2. INTRODUCTORY APPLICATIONS

In the previous chapter, we laid the foundations of singular perturbation theory and, although we will need to add some specific techniques for solving certain types of differential equations, we can already tackle simple examples. In addition, we will see that we can apply these ideas directly to other, more routine problems—and this is where we shall begin. Here, we will describe how to approach the problem of finding roots of equations (which contain a small parameter), and how to evaluate integrals of functions which are represented by asymptotic expansions with respect to a parameter. Finally, we begin our study of differential equations by examining a few important, fairly straightforward examples which are, nonetheless, not trivial.

2.1 ROOTS OF EQUATIONS

At some stage in many mathematical problems, it is not unusual to be faced with the need to solve an equation for specific values of an unknown. Such a problem might be as simple as solving a quadratic equation:

$$\varepsilon x^2 + x + 1 = 0,$$

or finding the solution of more complicated equations such as

$$\varepsilon x^3 + \sqrt{x} - 1 = 0 \quad \text{or} \quad \varepsilon \sin x = 1 + \varepsilon - \mathrm{e}^{-x/\varepsilon}.$$

In this section, we will describe a technique (for equations which contain a small parameter, as in those above) which is a natural extension of simply obtaining an asymptotic expansion of a function, examining its breakdown, rescaling, and so on.

We will begin by examining the simple quadratic equation

$$f(x;\varepsilon) = \varepsilon x^2 + x + 1 = 0 \tag{2.1}$$

and seek the solutions for $\varepsilon \to 0$. The essential idea is to obtain different asymptotic approximations for $f(x;\varepsilon)$, valid for different sizes of x, and see if these admit (approximate) roots. Given that $-\infty < x < \infty$, we could have roots anywhere on the real line, and so all sizes of x must be examined. (We will consider, first, only the real roots of equations; the extension to complex roots will be discussed in due course.) One further comment is required at this stage: we describe here a technique for finding roots that builds on the ideas of singular perturbation theory. In practice, other approaches are likely to be used in conjunction with ours to solve particular equations e.g. sketching or plotting the function, or using a standard numerical procedure (such as Newton-Raphson). There is no suggestion that this expansion technique should be used in isolation—it is simply one of a number of tools available.

Returning to (2.1), if $x = O(1)$, then

$$f(x;\varepsilon) \sim 1 + x \quad \text{as } \varepsilon \to 0 \tag{2.2}$$

and so we have a root $x = -1$ (approximately). In order to generate a better approximation, we may use any appropriate method. For example, we could invoke the familiar procedure of iteration, so we may write

$$x_{n+1} = -1 - \varepsilon x_n^2, \quad n = 0, 1, 2, \ldots,$$

with $x_0 = -1$. Then we obtain

$$x_1 = -1 - \varepsilon, \quad x_2 = -1 - \varepsilon - 2\varepsilon^2 - \varepsilon^3,$$

and so on (but note that iteration may not generate a correct asymptotic expansion at a *given* order in ε). It is clear from this approach that a complete representation of the root will be obtained if we use the asymptotic sequence $\{\varepsilon^n\}$, and so an alternative is to seek this form directly—and this is more in keeping with the ideas of perturbation theory. Thus we might seek a root in the form

$$x \sim -1 + \sum_{n=1}^{\infty} \varepsilon^n a_n \tag{2.3}$$

so that (2.1) can be written as

$$\varepsilon(1 - 2\varepsilon a_1 + \cdots) + (-1 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots) + 1 = 0$$

(where the use of '= 0' here is to imply 'equal to zero to all orders in ε'); thus $a_1 = -1$, $a_2 = -2$, and so on. We have one root

$$x \sim -1 - \varepsilon - 2\varepsilon^2 \quad \text{as } \varepsilon \to 0. \tag{2.4}$$

It is clear that (2.2) admits only one root; the other—it is a quadratic equation that we are solving—must appear for a different size of x.

The 'asymptotic expansion' (we treat the function as such)

$$f(x;\varepsilon) \sim 1 + x + \varepsilon x^2 \tag{2.5}$$

remains valid for $x \to 0$, and so there is no new root for $x = o(1)$; however, this expansion does break down where $\varepsilon x^2 = O(x)$ i.e. $x = O(\varepsilon^{-1})$. We define $x = X/\varepsilon$ and write

$$f(X/\varepsilon;\varepsilon) \equiv \varepsilon^{-1} F(X;\varepsilon) = 1 + \frac{X}{\varepsilon} + \frac{X^2}{\varepsilon} (= 0)$$

or

$$\begin{aligned} F(X;\varepsilon) &= X^2 + X + \varepsilon \\ &\sim X^2 + X \quad \text{for } X = O(1) \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{2.6}$$

This approximation admits the roots X = 0 and X = –1, so now the quadratic equation has a total of three roots! Of course, this cannot be the case; indeed, it is clear that the root X = 0 is inadmissible, because the 'asymptotic expansion' $F \sim X^2 + X + \varepsilon$ breaks down where $X = O(\varepsilon)$ (which is $x = O(1)$ and so returns us to (2.5)). The only available root is X = –1, and this is the second (approximate) root of the equation (leaving X = 0 as no more than a 'ghost' of the root $x \sim -1$). The expansion for F does not further breakdown (as $X \to \infty$) and so there are no other roots—not that we expected any more! We may seek a better approximation, as we did before, in the form

$$X \sim -1 + \sum_{n=1}^{\infty} \varepsilon^n A_n$$

which gives

$$\left(1 - 2\varepsilon A_1 + \varepsilon^2 A_1^2 - 2\varepsilon^2 A_2 + \cdots\right) + \left(-1 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots\right) + \varepsilon = 0$$

i.e. $A_1 = 1$, $A_2 = 1$; thus

$$X \sim -1 + \varepsilon + \varepsilon^2 \quad \text{or} \quad x \sim -\frac{1}{\varepsilon} + 1 + \varepsilon \quad \text{as } \varepsilon \to 0.$$

The two roots of the quadratic equation, (2.1), are therefore

$$x \sim -1 - \varepsilon - 2\varepsilon^2, \quad x \sim -\frac{1}{\varepsilon} + 1 + \varepsilon \quad \text{as } \varepsilon \to 0;$$

it is left as an exercise to confirm that these results can be obtained directly from the familiar solution of the quadratic equation, suitably approximated (by using the binomial expansion) for $\varepsilon \to 0$. (Similar problems based on quadratic equations can be found in exercise Q2.1.)

This simple introductory example covers the essentials of the technique: for $-\infty < x < \infty$, find all the different asymptotic forms of $f(x;\varepsilon)$ $(= 0)$, and investigate if roots exist for each (dominant) asymptotic representation. Let us now apply this to a slightly more difficult equation which, nevertheless, has a similar structure.

E2.1 A cubic equation

We are to find approximations to all the real roots of the cubic equation

$$f(x;\varepsilon) = \varepsilon x^3 - x^2 + \varepsilon x + 1 = 0,$$

for $\varepsilon \to 0$. First, for $x = \mathrm{O}(1)$, we have

$$f(x;\varepsilon) \sim 1 - x^2 \quad \text{as } \varepsilon \to 0,$$

and this approximation admits the roots $x = \pm 1$; a better approximation is then obtained by writing

$$x \sim \pm 1 + \sum_{n=1}^{\infty} \varepsilon^n a_n$$

so that we obtain

$$\varepsilon(\pm 1 + 3\varepsilon a_1 \cdots) - (1 \pm 2\varepsilon a_1 + \varepsilon^2 a_1^2 \pm 2\varepsilon^2 a_2 \cdots) + \varepsilon(\pm 1 + \varepsilon a_1 \cdots) + 1 = 0.$$

This equation requires that $a_1 = 1$, $a_2 = \pm 3/2$, and so on; two roots are therefore

$$x \sim 1 + \varepsilon + \frac{3}{2}\varepsilon^2, \quad x \sim -1 + \varepsilon - \frac{3}{2}\varepsilon^2 \quad \text{as } \varepsilon \to 0.$$

Now the 'asymptotic expansion'

$$f(x;\varepsilon) \sim 1 - x^2 + \varepsilon(x^3 + x)$$

remains valid as $x \to 0$ but not as $|x| \to \infty$; it breaks down where $\varepsilon x^3 = \mathrm{O}(x^2)$ or $x = \mathrm{O}(\varepsilon^{-1})$. We write $x = X/\varepsilon$ and then

$$f(X/\varepsilon;\varepsilon) \equiv \varepsilon^{-2} F(X;\varepsilon) = \frac{X^3}{\varepsilon^2} - \frac{X^2}{\varepsilon^2} + X + 1 \ (= 0)$$

or

$$F(X;\varepsilon) = X^3 - X^2 + \varepsilon^2(1 + X)$$
$$\sim X^3 - X^2 \quad \text{for } X = \mathrm{O}(1) \quad \text{as } \varepsilon \to 0.$$

The only relevant root is $X = 1$ (because the other two are the ghosts of the roots that appear for $x = O(1)$). To improve this approximation, we set

$$X \sim 1 + \sum_{n=1}^{\infty} \varepsilon^n A_n$$

to give

$$1 + 3\varepsilon A_1 + 3\varepsilon^2 A_1^2 + 3\varepsilon^2 A_2 \cdots - \left(1 + 2\varepsilon A_1 + \varepsilon^2 A_1^2 + 2\varepsilon^2 A_2 \cdots\right) + \varepsilon^2(1 + \varepsilon A_1 \cdots + 1) = 0$$

i.e. $A_1 = 0$, $A_2 = -2$, so that a third root is $X \sim 1 - 2\varepsilon^2$ as $\varepsilon \to 0$. Thus the three (real) roots are

$$x \sim \pm 1 + \varepsilon \pm \frac{3}{2}\varepsilon^2, \quad x \sim \frac{1}{\varepsilon} - 2\varepsilon \quad \text{as } \varepsilon \to 0.$$

Our introductory example, and the one above, have been rather conventional polynomial equations, but the technique is particularly powerful when we have to solve, for example, transcendental equations (which contain a small parameter). We will now see how the approach works in a problem of this type.

E2.2 A transcendental equation

We require the approximate (real) roots, as $\varepsilon \to 0^+$, of the equation

$$f(x;\varepsilon) = x^2 - 3\varepsilon x - 1 - \varepsilon + e^{-x/\varepsilon} = 0. \tag{2.7}$$

For $x = O(1)$, we now have two possibilities:

$$f(x;\varepsilon) \sim \begin{cases} x^2 - 1 & \text{for } x > 0 \\ e^{-x/\varepsilon} & \text{for } x < 0 \end{cases}$$

but only the first option admits any roots for real, finite x. Thus $x = \pm 1$ (approximately) and then only the choice $x = +1$ is acceptable (because we require $x > 0$); a better approximation follows directly:

$$x \sim 1 + 2\varepsilon + \varepsilon^2 \quad \text{as } \varepsilon \to 0^+.$$

The 'expansion' is written

$$f(x;\varepsilon) \sim x^2 - 1 - \varepsilon(3x + 1) + e^{-x/\varepsilon} \tag{2.8}$$

where the term $\exp(-x/\varepsilon)$ must be exponentially small for $x = O(1)$ or larger (because no roots exist if this term dominates). Now, for $x > 0$, there is no breakdown as

$x \to \infty$; thus any other roots that might exist must arise as $x \to 0$. Indeed, as $x \to 0$, we see that the expansion (2.8) breaks down where $x = O(\varepsilon)$, and so we set $x = \varepsilon X$, to give

$$\begin{aligned} f(\varepsilon X; \varepsilon) \equiv F(X; \varepsilon) &= \varepsilon^2 X^2 - 1 - \varepsilon(3\varepsilon X + 1) + e^{-X} \; (= 0) \\ &\sim e^{-X} - 1 \quad \text{for } X = O(1) \quad \text{as } \varepsilon \to 0^+. \end{aligned} \tag{2.9}$$

This approximation has one root at $X = 0$, but this cannot be used directly because the expansion, (2.9), itself breaks down as $X \to 0$. This occurs where $e^{-X} - 1 = O(\varepsilon)$ i.e. $X = O(\varepsilon)$, so a further scaling must be introduced: $X = \varepsilon\chi$, to produce

$$\begin{aligned} F(\varepsilon\chi; \varepsilon) \equiv \varepsilon\mathfrak{F}(\chi; \varepsilon) &= \varepsilon^4\chi^2 - 1 - \varepsilon(3\varepsilon^2\chi + 1) + e^{-\varepsilon\chi} \\ &\sim -\varepsilon\chi - \varepsilon + \frac{1}{2}\varepsilon^2\chi^2 - \frac{1}{6}\varepsilon^3\chi^3 - 3\varepsilon^3\chi \quad \text{for } \chi = O(1) \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Since we have

$$\mathfrak{F}(\chi; \varepsilon) \sim -\chi - 1$$

we have a root near $\chi = -1$; to obtain an improved approximation, we write

$$\chi \sim -1 + \sum_{n=1}^{\infty} \varepsilon^n a_n$$

and so obtain

$$1 - \varepsilon a_1 - \varepsilon^2 a_2 \cdots - 1 + \frac{1}{2}\varepsilon\left(1 - 2\varepsilon a_1 + \varepsilon^2 a_1^2 - 2\varepsilon^2 a_2 \cdots\right) - \frac{\varepsilon^2}{6}(-1\cdots) - 3\varepsilon^2(-1\cdots) = 0$$

i.e. $a_1 = 1/2$, $a_2 = 8/3$:
$$\chi \sim -1 + \frac{1}{2}\varepsilon + \frac{8}{3}\varepsilon^2.$$

As $\chi \to 0$, there is no further breakdown, and so we have found two real roots

$$x \sim 1 + 2\varepsilon + \varepsilon^2, \quad x \sim -\varepsilon^2 + \frac{1}{2}\varepsilon^3 + \frac{8}{3}\varepsilon^4 \quad \text{as } \varepsilon \to 0^+.$$

A number of other equations, both polynomial and transcendental, are discussed in the exercises Q2.2, 2.3 and 2.4. However, all these involve the search for *real* roots; we now turn, therefore, to a brief discussion of the corresponding problem of finding *all* roots, whether real or complex. It will soon become clear that we may often adopt precisely the same approach when any roots are being sought (although, sometimes, there may be an advantage in writing $x = \alpha + i\beta$ and working with two, coupled, real equations). The only small word of warning is that the size of the real and imaginary parts, measured in terms of ε, may be different e.g. $x = O(1)$ now implies that $|\alpha + i\beta| = O(1)$, which can be satisfied if either, but not necessarily both, α and β are O(1). Let us see how this arises in an example.

