

I. Introduction to Homotopy Theory

O.Ya. Viro, D.B. Fuchs

Translated from the Russian
by C.J. Shaddock

Contents

| | |
|--|----|
| Chapter 1. Basic Concepts | 4 |
| §1. Terminology and Notations | 4 |
| 1.1. Set Theory | 4 |
| 1.2. Logical Equivalence | 4 |
| 1.3. Topological Spaces | 5 |
| 1.4. Operations on Topological Spaces | 5 |
| 1.5. Operations on Pointed Spaces | 8 |
| §2. Homotopy | 10 |
| 2.1. Homotopies | 10 |
| 2.2. Paths | 10 |
| 2.3. Homotopy as a Path | 11 |
| 2.4. Homotopy Equivalence | 11 |
| 2.5. Retractions | 11 |
| 2.6. Deformation Retractions | 12 |
| 2.7. Relative Homotopies | 13 |
| 2.8. k -connectedness | 13 |
| 2.9. Borsuk Pairs | 14 |
| 2.10. CNRS Spaces | 15 |
| 2.11. Homotopy Properties of Topological Constructions | 15 |
| 2.12. Natural Group Structures on Sets of Homotopy Classes | 16 |
| §3. Homotopy Groups | 20 |
| 3.1. Absolute Homotopy Groups | 20 |

| | |
|---|----|
| 3.2. Digression: Local Systems | 22 |
| 3.3. Local Systems of Homotopy Groups of a Topological Space | 23 |
| 3.4. Relative Homotopy Groups | 25 |
| 3.5. The Homotopy Sequence of a Pair | 28 |
| 3.6. Splitting | 31 |
| 3.7. The Homotopy Sequence of a Triple | 32 |
| Chapter 2. Bundle Techniques | 33 |
| §4. Bundles | 33 |
| 4.1. General Definitions | 33 |
| 4.2. Locally Trivial Bundles | 34 |
| 4.3. Serre Bundles | 36 |
| 4.4. Bundles of Spaces of Maps | 37 |
| §5. Bundles and Homotopy Groups | 38 |
| 5.1. The Local System of Homotopy Groups of the Fibres of a Serre Bundle | 38 |
| 5.2. The Homotopy Sequence of a Serre Bundle | 39 |
| 5.3. Important Special Cases | 40 |
| §6. The Theory of Coverings | 41 |
| 6.1. Coverings | 41 |
| 6.2. The Group of a Covering | 42 |
| 6.3. Hierarchies of Coverings | 42 |
| 6.4. The Existence of Coverings | 43 |
| 6.5. Automorphisms of a Covering | 44 |
| 6.6. Regular Coverings | 44 |
| 6.7. Covering Maps | 45 |
| Chapter 3 Cellular Techniques | 45 |
| §7. Cellular Spaces | 45 |
| 7.1. Basic Concepts | 45 |
| 7.2. Gluing of Cellular Spaces from Balls | 48 |
| 7.3. Examples of Cellular Decompositions | 49 |
| 7.4. Topological Properties of Cellular Spaces | 52 |
| 7.5. Cellular Constructions | 53 |
| §8. Simplicial Spaces | 54 |
| 8.1. Basic Concepts | 54 |
| 8.2. Simplicial Schemes | 58 |
| 8.3. Simplicial Constructions | 59 |
| 8.4. Stars, Links, Regular Neighbourhoods | 62 |
| 8.5. Simplicial Approximation of a Continuous Map | 64 |
| §9. Cellular Approximation of Maps and Spaces | 64 |
| 9.1. Cellular Approximation of a Continuous Map | 64 |
| 9.2. Cellular k -connected Pairs | 65 |
| 9.3. Simplicial Approximation of Cellular Spaces | 66 |

| | |
|---|----|
| I. Introduction to Homotopy Theory | 3 |
| 9.4. Weak Homotopy Equivalence | 67 |
| 9.5. Cellular Approximation to Topological Spaces | 69 |
| 9.6. The Covering Homotopy Theorem | 71 |
| Chapter 4 The Simplest Calculations | 72 |
| §10. The Homotopy Groups of Spheres and Classical Manifolds | 72 |
| 10.1. Suspension in the Homotopy Groups of Spheres | 72 |
| 10.2. The Simplest Homotopy Groups of Spheres | 73 |
| 10.3. The Composition Product | 74 |
| 10.4. Homotopy Groups of Spheres | 75 |
| 10.5. Homotopy Groups of Projective Spaces and Lens Spaces | 77 |
| 10.6. Homotopy Groups of the Classical Groups | 78 |
| 10.7. Homotopy Groups of Stiefel Manifolds and Spaces | 79 |
| 10.8. Homotopy Groups of Grassmann Manifolds and Spaces | 80 |
| §11. Application of Cellular Techniques | 81 |
| 11.1. Homotopy Groups of a 1-dimensional Cellular Space | 81 |
| 11.2. The Effect of Attaching Balls | 81 |
| 11.3. The Fundamental Group of a Cellular Space | 83 |
| 11.4. Homotopy Groups of Compact Surfaces | 84 |
| 11.5. Homotopy Groups of Bouquets | 85 |
| 11.6. Homotopy Groups of a k -connected Cellular Pair | 86 |
| 11.7. Spaces with Given Homotopy Groups | 87 |
| §12. Appendix | 89 |
| 12.1. The Whitehead Product | 89 |
| 12.2. The Homotopy Sequence of a Triad | 91 |
| 12.3. Homotopy Excision, Quotient and Suspension Theorems | 93 |

