1

Introduction and Main Results

This book is devoted to the study of three interrelated subjects in analysis: semigroups, elliptic boundary value problems and Markov processes. The purpose of the book provides a careful and accessible exposition of the functional analytic approach to the problem of construction of Markov processes with boundary conditions in probability theory. We construct a Feller semigroup corresponding to such a physical phenomenon that a Markovian particle moves both by jumps and continuously in the state space until it "dies" at the time when it reaches the set where the particle is definitely absorbed. Our approach is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of partial differential equations. The following diagram gives a bird's eye view of Markov processes, semigroups and boundary value problems and how these relate to each other:

Probability	Functional Analysis	Boundary Value Problems
Markov process (X_t)	Feller semigroup $T_t f(\cdot) = \int p_t(\cdot, dy) f(y)$ $p_t(x, dy)$ Markov transition function	Infinitesimal generator $\mathfrak{W}$ $T_t = \exp[t\mathfrak{W}]$
Markov property Starting afresh property	Semigroup property $T_{t+s} = T_t \cdot T_s$	Waldenfels operator Ventcel' boundary condition

This introductory chapter is devoted to the functional analytic approach to a class of *degenerate* boundary value problems for second-order elliptic differential operators which includes as particular cases the Dirichlet and Robin problems. We prove that this class of boundary value problems provides a new example of *analytic semigroups* both in the L^p topology and in the topology of uniform convergence.

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, with smooth boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold with boundary.

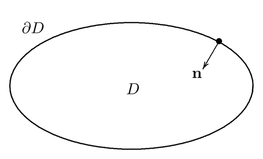

Fig. 1.1.

We let

$$Au(x) = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x)$$

be a second-order, *elliptic* differential operator with real coefficients such that

(1) $a^{ij}(x) \in C^\infty(\overline{D})$, $a^{ij}(x) = a^{ji}(x)$ and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \quad x \in \overline{D},\ \xi \in \mathbf{R}^N .$$

(2) $b^i(x) \in C^\infty(\overline{D})$.
(3) $c(x) \in C^\infty(\overline{D})$ and $c(x) \leq 0$ on $\overline{D}$, but $c(x) \not\equiv 0$ in D.

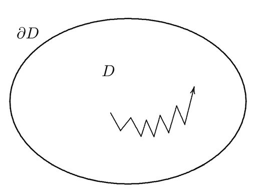

Fig. 1.2.

The functions $a^{ij}(x)$, $b^i(x)$ and $c(x)$ are called the diffusion coefficients, the drift coefficients and the termination coefficient, respectively. The differential operator A is called a *diffusion operator* which describes analytically a strong Markov process with continuous paths in the interior D such as Brownian motion (see Fig. 1.2).

Let L_0 be a first-order, boundary condition with real coefficients such that

$$L_0 u(x') = \mu(x') \frac{\partial u}{\partial \mathbf{n}}(x') + \gamma(x') u(x') \, .$$

Here:

(1) $\mu(x') \in C^\infty(\partial D)$ and $\mu(x') \geq 0$ on ∂D.
(2) $\gamma(x') \in C^\infty(\partial D)$ and $\gamma(x') \leq 0$ on ∂D.
(3) $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit interior normal to the boundary ∂D (see Fig. 1.1).

The boundary condition L_0 is a special case of general *Ventcel' boundary conditions* (cf. [We], [BCP]). The two terms of L_0

$$\mu(x') \frac{\partial u}{\partial \mathbf{n}}(x'), \quad \gamma(x') u(x')$$

are supposed to correspond to the reflection phenomenon and the absorption phenomenon, respectively (see Fig. 1.3).

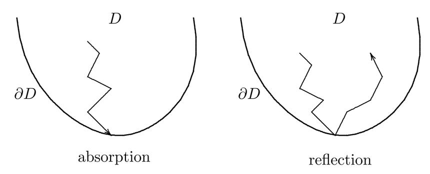

Fig. 1.3.

We consider the following boundary value problem: Given functions f and φ defined in D and on ∂D, respectively, find a function u in D such that

$$\begin{cases} Au = f & \text{in } D \, , \\ L_0 u = \mu(x') \dfrac{\partial u}{\partial \mathbf{n}} + \gamma(x') u = \varphi & \text{on } \partial D \, . \end{cases} \tag{1.1}$$

It should be noticed that if $\mu(x') \equiv 0$ and $\gamma(x') \equiv -1$ on ∂D (resp. $\mu(x') > 0$ on ∂D), then the boundary condition L_0 is the so-called Dirichlet (resp. Robin) condition. It is easy to see that problem (1.1) is non-degenerate (or coercive) if and only if either $\mu(x') > 0$ on ∂D or $\mu(x') \equiv 0$ and $\gamma(x') < 0$ on ∂D. The generation theorem for analytic semigroups is well established in the non-degenerate case both in the L^p topology (see Friedman [Fr1], Tanabe [Tn]) and in the topology of uniform convergence (see Masuda [Ma] for the Dirichlet case).