E2.3 An equation with complex roots

Here, using the more usual notation for a complex number, we consider

$$f(z;\varepsilon) = \varepsilon z^3 + z^2 + 1 = 0$$

and immediately we obtain

$$f(z;\varepsilon) \sim z^2 + 1 \quad \text{for } |z| = \mathrm{O}(1) \quad \text{as } \varepsilon \to 0,$$

and so we have roots $z = \pm\mathrm{i}$, approximately. Thus we write

$$z \sim \pm\mathrm{i} + \sum_{n=1}^{\infty} \varepsilon^n a_n$$

and so the equation becomes

$$\varepsilon(\mp\mathrm{i} - 3\varepsilon a_1 \cdots) + (-1 \pm 2\mathrm{i}\varepsilon a_1 \pm 2\mathrm{i}\varepsilon^2 a_2 + \varepsilon^2 a_1^2 \cdots) + 1 = 0.$$

This is satisfied if $a_1 = 1/2$, $a_2 = \mp 5\mathrm{i}/8$, etc., and thus we have two complex roots

$$z \sim \pm\mathrm{i} + \frac{1}{2}\varepsilon \mp \mathrm{i}\frac{5}{8}\varepsilon^2,$$

and we observe that the imaginary part is O(1), but that the real part is O(ε).

The full 'expansion' is clearly not uniformly valid as $|z| \to \infty$: there is a breakdown where $\varepsilon|z|^3 = \mathrm{O}(|z|^2)$ or $z = \mathrm{O}(\varepsilon^{-1})$. We introduce $z = Z/\varepsilon$ and write

$$f(Z/\varepsilon;\varepsilon) \equiv \varepsilon^{-2} F(Z;\varepsilon) = \frac{Z^3}{\varepsilon^2} + \frac{Z^2}{\varepsilon^2} + 1 (= 0)$$

and so
$$F(Z;\varepsilon) \sim Z^3 + Z^2 \quad \text{for } Z = \mathrm{O}(1) \quad \text{as } \varepsilon \to 0$$

which produces the single, available root near Z = – 1 and then, more accurately, we have $Z \sim -1 - \varepsilon^2$. The equation has three roots, two of which are complex:

$$z \sim \pm\mathrm{i} + \frac{1}{2}\varepsilon \mp \mathrm{i}\frac{5}{8}\varepsilon^2, \quad z \sim -\frac{1}{\varepsilon} - \varepsilon \quad \text{as } \varepsilon \to 0.$$

Finally, a class of equations for which this direct approach (for complex roots) is not useful is characterised by the appearance of terms such as $\exp(z)$ (or anything equivalent).

In this case, it is almost always convenient to formulate the problem in real and imaginary parts, and the appearance of a small parameter does not affect this approach in any significant way; we present an example of this type.

E2.4 A real-imaginary problem

We seek all the roots of the equation

$$\sin z = 1 + \frac{\varepsilon}{1 + |z|} \tag{2.9}$$

as $\varepsilon \to 0^+$; note that

$$\sin z = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})$$

and so the form of this problem does indeed exhibit this more complicated structure. Let us write $z = x + \mathrm{i}y$, and then (2.9) becomes

$$\sin x \cosh y + \mathrm{i} \cos x \sinh y = 1 + \frac{\varepsilon}{1 + \sqrt{x^2 + y^2}}$$

or

$$\sin x \cosh y = 1 + \frac{\varepsilon}{1 + \sqrt{x^2 + y^2}} \quad \text{and} \quad \cos x \sinh y = 0. \tag{2.10a,b}$$

We see immediately that the right-hand sides of these two equations do not presage a breakdown of these contributions, as x or y increases or decreases; thus we proceed with $x = O(1)$ and $y = O(1)$. Now equation (2.10b) possesses the solutions

$$x = \pi/2 + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots),$$

and this is the relevant choice (rather than $y = 0$) because we require $\sin x \cosh y > 1$ (from (2.10a)). Then equation (2.10a) gives

$$(-1)^n \cosh y = 1 + \frac{\varepsilon}{1 + \sqrt{\left(\frac{1}{2} + n\right)^2 \pi^2 + y^2}}$$

and this is consistent only if $n = 2m$ ($m = 0, \pm 1, \pm 2, \ldots$) because $\cosh y > 0$. Finally, the solution arises only for $y \to 0$; since $\cosh y \sim 1 + y^2/2$ as $y \to 0$, we see that a solution exists where $y^2 = O(\varepsilon)$ and so we introduce $y = \sqrt{\varepsilon} Y$. Thus we obtain

$$\frac{1}{2} Y^2 \sim \frac{1}{1 + \left|\frac{1}{2} + 2m\right| \pi} \quad \text{or} \quad Y \sim \frac{\pm 2}{\sqrt{2 + |1 + 4m|\pi}}$$

and so we have the set of (approximate) roots

$$z = x + \mathrm{i}y \sim \frac{\pi}{2} + 2m\pi \pm \mathrm{i}\frac{2\sqrt{\varepsilon}}{\sqrt{2 + |1 + 4m|\pi}}, \qquad m = 0, \pm 1, \pm 2, \ldots.$$

A few examples of other equations with complex roots (some of which may be real, of course) are set as exercises in Q2.5.

2.2 INTEGRATION OF FUNCTIONS REPRESENTED BY ASYMPTOTIC EXPANSIONS

Our second direct, and rather routine application of these ideas is to the evaluation of integrals. In particular, we consider integrals of functions that are represented by asymptotic expansions in a small parameter; this may involve one or more expansions, but if it is the latter—and it often is—then the expansions will satisfy the matching principle.

The procedure that we adopt calls upon two general properties: the first is the existence of an intermediate variable (valid in the overlap region; see §1.7), and the second is the familiar device of splitting the range of integration, as appropriate. We then express the integral as a sum of integrals over each of the asymptotic expansions of the integrand, the switch from one to the next being at a point which is in the overlap region. The expansions are then valid for each integration range selected and, furthermore, the value of the original integral (assuming that it exists) is *independent* of how we split the integral. Thus the particular choice of intermediate variable is unimportant; indeed, it may be quite general, satisfying only the necessary conditions for such a variable; see (1.56), for example. Let us apply this technique to a simple example.

E2.5 An elementary integral

We are given

$$f(x;\varepsilon) = \sqrt{x+\varepsilon} + \frac{\varepsilon}{\sqrt{1+x}} + \mathrm{e}^{-x/\varepsilon}, \qquad x \geq 0, \qquad \varepsilon > 0,$$

and we require the value, as $\varepsilon \to 0^+$, of the integral

$$I(\varepsilon) = \int_0^1 f(x;\varepsilon)\,\mathrm{d}x.$$

(Note that the integral here is elementary, to the extent that it may be evaluated directly, although we will integrate only the relevant asymptotic expansions; this example has been selected so that the interested reader may check the results against the expansion of the exact value.)

First, we expand $f(x;\varepsilon)$ for $x = \mathrm{O}(1)$ and for $x = \varepsilon X = \mathrm{O}(\varepsilon)$, to give

$$f(x;\varepsilon) \sim \sqrt{x} + \varepsilon\left(\frac{1}{\sqrt{1+x}} + \frac{1}{2\sqrt{x}}\right) - \frac{\varepsilon^2}{8x\sqrt{x}}, \quad x = \mathrm{O}(1),$$

and
$$f(\varepsilon X;\varepsilon) \equiv F(X;\varepsilon) \sim \mathrm{e}^{-X} + \sqrt{\varepsilon}\sqrt{1+X} + \varepsilon - \frac{\varepsilon^2 X}{2}, \quad X = \mathrm{O}(1),$$

both for $\varepsilon \to 0^+$; we have retained terms as far as $\mathrm{O}(\varepsilon^2)$ in each expansion. (You should confirm that these two expansions satisfy the matching principle.)

Now these two expansions are valid in the overlap region, represented by $x = \mathrm{O}(\delta)$, defined by

$$\delta(\varepsilon) \to 0 \quad \text{and} \quad \varepsilon/\delta(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0;$$

thus we express the integral as

$$I(\varepsilon) = \int_0^\delta F(X;\varepsilon)\,\mathrm{d}x + \int_\delta^1 f(x;\varepsilon)\,\mathrm{d}x \sim \varepsilon\int_0^{\delta/\varepsilon}\left\{\mathrm{e}^{-X} + \sqrt{\varepsilon}\sqrt{1+X} + \varepsilon - \frac{\varepsilon^2 X}{2}\right\}\mathrm{d}X$$
$$+ \int_\delta^1\left\{\sqrt{x} + \left(\frac{1}{\sqrt{1+x}} + \frac{1}{2\sqrt{x}}\right) - \frac{\varepsilon^2}{8x\sqrt{x}}\right\}\mathrm{d}x.$$

The only requirement, at this stage, is that we are able to perform the integration of the various functions that appear in the asymptotic expansions. Note that the first integral has been expressed as an integration in X—the most natural choice of integration variable in this context. To proceed, we obtain

$$I(\varepsilon) \sim \varepsilon\left[-\mathrm{e}^{-X} + \frac{2\sqrt{\varepsilon}}{3}(1+X)^{3/2} + \varepsilon X - \frac{\varepsilon^2 X^2}{4}\right]_0^{\delta/\varepsilon}$$
$$+ \left[\frac{2}{3}x^{3/2} + \varepsilon\{2(1+x)^{1/2} + x^{1/2}\} + \frac{\varepsilon^2 x^{-1/2}}{4}\right]_\delta^1$$
$$= \varepsilon\left\{-\mathrm{e}^{-\delta/\varepsilon} + \frac{2\sqrt{\varepsilon}}{3}\left(1 + \frac{\delta}{\varepsilon}\right)^{3/2} + \delta - \frac{\delta^2}{4} + 1 - \frac{2\sqrt{\varepsilon}}{3}\right\}$$
$$+ \left\{\frac{2}{3} + \varepsilon(2\sqrt{2}+1) + \frac{\varepsilon^2}{4} - \frac{2\delta^{3/2}}{3} - \varepsilon[2(1+\delta)^{1/2} + \delta^{1/2}] - \frac{\varepsilon^2}{4\sqrt{\delta}}\right\}$$

and this is to be expanded for $\varepsilon \to 0$, $\delta \to 0$ and $\varepsilon/\delta \to 0$ (note!). Thus we obtain

$$I(\varepsilon) \sim \varepsilon\left\{\frac{2\sqrt{\varepsilon}}{3}\left(\frac{\delta}{\varepsilon}\right)^{3/2}\left(1 + \frac{3}{2}\frac{\delta}{\varepsilon} + \frac{3}{8}\frac{\varepsilon^2}{\delta^2}\cdots\right) + \delta - \frac{\delta^2}{4} + 1 - \frac{2\sqrt{\varepsilon}}{3}\right\}$$
$$+ \left\{\frac{2}{3} + \varepsilon(1+2\sqrt{2}) + \frac{\varepsilon^2}{4} - \frac{2\delta^{3/2}}{3} - 2\varepsilon\left(1 + \frac{1}{2}\delta - \frac{1}{8}\delta^2\cdots\right) - \varepsilon\sqrt{\delta} - \frac{\varepsilon^2}{4\sqrt{\delta}}\right\},$$

where the ellipsis $(\cdots)$ indicates further terms in the various binomial expansions; we keep as many as required in order to demonstrate that $\delta(\varepsilon)$ vanishes identically (at this order), to leave

$$\varepsilon - \frac{2}{3}\varepsilon\sqrt{\varepsilon} + \frac{2}{3} + \varepsilon(1 + 2\sqrt{2}) + \frac{\varepsilon^2}{4} - 2\varepsilon \cdots .$$

Thus we have found that

$$I(\varepsilon) \sim \frac{2}{3} + \varepsilon 2\sqrt{2} - \frac{2\varepsilon^{3/2}}{3} + \frac{\varepsilon^2}{4}$$

as $\varepsilon \to 0^+$, as far as terms at $O(\varepsilon^2)$; here we see that the integration over $x = O(1)$ provides the dominant contribution to this value.