II. Homology and Cohomology

O.Ya. Viro, D.B. Fuchs

Translated from the Russian
by C.J. Shaddock

Contents

| | |
|--|-----|
| Chapter 1. Additive Theory | 98 |
| §1. Algebraic Preparation | 98 |
| 1.1. Complexes and Their Homology | 98 |
| 1.2. Maps and Homotopies | 99 |
| 1.3. Homology sequences | 100 |
| 1.4. The Euler characteristic and the Lefschetz number | 101 |
| 1.5. Change of coefficients | 103 |
| 1.6. Tensor products of complexes and the Künneth formula | 106 |
| §2. General singular homology theory | 107 |
| 2.1. Basic definitions | 107 |
| 2.2. The simplest calculations | 110 |
| 2.3. Natural transformations; refinement and approximation | 112 |
| 2.4. Excision, factorization, suspension | 113 |
| 2.5. Addition theorems | 115 |
| 2.6. Dependence on the coefficients | 117 |
| §3. Homology of cellular spaces | 119 |
| 3.1. The cellular complex | 119 |
| 3.2. Interrelations with the singular complex | 120 |
| 3.3. The simplicial case | 122 |
| 3.4. Examples of calculations | 122 |
| 3.5. Other applications | 123 |
| §4. Homology and homotopy | 124 |
| 4.1. Weak homotopy equivalence and homology | 124 |

| | | |
|---|--|-----|
| 4.2. | The Hurewicz theorems | 124 |
| 4.3. | The theorems of Poincaré and Hopf | 126 |
| 4.4. | Whitehead's theorem | 127 |
| 4.5. | Some instructive examples | 127 |
| §5. | Homology and fixed points | 127 |
| 5.1. | Lefschetz's theorem | 127 |
| 5.2. | Smith theory | 131 |
| §6. | Other homology and cohomology theories | 134 |
| 6.1. | The Eilenberg-Steenrod axioms | 134 |
| 6.2. | An alternative construction of the Eilenberg-Steenrod homology and cohomology theory: the Aleksandrov-Čech theory | 136 |
| 6.3. | Extraordinary theories | 139 |
| 6.4. | Homology and cohomology with local coefficients | 144 |
| 6.5. | Cohomology with coefficients in a sheaf | 148 |
| 6.6. | Conclusion | 152 |
| Chapter 2. Multiplicative theory | | 152 |
| §7. | Products | 152 |
| 7.1. | Introduction | 152 |
| 7.2. | Direct construction of the \cup -product | 154 |
| 7.3. | Application: the Hopf invariant | 155 |
| 7.4. | Other products | 156 |
| §8. | Homology and manifolds | 157 |
| 8.1. | Introduction | 157 |
| 8.2. | The fundamental class | 157 |
| 8.3. | The Poincaré isomorphisms | 159 |
| 8.4. | Intersection numbers and Poincaré duality | 161 |
| 8.5. | Linking coefficients | 163 |
| 8.6. | Inverse homomorphisms | 164 |
| 8.7. | The relation with the \cup -product | 166 |
| 8.8. | Generalizations of the Poincaré isomorphism and duality | 167 |
| Chapter 3. Obstructions, characteristic classes and cohomology operations | | 171 |
| §9. | Obstructions | 171 |
| 9.1. | Obstructions to extending a continuous map | 171 |
| 9.2. | The relative case | 172 |
| 9.3. | Application: cohomology and maps into $K(\pi, n)$ spaces | 173 |
| 9.4. | Another application: Hopf's theorems | 174 |
| 9.5. | Obstructions to the extension of sections | 175 |
| §10. | Characteristic classes of vector bundles | 176 |
| 10.1. | Vector bundles | 176 |
| 10.2. | Associated bundles and characteristic classes | 177 |
| 10.3. | Characteristic classes and classifying spaces | 179 |
| 10.4. | The most important properties of Stiefel-Whitney classes | 180 |

| | |
|--|-----|
| 10.5. The most important properties of Euler, Chern, and Pontryagin classes | 182 |
| 10.6. Characteristic classes in the topology of smooth manifolds | 184 |
| §11. Steenrod squares | 189 |
| 11.1. General theory of cohomology operations | 189 |
| 11.2. Steenrod squares and their properties | 190 |
| 11.3. Steenrod squares and Stiefel-Whitney classes | 191 |
| 11.4. Secondary obstructions | 193 |
| 11.5. The non-existence of spheroids with odd Hopf invariant | 194 |
| References | 195 |

