,6,8} \tilde{B} e_{\alpha i} u_i + \sum_{i=2,4,6,8} \tilde{C} e_{\alpha i} e_{\alpha j} u_i u_j + 4\tilde{D} u_i u_i = h. \end{aligned} \tag{3.23}$$

After evaluating the terms in the above equation with Eq. (3.11) and equating the coefficients of h and $u_i u_i$, respectively, we have

$$A_0 + 4\bar{A} + 4\tilde{A} = h, \tag{3.24}$$

and

$$D_0 + 2e^2 \bar{C} + 4e^2 \tilde{C} + 4\bar{D} + 4\tilde{D} = 0. \tag{3.25}$$

Inserting Eqs. (3.19) to Eq. (3.21) leads to

$$\begin{aligned} & A_0 e_{\alpha i} + D_0 e_{\alpha i} u_j u_j + \\ & \sum_{\alpha=1,3,5,7} (\bar{A} e_{\alpha i} + \bar{B} e_{\alpha i} e_{\alpha j} u_j + \bar{C} e_{\alpha i} e_{\alpha j} e_{\alpha k} u_j u_k + \bar{D} e_{\alpha i} u_j u_j) + \\ & \sum_{\alpha=2,4,6,8} (\tilde{A} e_{\alpha i} + \tilde{B} e_{\alpha i} e_{\alpha j} u_j + \tilde{C} e_{\alpha i} e_{\alpha j} e_{\alpha k} u_j u_k + \tilde{D} e_{\alpha i} u_j u_j) = h u_i, \end{aligned} \tag{3.26}$$

from which we can obtain

$$2e^2 \bar{B} + 4e^2 \tilde{B} = h. \tag{3.27}$$

Substituting Eqs. (3.19) into Eq. (3.22) results in

$$\begin{aligned} & \sum_{\alpha=1,3,5,7} (\bar{A} e_{\alpha i} e_{\alpha j} + \bar{B} e_{\alpha i} e_{\alpha j} e_{\alpha k} u_k + \bar{C} e_{\alpha i} e_{\alpha j} e_{\alpha k} e_{\alpha l} u_k u_l + \bar{D} e_{\alpha i} e_{\alpha j} u_k u_k) + \\ & \sum_{\alpha=2,4,6,8} (\tilde{A} e_{\alpha i} e_{\alpha j} + \tilde{B} e_{\alpha i} e_{\alpha j} e_{\alpha k} u_k + \tilde{C} e_{\alpha i} e_{\alpha j} e_{\alpha k} e_{\alpha l} u_k u_l + \tilde{D} e_{\alpha i} e_{\alpha j} u_k u_k) \\ & \qquad = \frac{1}{2} g h^2 \delta_{ij} + h u_i u_j \end{aligned} \tag{3.28}$$

Use of Eq. (3.11) to simplify the above equation leads to

$$\begin{aligned} & 2\bar{A} e^2 \delta_{ij} + 2\bar{C} e^4 u_i u_i + 2\bar{D} e^2 u_i u_i + 4\tilde{A} e^2 \delta_{ij} + \\ & 8\tilde{C} e^4 u_i u_j + 4\tilde{C} e^4 u_i u_i + 4\tilde{D} e^2 u_i u_i = \frac{1}{2} g h^2 \delta_{ij} + h u_i u_j, \end{aligned} \tag{3.29}$$

which provides the following four relations,

$$2e^2\bar{A} + 4e^2\tilde{A} = \frac{1}{2}gh^2, \tag{3.30}$$

$$8e^4\tilde{C} = h, \tag{3.31}$$

$$2e^4\bar{C} = h, \tag{3.32}$$

$$2e^2\bar{D} + 4e^2\tilde{D} + 4e^4\tilde{C} = 0. \tag{3.33}$$

Combining of Eq. (3.31) and Eq. (3.32) gives

$$\bar{C} = 4\tilde{C}. \tag{3.34}$$

From the symmetry of the lattice, based on Eq. (3.34), we have good reason to assume the three additional relations as follows,