This example has presented, *via* a fairly routine calculation, the essential idea that underpins this method for evaluating integrals. Of course, there is no need to exploit this technique if the integral can be evaluated directly (as was the case here); let us therefore examine another problem which is less elementary.

E2.6 Another integral

We wish to evaluate the integral

$$I(\varepsilon) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{\dfrac{x+\varepsilon}{(1+x)^2} + \varepsilon^2 \exp(-x/\varepsilon^2)}}$$

as $\varepsilon \to 0^+$; here, the expansion of the integrand requires three different asymptotic expansions (valid for $x = O(1)$, $x = O(\varepsilon)$, $x = O(\varepsilon^2)$). Thus we obtain

$$\begin{aligned} f(x;\varepsilon) &= \left\{ \frac{x+\varepsilon}{(1+x)^2} + \varepsilon^2 \exp(-x/\varepsilon^2) \right\}^{-1/2} \\ &\sim \frac{1+x}{\sqrt{x}}\left(1 - \frac{\varepsilon}{2x} + \frac{3\varepsilon^2}{8x^2}\right), \quad \text{for } x = O(1); \end{aligned}$$

$$\begin{aligned} f(\varepsilon X;\varepsilon) \equiv F(X;\varepsilon) &= \left\{ \frac{\varepsilon(1+X)}{(1+\varepsilon X)^2} + \varepsilon^2 e^{-X/\varepsilon} \right\}^{-1/2} \\ &\sim \frac{1}{\sqrt{\varepsilon}}\frac{1}{\sqrt{1+X}}(1+\varepsilon X), \quad \text{for } X = O(1); \end{aligned}$$

$$f(\varepsilon^2\chi;\varepsilon) \equiv \mathfrak{F}(\chi;\varepsilon) = \left\{\frac{\varepsilon(1+\varepsilon\chi)}{(1+\varepsilon^2\chi)^2} + \varepsilon^2 \mathrm{e}^{-\chi}\right\}^{-1/2}$$

$$\sim \frac{1}{\sqrt{\varepsilon}}\left\{1 - \frac{1}{2}\varepsilon(\chi + \mathrm{e}^{-\chi}) + \varepsilon^2\chi + \frac{3}{8}\varepsilon^2(\chi + \mathrm{e}^{-\chi})^2\right\}, \quad \text{for } \chi = \mathrm{O}(1).$$

In this problem, we require two intermediate variables; these are defined by

$$\delta(\varepsilon) \to 0 \quad \text{and} \quad \varepsilon/\delta(\varepsilon) \to 0; \quad \Delta(\varepsilon)/\varepsilon \to 0 \quad \text{and} \quad \varepsilon^2/\Delta(\varepsilon) \to 0,$$

all as $\varepsilon \to 0^+$. The integral is then written as

$$I(\varepsilon) = \int_0^\Delta \mathfrak{F}(\chi;\varepsilon)\,\mathrm{d}x + \int_\Delta^\delta F(X;\varepsilon)\,\mathrm{d}x + \int_\delta^1 f(x;\varepsilon)\,\mathrm{d}x$$

$$\sim \varepsilon^{3/2}\int_0^{\Delta/\varepsilon^2}\left\{1 - \frac{\varepsilon}{2}(\chi + \mathrm{e}^{-\chi}) + \varepsilon^2\chi + \frac{3\varepsilon^2}{8}(\chi + \mathrm{e}^{-\chi})^2\right\}\mathrm{d}\chi$$

$$+\,\varepsilon^{1/2}\int_{\Delta/\varepsilon}^{\delta/\varepsilon}\frac{1+\varepsilon X}{\sqrt{1+X}}\,\mathrm{d}X + \int_\delta^1 \frac{1+x}{\sqrt{x}}\left(1 - \frac{\varepsilon}{2x} + \frac{3\varepsilon^2}{8x^2}\right)\mathrm{d}x$$

and we will now retain terms that will enable us to find an expression for $I(\varepsilon)$ correct at $\mathrm{O}(\varepsilon^2)$. Thus we find that

$$I(\varepsilon) \sim \varepsilon^{3/2}\left[\chi\right]_0^{\Delta/\varepsilon^2} + \varepsilon^{1/2}\left[2\sqrt{1+X} + \frac{2\varepsilon}{3}(1+X)^{3/2} - 2\varepsilon\sqrt{1+X}\right]_{\Delta/\varepsilon}^{\delta/\varepsilon}$$

$$+\left[2\sqrt{x} + \frac{2}{3}x^{3/2} + \varepsilon\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right) + \frac{3\varepsilon^2}{8}\left(-\frac{2x^{-3/2}}{3} - \frac{2}{\sqrt{x}}\right)\right]_\delta^1$$

$$= \frac{\Delta}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}\left\{2\sqrt{1+\delta/\varepsilon} + \frac{2\varepsilon}{3}(1+\delta/\varepsilon)^{3/2} - 2\varepsilon\sqrt{1+\delta/\varepsilon}\right.$$

$$\left. - 2\sqrt{1+\Delta/\varepsilon} - \frac{2\varepsilon}{3}(1+\Delta/\varepsilon)^{3/2} + 2\varepsilon\sqrt{1+\Delta/\varepsilon}\right\}$$

$$+\left\{2 + \frac{2}{3} - \varepsilon^2 - 2\sqrt{\delta} - \frac{2\delta^{3/2}}{3} - \varepsilon\left(\frac{1}{\sqrt{\delta}} - \sqrt{\delta}\right) + \frac{3\varepsilon^2}{8}\left(\frac{2\delta^{-3/2}}{3} + \frac{2}{\sqrt{\delta}}\right)\right\}$$

$$= \frac{\Delta}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}\left\{2\sqrt{\frac{\delta}{\varepsilon}}\left(1 + \frac{\varepsilon}{2\delta} - \frac{\varepsilon^2}{8\delta^2}\cdots\right) + \frac{2\delta^{3/2}}{3\sqrt{\varepsilon}}\left(1 + \frac{3\varepsilon}{2\delta} + \frac{3\varepsilon^2}{8\delta^2}\cdots\right)\right.$$

$$\left. - 2\sqrt{\varepsilon\delta}\left(1 + \frac{\varepsilon}{2\delta}\cdots\right) - 2\left(1 + \frac{\Delta}{2\varepsilon}\cdots\right) - \frac{2\varepsilon}{3}\left(1 + \frac{3\Delta}{2\varepsilon}\cdots\right) + 2\varepsilon\left(1 + \frac{\Delta}{2\varepsilon}\cdots\right)\right\}$$

$$+ 2 + \frac{2}{3} - \varepsilon^2 - 2\sqrt{\delta} - \frac{2\delta^{3/2}}{3} - \frac{\varepsilon}{\sqrt{\delta}} + \varepsilon\sqrt{\delta} + \frac{\varepsilon^2}{4\delta^{3/2}} + \frac{3\varepsilon^2}{4\sqrt{\delta}}$$

$$\sim \frac{8}{3} - 2\sqrt{\varepsilon} + \frac{4}{3}\varepsilon\sqrt{\varepsilon} - \varepsilon^2 \quad \text{as } \varepsilon \to 0^+.$$

(You should confirm that, in the above, both δ and Δ cancel identically, to this order.)

Further examples that make use of these ideas can be found in exercises Q2.6, 2.7 and 2.8. With a little experience, it should not be too difficult to recognise how many terms need to be retained in each expansion in order to produce $I(\varepsilon)$ to a desired accuracy. The region that gives the dominant contribution is usually self-evident, and quite often this alone will provide an acceptable approximation to the value of the integral. Furthermore, terms that contain the overlap variables can be ignored altogether, because they must cancel (although there is a case for their retention—which was our approach above—as a check on the correctness of the details).

2.3 ORDINARY DIFFERENTIAL EQUATIONS: REGULAR PROBLEMS

We now turn to an initial discussion of how the techniques of singular perturbation theory can be applied to the problem of finding solutions of differential equations—unquestionably the most significant and far-reaching application that we encounter. The relevant ideas will be developed, first, for problems that turn out to be *regular* (but we will indicate how singular versions of these problems might arise, and we will discuss some simple examples of these later in this chapter). Clearly, we need to lay down the basic procedure that must be followed when we seek solutions of differential equations. However, these techniques are many and varied, and so we cannot hope to present, at this stage, an all-encompassing recipe. Nevertheless, the fundamental principles can be developed quite readily; to aid us in this, we consider the differential equation

$$\frac{dy}{dx} + y + \varepsilon y^2 = x, \quad 0 \le x \le 1, \quad \text{with} \quad y(1;\varepsilon) = 1, \tag{2.11}$$

for $\varepsilon \to 0$. This problem, we observe, is not trivial; it is an equation which, although first order, is nonlinear and with a forcing term on the right-hand side.

The first stage is to decide on a suitable asymptotic sequence for the representation of $y(x;\varepsilon)$. Here, we note that the process of iteration on the equation, which can be written for this purpose (with a prime for the derivative) as

$$y'_{n+1} + y_{n+1} - x = -\varepsilon y_n^2, \quad n \geq 0, \quad \text{with} \quad y_0 = 0,$$

gives
$$y'_1 + y_1 - x = 0; \quad y'_2 + y_2 - x = -\varepsilon y_1^2,$$

and so on, so that y_2 takes the form $y_2 = u_2(x) + \varepsilon v_2(x)$ (for appropriate functions u_2, v_2). When this solution is used to generate y_3, it is clear that we will produce terms in ε, ε^2 and ε^3, and so this pattern will continue: the equation implies the 'natural' asymptotic sequence $\{\varepsilon^n\}$, so this is what we will assume to initiate the solution method. (It should be noted that the boundary condition is consistent with this assumption, as is the alternative condition $y(1;\varepsilon) = 1 + \varepsilon$. On the other hand, a boundary value $y(1;\varepsilon) = 1 + \sqrt{\varepsilon}$ would force the asymptotic sequence to be adjusted to accommodate this i.e. $\{\varepsilon^{n/2}\}$, $n = 0, 1, 2, \ldots$.)

Thus we seek a solution of the problem (2.11) in the form

$$y(x;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n y_n(x), \tag{2.12}$$

for some $x \in D$ (and we do not know which xs will be allowed, at this stage). The expansion (2.12) is used in the differential equation to give

$$(y'_0 + \varepsilon y'_1 + \varepsilon^2 y'_2 \cdots) + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \cdots) + \varepsilon(y_0 + \varepsilon y_1 \cdots)^2 - x = 0,$$

where '= 0' means zero to all orders in ε; thus we require

$$y'_0 + y_0 - x = 0; \quad y'_1 + y_1 + y_0^2 = 0; \quad y'_2 + y_2 + 2y_0 y_1 = 0, \tag{2.13a,b,c}$$

and so on. Similarly, the boundary condition gives

$$y_0(1) + \varepsilon y_1(1) + \varepsilon^2 y_2(1) \cdots - 1 = 0$$

so that
$$y_0(1) = 1, \quad y_n(1) = 0 \quad \text{for } n \geq 1; \tag{2.14a,b}$$

of course, to evaluate on $x = 1$ implies that the asymptotic expansion, (2.12), is valid here—but we do not know this yet. This is written down because, if the problem turns out to be well-behaved i.e. regular, then we will have this ready for use; essentially, all we are doing is noting (2.14)—we can reject it if the expansion will not permit evaluation on $x = 1$.

