

# Table of Contents

<b>Introduction</b>	1
Basic notation	10
<b>1 Getting started</b>	11
1.1 The axiomatical system of Hrbáček set theory	12
1.1a The universe of <b>HST</b>	12
1.1b Axioms for the external universe	14
1.1c Axioms for standard and internal sets	14
1.1d Well-founded sets	16
1.1e The $\in$ -structure of internal and well-founded sets	17
1.1f Axioms for sets of standard size	19
1.1g Putting it all together	20
1.1h Zermelo – Fraenkel theory <b>ZFC</b>	20
1.2 Basic elements of the nonstandard universe	22
1.2a How to define fundamental set theoretic notions in <b>HST</b>	22
1.2b Closure properties and absoluteness	22
1.2c Ordinals and cardinals	24
1.2d Natural numbers, finite and $\ast$ -finite sets	25
1.2e Hereditarily finite sets	28
1.3 Sets of standard size	29
1.3a Cardinalities of sets of standard size	29
1.3b Saturation and the Hrbáček paradox	30
1.3c The principle of Extension	32
1.4 The class $\Delta_2^{ss}$	34
1.4a Basic properties of $\Delta_2^{ss}$	34
1.4b Cuts (initial segments) of $\ast$ -ordinals	35
1.4c Monads and transversals	37
1.4d On non-well-founded cardinalities	38
1.4e Small and large sets	40
1.5 Some finer points	42
1.5a Von Neumann hierarchy and Reflection in <b>ZFC</b>	42
1.5b Von Neumann hierarchy over internal sets in <b>HST</b>	44
1.5c Classes and structures	45
1.5d Interpretations	47
1.5e Models	48
1.5f Simulation of models of <b>ZFC</b>	49
1.5g Asterisk is an elementary embedding	50
<i>Historical and other notes to Chapter 1</i>	52

## XII Table of Contents

<b>2</b>	<b>Elementary real analysis in the nonstandard universe</b>	<b>53</b>
2.1	Hyperreal line	54
2.1a	Hyperreals	54
2.1b	Fundamentals of nonstandard real analysis	56
2.1c	Directed Saturation	57
2.1d	Nonstandard characterization of closed and compact sets	58
2.2	Sequences and functions	59
2.2a	Limits	60
2.2b	Continuous functions	61
2.2c	Intermediate value theorem	62
2.2d	Robinson's lemma and uniform limits	62
2.3	Topics in nonstandard real analysis	64
2.3a	Shadows and equivalences	64
2.3b	Near-standard elements	66
2.3c	Topology	69
2.4	Two special applications	73
2.4a	Euler factorization of the sine function	73
2.4b	Jordan curve theorem	76
	<i>Historical and other notes to Chapter 2</i>	81
<b>3</b>	<b>Theories of internal sets</b>	<b>83</b>
3.1	Introduction to internal set theories	84
3.1a	Internal set theory	84
3.1b	Bounded set theory	86
3.1c	Internal sets interpret <b>BST</b> in the external universe	87
3.1d	Basic internal set theory	88
3.1e	Standard natural numbers and standard finite sets	90
3.1f	Remarks on Basic Idealization and Saturation	92
3.2	Development of bounded set theory	93
3.2a	Half-bounded forms of Idealization	93
3.2b	Reduction to two "external" quantifiers	94
3.2c	Finite axiomatizability of <b>BST</b> and other corollaries	95
3.2d	Collection in <b>BST</b>	97
3.2e	Other basic theorems of <b>BST</b>	99
3.2f	Introduction to the problem of external sets	101
3.2g	More on "external sets" in <b>BST</b>	104
3.3	Internal theories with partial Saturation	105
3.3a	Two schemes of partially saturated internal theories	105
3.3b	$\kappa$ -deep Basic Idealization scheme	106
3.3c	$\kappa$ -size Basic Idealization scheme	109
3.4	Development of Nelson's internal set theory	111
3.4a	Bounded sets in <b>IST</b>	111
3.4b	Bounded formulas: reduction to two "external" quantifiers	113
3.4c	Collection in <b>IST</b>	114
3.4d	Uniqueness in <b>IST</b>	117
3.5	Truth definition in internal set theory	118
3.5a	Truth definition for the standard universe	118
3.5b	Connection with the ordinary truth	120
3.5c	Extension of the definition of formal truth	122