$$\bar{A} = 4\tilde{A}, \tag{3.35}$$

$$\bar{B} = 4\tilde{B}, \tag{3.36}$$

$$\bar{D} = 4\tilde{D}. \tag{3.37}$$

Solution of Eqs. (3.24), (3.25), (3.27) and (3.30) - (3.37) results in

$$A_0 = h - \frac{5gh^2}{6e^2}, \qquad D_0 = -\frac{2h}{3e^2}, \tag{3.38}$$

$$\bar{A} = \frac{gh^2}{6e^2}, \qquad \bar{B} = \frac{h}{3e^2}, \qquad \bar{C} = \frac{h}{2e^4}, \qquad \bar{D} = -\frac{h}{6e^2}, \tag{3.39}$$

$$\tilde{A} = \frac{gh^2}{24e^2}, \qquad \tilde{B} = \frac{h}{12e^2}, \qquad \tilde{C} = \frac{h}{8e^4}, \qquad \tilde{D} = -\frac{h}{24e^2}. \tag{3.40}$$

Substitution of the above equations (3.38) - (3.40) into Eq. (3.19) leads to the following local equilibrium function,

$$f_\alpha^{eq} = \begin{cases} h - \frac{5gh^2}{6e^2} - \frac{2h}{3e^2} u_i u_i, & \alpha = 0, \\ \frac{gh^2}{6e^2} + \frac{h}{3e^2} e_{\alpha i} u_i + \frac{h}{2e^4} e_{\alpha i} e_{\alpha j} u_i u_j - \frac{h}{6e^2} u_i u_i, & \alpha = 1, 3, 5, 7, \\ \frac{gh^2}{24e^2} + \frac{h}{12e^2} e_{\alpha i} u_i + \frac{h}{8e^4} e_{\alpha i} e_{\alpha j} u_i u_j - \frac{h}{24e^2} u_i u_i, & \alpha = 2, 4, 6, 8, \end{cases} \tag{3.41}$$

which is used in the lattice Boltzmann equation (3.16) for solution of the shallow water equations (2.53) and (2.54).

3.5 Macroscopic Properties

The lattice Boltzmann equation (3.16) with the local equilibrium function (3.41) form a lattice Boltzmann model for shallow water flows (LABSWE) on the square lattices, which is described by Zhou [15]. The remaining task is how to determine the physical quantities, water depth h and velocity u_i, as the solution to the shallow water equations (2.53) and (2.54). For this purpose, we examine the macroscopic properties of the lattice Boltzmann equation (3.16). Taking the sum of the zeroth moment of the distribution function in Eq. (3.16) over the lattice velocities leads to

$$\sum_\alpha [f_\alpha(\mathbf{x}+\mathbf{e}_\alpha \Delta t, t+\Delta t) - f_\alpha(\mathbf{x}, t)] = -\frac{1}{\tau}\sum_\alpha (f_\alpha - f_\alpha^{eq}) + \frac{\Delta t}{6e^2}\sum_\alpha e_{\alpha i} F_i. \quad (3.42)$$

Notice $\sum_\alpha e_{\alpha i} F_i = 0$, Eq. (3.42) becomes

$$\sum_\alpha [f_\alpha(\mathbf{x}+\mathbf{e}_\alpha \Delta t, t+\Delta t) - f_\alpha(\mathbf{x}, t)] = -\frac{1}{\tau}\sum_\alpha (f_\alpha - f_\alpha^{eq}). \quad (3.43)$$

In the lattice Boltzmann method, we have an explicit constraint to preserve conservative property, i.e. the cumulative mass and momentum which are the corresponding summations of the microdynamic mass and momentum are conserved. The mass conservation requires the following identity,

$$\sum_\alpha f_\alpha(\mathbf{x}+\mathbf{e}_\alpha \Delta t, t+\Delta t) \equiv \sum_\alpha f_\alpha(\mathbf{x}, t), \quad (3.44)$$

which is the continuity equation with microdynamic variables.