The next step is simply to solve each equation (for y_0, $y_1, \ldots$) in turn; we see directly (from (2.13a)) that the general solution for y_0 is

$$y_0(x) = x - 1 + A_0 e^{-x}, \tag{2.15}$$

where A_0 is an arbitrary constant. Then, from (2.13b), we have

$$y_1' + y_1 = -y_0^2 = -(x-1)^2 - 2A_0(x-1)e^{-x} - A_0^2 e^{-2x}$$

which can be written (on using the integrating factor e^x) as

$$(e^x y_1)' = -(x-1)^2 e^x - 2A_0(x-1) - A_0^2 e^{-x}.$$

This produces the general solution

$$y_1(x) = -(x-1)^2 + 2x - 4 - A_0(x-1)^2 e^{-x} + A_0^2 e^{-2x} + A_1 e^{-x}, \qquad (2.16)$$

where A_1 is a second arbitrary constant. It is immediately clear that these first two terms in the expansion are defined (and well-behaved i.e. no hint of a non-uniformity) for $0 \le x \le 1$, so we may impose the boundary conditions, (2.14a,b); these produce

$$y_0(x) = x - 1 + e^{1-x}; \quad y_1(x) = -x^2 + 4x - 5 - (x^2 - 2x)e^{1-x} + e^{2(1-x)}.$$

Thus our asymptotic expansion, so far, is

$$y(x;\varepsilon) \sim x - 1 + e^{1-x} + \varepsilon\left\{-x^2 + 4x - 5 - (x^2 - 2x)e^{1-x} + e^{2(1-x)}\right\}, \qquad (2.17)$$

and this is certainly uniformly valid for $0 \le x \le 1$: we have a 2-term expansion of the solution. (Note that the specification of the domain is critical here; if, for example, we were seeking the solution with the same boundary condition, but in $x \ge 1$, then (2.17) would *not* be uniformly valid: there is a breakdown where $x = O(\varepsilon x^2)$ i.e. $x = O(\varepsilon^{-1})$; see the problem in (2.34), below.) The evidence in (2.17) suggests that we have the beginning of a uniformly valid asymptotic expansion i.e. (2.12) is valid for $\forall N \ge 0$ and for $\forall x \in D$ (and it is left as an exercise to find $y_2(x)$ and to check that the inclusion of this term does not alter this proposition).

In order to investigate the uniform validity, or otherwise, of (2.12), one approach is to examine the general term in the expansion; this is the solution of

$$y_{n+1}' + y_{n+1} = f_n \quad \text{with} \quad y_{n+1}(1) = 0 \quad \text{for } n \ge 0, \qquad (2.18)$$

where $f_n = f_n(y_0, y_1, \ldots, y_n)$ with $f_0 = -y_0^2$, $f_1 = -2y_0y_1$. The solution to (2.18) is

$$y_{n+1}(x) = -e^{-x}\int_x^1 f_n e^x \,dx;$$

but $y_0(x)$ and $y_1(x)$ are bounded functions for $x \in [0, 1]$, and hence so is $y_2(x)$, and then so is $y_3(x)$ and hence all the y_ns. In particular, $y_n(x) \to 0$ as $x \to 1$ $(n \ge 1)$, and $y_n(x) \to c_n$ (c_n constants) as $x \to 0$ $(n \ge 0)$: there is no breakdown of the asymptotic expansion. The problem posed in (2.11) is therefore *regular,* resulting in a uniformly valid asymptotic expansion.

More complete, formal and rigorous discussions of uniform validity, in the context of differential equations, can be found in other texts, such as Smith (1985), O'Malley (1991) and Eckhaus (1979). Typically, these arguments involve writing

$$y(x;\varepsilon) = Y_N(x;\varepsilon) + \varepsilon^{N+1} R_{N+1}(x;\varepsilon),$$

where $Y_N(x;\varepsilon) = \sum_{n=0}^{N} \varepsilon^n y_n(x)$, and then showing that R_{N+1} remains bounded for $\forall N \geq 0$ and for $\forall x \in D$. We will outline how this can be applied to our problem, (2.11); first, we obtain

$$Y_N' + \varepsilon^{N+1} R_{N+1}' + Y_N + \varepsilon^{N+1} R_{N+1} + \varepsilon(Y_N + \varepsilon^{N+1} R_{N+1})^2 = x$$

with $R_{N+1}(1;\varepsilon) = 0$ for $N \geq 0$. Since each $y_n(x)$ satisfies an appropriate differential equation and boundary condition, this gives

$$R_{N+1}' + R_{N+1} + F_N(y_0, y_1, \ldots, y_N; \varepsilon) + 2\varepsilon Y_N R_{N+1} + \varepsilon^{N+2} R_{N+1}^2 = 0, \tag{2.19}$$

where F_N comprises the $\mathrm{O}(\varepsilon^N)$ terms, and smaller, from the expansion of Y_N^2 (after division by ε^{N+1}). A uniform asymptotic expansion requires that R_{N+1} is bounded as $\varepsilon \to 0$, for $\forall x \in D$ and $\forall N \geq 0$. To prove such a result is rarely an elementary exercise in general, and it is not trivial here, although a number of approaches are possible. One method is based on Picard's iterative scheme (which is a standard technique for proving the existence of solutions of first order ordinary differential equations in some appropriate region of (x, y)-space); this will be described in any good basic text on ordinary differential equations (e.g. Boyce & DiPrima, 2001). Another possibility, closely related to Picards method, is formally to integrate the equation for R_{N+1}, thereby obtaining an integral equation, and then to derive estimates for the integral term (and hence for R_{N+1}). We will outline a third technique, which involves the construction of estimates directly for the differential equation, and then integrating a reduced version of the equation for R_{N+1}.

At this stage we do not know if R_{N+1} is of one sign, for $0 \leq x \leq 1$, or if it changes sign on this interval; however, we may proceed without specifying or assuming the nature of this property, but it will affect the details; first we write

$$R_{N+1}' + R_{N+1} + 2\varepsilon Y_N R_{N+1} + \varepsilon^{N+2} R_{N+1}^2 = -F_N.$$

But we do know that each y_n is bounded (for $x \in [0, 1]$) and hence so is F_N, which we will express in the form $|F_N| \leq k$ (a constant independent of ε), and so

$$-k \leq R_{N+1}' + R_{N+1} + 2\varepsilon Y_N R_{N+1} + \varepsilon^{N+2} R_{N+1}^2 \leq k.$$

This same property of the functions y_n leads to a corresponding statement for Y_N: $|Y_N| \leq c$ (again, independent of ε), which now gives

$$-k \mp 2\varepsilon c\, R_{N+1} \leq R'_{N+1} + R_{N+1} + \varepsilon^{N+2} R^2_{N+1} \leq k \pm 2\varepsilon c\, R_{N+1}, \tag{2.20}$$

where the upper sign applies if $R_{N+1} \geq 0$, and the lower if $R_{N+1} \leq 0$. (Often in arguments of this type, we cannot incorporate both signs, and we are reduced to working with the modulus of the function; we will see here that we can allow the form given in (2.20).) Between the two inequalities, we have an expression associated with a constant coefficient Riccati equation; let us therefore consider

$$R'_{N+1} + R_{N+1} + \varepsilon^{N+2} R^2_{N+1} = \lambda + 2\varepsilon\mu R_{N+1}, \tag{2.21}$$

where $-k \leq \lambda(x) \leq k$ and $-c \leq \mu(x) \leq c$ for arbitrary (bounded) functions λ and μ. If an appropriate unique solution of (2.21), satisfying $R_{N+1}(1;\varepsilon) = 0$, exists for all λ and all μ as specified, then we will certainly have satisfied (2.20). However, we will, in this text, give only the flavour of how the development proceeds, by considering a restricted version of the problem with the special choice: λ and μ constant (but satisfying the given bounds).

To solve (2.21), we introduce $R_{N+1} = \varepsilon^{-N-2}\phi'/\phi$ to obtain

$$\phi'' + (1 - 2\varepsilon\mu)\phi' - \varepsilon^{N+2}\lambda\phi = 0$$

which has, in our special case, the general solution

$$A \exp[-(1 - \varepsilon\alpha_1)x] + B \exp\left[\varepsilon^{N+2}\alpha_2 x\right]$$

where the arbitrary constants are A and B, and the auxiliary equation for the exponents is

$$\alpha^2 + (1 - 2\varepsilon\mu)\alpha - \varepsilon^{N+2}\lambda = 0.$$

The roots of this equation have been written as

$$\alpha = -1 + \varepsilon\alpha_1, \quad \varepsilon^{N+2}\alpha_2,$$

where $\alpha_i(\varepsilon) = O(1)$, for $i = 1, 2$, as $\varepsilon \to 0$. Finally, the solution which satisfies the condition on $x = 1$ is

$$R_{N+1}(x;\varepsilon) = -\alpha_2 \frac{\left\{\exp\left[(1 - \varepsilon\alpha_1 + \varepsilon^{N+2}\alpha_2)(1 - x)\right] - 1\right\}}{1 + \varepsilon^{N+2}\alpha_2 \exp\left[(1 - \varepsilon\alpha_1 + \varepsilon^{N+2}\alpha_2)(1 - x)\right]},$$

which is bounded for $\forall\lambda \in [-k,\ k]$, $\forall\mu \in [-c,\ c]$, $\forall x \in [0,\ 1]$ as $\varepsilon \to 0$. Thus the error is $\varepsilon^{N+1} R_{N+1} = O(\varepsilon^{N+1})$ for $\forall N \geq 0$, as required. (A comprehensive discussion requires the analysis of the equation for ϕ with general, bounded $\lambda = \lambda(x)$ and $\mu = \mu(x)$, which is possible, but beyond the aims of this text.)

As should be clear from this calculation, it is to be anticipated that special properties, relevant to a particular problem, may have to be invoked. Here, for example, we took advantage of the underlying Riccati equation; other problems may require quite different approaches. However, we must also emphasise that, for many practical and important problems encountered in applied mathematics, these calculations are often too difficult to succumb to such a general analysis. Indeed, the conventional wisdom is that, if breakdowns have been identified, rescaling employed and asymptotic solutions found (and matched, as required), then we have produced a sufficiently robust description. It should be noted that the process of rescaling might involve a consideration of all possible scalings allowed by the governing equation, which will then greatly strengthen our trust in the results obtained. Those readers who prefer the more rigorous approach that such discussions afford are encouraged to study the texts previously mentioned. In this text, however, we shall proceed without much further consideration of these more formal aspects of the asymptotic solution of differential equations.

Now that we have presented the salient features of the method of constructing solutions, we apply it to another example.

E2.7 A regular second-order problem

We seek an asymptotic solution, as $\varepsilon \to 0$, of

$$y'' + y' + \varepsilon y^2 = 0, \quad 0 \le x \le 1, \tag{2.22}$$

with $y(0; \varepsilon) = 1$ and $y(1; \varepsilon) = \mathrm{e}^{-1}$; the primes here denote derivatives. First, we assume that there is a solution, for some $x \in D$, of the form

$$y(x; \varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n y_n(x).$$

Thus we obtain

$$y_0'' + y_0' = 0; \quad y_1'' + y_1' + y_0^2 = 0, \tag{2.23a,b}$$

and so on, with

$$y_0(0) = 1, \quad y_0(1) = \mathrm{e}^{-1}; \quad y_n(0) = y_n(1) = 0 \quad \text{for } n \ge 1 \tag{2.24}$$

(if the expansion is valid at the end-points). The general solution of (2.23a) is

$$y_0(x) = A_0 + B_0 \mathrm{e}^{-x} \quad (\text{where } A_0,\ B_0 \text{ are the arbitrary constants})$$

and then (2.23b) becomes

$$y_1'' + y_1' = -y_0^2 = -A_0^2 - 2A_0 B_0 e^{-x} - B_0^2 e^{-2x}$$

which, in turn, has the general solution

$$y_1(x) = A_1 + B_1 e^{-x} - A_0^2(x-1) + 2A_0 B_0 x e^{-x} - \tfrac{1}{2} B_0^2 e^{-2x}$$

where A_1, B_1 are the new arbitrary constants. The functions $y_0(x)$ and $y_1(x)$ are clearly defined for $0 \leq x \leq 1$, and there is no suggestion of a breakdown, so we impose the boundary conditions (2.24) to give

$$y_0(x) = e^{-x}; \quad y_1(x) = \frac{1}{2}(e^{-x} - e^{-1} + e^{-1-x} - e^{-2x})$$

and then our asymptotic expansion (to this order) is

$$y(x;\varepsilon) \sim e^{-x} + \frac{\varepsilon}{2}\{e^{-x} - e^{-1} + e^{-1-x} - e^{-2x}\}. \tag{2.25}$$

Now that we have obtained the expansion, (2.25), we are able to confirm that we have a 2-term uniformly valid representation of the solution. In order to examine the general term in this asymptotic expansion, if this is deemed necessary, we can follow the method described earlier. Thus we may write

$$y_n'' + y_n' = f_{n-1}(y_0, \ldots, y_{n-1}), \quad n \geq 1,$$

where, in particular, we have $f_0 = -y_0^2$, and $y_n(0) = y_n(1) = 0$ for $n \geq 1$; the general solution for y_n is

$$y_n(x) = e^{-x} \int_0^x e^x \left(\int_0^x f_{n-1}\,dx \right) dx + A_n + B_n e^{-x},$$

where A_n and B_n are determined to satisfy the two boundary conditions. The essentials of the argument are then as we have already outlined in our first, simple, presentation: y_0 is bounded (on [0, 1]), so is y_1, and hence so is y_2, etc., for all $y_n(x)$. Further, $y_n(x) \to 0$ as $x \to 0$ and as $x \to 1$, for $\forall n \geq 1$: the asymptotic expansion is uniformly valid.