Our fundamental hypothesis is the following:
(H) $\mu(x') + |\gamma(x')| > 0$ on ∂D.
The intuitive meaning of hypothesis (H) is that either the reflection phenomenon or the absorption phenomenon occurs at each point of the boundary ∂D. More precisely, condition (H) implies that absorption phenomenon occurs at each point of the set $M = \{x' \in \partial D : \mu(x') = 0\}$, while reflection phenomenon occurs at each point of the set $\partial D \setminus M = \{x' \in \partial D : \mu(x') > 0\}$. In other words, a Markovian particle moves continuously in the space $\overline{D} \setminus M$ until it "dies" at which time it reaches the set M (see Fig. 1.4).

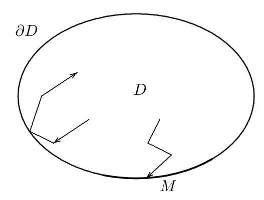

Fig. 1.4.

The first purpose of this book is to prove existence and uniqueness theorems for problem (1.1) in the framework of L^p spaces and Hölder spaces. The crucial point is how to define function subspaces in which problem (1.1) is uniquely solvable.

If k is a positive integer and $1 < p < \infty$, we define the Sobolev space of L^p style

$$H^{k,p}(D) = \text{the space of (equivalence classes of) functions } u \in L^p(D) \text{ whose derivatives } D^\alpha u,\ |\alpha| \le k, \text{ in the sense of distributions are in } L^p(D),$$

and the space

$$B^{k-1/p,p}(\partial D) = \text{the space of the boundary values } \varphi \text{ of functions } u \in H^{k,p}(D).$$

In the space $B^{k-1/p,p}(\partial D)$, we introduce a norm

$$|\varphi|_{B^{k-1/p,p}(\partial D)} = \inf\left\{\|u\|_{H^{k,p}(D)} : u \in H^{k,p}(D),\ u|_{\partial D} = \varphi\right\} .$$

The space $B^{k-1/p,p}(\partial D)$ is a Banach space with respect to this norm $|\cdot|_{B^{k-1/p,p}(\partial D)}$. More precisely, it is a Besov space (see Bergh–Löfström [BL], Taibleson [Tb], Triebel [Tr]).

We introduce a subspace of $B^{k-1/p,p}(\partial D)$ which is associated with the boundary condition L_0 in the following way: We let

$$\begin{aligned} &B^{1-1/p,p}_{L_0}(\partial D) \\ &= \left\{\varphi = \mu(x')\varphi_1 - \gamma(x')\varphi_2 : \varphi_1 \in B^{1-1/p,p}(\partial D),\ \varphi_2 \in B^{2-1/p,p}(\partial D)\right\} , \end{aligned}$$

and define a norm

$$\begin{aligned} &|\varphi|_{B^{1-1/p,p}_{L_0}(\partial D)} \\ &= \inf\left\{|\varphi_1|_{B^{1-1/p,p}(\partial D)} + |\varphi_2|_{B^{2-1/p,p}(\partial D)} : \varphi = \mu(x')\varphi_1 - \gamma(x')\varphi_2\right\} . \end{aligned}$$

Then it is easy to verify that the space $B^{1-1/p,p}_{L_0}(\partial D)$ is a Banach space with respect to this norm $|\cdot|_{B^{1-1/p,p}_{L_0}(\partial D)}$. It should be noticed that the space $B^{1-1/p,p}_{L_0}(\partial D)$ is an "interpolation space" between the Besov spaces $B^{2-1/p,p}(\partial D)$ and $B^{1-1/p,p}(\partial D)$. More precisely, we have

$$\begin{cases} B^{1-1/p,p}_{L_0}(\partial D) = B^{2-1/p,p}(\partial D) & \text{if } \mu(x') \equiv 0 \text{ on } \partial D , \\ B^{1-1/p,p}_{L_0}(\partial D) = B^{1-1/p,p}(\partial D) & \text{if } \mu(x') > 0 \text{ on } \partial D . \end{cases}$$

Now we can state an existence and uniqueness theorem for the boundary value problem (1.1) in the framework of L^p spaces:

Theorem 1.1. *Let $1 < p < \infty$. Assume that condition (H) is satisfied:*
(H) $\mu(x') + |\gamma(x')| > 0$ on ∂D.
Then the mapping

$$(A, L_0) \colon H^{2,p}(D) \longrightarrow L^p(D) \bigoplus B^{1-1/p,p}_{L_0}(\partial D)$$

is an algebraic and topological isomorphism. *In particular, for any $f \in L^p(D)$ and any $\varphi \in B^{1-1/p,p}_{L_0}(\partial D)$, there exists a unique solution $u \in H^{2,p}(D)$ of problem (1.1).*

Furthermore, in order to study problem (1.1) in the framework of Hölder spaces, we introduce a subspace of $C^{1+\theta}(\partial D)$, $0 < \theta < 1$, which is a Hölder space version of $B^{1-1/p,p}_{L_0}(\partial D)$. We let

$$C^{1+\theta}_{L_0}(\partial D) = \left\{\varphi = \mu(x')\varphi_1 - \gamma(x')\varphi_2 : \varphi_1 \in C^{1+\theta}(\partial D),\ \varphi_2 \in C^{2+\theta}(\partial D)\right\} ,$$

and define a norm

$$|\varphi|_{C^{1+\theta}_{L_0}(\partial D)} = \inf \left\{ |\varphi_1|_{C^{1+\theta}(\partial D)} + |\varphi_2|_{C^{2+\theta}(\partial D)} : \varphi = \mu(x')\varphi_1 - \gamma(x')\varphi_2 \right\} .$$

