

---

# Contents

<b>The Objective Method: Probabilistic Combinatorial Optimization and Local Weak Convergence</b> <i>David Aldous, J. Michael Steele</i> .....	1
<b>The Random-Cluster Model</b> <i>Geoffrey Grimmett</i> .....	73
<b>Models of First-Passage Percolation</b> <i>C. Douglas Howard</i> .....	125
<b>Relaxation Times of Markov Chains in Statistical Mechanics and Combinatorial Structures</b> <i>Fabio Martinelli</i> .....	175
<b>Random Walks on Finite Groups</b> <i>Laurent Saloff-Coste</i> .....	263
<b>Index</b> .....	347

---

# The Objective Method: Probabilistic Combinatorial Optimization and Local Weak Convergence

David Aldous and J. Michael Steele

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	A Motivating Example: the Assignment Problem	3
1.2	A Stalking Horse: the Partial Matching Problem	4
1.3	Organization of the Survey	4
<b>2</b>	<b>Geometric Graphs and Local Weak Convergence</b>	<b>6</b>
2.1	Geometric Graphs	6
2.2	$\mathcal{G}_*$ as a Metric Space	7
2.3	Local Weak Convergence	8
2.4	The Standard Construction	8
2.5	A Prototype: The Limit of Uniform Random Trees	9
<b>3</b>	<b>Maximal Weight Partial Matching on Random Trees</b>	<b>12</b>
3.1	Weighted Matchings of Graphs in General	12
3.2	Our Case: Random Trees with Random Edge Weights	12
3.3	Two Obvious Guesses: One Right, One Wrong	12
3.4	Not Your Grandfather's Recursion	13
3.5	A Direct and Intuitive Plan	14
3.6	Characterization of the Limit of $B(T_n^{small})$	16
3.7	Characterization of the Limit of $B(T_n^{big})$	19
3.8	The Limit Theorem for Maximum Weight Partial Matchings	21
3.9	Closing the Loop: Another Probabilistic Solution of a Fixed-Point Equation	24
3.10	From Coupling to Stability – Thence to Convergence	26
3.11	Looking Back: Perspective on a Case Study	28
<b>4</b>	<b>The Mean-Field Model of Distance</b>	<b>29</b>
4.1	From Poisson Points in $\mathbb{R}^d$ to a Simple Distance Model	29
4.2	The Poisson Weighted Infinite Tree – or, the PWIT	31
4.3	The Cut-off Components of a Weighted Graph and a PWIT	32

4.4	The Minimum Spanning Forests of an Infinite Graph . . . . .	33
4.5	The Average Length Per Vertex of the MSF of a PWIT . . . . .	34
4.6	The Connection to Frieze's $\zeta(3)$ Theorem . . . . .	35
<b>5</b>	<b>Minimal Cost Perfect Matchings</b> . . . . .	<b>37</b>
5.1	A Natural Heuristic – Which Fails for a Good Reason . . . . .	38
5.2	Involution Invariance and the Standard Construction . . . . .	39
5.3	Involution Invariance and the Convergence of MSTs . . . . .	42
5.4	A Heuristic That Works by Focusing on the Unknown . . . . .	46
5.5	A Distributional Identity with a Logistic Solution . . . . .	47
5.6	A Stochastic Process that Constructs a Matching . . . . .	49
5.7	Calculation of a Limiting Constant: $\pi^2/6$ . . . . .	52
5.8	Passage from a PWIT Matching to a $K_n$ Matching . . . . .	53
5.9	Finally – Living Beyond One's Means . . . . .	55
<b>6</b>	<b>Problems in Euclidean Space</b> . . . . .	<b>56</b>
6.1	A Motivating Problem . . . . .	57
6.2	Far Away Places and Their Influence . . . . .	59
6.3	Euclidean Methods and Some Observations in Passing . . . . .	62
6.4	Recurrence of Random Walks in Limits of Planar Graphs . . . . .	65
<b>7</b>	<b>Limitations, Challenges, and Perspectives</b> . . . . .	<b>66</b>
	<b>References</b> . . . . .	<b>69</b>

## 1 Introduction

This survey describes a general approach to a class of problems that arise in combinatorial probability and combinatorial optimization. Formally, the method is part of weak convergence theory, but in concrete problems the method has a flavor of its own. A characteristic element of the method is that it often calls for one to introduce a new, infinite, probabilistic *object* whose *local properties* inform us about the *limiting properties* of a sequence of finite problems.

The name *objective method* hopes to underscore the value of shifting ones attention to the new, large random object with fixed distributional properties and way from the sequence of objects with changing distributions. The new object always provides us with some new information on the asymptotic behavior of the original sequence, and, in the happiest cases, the constants associated with the infinite object even permit us to find the elusive limit constants for that sequence.