Some further examples of regular expansions can be found in Q2.9 and 2.10, and an interesting variant of E2.7 is discussed in Q2.15.

2.4 ORDINARY DIFFERENTIAL EQUATIONS: SIMPLE SINGULAR PROBLEMS

Now that we have introduced the simplest ideas that enable solutions of differential equations to be constructed, we must extend our horizons. The first point to record is that, only quite rarely, do we encounter problems that can be represented by uniformly valid expansions (although, somewhat after the event, we can often construct such expansions—in the form of a composite expansion, for example; see §1.10). The more common equations exhibit singular behaviour, in one form or another; the simplest situation, we suggest, is when the techniques used above (§2.3) produce asymptotic expansions that break down, resulting in the need to rescale, expand again and (probably) invoke the matching principle. (Other types of singularity can arise, and these will be described in due course.) To see how this approach is a natural extension of what we have done thus far, we will present a problem based on the equation given in (2.11).

We consider

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1+\frac{\varepsilon^2}{x^2+\varepsilon^2}\right)y+\varepsilon y^2=0, \quad 0\le x\le 1, \quad \text{with} \quad y(1;\varepsilon)=1, \tag{2.26}$$

for $\varepsilon \to 0$; the important new ingredient here is the variable coefficient (which, we note, is $1 + \mathrm{O}(\varepsilon^2)$ for $x = \mathrm{O}(1)$). We seek a solution in the form

$$y(x;\varepsilon)\sim\sum_{n=0}^{N}\varepsilon^n y_n(x)$$

and we will need to find the terms y_0, y_1 and y_2 (at least) in order to include a contribution from the new part of the coefficient. The equations for the $y_n(x)$ are

$$y_0'+y_0=0; \quad y_1'+y_1+y_0^2=0; \quad y_2'+y_2+\frac{y_0}{x^2}+2y_0y_1=0,$$

and so on; the boundary condition requires that

$$y_0(1)=1, \quad y_n(1)=0 \quad \text{for } n\ge 1.$$

In this problem, we should expect that evaluation of the expansion on $x = 1$ is allowed—all terms are defined for $x = \mathrm{O}(1)$—but we must anticipate difficulties as $x \to 0$.

The solutions for the functions $y_0(x)$ and $y_1(x)$ follow from the results given in (2.15) and (2.16), respectively, but with the particular integral omitted; thus

$$y_0(x)=\mathrm{e}^{1-x}; \quad y_1(x)=\mathrm{e}^{2-2x}-\mathrm{e}^{1-x}.$$

The 2-term asymptotic expansion, $y \sim y_0 + \varepsilon y_1$ is uniformly valid for $\forall x \in [0,\ 1]$. Let us find the next term in the expansion; this is the solution of

$$y_2' + y_2 + \frac{e^{1-x}}{x^2} + 2e^{1-x}(e^{2-2x} - e^{1-x}) = 0 \quad \text{with} \quad y_2(1) = 0.$$

Thus, introducing the integrating factor e^x, we have

$$(e^x y_2)' + \frac{e}{x^2} + 2e(e^{2-2x} - e^{1-x}) = 0$$

and so
$$y_2(x) = \frac{e^{1-x}}{x} + e^{1-x}(e^{2-2x} - 2e^{1-x}) + Ae^{-x}$$

where the arbitrary constant must be $A = 0$ (to satisfy $y_2(1) = 0$). This third term in the asymptotic expansion is very different from the first two: it is not defined on $x = 0$, so we must expect a breakdown. The expansion, to this order, is now

$$y(x;\varepsilon) \sim e^{1-x} + \varepsilon\left\{e^{2-2x} - e^{1-x}\right\} + \varepsilon^2\left\{\frac{e^{1-x}}{x} + e^{3-3x} - 2e^{2-2x}\right\} \tag{2.27}$$

as $\varepsilon \to 0$ for $x = O(1)$; as $x \to 0$, we clearly have a breakdown where the second and third terms in the expansion become the same size i.e. $\varepsilon = O(\varepsilon^2/x)$ or $x = O(\varepsilon)$. Note that this breakdown occurs for a larger size of x (as x is decreased from $O(1)$) than the breakdown associated with the first and third terms, so we must consider $x = O(\varepsilon)$.

The problem for $x = O(\varepsilon)$ is formulated by writing

$$x = \varepsilon X \quad \text{and} \quad y(\varepsilon X;\varepsilon) \equiv Y(X;\varepsilon),$$

where the relabelling of y is an obvious convenience (and we note that $y = O(1)$ for $x = O(\varepsilon)$). The original equation, in (2.26), expressed in terms of X and Y, requires the identity

$$\frac{dy}{dx} = \frac{dY}{dx} = \frac{d}{dx}Y(x/\varepsilon;\varepsilon) = \varepsilon^{-1}\frac{dY}{dX}$$

and then we obtain

$$\varepsilon^{-1}\frac{dY}{dX} + \left(1 + \frac{\varepsilon^2}{\varepsilon^2 X^2 + \varepsilon^2}\right)Y + \varepsilon Y^2 = 0 \quad \text{or} \quad \frac{dY}{dX} + \varepsilon\left(1 + \frac{1}{1 + X^2}\right)Y + \varepsilon^2 Y^2 = 0, \tag{2.28}$$

but the boundary condition is not available, because this is specified where $x = O(1)$. Equation (2.28) suggests that we seek a solution in the form

$$Y(X;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n Y_n(X),$$

which gives

$$Y_0' = 0; \quad Y_1' + \left(1 + \frac{1}{1+X^2}\right) Y_0 = 0, \quad \text{and so on.} \tag{2.29a,b}$$

Immediately we obtain $Y_0 = A_0$ (an arbitrary constant), and then equation (2.29b) becomes

$$Y_1' = -\left(1 + \frac{1}{1+X^2}\right) A_0$$

which integrates to give

$$Y_1(X) = -A_0(X + \arctan X) + A_1,$$

where A_1 is a second arbitrary constant.

The resulting 2-term expansion is therefore

$$Y(X;\varepsilon) \sim A_0 + \varepsilon \left\{A_1 - A_0(X + \arctan X)\right\}, \quad X = O(1); \tag{2.30}$$

the two arbitrary constants are determined by invoking the matching principle: (2.30) and (2.27) are to match. Thus we write the terms in (2.27) as functions of X, let $\varepsilon \to 0$ (for $X = O(1)$) and retain terms $O(1)$ and $O(\varepsilon)$ (which are used in (2.30)); conversely, write (2.30) as a function of x, expand and retain terms $O(1)$, $O(\varepsilon)$ and $O(\varepsilon^2)$. From (2.27) we construct

$$\begin{aligned} &\mathrm{e}^{1-\varepsilon X} + \varepsilon\left\{\mathrm{e}^{2-2\varepsilon X} - \mathrm{e}^{1-\varepsilon X}\right\} + \varepsilon^2\left\{\frac{\mathrm{e}^{1-\varepsilon X}}{\varepsilon X} + \mathrm{e}^{3-3\varepsilon X} - 2\mathrm{e}^{2-2\varepsilon X}\right\} \\ &\qquad \sim \mathrm{e} + \varepsilon\left(\frac{\mathrm{e}}{X} + \mathrm{e}^2 - \mathrm{e} - \mathrm{e}X\right) \quad \text{for } X = O(1); \end{aligned} \tag{2.31}$$

and from (2.30) we write

$$\begin{aligned} &A_0 + \varepsilon\left\{A_1 - A_0\left(\frac{x}{\varepsilon} + \arctan\frac{x}{\varepsilon}\right)\right\} \\ &\qquad \sim A_0 - A_0 x + \varepsilon\left(A_1 - A_0\frac{\pi}{2}\right) + \frac{\varepsilon^2 A_0}{x} \quad \text{for } x = O(1). \end{aligned} \tag{2.32}$$

(This expansion requires the standard result:

$$\arctan X \sim \frac{\pi}{2} - \frac{1}{X} + \frac{1}{3X^3} \quad \text{as } X \to +\infty,$$

which the interested reader may wish to derive.) The two 'expansions of expansions', (2.31) and (2.32), match when we choose $A_0 = \mathrm{e}$ and $A_1 = \mathrm{e}^2 - \mathrm{e} + \mathrm{e}\pi/2$; the asymptotic expansion for $X = \mathrm{O}(1)$ is therefore

$$Y(X;\varepsilon) \sim \mathrm{e} + \varepsilon\left\{\frac{\mathrm{e}\pi}{2} + \mathrm{e}^2 - \mathrm{e} - \mathrm{e}(X + \arctan X)\right\}. \tag{2.33}$$

We now observe that, although the expansion (2.27) is not defined on $x = 0$, the expansion valid for $x = \mathrm{O}(\varepsilon)$ does allow evaluation on $x = 0$ i.e. $X = 0$; indeed, from (2.33), we see that

$$y(0;\varepsilon) = Y(0;\varepsilon) \sim \mathrm{e}\left\{1 + \varepsilon\left(\tfrac{1}{2}\pi + \mathrm{e} - 1\right)\right\} \quad \text{as } \varepsilon \to 0.$$

In summary, the procedure involves the construction of an asymptotic expansion valid for $x = \mathrm{O}(1)$ and applying the boundary condition(s) if the expansion remains valid here. The expansion is then examined for $\forall x \in D$, seeking any breakdowns, rescaling and hence rewriting the equation in terms of the new, scaled variable; this problem is then solved as another asymptotic expansion, matching as necessary. A couple of general observations are prompted by this example. First, the matching principle has been used to determine the arbitrary constants of integration because the boundary condition does not sit where $x = \mathrm{O}(\varepsilon)$; thus the process of matching is equivalent, here, to imposing boundary conditions (and thereby obtaining unique solutions for $\forall x \in D$). In the context of differential equations, this is the usual role of the matching principle, and it is fundamental in seeking complete solutions.

The second issue is rather more general. In this example, the expansion for $x = \mathrm{O}(1)$, (2.27), had to be taken to the term at $\mathrm{O}(\varepsilon^2)$ before the non-uniformity (as $x \to 0$) became evident. This prompts the obvious question: how many terms should be determined so that we can be (reasonably) sure that all possible contributions to a breakdown have been identified? A very good rule of thumb is to ensure that the asymptotic expansion contains information generated by every term in the differential equation. Thus our recent example, (2.26),

$$y' + \left(1 + \frac{\varepsilon^2}{x^2 + \varepsilon^2}\right) y + \varepsilon y^2 = 0$$

requires $\mathrm{O}(\varepsilon)$ terms to include the nonlinearity (εy^2) and $\mathrm{O}(\varepsilon^2)$ terms for the dominant representation of the varying part of the variable coefficient. In a physically based problem, the interpretation of this rule is simply to ensure that every different physical effect is included at some stage in the expansion. As an example of this idea, consider the nonlinear, damped oscillator with variable frequency described by the equation:

$$\ddot{x} + \varepsilon^3 \dot{x} + \left(1 + \frac{\varepsilon}{t + \varepsilon}\right) x + \varepsilon^2 x^2 = 0, \quad t \geq 0.$$

At O(1) we have the basic oscillator; at $O(\varepsilon)$ the variable frequency; at $O(\varepsilon^2)$ the nonlinearity; at $O(\varepsilon^3)$ the damping. Thus, in order to investigate the leading contributions (at least) to each of these properties of the oscillation, the asymptotic expansion must be taken as far as the inclusion of terms $O(\varepsilon^3)$. (There is no suggestion that each will necessarily lead to a breakdown, and an associated scaling, but each needs to be examined.) One further important observation will be discussed in the next section; we conclude this section with two examples that exploit all these ideas.