Then it is easy to verify that the space $C^{1+\theta}_{L_0}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{C^{1+\theta}_{L_0}(\partial D)}$. We remark that

$$\begin{cases} C^{1+\theta}_{L_0}(\partial D) = C^{2+\theta}(\partial D) & \text{if } \mu(x') \equiv 0 \text{ on } \partial D , \\ C^{1+\theta}_{L_0}(\partial D) = C^{1+\theta}(\partial D) & \text{if } \mu(x') > 0 \text{ on } \partial D . \end{cases}$$

The next theorem is a Hölder space version of Theorem 1.1:

Theorem 1.2. *Let $0 < \theta < 1$. If condition (H) is satisfied, then the mapping*

$$(A, L_0) \colon C^{2+\theta}(\overline{D}) \longrightarrow C^{\theta}(\overline{D}) \bigoplus C^{1+\theta}_{L_0}(\partial D)$$

is an algebraic and topological isomorphism. *In particular, for any $f \in C^{\theta}(\overline{D})$ and any $\varphi \in C^{1+\theta}_{L_0}(\partial D)$, there exists a unique solution $u \in C^{2+\theta}(\overline{D})$ of problem (1.1).*

The second purpose of this book is to study the boundary value problem (1.1) from the point of view of analytic semigroup theory in functional analysis, and is to generalize the generation theorem for analytic semigroups to the *degenerate* case.

We associate with problem (1.1) a unbounded linear operator A_p from $L^p(D)$ into itself as follows:

(a) The domain of definition $\mathcal{D}(A_p)$ of A_p is the set

$$\mathcal{D}(A_p) = \left\{ u \in H^{2,p}(D) : L_0 u = \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \gamma(x') u = 0 \text{ on } \partial D \right\} .$$

(b) $A_p u = Au$, $u \in \mathcal{D}(A_p)$.

Then we can prove a generation theorem for analytic semigroups in the framework of L^p spaces:

Theorem 1.3. *Let $1 < p < \infty$. If condition (H) is satisfied, then we have the following two assertions:*

(i) For every $\varepsilon > 0$, there exists a constant $r_p(\varepsilon) > 0$ such that the resolvent set of A_p contains the set

$$\Sigma_p(\varepsilon) = \left\{ \lambda = r^2 e^{i\vartheta} : r \geq r_p(\varepsilon), -\pi + \varepsilon \leq \vartheta \leq \pi - \varepsilon \right\} ,$$

and that the resolvent $(A_p - \lambda I)^{-1}$ satisfies the estimate

$$\left\| (A_p - \lambda I)^{-1} \right\| \leq \frac{c_p(\varepsilon)}{|\lambda|}, \quad \lambda \in \Sigma_p(\varepsilon) , \tag{1.2}$$

where $c_p(\varepsilon) > 0$ is a constant depending on ε.

(ii) The operator A_p generates a semigroup e^{zA_p} on the space $L^p(D)$ which is analytic *in the sector $\Delta_\varepsilon = \{z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$ for any $0 < \varepsilon < \pi/2$ (see Fig. 1.5).*

It should be noticed that Theorem 1.3 for $p = 2$ is proved by Taira [Ta1, Theorem 1].

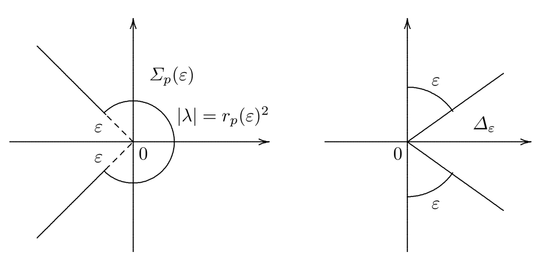

Fig. 1.5.

Secondly, we state a generation theorem for analytic semigroups in the topology of uniform convergence.

Let $C(\overline{D})$ be the space of real-valued, continuous functions f on $\overline{D}$. We equip the space $C(\overline{D})$ with the topology of uniform convergence on the whole $\overline{D}$; hence it is a Banach space with the maximum norm

$$\|f\|_\infty = \max_{x \in \overline{D}} |f(x)| \,.$$

We introduce a subspace of $C(\overline{D})$ which is associated with the boundary condition L_0. We remark that the boundary condition

$$L_0 u = \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \gamma(x') u = 0 \quad \text{on } \partial D$$

includes the condition

$$u = 0 \quad \text{on } M = \{x' \in \partial D : \mu(x') = 0\} \,,$$

if $\gamma(x') \neq 0$ on M. With this fact in mind, we let

$$C_0(\overline{D} \setminus M) = \left\{u \in C(\overline{D}) : u = 0 \text{ on } M\right\} \,.$$

The space $C_0(\overline{D} \setminus M)$ is a closed subspace of $C(\overline{D})$; hence it is a Banach space. Further we introduce a linear operator $\mathfrak{A}$ from $C_0(\overline{D} \setminus M)$ into itself as follows:

(a) The domain of definition $\mathcal{D}(\mathfrak{A})$ of $\mathfrak{A}$ is the set

$$\mathcal{D}(\mathfrak{A}) = \left\{u \in C_0(\overline{D} \setminus M) : Au \in C_0(\overline{D} \setminus M),\ L_0 u = 0 \text{ on } \partial D\right\} . \quad (1.3)$$

(b) $\mathfrak{A}u = Au$, $u \in \mathcal{D}(\mathfrak{A})$.