---

# The Random-Cluster Model

Geoffrey Grimmett

**Abstract.** The class of random-cluster models is a unification of a variety of stochastic processes of significance for probability and statistical physics, including percolation, Ising, and Potts models; in addition, their study has impact on the theory of certain random combinatorial structures, and of electrical networks. Much (but not all) of the physical theory of Ising/Potts models is best implemented in the context of the random-cluster representation. This systematic summary of random-cluster models includes accounts of the fundamental methods and inequalities, the uniqueness and specification of infinite-volume measures, the existence and nature of the phase transition, and the structure of the subcritical and supercritical phases. The theory for two-dimensional lattices is better developed than for three and more dimensions. There is a rich collection of open problems, including some of substantial significance for the general area of disordered systems, and these are highlighted when encountered. Amongst the major open questions, there is the problem of ascertaining the exact nature of the phase transition for general values of the cluster-weighting factor  $q$ , and the problem of proving that the critical random-cluster model in two dimensions, with  $1 \leq q \leq 4$ , converges when re-scaled to a stochastic Löwner evolution (SLE). Overall the emphasis is upon the random-cluster model for its own sake, rather than upon its applications to Ising and Potts systems.

<b>1</b>	<b>Introduction</b>	74
<b>2</b>	<b>Potts and random-cluster processes</b>	77
2.1	Random-cluster measures	77
2.2	Ising and Potts models	78
2.3	Random-cluster and Ising–Potts coupled	80
2.4	The limit as $q \downarrow 0$	82
2.5	Rank-generating functions	83
<b>3</b>	<b>Infinite-volume random-cluster measures</b>	84
3.1	Stochastic ordering	84
3.2	A differential formula	85

3.3	Conditional probabilities . . . . .	85
3.4	Infinite-volume weak limits . . . . .	86
3.5	Random-cluster measures on infinite graphs . . . . .	88
3.6	The case $q < 1$ . . . . .	89
<b>4</b>	<b>Phase transition, the big picture</b> . . . . .	<b>91</b>
4.1	Infinite open clusters . . . . .	91
4.2	First- and second-order phase transition . . . . .	92
<b>5</b>	<b>General results in <math>d (\geq 2)</math> dimensions</b> . . . . .	<b>95</b>
5.1	The subcritical phase, $p < p_c(q)$ . . . . .	95
5.2	The supercritical phase, $p > p_c(q)$ . . . . .	96
5.3	Near the critical point, $p \simeq p_c(q)$ . . . . .	98
<b>6</b>	<b>In two dimensions</b> . . . . .	<b>101</b>
6.1	Graphical duality . . . . .	101
6.2	Value of the critical point . . . . .	103
6.3	First-order phase transition . . . . .	104
6.4	SLE limit when $q \leq 4$ . . . . .	105
<b>7</b>	<b>On complete graphs and trees</b> . . . . .	<b>108</b>
7.1	On complete graphs . . . . .	108
7.2	On trees and non-amenable graphs . . . . .	110
<b>8</b>	<b>Time-evolutions of random-cluster models</b> . . . . .	<b>111</b>
8.1	Reversible dynamics . . . . .	111
8.2	Coupling from the past . . . . .	113
8.3	Swendsen–Wang dynamics . . . . .	114
	<b>References</b> . . . . .	<b>116</b>

## 1 Introduction

During a classical period, probabilists studied the behaviour of *independent* random variables. The emergent theory is rich, and is linked through theory and application to areas of pure/applied mathematics and to other sciences. It is however unable to answer important questions from a variety of sources concerning large families of *dependent* random variables. Dependence comes in many forms, and one of the targets of modern probability theory has been to derive robust techniques for studying it. The voice of statistical physics has been especially loud in the call for rigour in this general area. In a typical scenario, we are provided with an infinity of random variables, indexed by the vertices of some graph such as the cubic lattice, and which have some dependence structure governed by the geometry of the graph. Thus mathematicians and physicists have had further cause to relate probability and geometry. One