E2.8 Problem (2.11) extended

We consider the problem

$$\frac{dy}{dx} + y + \varepsilon y^2 = x, \quad x \geq 1, \quad \text{with} \quad y(1;\varepsilon) = 1, \tag{2.34}$$

which is the same equation and boundary condition as we introduced in (2.11), but now the domain is $x \geq 1$. The asymptotic solution for $x = O(1)$, which satisfies the boundary condition, is (2.17) i.e.

$$y(x;\varepsilon) \sim x - 1 + e^{1-x} + \varepsilon\left\{-x^2 + 4x - 5 - (x^2 - 2x)e^{1-x} + e^{2(1-x)}\right\} \tag{2.35}$$

and this breaks down where $x = O(\varepsilon x^2)$ or $x = O(\varepsilon^{-1})$. Thus we introduce $x = X/\varepsilon$; but for this size of x, we observe that $y = O(\varepsilon^{-1})$ and so this also must be scaled: we write

$$y(X/\varepsilon;\varepsilon) \equiv \varepsilon^{-1} Y(X;\varepsilon).$$

Equation (2.34) becomes

$$\frac{\varepsilon}{\varepsilon}\frac{dY}{dX} + \frac{1}{\varepsilon}Y + \frac{\varepsilon}{\varepsilon^2}Y^2 = \frac{X}{\varepsilon} \quad \text{or} \quad \varepsilon\frac{dY}{dX} + Y + Y^2 = X,$$

and we seek a solution

$$Y(X;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n Y_n(X),$$

which gives

$$Y_0 + Y_0^2 = X; \quad Y_1 + 2Y_0Y_1 + Y_0' = 0, \tag{2.36a,b}$$

and so on. This result may cause some surprise: this sequence of problems is purely *algebraic*—there is no integration of differential equations required at any stage.

Equation (2.36a) has the solution

$$Y_0(X) = \frac{1}{2}\left(-1 \pm \sqrt{1+4X}\right), \tag{2.37}$$

and we will need to invoke the matching principle to decide which sign is appropriate. Thus from the first term in (2.35), we see that

$$\frac{Y}{\varepsilon} \sim \frac{X}{\varepsilon} - 1 + \mathrm{e}^{1-X/\varepsilon} \quad \text{and so} \quad Y \sim X; \tag{2.38}$$

from (2.37) we obtain

$$\varepsilon y \sim \frac{1}{2}\left(-1 \pm \sqrt{1+4\varepsilon x}\right) \quad \text{so} \quad y \sim \frac{1}{2\varepsilon}(-1 \pm 1) \pm x \tag{2.39}$$

and then (2.38) and (2.39) match only for the positive sign. (Note that, because y has also been scaled, this must be included in the construction which enables the matching to be completed.) The solution for the first term is therefore

$$Y_0(X) = \frac{1}{2}\left(\sqrt{1+4X} - 1\right),$$

and then the second term is obtained directly as

$$Y_1(X) = -\frac{1}{1+4X};$$

the resulting 2-term asymptotic expansion is

$$Y(X;\varepsilon) \sim \frac{1}{2}\left(\sqrt{1+4X} - 1\right) - \frac{\varepsilon}{1+4X} \quad \text{for } X = \mathrm{O}(1). \tag{2.40}$$

We have found that this problem, (2.34), requires an asymptotic expansion for $x = \mathrm{O}(1)$ and another for $x = \mathrm{O}(\varepsilon^{-1})$. In addition, it is clear that (2.40) does not further break down as $X \to \infty$ (and it is fairly easy to see that no later terms in the expansion will alter this observation): two asymptotic expansions are sufficient. The appearance of an algebraic problem implies that all solutions are the same—any variation by virtue of different boundary values $y(1;\varepsilon)$ is lost for $x = \mathrm{O}(\varepsilon^{-1})$; how is this possible? The explanation becomes clear when (2.35) is examined more closely; the terms associated with the arbitrary constants (at each order) are exponential functions, and for $x = X/\varepsilon$ these are all proportional to $\exp(-nX/\varepsilon)$: they are exponentially small. Such terms have been omitted from the asymptotic expansion for $Y(X;\varepsilon)$; if they had been included, then the matching of these terms would have ensured that information about the boundary values would have been transmitted to the solution valid for $x = \mathrm{O}(\varepsilon^{-1})$, albeit in exponentially small terms.

E2.9 An equation with an interesting behaviour near $x = 0$

We consider the problem

$$(x + \varepsilon y)\frac{dy}{dx} + y + \varepsilon y^2 = 0, \quad 0 \le x \le 1, \quad \text{with} \quad y(1;\varepsilon) = 1, \tag{2.41}$$

as $\varepsilon \to 0^+$;we assume that

$$y(x;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n y_n(x)$$

for some $x = O(1)$. We obtain the sequence of equations

$$xy_0' + y_0 = 0; \quad xy_1' + y_1 + y_0 y_0' + y_0^2 = 0, \tag{2.42a,b}$$

and so on; the boundary condition (if available here) gives

$$y_0(1) = 1; \quad y_n(1) = 0 \quad \text{for } n \ge 1. \tag{2.43}$$

The general solution of (2.42a) is simply

$$y_0 = \frac{A_0}{x}$$

(which is not defined on $x = 0$, so we anticipate the need for a scaling as $x \to 0$, if a bounded solution exists), and then (2.42b) gives

$$xy_1' + y_1 = -y_0 y_0' - y_0^2 = \frac{A_0^2}{x^3} - \frac{A_0^2}{x^2}$$

or

$$xy_1 = -\frac{1}{2}\frac{A_0^2}{x^2} + \frac{A_0^2}{x} + A_1.$$

The asymptotic expansion is therefore

$$y(x;\varepsilon) \sim \frac{A_0}{x} + \varepsilon\left\{\frac{A_1}{x} - \frac{1}{2}\frac{A_0^2}{x^3} + \frac{A_0^2}{x^2}\right\}$$

and this is defined for $x = O(1)$, including $x = 1$, but not as $x \to 0$. So the boundary condition, (2.43), can be applied, requiring the arbitrary constants to be $A_0 = 1$ and $A_1 = -1/2$ i.e.

$$y(x;\varepsilon) \sim \frac{1}{x} + \varepsilon\left\{\frac{1}{x^2} - \frac{1}{2x} - \frac{1}{2x^3}\right\} \quad \text{for } x = O(1). \tag{2.44}$$

As $x \to 0$, expansion (2.44) breaks down where $x^{-1} = \mathrm{O}(\varepsilon x^{-3})$ or $x = \mathrm{O}(\varepsilon^{1/2})$, and then $y = \mathrm{O}(\varepsilon^{-1/2})$; we introduce the scaled variables

$$x = \sqrt{\varepsilon} X, \quad y(\sqrt{\varepsilon} X; \varepsilon) \equiv \frac{1}{\sqrt{\varepsilon}} Y(X; \varepsilon)$$

and then the equation in (2.41) becomes

$$\left(\sqrt{\varepsilon} X + \varepsilon \frac{Y}{\sqrt{\varepsilon}}\right) \frac{1}{\varepsilon} \frac{\mathrm{d}Y}{\mathrm{d}X} + \frac{Y}{\sqrt{\varepsilon}} + \varepsilon \frac{Y^2}{\varepsilon} = 0 \quad \text{or} \quad (X + Y)\frac{\mathrm{d}Y}{\mathrm{d}X} + Y + \sqrt{\varepsilon}\, Y^2 = 0. \tag{2.45}$$

For this equation, it is clear that we must seek a solution in the form

$$Y(X; \varepsilon) \sim \sum_{n=0}^{N} \varepsilon^{n/2} Y_n(X)$$

and then (2.45) yields

$$(X + Y_0)Y_0' + Y_0 = 0; \quad (X + Y_0)Y_1' + Y_0 Y_1' + Y_1 + Y_0^2 = 0, \tag{2.46a,b}$$

and so on. The first equation here can be written as

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(XY_0 + \frac{1}{2}Y_0^2\right) = 0 \quad \text{and so} \quad XY_0 + \frac{1}{2}Y_0^2 = \frac{1}{2}B_0,$$

where B_0 is an arbitrary constant; thus

$$Y_0(X) = -X \pm \sqrt{X^2 + B_0}. \tag{2.47}$$

and both B_0 and the choice of sign are to be determined by matching.

From the first term in (2.44) we obtain

$$\frac{Y}{\sqrt{\varepsilon}} \sim \frac{1}{\sqrt{\varepsilon} X} \quad \text{or} \quad Y \sim \frac{1}{X}, \tag{2.48}$$

and from (2.47) we have

$$\sqrt{\varepsilon} y \sim -\frac{x}{\sqrt{\varepsilon}} \pm \sqrt{\frac{x^2}{\varepsilon} + B_0} \quad \text{so} \quad y \sim \frac{1}{\varepsilon}\left[-x \pm x\left(1 + \frac{1}{2}\frac{\varepsilon B_0}{x^2}\right)\right]$$

which matches with (2.48) only for the positive sign and then with $B_0 = 2$. Thus (2.47) becomes

$$Y_0(X) = -X + \sqrt{2 + X^2},$$

and then equation (2.46b) is

$$(XY_1 + Y_0Y_1)' = -Y_0^2 = -\left(X^2 - 2X\sqrt{2+X^2} + 2 + X^2\right)$$

or

$$XY_1 + Y_0Y_1 = -\frac{2}{3}X^3 + \frac{2}{3}(2+X^2)^{3/2} - 2X + B_1.$$

The general expression for Y_1 is therefore

$$Y_1(X) = \frac{2}{3}(2+X^2) - \frac{2X}{3}\frac{(3+X^2)}{\sqrt{2+X^2}} + \frac{B_1}{\sqrt{2+X^2}}$$

and so we have the expansion

$$Y(X;\varepsilon) \sim -X + \sqrt{2+X^2} + \sqrt{\varepsilon}\left\{\frac{2}{3}(2+X^2) - \frac{2X}{3}\frac{(3+X^2)}{\sqrt{2+X^2}} + \frac{B_1}{\sqrt{2+X^2}}\right\} \qquad (2.49)$$

and this is to be matched with (2.44). From (2.44) we obtain

$$\frac{Y}{\sqrt{\varepsilon}} \sim \frac{1}{\sqrt{\varepsilon}X} + \varepsilon\left\{\frac{1}{\varepsilon X^2} - \frac{1}{2\varepsilon\sqrt{\varepsilon}X^3} - \frac{1}{2\sqrt{\varepsilon}X}\right\} \quad \text{or} \quad Y \sim \frac{1}{X} - \frac{1}{2X^3} + \frac{\sqrt{\varepsilon}}{X^2} \qquad (2.50)$$

and, correspondingly from (2.49), we have

$$\sqrt{\varepsilon}y \sim -\frac{x}{\sqrt{\varepsilon}} + \sqrt{2+\frac{x^2}{\varepsilon}} + \sqrt{\varepsilon}\left\{\frac{2}{3}\left(2+\frac{x^2}{\varepsilon}\right) - \frac{2x}{3\sqrt{\varepsilon}}\frac{(3+x^2/\varepsilon)}{\sqrt{2+x^2/\varepsilon}} + \frac{B_1}{\sqrt{2+x^2/\varepsilon}}\right\}$$

or

$$y \sim \frac{1}{x} - \frac{\varepsilon}{2x^3} + \frac{\varepsilon}{x^2} + \frac{B_1\sqrt{\varepsilon}}{x}$$

which matches only if $B_1 = 0$ (because the term $O(\sqrt{\varepsilon})$ in B_1 must be eliminated). The solution valid for $x = O(\sqrt{\varepsilon})$ is therefore

$$Y(X;\varepsilon) \sim -X + \sqrt{2+X^2} + \sqrt{\varepsilon}\left\{\frac{2}{3}(2+X^2) - \frac{2X}{3}\frac{(3+X^2)}{\sqrt{2+X^2}}\right\} \qquad (2.51)$$

and this expansion is defined on $X = 0$, yielding

$$Y(0;\varepsilon) \sim \sqrt{2} + \frac{4}{3}\sqrt{\varepsilon} \quad \text{or} \quad y(0;\varepsilon) \sim \sqrt{\frac{2}{\varepsilon}} + \frac{4}{3} \quad \text{as } \varepsilon \to 0^+.$$

We observe, in this example, that the value of the function on $x = 0$ is well-defined from (2.51), but that it diverges as $\varepsilon \to 0^+$. This demonstrates the important property that we require, for a solution to exist, that the asymptotic representation be defined for $\forall x \in D$, but that solutions obtained from these expansions may diverge as $\varepsilon \to 0$: we may have $x = 0$ in the domain, but $\varepsilon \neq 0$.

The examples that we have presented thus far (and others can be found in Q2.11-2.15), and particularly those that involve a rescaling after a breakdown, possess an important but rather less obvious property. This relates to the existence of general scalings of the differential equation, and the resulting 'balance' of (dominant) terms in the equation; this leads us to the introduction of an additional fundamental tool. This idea will now be explored in some detail, and use made of it in some further examples.