Here Au and $L_0 u$ are taken in the sense of *distributions* (see Chap. 13).

Then Theorem 1.3 remains valid with $L^p(D)$ and A_p replaced by $C_0(\overline{D} \setminus M)$ and $\mathfrak{A}$, respectively. More precisely, we can prove the following:

Theorem 1.4. *If condition (H) is satisfied, then we have the following two assertions:*

(i) For every $\varepsilon > 0$, there exists a constant $r(\varepsilon) > 0$ such that the resolvent set of $\mathfrak{A}$ contains the set

$$\Sigma(\varepsilon) = \left\{\lambda = r^2 e^{i\vartheta} : r \geq r(\varepsilon), -\pi + \varepsilon \leq \vartheta \leq \pi - \varepsilon\right\} ,$$

and that the resolvent $(\mathfrak{A} - \lambda I)^{-1}$ satisfies the estimate

$$\|(\mathfrak{A} - \lambda I)^{-1}\| \leq \frac{c(\varepsilon)}{|\lambda|}, \quad \lambda \in \Sigma(\varepsilon) , \quad (1.4)$$

where $c(\varepsilon) > 0$ is a constant depending on ε.

(ii) The operator $\mathfrak{A}$ generates a semigroup $e^{z\mathfrak{A}}$ on the space $C_0(\overline{D} \setminus M)$ which is analytic *in the sector $\Delta_\varepsilon = \{z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$ for any $0 < \varepsilon < \pi/2$ (see Fig. 1.6).*

Moreover, the operators $\{e^{t\mathfrak{A}}\}_{t \geq 0}$ are non-negative and contractive on the space $C_0(\overline{D} \setminus M)$:

$$f \in C_0(\overline{D} \setminus M),\ 0 \leq f(x) \leq 1 \text{ on } \overline{D} \setminus M \implies 0 \leq e^{t\mathfrak{A}} f(x) \leq 1 \text{ on } \overline{D} \setminus M .$$

Theorems 1.3 and 1.4 express a *regularizing effect* for the parabolic differential operator $\partial/\partial t - A$ with homogeneous boundary condition L_0.

A strongly continuous, non-negative and contraction semigroup $\{T_t\}_{t \geq 0}$ on the Banach space $C_0(\overline{D} \setminus M)$ is called a *Feller semigroup* on $\overline{D} \setminus M$. Thus part (ii) of Theorem 1.4 may be reformulated as follows:

Theorem 1.5. *If condition (H) is satisfied, then the operator $\mathfrak{A}$ generates a Feller semigroup $\{e^{t\mathfrak{A}}\}_{t \geq 0}$ on $\overline{D} \setminus M$.*

Theorem 1.5 is a generalization of Bony–Courrège–Priouret [BCP, Théorème XIX] to the degenerate case where $\mu(x') \geq 0$ on ∂D (see also [Ta4, Theorem 4]).

It is known (see Dynkin [Dy, Chap. III], Taira [Ta2, Chap. 9]) that if $\{T_t\}_{t \geq 0}$ is a Feller semigroup on $\overline{D} \setminus M$, then there exists a unique Markov transition function $p_t(x, \cdot)$ on $\overline{D} \setminus M$ such that

$$T_t f(x) = \int_{\overline{D} \setminus M} p_t(x, dy) f(y), \quad f \in C_0(\overline{D} \setminus M) .$$

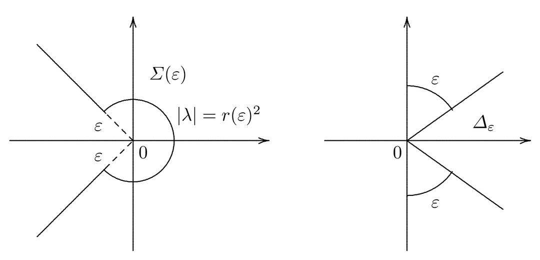

Fig. 1.6.

Furthermore, it can be shown that the function $p_t(x, \cdot)$ is the transition function of some strong *Markov process*; hence the value $p_t(x, E)$ expresses the transition probability that a Markovian particle starting at position x will be found in the set E at time t.

Rephrased, Theorem 1.5 asserts that there exists a Feller semigroup corresponding to such a physical phenomenon that a Markovian particle moves continuously in the space $\overline{D} \setminus M$ until it "dies" at which time it reaches the set M, as in Fig. 1.4.

It is worth while pointing out here that the condition

(H.1) $\mu(x') \geq 0$ and $\gamma(x') \leq 0$ on ∂D

is necessary in order that the operator $\mathfrak{A}$ be the infinitesimal generator of a Feller semigroup $\{e^{t\mathfrak{A}}\}_{t\geq 0}$ on $\overline{D} \setminus M$ (see [Ta2, Sect. 9.5]). Moreover, if condition (H.1) is satisfied, then it is easy to see that condition (H) is equivalent to the condition

(H.2) $\gamma(x') < 0$ on $M = \{x' \in \partial D : \mu(x') = 0\}$.