3.6	Second edition of <b>IST</b> .....	124
3.6a	Standard and nonstandard theories of Nelson's system ....	124
3.6b	The background nonstandard universe .....	125
3.6c	Three "myths" of <b>IST</b> .....	127
	<i>Historical and other notes to Chapter 3</i> .....	129
4	<b>Metamathematics of internal theories</b> .....	131
4.1	Outline of metamathematical properties .....	132
4.1a	Nonstandard extensions of structures .....	132
4.1b	Nonstandard extensions of theories .....	133
4.1c	Comments .....	134
4.1d	Metamathematics of internal theories: the main results ....	136
4.2	Ultrapowers and saturated extensions .....	138
4.2a	Saturated structures and nonstandard set theories .....	138
4.2b	Quotient power extensions .....	140
4.2c	Adequate and good ultrafilters and ultrapowers .....	142
4.2d	Elementary chains of structures .....	144
4.3	Metamathematics of <b>BST</b> .....	146
4.3a	Warmup: several examples .....	146
4.3b	Infinite Fubini products of adequate ultrafilters .....	148
4.3c	Standard core interpretation of <b>BST</b> in <b>ZFC</b> .....	150
4.3d	Saturated standard core interpretation .....	152
4.4	The conservativity and equiconsistency of <b>IST</b> .....	154
4.4a	Good extensions of von Neumann sets in <b>ZFC</b> universe ...	154
4.4b	Iterated adequate extensions of von Neumann sets .....	155
4.4c	Iterated adequate extensions in the $\vartheta$ -version of <b>ZFC</b> ....	156
4.4d	Long iterated quotient power chains .....	156
4.4e	Conservativity of <b>IST</b> by inner models .....	157
4.5	Non-reducibility of <b>IST</b> .....	159
4.5a	The minimality axiom .....	159
4.5b	The source of counterexamples .....	160
4.5c	The ultrafilter .....	161
4.5d	"Definable" adequate quotient power .....	163
4.5e	Corollaries and remarks .....	164
4.6	Interpretability of <b>IST</b> in a standard theory .....	166
4.6a	Standard theory with a global choice and a truth predicate	166
4.6b	Formally definable classes .....	168
4.6c	A nonstandard theory extending <b>IST</b> .....	169
4.6d	The ultrafilter .....	170
4.6e	The interpretation .....	173
4.6f	Extendibility of standard models .....	175
	<i>Historical and other notes to Chapter 4</i> .....	176
5	<b>Definable external sets and metamathematics of HST</b> .....	179
5.1	Introduction to metamathematics of <b>HST</b> .....	180
5.1a	Internal core embeddings and interpretability .....	180
5.1b	Metamathematics of <b>HST</b> : an overview .....	181
5.2	From internal to elementary external sets .....	184
5.2a	Interpretation of <b>EEST</b> in <b>BST</b> .....	184

# XIV Table of Contents

5.2b	Elementary external sets in external theories.....	186
5.2c	Some basic theorems of <b>EEST</b> .....	188
5.2d	Standard size, natural numbers, finiteness in <b>EEST</b> .....	189
5.3	Assembling of external sets in <b>HST</b> .....	191
5.3a	Well-founded trees .....	191
5.3b	Coding of the assembling construction .....	192
5.3c	Examples of codes .....	193
5.3d	Regular codes .....	195
5.4	From elementary external to all external sets .....	196
5.4a	The domain of the interpretation .....	196
5.4b	Basic relations between codes .....	198
5.4c	The structure of basic relations .....	200
5.4d	The interpretation and the embedding .....	202
5.4e	Verification of the <b>HST</b> axioms .....	204
5.4f	Superposition of interpretations .....	207
5.4g	The problem of external sets revisited .....	209
5.5	The class $\mathbb{L}[\emptyset]$ : sets constructible from internal sets .....	211
5.5a	Sets constructible from internal sets .....	211
5.5b	Proof of the theorem on I-constructible sets .....	212
5.5c	The axiom of $\mathbb{L}$ -constructibility .....	214
5.5d	Transfinite constructions in $\mathbb{L}[\emptyset]$ .....	215
	<i>Historical and other notes to Chapter 5</i> .....	217
6	<b>Partially saturated universes and the Power Set problem</b> .....	219
6.1	Internal subuniverses .....	220
6.1a	Some basic definitions and results .....	220
6.1b	Relative standardness .....	221
6.1c	Simple relative standardness .....	222
6.1d	Gordon classes .....	224
6.1e	Associated structures .....	225
6.1f	More on internal subuniverses .....	228
6.1g	Appendix: Kunen's theorem .....	229
6.2	Partially saturated internal universes .....	230
6.2a	Partially saturated classes $\mathbb{I}_\kappa$ .....	230
6.2b	Good internal subuniverses .....	232
6.2c	Internal universes over complete sets .....	233
6.3	External universes .....	237
6.3a	External universes and internal core extensions .....	237
6.3b	Von Neumann construction over non-transitive classes ....	239
6.3c	Absoluteness for external subuniverses .....	240
6.4	Partially saturated external universes .....	241
6.4a	Partially saturated external theories .....	241
6.4b	Extensions of thin classes .....	243
6.4c	Constructible extensions .....	244
6.4d	Constructible extensions of self-definable classes .....	246
6.4e	The classes $\mathbb{L}[\mathbb{I}_\kappa]$ .....	248
6.4f	External universes over complete sets .....	249
6.4g	Collapse onto a transitive class .....	251
6.4h	Outline of applications: subuniverses satisfying Power Set..	252