Substitution of the above equation into Eq. (3.43) leads to

$$\sum_\alpha f_\alpha(\mathbf{x}, t) = \sum_\alpha f_\alpha^{eq}(\mathbf{x}, t). \quad (3.45)$$

With reference to Eq. (3.20), the above expression in fact results in the definition for the physical quantity, water depth h, as

$$h(\mathbf{x}, t) = \sum_\alpha f_\alpha(\mathbf{x}, t). \quad (3.46)$$

Now, in order to find the expression for the velocity, we take the sum of the first moment of the distribution function in Eq. (3.16) over the lattice velocities,

$$\sum_\alpha e_{\alpha i}[f_\alpha(\mathbf{x}+\mathbf{e}_\alpha \Delta t, t+\Delta t) - f_\alpha(\mathbf{x}, t)] = -\frac{1}{\tau}\sum_\alpha e_{\alpha i}(f_\alpha - f_\alpha^{eq}) + \frac{\Delta t}{6e^2}\sum_\alpha e_{\alpha i} e_{\alpha j} F_j, \quad (3.47)$$

which can be simplified with Eq. (3.13) and rearranged as

$$\sum_\alpha e_{\alpha i}[f_\alpha(\mathbf{x}+\mathbf{e}_\alpha \Delta t, t+\Delta t) - f_\alpha(\mathbf{x}, t)] = F_i \Delta t - \frac{1}{\tau}\sum_\alpha e_{\alpha i}(f_\alpha - f_\alpha^{eq}). \quad (3.48)$$

which reflects the evolution of cumulative momentum in the distribution function. Again, the momentum conservation requires the following identity,

$$\sum_\alpha e_{\alpha i}[f_\alpha(\mathbf{x}+\mathbf{e}_\alpha \Delta t, t+\Delta t) - f_\alpha(\mathbf{x}, t)] \equiv F_i \Delta t. \quad (3.49)$$

which is the momentum equation with microdynamic variables, representing the Newton's second law.

Substitution of Eq. (3.49) into Eq. (3.48) provides

$$\sum_\alpha e_{\alpha i} f_\alpha(\mathbf{x}, t) = \sum_\alpha e_{\alpha i} f_\alpha^{eq}(\mathbf{x}, t). \quad (3.50)$$

The use of Eq. (3.21) in the above equation leads to the definition for another physical variable, velocity u_i, as

$$u_i(\mathbf{x}, t) = \frac{1}{h(\mathbf{x}, t)}\sum_\alpha e_{\alpha i} f_\alpha(\mathbf{x}, t). \quad (3.51)$$

As can be seen from Eqs. (3.20), (3.21), (3.46) and (3.51), it seems that there are redundant definitions for the physical variables h and u_i. However, a careful examination indicates that this is just an important feature which is peculiar to the lattice Boltzmann method. First of all, the local equilibrium function f_α^{eq} given by (3.41) satisfy Eqs. (3.20) and (3.21). Secondly, the distribution function f_α relaxes to its local equilibrium function f_α^{eq} via the lattice Boltzmann equation (3.10). Finally, the physical variables determined from Eqs. (3.46) and (3.51) will guarantee that both Eqs. (3.45) and (3.50) hold true, hence preserving the two identities (3.44) and (3.49) during the numerical procedure. This makes the lattice Boltzmann method very elegant and effectively explains why the method is accurate and conservative.

3.6 Recovery of the Shallow Water Equations

In order to prove that the depth and the velocities calculated from Eqs. (3.46) and (3.51) are the solution to the shallow water equations, we perform the Chapman-Enskog expansion to the lattice Boltzmann equation (3.16) that recovers the macroscopic equations (2.53) and (2.54).