Furthermore, it should be emphasized that Theorem 1.2 through Theorem 1.5 may be extended to the *integro-differential operator* case. For simplicity, we assume that the domain D is . *convex*.

Let W be a second-order, elliptic integro-differential operator with real coefficients such that

$$\begin{aligned} Wu(x) &= Au(x) + Su(x) \\ &:= \left(\sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) \right) \\ &\quad + \int_{\mathbf{R}^N \setminus \{0\}} \left(u(x+z) - u(x) - \sum_{j=1}^{N} z_j \frac{\partial u}{\partial x_j}(x) \right) s(x,z)\, m(dz) \, . \end{aligned}$$

Here:

(1) $a^{ij}(x) \in C^\infty(\overline{D})$, $a^{ij}(x) = a^{ji}(x)$ and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2\,, \quad x \in \overline{D},\ \xi \in \mathbf{R}^N\,.$$

(2) $b^i(x) \in C^\infty(\overline{D})$.
(3) $c(x) \in C^\infty(\overline{D})$, and $c(x) \leq 0$ in D, but $c(x) \not\equiv 0$ in D.
(4) $s(x,z) \in C(\overline{D} \times \mathbf{R}^N)$ and $0 \leq s(x,z) \leq 1$ on $\overline{D} \times \mathbf{R}^N$, and there exist constants $C_0 > 0$ and $0 < \theta < 1$ such that

$$|s(x,z) - s(y,z)| \leq C_0|x-y|^\theta, \quad x,\ y \in \overline{D},\ z \in \mathbf{R}^N\,, \tag{1.5}$$

and

$$s(x,z) = 0 \quad \text{if } x + z \notin \overline{D}\,. \tag{1.6}$$

Condition (1.6) implies that the integral operator S may be considered as an operator acting on functions u defined on the closure $\overline{D}$ (see Garroni–Menaldi [GM, Chap. II, Remark 1.19]).

(5) The measure $m(dz)$ is a Radon measure on the space $\mathbf{R}^N \setminus \{0\}$ which satisfies the *moment condition*

$$\int_{\{|z|\leq 1\}} |z|^2\, m(dz) + \int_{\{|z|>1\}} |z|\, m(dz) < \infty\,. \tag{1.7}$$

Condition (1.7) is a standard condition for the measure $m(dz)$, and it implies that a Markovian particle does not move by jumps so far.

The operator W is called a second-order *Waldenfels operator* (cf. [BCP], [Wa]). The differential operator A is called a diffusion operator which describes analytically a strong Markov process with continuous paths in the interior D such as Brownian motion. The integral operator S is called a second-order *Lévy operator* which is supposed to correspond to the jump phenomenon in the closure $\overline{D}$ (see Fig. 1.7). In this context, condition (1.6) implies that any Markovian particle does not move by jumps from $x \in D$ to the outside of $\overline{D}$.

First, we consider (instead of problem (1.1)) the following boundary value problem: Given functions f and φ defined in D and on ∂D, respectively, find a function u in D such that

$$\begin{cases} Wu = f & \text{in } D\,, \\ L_0 u = \varphi & \text{on } \partial D\,. \end{cases} \tag{1.8}$$

The next theorem is a generalization of Theorem 1.2 to the integro-differential operator case:

Theorem 1.6. *Assume that condition (H) is satisfied:*
(H) $\mu(x') + |\gamma(x')| > 0$ *on* ∂D.
Then the mapping

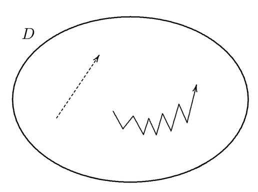

Fig. 1.7.

$$(W, L_0)\colon C^{2+\theta}(\overline{D}) \longrightarrow C^{\theta}(\overline{D}) \bigoplus C^{1+\theta}_{L_0}(\partial D)$$

is an algebraic and topological isomorphism. *In particular, for any $f \in C^{\theta}(\overline{D})$ and any $\varphi \in C^{1+\theta}_{L_0}(\partial D)$, there exists a unique solution $u \in C^{2+\theta}(\overline{D})$ of problem (1.8).*

As an application of Theorem 1.6, we can construct a Feller semigroup corresponding to such a physical phenomenon that a Markovian particle moves both by jumps and continuously in the state space until it "dies" at the time when it reaches the set where the particle is definitely absorbed, generalizing Theorem 1.5.

To do this, we define a linear operator $\mathcal{W}$ from the Banach space $C_0(\overline{D} \setminus M)$ into itself as follows:

(a) The domain of definition $\mathcal{D}(\mathcal{W})$ is the set

$$\begin{aligned}\mathcal{D}(\mathcal{W}) = \{u \in C^2(\overline{D}) \cap C_0(\overline{D} \setminus M) : Wu \in C_0(\overline{D} \setminus M)\,, \\ L_0 u = 0 \text{ on } \partial D\}\,.\end{aligned}$$

(b) $\mathcal{W}u = Wu$, $u \in \mathcal{D}(\mathcal{W})$.