	<i>Historical and other notes to Chapter 6</i> .....	254
<b>7</b>	<b>Forcing extensions of the nonstandard universe</b> .....	257
7.1	Generic extensions of models of <b>HST</b> .....	258
7.1a	Ground model .....	258
7.1b	Regular extensions .....	259
7.1c	Forcing notions and names .....	260
7.1d	Adding a set .....	261
7.1e	Forcing relation .....	263
7.1f	Generic extensions and the truth lemma .....	266
7.1g	The extension models <b>HST</b> .....	267
7.2	Applications: collapse maps and isomorphisms .....	270
7.2a	Making two internal sets equinumerous .....	270
7.2b	Internal preserving bijections .....	272
7.2c	Making elementarily equivalent structures isomorphic .....	273
7.2d	The forcing notion .....	274
7.2e	Key lemma .....	276
7.2f	Generic isomorphisms .....	278
7.3	Consistency of the isomorphism property .....	279
7.3a	The product forcing notion .....	280
7.3b	Externalization .....	281
7.3c	Restricted forcing relations .....	282
7.3d	Automorphisms and the restriction property .....	283
7.3e	The product generic extension .....	284
	<i>Historical and other notes to Chapter 7</i> .....	287
<b>8</b>	<b>Other nonstandard theories</b> .....	289
8.1	Nonstandard set theory of Kawai .....	290
8.1a	The axioms of Kawai's theory .....	290
8.1b	Metamathematical properties .....	292
8.1c	Special model axiom .....	293
8.2	"Nonstandard set theory" of Hrbáček .....	295
8.2a	Axioms .....	295
8.2b	Additional axioms of Collection .....	297
8.2c	Conservativity and consistency .....	298
8.2d	Remarks and exercises .....	301
8.3	Non-well-founded set theories .....	303
8.3a	Boffa's non-well-founded set theory .....	303
8.3b	Extensions of proper classes .....	305
8.3c	Applications to nonstandard analysis .....	306
8.3d	Alpha theory .....	307
8.3e	Interpretation of Alpha theory in <b>ZFBC</b> .....	311
8.4	Miscellanea: some other theories .....	312
8.4a	A theory with "definable" Saturation .....	312
8.4b	Stratified nonstandard set theories .....	313
8.4c	Nonstandard class theories .....	314
	<i>Historical and other notes to Chapter 8</i> .....	315

<b>9</b>	<b>"Hyperfinite" descriptive set theory</b>	<b>317</b>
9.1	Introduction to "hyperfinite" DST	319
9.1a	General set-up	319
9.1b	Comments on notation	320
9.1c	Borel and projective sets in a nonstandard domain	321
9.1d	Some applications of countable Saturation	323
9.1e	Operation A and Souslin sets	324
9.2	Operations, countably determined sets, shadows	325
9.2a	Operations and quantifiers	325
9.2b	Countably determined sets	327
9.2c	Shadows or standard part maps	329
9.3	Structure of the hierarchies	331
9.3a	Operations associated with Borel and projective classes	331
9.3b	The "shadow" theorem	332
9.3c	Closure properties of the classes	335
9.4	Some classical questions	338
9.4a	Separation and reduction	338
9.4b	Countably determined sets with countable cross-sections	340
9.4c	Countably determined sets with internal and $\Sigma_1^0$ cross-sections	343
9.4d	Uniformization	344
9.4e	Variations on Louveau's theme	347
9.4f	On sets with $\Pi_1^0$ cross-sections	350
9.5	Loeb measures	351
9.5a	Definitions and examples	351
9.5b	Loeb measurability of projective sets	353
9.5c	Approximations almost everywhere	354
9.5d	Randomness in a hyperfinite domain	356
9.5e	Law of Large Numbers	358
9.5f	Random sequences and hyperfinite gambling	359
9.6	Borel and countably determined cardinalities	362
9.6a	Preliminaries	362
9.6b	Borel cardinals and cuts	364
9.6c	Proof of the theorem on Borel cardinalities	366
9.6d	Complete classification of Borel cardinalities	368
9.6e	Countably determined cardinalities	368
9.7	Equivalence relations and quotients	370
9.7a	Silver's theorem for countably determined relations	371
9.7b	Application: nonstandard partition calculus	373
9.7c	Generalization	375
9.7d	Transversals of "countable" equivalence relations	376
9.7e	Equivalence relations of monad partitions	378
9.7f	Borel and countably determined reducibility	380
9.7g	Reducibility structure of monad partitions	382
	<i>Historical and other notes to Chapter 9</i>	386
	<b>References</b>	<b>389</b>
	<b>Index</b>	<b>397</b>



<http://www.springer.com/978-3-540-22243-9>

Nonstandard Analysis, Axiomatically

Kanovei, V.; Reeken, M.

2004, XVI, 410 p., Hardcover

ISBN: 978-3-540-22243-9