Assuming Δt is small and is equal to ε,

$$\Delta t = \varepsilon, \quad (3.52)$$

we have the equation (3.16) expressed as

$$f_\alpha(\mathbf{x}+\mathbf{e}_\alpha\varepsilon, t+\varepsilon) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tau}(f_\alpha - f_\alpha^{eq}) + \frac{\varepsilon}{6e^2} e_{\alpha j} F_j. \tag{3.53}$$

Taking a Taylor expansion to the first term on the left-hand side of the above equation in time and space around point $(\mathbf{x}, t)$ leads to

$$\varepsilon(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}) f_\alpha + \frac{1}{2}\varepsilon^2(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j})^2 f_\alpha + \mathcal{O}(\varepsilon^2) = -\frac{1}{\tau}(f_\alpha - f_\alpha^{(0)}) + \frac{\varepsilon}{6e^2} e_{\alpha j} F_j. \tag{3.54}$$

We can also expand f_α around $f_\alpha^{(0)}$,

$$f_\alpha = f_\alpha^{(0)} + \varepsilon f_\alpha^{(1)} + \varepsilon^2 f_\alpha^{(2)} + \mathcal{O}(\varepsilon^2), \tag{3.55}$$

where $f_\alpha^{(0)} = f_\alpha^{eq}$.

The equation (3.54) to order ε is

$$(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}) f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(1)} + \frac{1}{6e^2} e_{\alpha j} F_j \tag{3.56}$$

and to order ε^2 is

$$(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}) f_\alpha^{(1)} + \frac{1}{2}(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j})^2 f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(2)}. \tag{3.57}$$

Substitution of Eq. (3.56) into Eq. (3.57), after rearranged, gives

$$(1 - \frac{1}{2\tau})(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j}) f_\alpha^{(1)} = -\frac{1}{\tau} f_\alpha^{(2)} - \frac{1}{2}(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j})(\frac{1}{6e^2} e_{\alpha k} F_k). \tag{3.58}$$

Taking $\sum$ [(3.56) + $\varepsilon\times$ (3.58)] about α provides

$$\frac{\partial}{\partial t}(\sum_\alpha f_\alpha^{(0)}) + \frac{\partial}{\partial x_j}(\sum_\alpha e_{\alpha j} f_\alpha^{(0)}) = -\varepsilon\frac{1}{12e^2}\frac{\partial}{\partial x_j}(\sum_\alpha e_{\alpha j} e_{\alpha k} F_k). \tag{3.59}$$

If the first-order accuracy for the force term is applied, evaluation of the other terms in the above equation using Eqs. (3.11) and (3.41) results in

$$\frac{\partial h}{\partial t} + \frac{\partial (h u_j)}{\partial x_j} = 0, \tag{3.60}$$

which is the continuity equation (2.53) for shallow water flows.

From $\sum e_{\alpha i}$[(3.56) + $\varepsilon\times$ (3.58)] about α, we have

$$\frac{\partial}{\partial t}(\sum_\alpha e_{\alpha i} f_\alpha^{(0)}) + \frac{\partial}{\partial x_j}(\sum_\alpha e_{\alpha i} e_{\alpha j} f_\alpha^{(0)}) + \varepsilon(1 - \frac{1}{2\tau})\frac{\partial}{\partial x_j}(\sum_\alpha e_{\alpha i} e_{\alpha j} f_\alpha^{(1)}) = F_j \delta_{ij} - \varepsilon\frac{1}{2}\sum_\alpha e_{\alpha i}(\frac{\partial}{\partial t} + e_{\alpha j}\frac{\partial}{\partial x_j})(\frac{1}{6e^2} e_{\alpha j} F_j). \tag{3.61}$$

Again, if the first-order accuracy for the force term is used, after the other terms is simplified with Eqs. (3.11) and (3.41), the above equation becomes

$$\frac{\partial(hu_i)}{\partial t} + \frac{\partial(hu_iu_j)}{\partial x_j} = -g\frac{\partial}{\partial x_i}(\frac{h^2}{2}) - \frac{\partial}{\partial x_j}\Lambda_{ij} + F_i, \tag{3.62}$$

with

$$\Lambda_{ij} = \frac{\varepsilon}{2\tau}(2\tau - 1)\sum_\alpha e_{\alpha i}e_{\alpha j}f_\alpha^{(1)}. \tag{3.63}$$