The next theorem asserts that there exists a Feller semigroup on $\overline{D} \setminus M$ corresponding to such a physical phenomenon that a Markovian particle moves both by jumps and continuously in the state space $\overline{D} \setminus M$ until it "dies" at the time when it reaches the set M as in Fig. 1.4:

Theorem 1.7. *If condition (H) is satisfied, then the operator $\mathcal{W}$ is closable in the space $C_0(\overline{D} \setminus M)$, and its minimal closed extension $\overline{\mathcal{W}}$ is the infinitesimal generator of some Feller semigroup $\{e^{t\overline{\mathcal{W}}}\}_{t \geq 0}$ on $\overline{D} \setminus M$.*

Remark 1.1. For the non-degenerate case, the reader is referred to Komatsu [Ko, Theorem 5.2], Stroock [St, Theorem 2.2], Garroni–Menaldi [GM, Chap. VIII, Theorem 3.3] and also Galakhov–Skubachevskiĭ [GB, Theorem 5.1].

Secondly, we study the boundary value problem (1.8) from the point of view of analytic semigroup theory, generalizing Theorems 1.3 and 1.4.

To do this, we associate with problem (1.8) an unbounded linear operator W_p from $L^p(D)$ into itself as follows:

(a) The domain of definition $\mathcal{D}(W_p)$ is the set

$$\mathcal{D}(W_p) = \{u \in H^{2,p}(D) : L_0 u = 0 \text{ on } \partial D\} .$$

(b) $W_p u = W u$, $u \in \mathcal{D}(W_p)$.

The next theorem is a generalization of Theorem 1.3 to the integro-differential operator case:

Theorem 1.8. *Let $1 < p < \infty$. Assume that condition (H) is satisfied. Then we have the following two assertions:*

(i) For every $\varepsilon > 0$, there exists a constant $r_p(\varepsilon) > 0$ such that the resolvent set of W_p contains the set

$$\Sigma_p(\varepsilon) = \left\{\lambda = r^2 e^{i\vartheta} : r \geq r_p(\varepsilon), -\pi + \varepsilon \leq \vartheta \leq \pi - \varepsilon\right\} ,$$

and that the resolvent $(W_p - \lambda I)^{-1}$ satisfies the estimate

$$\left\|(W_p - \lambda I)^{-1}\right\| \leq \frac{c_p(\varepsilon)}{|\lambda|}, \quad \lambda \in \Sigma_p(\varepsilon) , \tag{1.9}$$

where $c_p(\varepsilon) > 0$ is a constant depending on ε.

(ii) The operator W_p generates a semigroup e^{zW_p} on the space $L^p(D)$ which is analytic *in the sector $\Delta_\varepsilon = \{z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$ for any $0 < \varepsilon < \pi/2$.*

Moreover, we introduce a linear operator $\mathfrak{W}$ from $C_0(\overline{D} \setminus M)$ into itself as follows:

(a) The domain of definition $\mathcal{D}(\mathfrak{W})$ is the set

$$\begin{aligned} \mathcal{D}(\mathfrak{W}) = \{u \in C_0(\overline{D} \setminus M) \cap H^{2,p}(D) : W u \in C_0(\overline{D} \setminus M) , \\ L_0 u = 0 \text{ on } \partial D\} , \qquad N < p < \infty . \end{aligned}$$

(b) $\mathfrak{W} u = W u$, $u \in \mathcal{D}(\mathfrak{W})$.

Here it should be noticed that the domain $\mathcal{D}(\mathfrak{W})$ is *independent* of $N < p < \infty$ (see the proof of Lemma 14.10).

The next theorem is a generalization of Theorem 1.4 to the integro-differential operator case:

Theorem 1.9. *Let $N < p < \infty$. If condition (H) is satisfied, then we have the following two assertions:*

(i) For every $\varepsilon > 0$, *there exists a constant* $r(\varepsilon) > 0$ *such that the resolvent set of* $\mathfrak{W}$ *contains the set*

$$\Sigma(\varepsilon) = \left\{\lambda = r^2 e^{i\vartheta} : r \geq r(\varepsilon), -\pi + \varepsilon \leq \vartheta \leq \pi - \varepsilon\right\} ,$$

and that the resolvent $(\mathfrak{W} - \lambda I)^{-1}$ *satisfies the estimate*

$$\|(\mathfrak{W} - \lambda I)^{-1}\| \leq \frac{c(\varepsilon)}{|\lambda|}, \quad \lambda \in \Sigma(\varepsilon) , \tag{1.10}$$

where $c(\varepsilon) > 0$ *is a constant depending on* ε.

(ii) The operator $\mathfrak{W}$ *generates a semigroup* $e^{z\mathfrak{W}}$ *on the space* $C_0(\overline{D} \setminus M)$ *which is* analytic *in the sector* $\Delta_\varepsilon = \{z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$ *for any* $0 < \varepsilon < \pi/2$.

Remark 1.2. By combining Theorems 1.7 and 1.9, we can prove that the operator $\mathfrak{W}$ coincides with the minimal closed extension $\overline{W}$: $\mathfrak{W} = \overline{W}$ (see Sect. 14.6, Theorem 14.14).