---

# Models of First-Passage Percolation

C. Douglas Howard\*

<b>1</b>	<b>Introduction</b>	126
1.1	The Basic Model and Some Fundamental Questions	126
1.2	Notation	128
<b>2</b>	<b>The Time Constant</b>	129
2.1	The Fundamental Processes of Hammersley and Welsh	129
2.2	About $\mu$	131
2.3	Minimizing Paths	133
<b>3</b>	<b>Asymptotic Shape and Shape Fluctuations</b>	134
3.1	Shape Theorems for Standard FPP	134
3.2	About the Asymptotic Shape for Lattice FPP	138
3.3	FPP Based on Poisson Point Processes	140
3.4	Upper Bounds on Shape Fluctuations	143
3.5	Some Related Longitudinal Fluctuation Exponents	150
3.6	Monotonicity	151
<b>4</b>	<b>Transversal Fluctuations and the Divergence of Shape Fluctuations</b>	154
4.1	Transversal Fluctuation Exponents	154
4.2	Upper Bounds on $\xi$	155
4.3	Lower Bounds on $\chi$	157
4.4	Lower Bounds on $\xi$	158
4.5	Fluctuations for Other Related Models	160
<b>5</b>	<b>Infinite Geodesics and Spanning Trees</b>	161
5.1	Semi-Infinite Geodesics and Spanning Trees	161
5.2	Coalescence and Another Spanning Tree in 2 Dimensions	165
5.3	Doubly-Infinite Geodesics	167
<b>6</b>	<b>Summary of Some Open Problems</b>	168
	<b>References</b>	170

---

\* Research supported by NSF Grant DMS-02-03943.

---

# Relaxation Times of Markov Chains in Statistical Mechanics and Combinatorial Structures

Fabio Martinelli

**Abstract.** In Markov chain Monte Carlo theory a particular Markov chain is run for a very long time until its distribution is close enough to the equilibrium measure. In recent years, for models of statistical mechanics and of theoretical computer science, there has been a flourishing of new mathematical ideas and techniques to rigorously control the time it takes for the chain to equilibrate. This has provided a fruitful interaction between the two fields and the purpose of this paper is to provide a comprehensive review of the state of the art.

<b>1</b>	<b>Introduction</b>	177
<b>2</b>	<b>Mixing times for reversible, continuous-time Markov chains</b>	180
2.1	Analytic methods	182
2.2	Tensorization of the Poincaré and logarithmic Sobolev inequalities	186
2.3	Geometric tools	188
2.4	Comparison methods	190
2.5	Coupling methods and block dynamics	192
<b>3</b>	<b>Statistical mechanics models in <math>\mathbb{Z}^d</math></b>	194
3.1	Notation	194
3.2	Grand canonical Gibbs measures	195
3.3	Mixing conditions and absence of long-range order	197
3.4	Canonical Gibbs measures for lattice gases	201
3.5	The ferromagnetic Ising and Potts models	202
3.6	FK representation of Potts models	202
3.7	Antiferromagnetic models on an arbitrary graph: Potts and hard-core models	204
3.8	Model with random interactions	206
3.9	Unbounded spin systems	207

3.10	Ground states of certain quantum Heisenberg models as classical Gibbs measures . . . . .	208
<b>4</b>	<b>Glauber dynamics in <math>\mathbb{Z}^d</math></b> . . . . .	211
4.1	The dynamics in a finite volume . . . . .	211
4.2	The dynamics in an infinite volume . . . . .	213
4.3	Graphical construction . . . . .	214
4.4	Uniform ergodicity and logarithmic Sobolev constant . . . . .	215
<b>5</b>	<b>Mixing property versus logarithmic Sobolev constant in <math>\mathbb{Z}^d</math></b> . . . .	218
5.1	The auxiliary chain and sweeping out relations method . . . . .	219
5.2	The renormalization group approach . . . . .	220
5.3	The martingale method . . . . .	222
5.4	The recursive analysis . . . . .	225
5.5	Rapid mixing for unbounded spin systems . . . . .	226
<b>6</b>	<b>Torpid mixing in the phase coexistence region</b> . . . . .	227
6.1	Torpid mixing for the Ising model in $\Lambda \subset \mathbb{Z}^d$ with free boundary conditions . . . . .	227
6.2	Interface driven mixing inside one phase . . . . .	229
6.3	Torpid mixing for Potts model in $\mathbb{Z}^d$ . . . . .	231
<b>7</b>	<b>Glauber dynamics for certain random systems in <math>\mathbb{Z}^d</math></b> . . . . .	231
7.1	Combination of torpid and rapid mixing: the dilute Ising model . . . . .	231
7.2	Relaxation to equilibrium for spin glasses . . . . .	233
<b>8</b>	<b>Glauber dynamics for more general structures</b> . . . . .	234
8.1	Glauber dynamics on trees and hyperbolic graphs . . . . .	235
8.2	Glauber dynamics for the hard-core model . . . . .	236
8.3	Cluster algorithms: the Swendsen–Wang dynamics for Potts models . . . . .	237
<b>9</b>	<b>Mixing time for conservative dynamics</b> . . . . .	238
9.1	Random transposition, Bernoulli–Laplace and symmetric simple exclusion . . . . .	239
9.2	The asymmetric simple exclusion . . . . .	240
9.3	The Kac model for the Boltzmann equation . . . . .	245
9.4	Adsorbing staircase walks . . . . .	247
<b>10</b>	<b>Kawasaki dynamics for lattice gases</b> . . . . .	248
10.1	Diffusive scaling of the mixing time in the one-phase region . . . . .	249
10.2	Torpid mixing in the phase coexistence region . . . . .	252
	<b>References</b> . . . . .	253