With reference to Eq. (3.56), using Eq. (3.41), after some algebra, we obtain

$$\Lambda_{ij} \approx -\nu\left[\frac{\partial(hu_i)}{\partial x_j} + \frac{\partial(hu_j)}{\partial x_i}\right]. \tag{3.64}$$

Substitution of Eq. (3.64) into Eq. (3.62) leads to

$$\frac{\partial(hu_i)}{\partial t} + \frac{\partial(hu_iu_j)}{\partial x_j} = -g\frac{\partial}{\partial x_i}(\frac{h^2}{2}) + \nu\frac{\partial^2(hu_i)}{\partial x_j\partial x_j} + F_i, \tag{3.65}$$

with the kinematic viscosity ν defined by

$$\nu = \frac{e^2\Delta t}{6}(2\tau - 1) \tag{3.66}$$

and the force term F_i expressed as

$$F_i = -gh\frac{\partial z_b}{\partial x_i} + \frac{\tau_{wi}}{\rho} - \frac{\tau_{bi}}{\rho} + E_i. \tag{3.67}$$

The equation (3.65) is just the momentum equation (2.54) for the shallow water equations.

It should be pointed that the above proof shows that the lattice Boltzmann equation (3.16) is only first-order accurate for the recovered macroscopic continuity and momentum equations. We can prove that the use of a suitable form for the force term can make Eq. (3.16) second-order accurate for the recovered macroscopic continuity and momentum equations (the detail is given in Section 4).

3.7 Stability

The lattice Boltzmann equation (3.16) is a discrete form of a numerical method. It may suffer from a numerical instability like any other numerical methods. Theoretically, the stability conditions are not generally known for the method. In practice, a lot of computations have shown that the method is often stable if some basic conditions are satisfied. They are described now.

First of all, if a solution from the lattice Boltzmann equation (3.16) represents a physical water flow, there must be diffusion phenomena. This implies that the kinematic viscositykinematic viscosity ν should be positive [42], i.e. from Eq. (3.66) we have

$$\nu = \frac{e^2 \Delta t}{6}(2\tau - 1) > 0. \tag{3.68}$$

Thus an apparent constraint on the relaxation time is

$$\tau > \frac{1}{2}. \tag{3.69}$$

Secondly, the magnitude of the resultant velocity is smaller than the speed calculated with the lattice size divided by the time step,

$$\frac{u_j u_j}{e^2} < 1, \tag{3.70}$$

and so is the celerity,

$$\frac{gh}{e^2} < 1. \tag{3.71}$$

Finally, since the lattice Boltzmann method is limited to low speed flows, this suggests that the current lattice Boltzmann method is suitable for subcritical shallow water flows, which brings the final constraint on the method as

$$\frac{u_j u_j}{gh} < 1. \tag{3.72}$$

The first three conditions (3.69) - (3.71) can be easily satisfied by using suitable values for the relaxation time, the lattice size and the time step. The last condition (3.72) is the limitation of the lattice Boltzmann method.

3.8 Closure

In the lattice Boltzmann method, the central quantity is the particle distribution function f_α which is governed by the lattice Boltzmann equation (3.16). Its evolution is carried out on the square lattices in the computational domain. The physical variables, water depth and velocities, are calculated in terms of the distribution function via Eqs. (3.46) and (3.51). It has shown that the depth and the velocities determined in this way satisfy the shallow water equations. The four conditions for stability are described, which are very useful in practical computations. Apparently, the lattice Boltzmann method is a numerical technique for an indirect solution of flow equations through a microscopic approach to a macroscopic phenomenon. This bridges the gap between discrete microscopic and continuum macroscopic phenomena, providing a very powerful tool for modeling a wide variety of complex fluid flows.

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