Theorems 1.8 and 1.9 express a *regularizing effect* for the parabolic integro-differential operator $\partial/\partial t - W$ with homogeneous boundary condition L_0 (cf. [GM, Chap. VIII, Theorem 3.1]).

The rest of this book is organized as follows.

The first part (Chaps. 2–5) provides the elements of semigroups, distributions and pseudo-differential operators which are used throughout the book. The material in these preparatory chapters is given for completeness, to minimize the necessity of consulting too many outside references. This makes the book fairly self-contained.

Chapter 2 is devoted to a review of standard topics from the theory of semigroups such as contraction semigroups and analytic semigroups. These topics form a necessary background for the proof of Theorems 1.3 and 1.4 (Theorem 2.11).

In Chap. 3 we introduce a class of semigroups associated with Markov processes, called Feller semigroups, and prove generation theorems for Feller semigroups (Theorems 3.3 and Theorem 3.5) which form a functional analytic background for the proof of Theorems 1.5 and 1.7. Moreover, following Bony–Courrège–Priouret [BCP] we give useful characterizations of linear operators which satisfy the positive maximum principle (Theorems 3.7 and 3.9).

In Chap. 4 we present a brief description of the basic concepts and results of the theory of distributions or generalized functions which will be used in subsequent chapters. Distribution theory has become a convenient tool in the study of partial differential equations. Many problems in partial differential equations can be formulated in terms of abstract operators acting between suitable spaces of distributions, and these operators are then analyzed by the methods of functional analysis.

Several recent developments in the theory of partial differential equations have made possible further progress in the study of boundary value problems and hence the study of Markov processes. The main technique used is the calculus of pseudo-differential operators which may be considered as a modern theory of potentials. The presentation of these new results is the main purpose of Chap. 5.

In Appendix A, by using Peetre's equivalent definition of Besov spaces and Sobolev spaces of L^p style we prove an L^p boundedness theorem for pseudo-differential operators (a global version of Theorem 5.14) as Theorem A.6 which plays a fundamental role throughout the book.

In Chap. 6, by using the L^p theory of pseudo-differential operators we study the boundary value problem (1.1) in the framework of Sobolev spaces of L^p style. We confine ourselves to the simple but important boundary condition. This makes it possible to develop our basic machinery with a minimum of bother and the principal ideas can be presented concretely and explicitly.

The idea of our approach is stated as follows. First, we consider the following *Neumann* problem:

$$\begin{cases} Av = f & \text{in } D\,, \\ \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial D\,. \end{cases} \tag{1.11}$$

The existence and uniqueness theorem for problem (1.11) is well established in the framework of Sobolev spaces of L^p style (see Agmon–Douglis–Nirenberg [ADN], Lions–Magenes [LM], Gilbarg–Trudinger [GT]). We let

$$v = G_N f\,.$$

The operator G_N is the Green operator for the Neumann problem. Then it follows that a function u is a solution of the problem

$$\begin{cases} Au = f & \text{in } D\,, \\ L_0 u = 0 & \text{on } \partial D \end{cases} \tag{1.12}$$

if and only if the function $w = u - v$ is a solution of the problem

$$\begin{cases} Aw = 0 & \text{in } D\,, \\ L_0 w = -L_0 v = -\mu(x') \dfrac{\partial v}{\partial \mathbf{n}} - \gamma(x')v = -\gamma(x')v & \text{on } \partial D\,. \end{cases}$$

However, we know that every solution w of the homogeneous equation

$$Aw = 0 \quad \text{in } D$$

can be expressed by means of a single layer potential as follows:

$$w = P\psi\,.$$

The operator P is the Poisson operator for the Dirichlet problem. Thus, by using the operators G_N and P we can reduce the study of problem (1.12) to that of the equation

$$T\psi := L_0 P\psi = -\gamma(x')v \quad \text{on } \partial D\,.$$

This is a generalization of the classical Fredholm integral equation.

It is well known (see Hörmander [Ho1], Seeley [Se2], Chazarain–Piriou [CP], Èskin [Es], Kumano-go [Ku], Taylor [Ty], Rempel–Schulze [RS]) that the operator $T = L_0 P$ is a pseudo-differential operator of first order on the boundary ∂D. We prove that the study of problem (1.12) can be reduced to that of the operator T (Theorems 6.9, 6.10 and 6.11).

In the early 1950's, W. Feller characterized completely the analytic structure of one-dimensional diffusion processes; he gave an intrinsic representation of the infinitesimal generator $\mathfrak{A}$ of a one-dimensional diffusion process and determined all possible boundary conditions which describe the domain $D(\mathfrak{A})$. The probabilistic meaning of Feller's work was clarified by E. B. Dynkin, K. Itô, H. P. McKean, Jr., D. B. Ray and others. One-dimensional diffusion processes are completely studied both from analytic and probabilistic viewpoints. The purpose of Chap. 7 is to generalize Feller's work to the multidimensional case. In 1959, A. D. Ventcel' (Wentzell) studied the problem of determining all possible boundary conditions for multidimensional diffusion processes. The main results (Theorems 7.1 and 7.20) discussed there are adapted from Bony–Courrège–Priouret [BCP], Taira [Ta2] and [Ta4]. Our proof is based on the generation theorems for Feller semigroups discussed in Chap. 3. Moreover, a unique solvability theorem for pseudo-differential operators (Theorem 7.16) plays an essential role in the construction of Feller semigroups.