III. Classical Manifolds

D.B. Fuchs

Translated from the Russian
by the author

Contents

| | |
|---|-----|
| Introduction | 199 |
| Chapter 1. Spheres | 199 |
| §1. Homotopy Groups | 199 |
| 1.1. Generalities | 199 |
| 1.2. Tables and Related Information | 203 |
| 1.3. The Groups $\pi_{n+1}(S^n)$ | 204 |
| 1.4. The Groups $\pi_{n+2}(S^n)$ | 205 |
| 1.5. The Whitehead J -Homomorphism | 206 |
| §2. Differential Structures | 207 |
| 2.1. Generalities | 207 |
| 2.2. Explicit Constructions of Exotic Spheres | 208 |
| §3. Appendix | 209 |
| 3.1. Structures | 209 |
| 3.2. Vector Fields and Plane Fields | 210 |
| 3.3. Foliations | 210 |
| Chapter 2. Lie Groups and Stiefel Manifolds | 210 |
| §1. Lie Groups: Geometric Information | 210 |
| 1.1. Generalities | 210 |
| 1.2. Some Lie Groups of Low Dimension | 212 |
| 1.3. Homotopy Groups | 213 |
| §2. Lie Groups: Homological Information | 215 |
| 2.1. Real Cohomology | 215 |
| 2.2. Cohomology Modulo "Good Primes". Integer Cohomology of $U(n)$ and $Sp(n)$ | 215 |
| 2.3. Modulo 2 Cohomology of Orthogonal and Spinor Groups | 216 |

| | |
|--|-----|
| 2.4. Cohomology of the Exceptional Groups | 216 |
| 2.5. The K -functor | 217 |
| §3. Stiefel Manifolds | 217 |
| 3.1. Definitions. Geometrical and Homotopical Information | 217 |
| 3.2. Cohomology | 218 |
| Chapter 3. Grassmann Manifolds and Spaces | 219 |
| §1. Geometric Information | 219 |
| 1.1. Definitions | 219 |
| 1.2. General Information | 220 |
| 1.3. Embeddings of the Manifolds $G(m, n)$, $\mathbb{C}G(m, n)$, $G_+(m, n)$ in Euclidean and Projective Spaces | 221 |
| §2. Homology Information | 223 |
| 2.1. Cell Decomposition | 223 |
| 2.2. Homology and Cohomology: Cellular Calculations | 225 |
| 2.3. The Cohomology Rings | 229 |
| 2.4. The K -functor | 232 |
| Chapter 4. Some Other Important Homogeneous Spaces | 233 |
| §1. Flag Manifolds | 233 |
| 1.1. Generalities | 233 |
| 1.2. Cell Decompositions | 234 |
| 1.3. Homology and Cohomology | 235 |
| 1.4. The Case of Complete Flag Manifolds | 235 |
| 1.5. Generalizations | 236 |
| §2. The Manifolds $U(n)/SO(n)$ and $U(n)/O(n)$ | 237 |
| 2.1. Generalities | 237 |
| 2.2. Cellular Decompositions | 238 |
| 2.3. Cellular Computation of Homology | 239 |
| 2.4. The Cohomology Rings | 240 |
| §3. The Manifolds $SO(2n)/U(n)$ and $U(2n)/Sp(n)$ | 241 |
| Chapter 5. Some Manifolds of Low Dimension | 242 |
| §1. Closed Surfaces | 242 |
| 1.1. The Standard Surfaces | 242 |
| 1.2. Homotopy Properties | 243 |
| 1.3. Automorphisms | 244 |
| 1.4. Complex Structures | 245 |
| §2. Some Three-dimensional Manifolds | 246 |
| 2.1. Lens Spaces | 246 |
| 2.2. The Poincaré Sphere | 248 |
| §3. Some Four-dimensional Manifolds | 249 |
| References | 251 |

Topology II

Homotopy and Homology. Classical Manifolds

Fuchs, D.B.; Viro, O.Y. - Rokhlin, V.A.; Novikov, S.P.

(Eds.)

2004, X, 258 p., Hardcover

ISBN: 978-3-540-51996-6