2.5 SCALING OF DIFFERENTIAL EQUATIONS

Let us first return to our most recent example

$$(x + \varepsilon y)\frac{\mathrm{d}y}{\mathrm{d}x} + y + \varepsilon y^2 = 0 \tag{2.52}$$

given in (2.41). We may, if it is convenient or expedient, choose to use new variables defined by

$$x = \delta X, \quad y = \Delta Y,$$

where δ and Δ are arbitrary positive constants; X and Y are now scaled versions of x and y, respectively. Thus, with $\mathrm{d}y/\mathrm{d}x = (\Delta/\delta)\mathrm{d}Y/\mathrm{d}X$, equation (2.52) becomes

$$\left(X + \frac{\varepsilon\Delta}{\delta}Y\right)\frac{\mathrm{d}Y}{\mathrm{d}X} + Y + \varepsilon\Delta Y^2 = 0, \tag{2.53}$$

and then a choice for δ and Δ might be driven by the requirement to find a new asymptotic expansion valid in an appropriate region of the domain. In this example, the first term of the asymptotic expansion valid for $x = \mathrm{O}(1)$ is $y \sim 1/x$ (see (2.44)) and so any scaling that is to produce a solution which matches to this must satisfy $Y \sim 1/X$ i.e. $\Delta = \delta^{-1}$. With this choice, equation (2.53) becomes

$$\left(X + \frac{\varepsilon}{\delta^2}Y\right)\frac{\mathrm{d}Y}{\mathrm{d}X} + Y + \frac{\varepsilon}{\delta}Y^2 = 0, \tag{2.54}$$

and from our previous analysis of this problem, we know that the breakdown of the asymptotic expansion valid for $x = \mathrm{O}(1)$ occurs for $x = \mathrm{O}(\sqrt{\varepsilon})$ i.e. $\delta = \sqrt{\varepsilon}$; the issue here is whether this can be deduced directly from the (scaled) equation.

The clue to the way forward can be found when we examine the terms, in the differential equation, that produce the leading terms in the two asymptotic expansions, one valid for $x = \mathrm{O}(1)$ and the other for $x = \mathrm{O}(\sqrt{\varepsilon})$. From (2.41) and (2.45), these are

$$(\underline{\hat{x}} + \varepsilon \hat{y}) \frac{\mathrm{d}y}{\mathrm{d}x} + \underline{\hat{y}} + \varepsilon y^2 = 0,$$

where '_' denotes terms used, in the first approximation, with $x = \mathrm{O}(1)$, and '^' denotes, correspondingly, the terms used where $x = \mathrm{O}(\sqrt{\varepsilon})$. (The derivative has not been labelled, but it will automatically be retained, by virtue of the multiplication of terms, when labelled terms are used in an approximation.) The important interpretation is that some terms—here all—used where $x = \mathrm{O}(1)$ are balanced against some terms *not* used previously (in the first approximation), but now required where $x = \mathrm{O}(\sqrt{\varepsilon})$. When we impose this requirement on (2.54), and note that the breakdown is as $x \to 0$ i.e. $\delta \to 0$, then the only balance occurs when we choose $\varepsilon/\delta^2 = \mathrm{O}(1)$ or, because we may define δ in any appropriate way, simply $\delta = \sqrt{\varepsilon}$. It is impossible to balance the term in ε/δ against the O(1) terms here (to give different leading terms) because, when we do this, the dominant term then becomes $\varepsilon/\delta^2 = 1/\varepsilon$, which is plainly inconsistent. Note that ε/δ^2 and ε/δ, as $\delta \to 0$, can never be balanced.

Thus, armed only with the general scaling property, the behaviour $y \sim 1/x$ as $x \to 0$ and the requirement to balance terms, we are led to the choice $\delta = \sqrt{\varepsilon}$ $(= \Delta^{-1})$; this does not involve any discussion of the *nature* of the breakdown of the asymptotic expansion. This new procedure is very easily applied, is very powerful and is the most immediate and natural method for finding the relevant scaled regions for the solution of a differential equation. We use this technique to explore two examples that we have previously discussed, and then we apply it to a new problem.

E2.10 Scaling for problem (2.11) (see also (2.34))

Consider the equation

$$y' + y + \varepsilon y^2 = x \tag{2.55}$$

with a boundary condition given at $x = x_0 > 0$, where $x_0 = \mathrm{O}(1)$ as $\varepsilon \to 0$; the domain is either $0 \le x \le x_0$ or $x \ge x_0$. The solution of the first term in the asymptotic expansion valid for $x = \mathrm{O}(1)$ is (see (2.15))

$$y(x;\varepsilon) \sim x - 1 + A_0 \mathrm{e}^{-x}. \tag{2.56}$$

The general scaling, $x = \delta X$, $y = \Delta Y$, in (2.55) gives

$$\frac{1}{\delta}\frac{\mathrm{d}Y}{\mathrm{d}X} + Y + \varepsilon \Delta Y^2 = \frac{\delta}{\Delta} X; \tag{2.57}$$

if the domain is $0 \geq x \geq x_0$ then from (2.56) we see that $y = O(1)$ as $x \to 0$ (unless $A_0 = 1$; see below). So we select $\Delta = 1$, (2.57) becomes

$$\frac{1}{\delta} Y' + Y + \varepsilon Y^2 = \delta X,$$

and there is no choice of scaling, as $\delta \to 0$, which balances the term δ^{-1} against εY^2. We conclude that a second asymptotic region does not exist, and hence that the expansion for $x = O(1)$ is uniformly valid on this (bounded) domain, which agrees with the discussion following (2.17). (In the special case $A_0 = 1$, $y = O(x^2)$ as $x \to 0$, and so a matched solution now requires $\Delta = \delta^2$, producing

$$\frac{1}{\delta} Y' + Y + \varepsilon \delta^2 Y^2 = \frac{1}{\delta} X;$$

again, there is no choice of δ, as $\delta \to 0$, which balances δ^{-1} against $\varepsilon\delta^2$.)

On the other hand, if the domain is $x \geq x_0$, then $y = O(x)$ as $x \to \infty$, and we require $\Delta = \delta$ for a matched solution to exist; equation (2.57) now becomes

$$\frac{1}{\delta} Y' + Y + \varepsilon \delta^2 Y^2 = X.$$

This time, with $\delta \to \infty$ (because $x \to \infty$), the O(1) terms balance $\varepsilon\delta^2 Y^2$ if $\varepsilon\delta = O(1)$ e.g. $\varepsilon\delta = 1$ or $\delta = \varepsilon^{-1}$, which recovers the scaling used to give (2.40) in E2.8.

E2.11 Scaling for problem (2.22)

Consider the equation

$$y'' + y' + \varepsilon y^2 = 0 \tag{2.58}$$

with $0 \leq x \leq 1$ and suitable boundary conditions (which may both be at one end, or one at each end). The general solution of the dominant terms from (2.58), with $x = O(1)$ as $\varepsilon \to 0$, is

$$y(x;\varepsilon) \sim A + B\mathrm{e}^{-x} \tag{2.59}$$

(as used to generate (2.25)). In this example, the asymptotic expansion, of which (2.59) is the first term, may break down as $x \to 0$ or as $x \to 1$ or, just possibly, as $x \to x_0$ with $0 < x_0 < 1$. All these may be subsumed into one calculation by introducing a simple extension of our method of scaling: let $x = x_0 + \delta X$, where we may allow $0 \leq x_0 \leq 1$ in this formulation. Then with the usual $y = \Delta Y$, (2.58) becomes

$$\frac{\Delta}{\delta^2} Y'' + \frac{\Delta}{\delta} Y' + \varepsilon \Delta^2 Y^2 = 0 \quad \text{or} \quad Y'' + \delta Y' + \varepsilon \Delta \delta^2 Y^2 = 0.$$

(Note that we have used the identity

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \Delta\frac{\mathrm{d}}{\mathrm{d}x}Y(X) = \Delta\frac{\mathrm{d}}{\mathrm{d}x}Y\left(\frac{x-x_0}{\delta}\right) = \frac{\Delta}{\delta}\frac{\mathrm{d}Y}{\mathrm{d}X},$$

and similarly for the second derivative; this is valid for $\forall x_0$.) For general A and B, (2.59) implies that $y = \mathrm{O}(1)$ as $x \to x_0$, so we select $\Delta = 1$:

$$Y'' + \delta Y' + \varepsilon\delta^2 Y^2 = 0,$$

and no balance exists (that is, between Y'' and $\varepsilon\delta^2 Y^2$) as $\delta \to 0$. Again we deduce, on the basis of this scaling argument, that the asymptotic expansion is uniformly valid for $0 \le x \le 1$ (exactly as we found was the case for expansion (2.25)). The special case in which $A + B\mathrm{e}^{-x_0} = 0$ produces $y = \mathrm{O}(x - x_0)$ as $x \to x_0$, and so we now require $\Delta = \delta$, but any balance is still impossible.

E2.12 Scaling procedure applied to a new equation

For our final example, we consider the equation

$$\varepsilon y'' + y' + y = r(x), \quad 0 \le x \le 1, \tag{2.60}$$

where $r(x)$ is either zero or $r(x) = x$; the two boundary conditions are either one at each end of the domain, or both at one end—it is immaterial in this discussion. The general solution of the dominant terms in (2.60) as $\varepsilon \to 0$, for $x = \mathrm{O}(1)$, is

$$y = A\mathrm{e}^{-x} \quad \text{or} \quad y = A\mathrm{e}^{-x} + x - 1, \tag{2.61a,b}$$

the latter applying when $r(x) = x$. The general scaling in the neighbourhood of any x_0, $0 \ge x_0 \ge 1$, is $x = x_0 + \delta X$ and $y = \Delta Y$, which gives

$$\frac{\varepsilon\Delta}{\delta^2}Y'' + \frac{\Delta}{\delta}Y' + \Delta Y = 0 \quad \text{or} \quad \frac{\varepsilon}{\delta^2}Y'' + \frac{1}{\delta}Y' + Y = 0, \tag{2.62}$$

for the equation with $r(x) \equiv 0$. Because this equation is linear and homogeneous, the scaling in y is redundant: it cancels identically. (We may still require Δ to measure the size of y, but it can play no rôle in the determination, from the equation, of any appropriate scaling near $x = x_0$.) The balance of terms, as $\delta \to 0$, requires either that $\varepsilon/\delta^2 = \mathrm{O}(1)$ or that $\varepsilon/\delta^2 = O(\delta^{-1})$, and only the latter is consistent, so $\delta = \varepsilon$; the former balances the terms Y'' and Y, but then Y' is the dominant contributor! Thus any scaled region must be described by $x = x_0 + \varepsilon X$, although this analysis cannot help us decide if an x_0 exists, or what x_0 might be; we will examine this issue in the next section.

The case of $r(x) = x$ is slightly different, because the equation is no longer homogeneous: there is a right-hand side. The same scaling now produces

$$\frac{\varepsilon}{\delta^2}Y'' + \frac{1}{\delta}Y' + Y = \frac{x_0}{\Delta} + \frac{\delta}{\Delta}X,$$

but Δ can be found by the condition that any solution we seek for Y must match to (2.61b). If $y = \mathrm{O}(1)$ as $x \to x_0$, then $\Delta = 1$ and the result is as before: the only balance is provided by $\delta = \varepsilon$. On the other hand, if $Ae^{-x_0} + x_0 - 1 = 0$, then $y = \mathrm{O}[(x - x_0)]$ as $x \to x_0$ $(x_0 \neq 0)$, so $\Delta = \delta$ and then we have

$$\frac{\varepsilon}{\delta^2}Y'' + \frac{1}{\delta}Y' + Y = \frac{x_0}{\delta} + X.$$

But the new term is the same size as an existing term, and so the same result follows yet again: $\delta = \varepsilon$ is the only available choice. (Taking $x_0 = 0$ gives the same result.)

The technique of scaling differential equations, coupled with the required behaviour necessary if matching is to be possible, is simple but powerful (as the above examples demonstrate). It is often incorporated at an early stage in most calculations, and that is how we will view it in the final introductory examples that we present; additional examples are available in Q2.16. We will shortly turn to a discussion of a classical type of problem: those that exhibit a boundary-layer behaviour (a phenomenon that we have already met in E1.4; see (1.16)). However, before we start this, a few comments of a rather more formal mathematical nature are in order, and may be of interest to some readers.

Let us suppose that we have scaled an equation according to $x = \delta X$ and $y = \Delta Y$, and that we have chosen $\Delta(\delta)$ to satisfy matching requirements; we will express this as $\Delta = \delta^n$, for some known n. The scaling, or *transformation,*

$$x = \delta X, \quad y = \delta^n Y \quad (\text{for real } \delta \neq 0)$$

will be represented by T_δ; this transformation of variables belongs to a *continuous group* or *Lie group.* (Note that this discussion has not invoked the balance of terms, which then leads to a choice $\delta(\varepsilon)$; this would constitute a selection of one member of the group.) We now explore the properties of this transformation. First we apply, successively, the transformation T_δ and then T_γ; this is equivalent to the single transformation $T_{\delta\gamma}$, and so we have the multiplication rule:

$$T_\delta T_\gamma = T_{\delta\gamma}.$$

But we also have $T_\gamma T_\delta = T_{\gamma\delta} = T_{\delta\gamma}$, so this law is *commutative*. Furthermore, the *associative* law is satisfied:

$$T_\delta(T_\gamma T_\beta) = T_\delta T_{\gamma\beta} = T_{\delta\gamma\beta} = T_{\delta\gamma} T_\beta = (T_\delta T_\gamma) T_\beta;$$

in addition, we have the identity transformation T_1 i.e. $T_1 T_\delta = T_{\delta 1} = T_\delta$ for all $\delta \neq 0$. Finally, we form

$$T_{1/\delta} T_\delta = T_1 \quad \text{and} \quad T_\delta T_{1/\delta} = T_1,$$

and so $T_{1/\delta}$ is both the left and right inverse of T_δ. Thus the elements of T_δ, for all real $\delta \neq 0$, form an *infinite group,* where δ is the *parameter* of this continuous group. (If n is fractional, then we may have to restrict the parameter to $\delta > 0$.)