---

# Random Walks on Finite Groups

Laurent Saloff-Coste\*

**Summary.** Markov chains on finite sets are used in a great variety of situations to approximate, understand and sample from their limit distribution. A familiar example is provided by card shuffling methods. From this viewpoint, one is interested in the “mixing time” of the chain, that is, the time at which the chain gives a good approximation of the limit distribution. A remarkable phenomenon known as the cut-off phenomenon asserts that this often happens abruptly so that it really makes sense to talk about “the mixing time”. Random walks on finite groups generalize card shuffling models by replacing the symmetric group by other finite groups. One then would like to understand how the structure of a particular class of groups relates to the mixing time of natural random walks on those groups. It turns out that this is an extremely rich problem which is very far to be understood. Techniques from a great variety of different fields – Probability, Algebra, Representation Theory, Functional Analysis, Geometry, Combinatorics – have been used to attack special instances of this problem. This article gives a general overview of this area of research.

<b>1</b>	<b>Introduction</b>	264
<b>2</b>	<b>Background and Notation</b>	267
2.1	Finite Markov Chains	267
2.2	Invariant Markov Chains on Finite Groups	270
<b>3</b>	<b>Shuffling Cards and the Cut-off Phenomenon</b>	272
3.1	Three Examples of Card Shuffling	272
3.2	Exact Computations	274
3.3	The Cut-off Phenomenon	277
<b>4</b>	<b>Probabilistic Methods</b>	281
4.1	Coupling	281
4.2	Strong Stationary Times	285
<b>5</b>	<b>Spectrum and Singular Values</b>	289

---

\* Research supported in part by NSF grant DMS 0102126

5.1	General Finite Markov Chains .....	289
5.2	The Random Walk Case .....	292
5.3	Lower Bounds .....	293
<b>6</b>	<b>Eigenvalue Bounds Using Paths .....</b>	<b>296</b>
6.1	Cayley Graphs .....	296
6.2	The Second Largest Eigenvalue.....	297
6.3	The Lowest Eigenvalue .....	300
6.4	Diameter Bounds, Isoperimetry and Expanders.....	302
<b>7</b>	<b>Results Involving Volume Growth Conditions .....</b>	<b>308</b>
7.1	Moderate Growth .....	308
7.2	Nilpotent Groups .....	311
7.3	Nilpotent Groups with many Generators .....	312
<b>8</b>	<b>Representation Theory for Finite Groups .....</b>	<b>315</b>
8.1	The General Set-up .....	315
8.2	Abelian Examples .....	317
8.3	Random Random Walks .....	323
<b>9</b>	<b>Central Measures and Bi-invariant Walks .....</b>	<b>325</b>
9.1	Characters and Bi-invariance .....	325
9.2	Random Transposition on the Symmetric Group.....	326
9.3	Walks Based on Conjugacy Classes of the Symmetric Group .....	328
9.4	Finite Classical Groups .....	331
9.5	Fourier Analysis for Non-central Measures.....	334
<b>10</b>	<b>Comparison Techniques .....</b>	<b>335</b>
10.1	The min-max Characterization of Eigenvalues .....	335
10.2	Comparing Dirichlet Forms Using Paths .....	336
10.3	Comparison for Non-symmetric Walks .....	339
	<b>References .....</b>	<b>340</b>

## 1 Introduction

This article surveys what is known about the convergence of random walks on finite groups, a subject to which Persi Diaconis gives a marvelous introduction in [27]. In the early twentieth century, Markov, Poincaré and Borel discussed the special instance of this problem associated with card shuffling where the underlying group is the symmetric group  $S_{52}$ . Two early references are to Emile Borel [15] and K.D. Kosambi and U.V.R. Rao [95]. The early literature focuses mostly on whether or not a given walk is ergodic: for card shuffling, ergodicity means that the deck gets mixed up after many shuffles.



<http://www.springer.com/978-3-540-00845-3>

Probability on Discrete Structures

Kesten, H. (Ed.)

2004, IX, 351 p., Hardcover

ISBN: 978-3-540-00845-3