In Appendix B, we give a sketch of the proof of a unique solvability theorem for pseudo-differential operators (Theorem B.1) which is an essential step for the construction of Feller semigroups in Chap. 7 (Theorem 7.16).

Chapter 8 is devoted to the proof of Theorem 1.1. We study the pseudo-differential operator T in question, and prove that if condition (H) is satisfied, then there exists a *parametrix* S for T in the Hörmander class $L^0_{1,1/2}(\partial D)$ (Lemma 8.4). Theorem 1.1 follows by applying a *Besov-space boundedness theorem* (Theorem 5.14) to the parametrix S (Theorems 8.2, 8.6 and 8.8). In the proof of surjectivity of the operator T we make use of Agmon's method (Proposition 8.11). This is a technique of treating a spectral parameter as a second-order differential operator of an extra variable and relating the old problem to a new one with the additional variable. More precisely, we introduce an auxiliary variable y of the unit circle

$$S = \mathbf{R}/2\pi\mathbf{Z}\,,$$

and consider instead of problem (1.1) the following boundary value problem:

$$\begin{cases} \widetilde{\Lambda}\tilde{u} := \left(A + \dfrac{\partial^2}{\partial y^2}\right)\tilde{u} = \tilde{f} & \text{in } D \times S\,, \\[2ex] L_0\tilde{u} = \mu(x')\dfrac{\partial \tilde{u}}{\partial \mathbf{n}} + \gamma(x')\tilde{u} = \tilde{\varphi} & \text{on } \partial D \times S\,. \end{cases}$$

In Chap. 9 we study the boundary value problem (1.1) in the framework of Hölder spaces, and prove Theorem 1.2 (Theorem 9.1). Theorem 1.2 follows from Theorem 1.1 by using the Hölder space theory of pseudo-differential operators (Proposition 9.2).

In Chap. 10 we study the operator A_p, and prove estimate (1.2) for $A_p - \lambda I$ (Theorem 10.3). Once again Agmon's method plays an important role in the proof of estimate (1.2) (Proposition 10.4) just as in Chap. 8, but we replace the differential operator $A - \lambda$, $\lambda = r^2 e^{i\vartheta}$, $-\pi < \vartheta < \pi$, by the differential operator

$$\widetilde{\Lambda}(\vartheta) := A + e^{i\vartheta} \frac{\partial^2}{\partial y^2} .$$

Chapter 11 is devoted to the proof of Theorem 1.3 (Theorems 11.1 and 11.3). Just as in Chap. 8, we make use of Agmon's method to prove the surjectivity of the operator $A_p - \lambda I$ (Proposition 11.2).

Chapters 12 and 13 are devoted to the proof of Theorem 1.4 and Theorem 1.5. In Chap. 12 we prove part (i) of Theorem 1.4. Part (i) of Theorem 1.4 follows from an application of Theorem 1.3 by using Sobolev's imbedding theorems (Theorems 12.1 and 12.2) and a λ-*dependent localization* argument essentially due to Masuda [Ma] (Lemma 12.4). An essential point in the proof is how to construct a localizing function φ_0 which is adapted to the boundary condition L_0 (see the proof of Claim 12.5). It should be emphasized here that if $\mu(x') \equiv 0$ and $\gamma(x') \equiv -1$ on ∂D (the Dirichlet case), then a proof of Theorem 1.4 is stated in Pazy [Pa, Chap. 7, Theorem 3.7]. However, his proof of estimate (1.4) is incomplete (see [Pa, p. 217, estimate (3.22)]. Moreover, if $\mu(x') > 0$ on ∂D (the Robin case), then a proof of Theorem 1.4 is given by Stewart [Sw, Theorem 2]. However, his λ-dependent localization argument concerning the boundary term is not correct (see the proof of [Sw, Theorem 1]).

In Chap. 13 we prove Theorem 1.5 and part (ii) of Theorem 1.4. The main idea of our approach is essentially the same as in Chap. 7. More precisely, we study Feller semigroups with reflecting barrier (Theorem 13.14) and then, by using these Feller semigroups we construct Feller semigroups corresponding to such a physical phenomenon that either absorption or reflection phenomenon occurs at each point of the boundary (Theorem 13.18). Part (i) of Theorem 1.3, together with Theorem 1.5, proves part (ii) of Theorem 1.4.

The final Chap. 14 is devoted to the proof of Theorem 1.6 through Theorem 1.9. The essential point in the proof is to estimate the integral operator S in terms of Hölder norms (Lemmas 14.2 and 14.3). We show that the operator (W, L_0) may be considered as a perturbation of a *compact* operator to the operator (A, L_0) in the framework of Hölder spaces. Thus the proof is reduced to the differential operator case which is studied in Chaps. 8 through 13.

In Appendix C, we prove various maximum principles for second-order elliptic Waldenfels operators W such as the weak and strong maximum principles and the boundary point lemma (Theorems C.1, C.2 and C.3 and Lemma C.5) which play an important role in Chaps. 13 and 14.

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