Although we have not used the full power of this continuous group—we eventually select only one member for a given ε—there are other significant applications of this fundamental property in the theory of differential equations. For example, if a particular scaling transformation leaves the equation unchanged (except for a change of the symbols!) i.e. the equation is *invariant,* then we may seek solutions which satisfy the same invariance. Such solutions are, typically, *similarity solutions* (if they exist) of the equation; this aspect of differential equations is generally outside the considerations of singular perturbation theory (although these solutions may be the relevant ones in certain regions of the domain, in particular problems).

2.6 EQUATIONS WHICH EXHIBIT A BOUNDARY-LAYER BEHAVIOUR

There are many problems, posed in terms of either ordinary or partial differential equations, that have solutions which include a thin region near a boundary of the domain which is required to accommodate the boundary value there. Such regions are thin by virtue of a scaling of the variables in the appropriate parameter and, typically, this involves large values of the derivatives near the boundary. The terminology—boundary layer—is rather self-evident, although it was first associated with the viscous boundary layer in fluid mechanics (which we will describe in Chapter 5). Here, we will introduce the essential idea *via* some appropriate ordinary differential equations, and make use of the relevant scaling property of the equation.

The nature of this problem is best described, first, by an analysis of equation (1.16):

$$\varepsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1+\varepsilon)\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0, \quad 0 \leq x \leq 1, \tag{2.63}$$

with $y(0;\varepsilon) = \alpha$ and $y(1;\varepsilon) = \beta$ (and we will assume that α and β are not functions of ε, and that $\alpha \neq \beta\mathrm{e}$). In this presentation of the construction of the asymptotic solution, valid as $\varepsilon \to 0^+$, we will work directly from (2.63) (although the exact solution is available in (1.22), which may be used as a check, if so desired). Because we wish to incorporate an application of the scaling property, we need to know *where* the boundary layer (interpreted as a scaled region) is situated: is it near $x = 0$ or near

$x = 1$ (or, possibly, both)? Here, we will assume that there is a single boundary layer near $x = 0$; the problem of finding the position of a boundary layer will be addressed in the next section, at least for a particular class of ordinary differential equations. Let us now return to equation (2.63).

As should be evident from example E1.4, and will become very clear in what follows, it is the appearance of the small parameter multiplying the highest derivative that is critical here. The presence of ε in another coefficient is altogether irrelevant to the general development; it is retained only to allow direct comparison with E1.4. We seek a solution of (2.63) in the form

$$y(x;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n y_n(x), \quad \text{for } x = \mathrm{O}(1),$$

and then we obtain

$$y_0' + y_0 = 0; \quad y_1' + y_1 + y_0'' + y_0' = 0, \tag{2.64a,b}$$

and so on. The only boundary condition available to us (because of the assumption about the position of the boundary layer) is

$$y(1;\varepsilon) = \beta \quad \text{i.e.} \quad y_0(1) = \beta,\ y_n(1) = 0 \quad \text{for } n \geq 1.$$

Thus

$$y_0(x) = \beta e^{1-x}, \quad y_1(x) \equiv 0,$$

and indeed, in this problem, we then have $y_n(x) \equiv 0$, $n \geq 1$, although exponentially small terms would be required for a more complete description of the asymptotic solution valid for $x = \mathrm{O}(1)$; so we have

$$y(x;\varepsilon) \sim \beta e^{1-x}. \tag{2.65}$$

The scaled version of (2.63) is obtained by writing $x = \delta X$, for $\delta \to 0$, and $y \equiv Y(X;\varepsilon)$ (because $y = \mathrm{O}(1)$ as $x \to 0$, although any scaling on Y will vanish identically from the equation); thus

$$\frac{\varepsilon}{\delta^2} Y'' + (1+\varepsilon)\frac{1}{\delta} Y' + Y = 0.$$

The relevant balance, as we have already seen in (2.62), is $\varepsilon/\delta^2 = \mathrm{O}(\delta^{-1})$ or $\delta = \varepsilon$, giving

$$Y'' + (1+\varepsilon)Y' + \varepsilon Y = 0. \tag{2.66}$$

We seek a solution of this equation in the usual form:

$$Y(X;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n Y_n(X)$$

which gives

$$Y_0'' + Y_0' = 0; \quad Y_1'' + Y_1' + Y_0' + Y_0 = 0, \tag{2.67a,b}$$

and so on. We have available the one boundary condition prescribed at $x = 0$ (which is in the region of the boundary layer) i.e. at $X = 0$, so

$$Y(0;\varepsilon) = \alpha \quad \text{or} \quad Y_0(0) = \alpha,\ Y_n(0) = 0 \quad \text{for } n \geq 1. \tag{2.68}$$

From (2.67a) and (2.68) we obtain

$$Y_0(X) = A_0 + (\alpha - A_0)\mathrm{e}^{-X} \tag{2.69}$$

where A_0 is an arbitrary constant, and then (2.67b) becomes

$$Y_1'' + Y_1' - (\alpha - A_0)\mathrm{e}^{-X} + A_0 + (\alpha - A_0)\mathrm{e}^{-X} = 0.$$

Thus

$$Y_1'' + Y_1' = -A_0 \quad \text{and so} \quad Y_1(X) = A_1 + B_1\mathrm{e}^{-X} - A_0 X,$$

where A_1 and B_1 are the arbitrary constants, which must satisfy $A_1 + B_1 = 0$ from $Y_1(0) = 0$; this gives the solution

$$Y_1(X) = A_1(1 - \mathrm{e}^{-X}) - A_0 X.$$

The 2-term asymptotic expansion valid for $x = \mathrm{O}(\varepsilon)$ is therefore

$$Y(X;\varepsilon) \sim A_0 + (\alpha - A_0)\mathrm{e}^{-X} + \varepsilon\left\{A_1(1 - \mathrm{e}^{-X}) - A_0 X\right\}, \quad X = \mathrm{O}(1), \tag{2.70}$$

which is to be matched to (2.65); this should uniquely determine A_0 and A_1. We write (2.65) as

$$\beta\mathrm{e}^{1-\varepsilon X} \sim \beta\mathrm{e}(1 - \varepsilon X), \quad X = \mathrm{O}(1), \tag{2.71}$$

retaining terms O(1) and $\mathrm{O}(\varepsilon)$ (as used in (2.70)); correspondingly, we write (2.70) as

$$A_0 + (\alpha - A_0)e^{-x/\varepsilon} + \varepsilon\left\{A_1(1 - e^{-x/\varepsilon}) - A_0\frac{x}{\varepsilon}\right\} \sim A_0 + \varepsilon A_1 - A_0 x, \quad x = \mathrm{O}(1), \tag{2.72}$$

and the $O(\varepsilon)$ term is included because that was obtained for (2.65)—although it had a zero coefficient. In order to match (2.71) and (2.72), we require $A_0 = \beta e$ and $A_1 = 0$ i.e.

$$Y(X;\varepsilon) \sim \beta e + (\alpha - \beta e)e^{-X} - \varepsilon\beta e X. \tag{2.73}$$

(It is left as an exercise to show that (2.65) and (2.73) are recovered from suitable expansions of (1.22).) We should note that (2.65) exhibits no breakdown—there is only one term here, after all—but (2.73) does break down where $\varepsilon X = O(1)$, i.e. $x = O(1)$, as we would expect. Any indication of a breakdown in the asymptotic expansion valid for $x = O(1)$ will come from the exponentially small terms; let us briefly address this aspect of the problem.

The first point to note is that, from (2.70) and (2.73), we would require not only $O(1)$ and $O(\varepsilon)$, but also $O(e^{-x/\varepsilon})$ terms (and others), in order to complete the matching procedure; this is simply because we need, at least in principle, to match to all the terms (cf. (2.72))

$$A_0 - A_0 x + \varepsilon A_1 + (\alpha - A_0)e^{-x/\varepsilon} \dots .$$

Thus the expansion valid for $x = O(1)$ must include a term $e^{-x/\varepsilon}$ to allow matching to this order (and then it is not too difficult to see that a *complete* asymptotic expansion requires all the terms in the sequence $\{\varepsilon^n e^{-mx/\varepsilon}\}$, $n = 0, 1, 2, \dots$, $m = 0, 1, 2, \dots$). In passing, we observe that this use of the matching principle is new in the context of our presentation here. We are using it, first, in a general sense, to determine the *type(*s*)* of term(s) required in the expansion valid in an adjacent region in order to allow matching. Then, with these terms included, the 'full' matching procedure may be employed to check the details and fix the values of any arbitrary constants left undetermined. We will find the first of the exponentially small terms here.

For $x = O(1)$, we seek a solution

$$y(x;\varepsilon) \sim \tilde{y}_N(x;\varepsilon) + \hat{Y}(x;\varepsilon)e^{-x/\varepsilon} \tag{2.74}$$

where $\tilde{y}_N(x;\varepsilon) \equiv \sum_{n=0}^{N} \varepsilon^n y_n(x)$ ($\sim \beta e^{1-x}$ in this example); thus equation (2.63) becomes

$$\varepsilon\tilde{y}_N'' + (1+\varepsilon)\tilde{y}_N' + \tilde{y}_N + \{\varepsilon\hat{Y}'' - (1-\varepsilon)\hat{Y}'\}\, e^{-x/\varepsilon} = 0.$$

(Again '= 0' means zero to all orders in ε.) We already have that $\tilde{y}_N(x;\varepsilon)$ satisfies the original equation, and so $\hat{Y}(x;\varepsilon)$ must satisfy

$$\varepsilon\hat{Y}'' - (1-\varepsilon)\hat{Y}' = 0;$$

with $\hat{Y} \sim \hat{Y}_M \equiv \sum_{n=0}^{M} \varepsilon^n \hat{y}_n(x)$ we obtain $\hat{y}_0' = 0$ or $\hat{y}_0 = C_0$ (constant). Thus the expansion (2.74) becomes

$$y(x;\varepsilon) \sim \beta \mathrm{e}^{1-x} + C_0 \mathrm{e}^{-x/\varepsilon} \tag{2.75}$$

and this is to be matched to the asymptotic expansion (2.73); from (2.75) we obtain

$$\beta \mathrm{e}(1 - \varepsilon X) + C_0 \mathrm{e}^{-X} \quad \text{for } X = \mathrm{O}(1)$$

and from (2.73) we have

$$\beta \mathrm{e}(1 - x) + (\alpha - \beta \mathrm{e})\mathrm{e}^{-x/\varepsilon} \quad \text{for } x = \mathrm{O}(1),$$

which match if $C_0 = \alpha - \beta \mathrm{e}$. Thus the solution valid for $x = \mathrm{O}(1)$, i.e. away from the boundary layer near $x = 0$, incorporating the first exponentially small term, is

$$y(x;\varepsilon) \sim \beta \mathrm{e}^{1-x} + (\alpha - \beta \mathrm{e})\mathrm{e}^{-x/\varepsilon}. \tag{2.76}$$

One final comment: this solution, (2.76), produces the value $y(1;\varepsilon) \sim \beta + (\alpha - \beta \mathrm{e})\mathrm{e}^{-1/\varepsilon}$, so now the boundary value is in error by $\mathrm{O}(\mathrm{e}^{-1/\varepsilon})$. To correct this, a further term is required; we must write (at the order of all the first exponentially small terms)

$$y(x;\varepsilon) \sim \tilde{y}_N(x;\varepsilon) + \hat{Y}_M(x;\varepsilon)\mathrm{e}^{-x/\varepsilon} + \overline{Y}_M(x;\varepsilon)\mathrm{e}^{-1/\varepsilon}$$

where $\overline{Y}_M(x;\varepsilon) \sim \sum_{n=0}^{M} \varepsilon^n \overline{y}_n(x)$. It is left as an exercise to show that $\overline{y}_0' + \overline{y}_0 = 0$, with the solution $\overline{y}_0(x) = -(\alpha - \beta \mathrm{e})\mathrm{e}^{1-x}$, ensures that the boundary condition on $x = 1$ is correct at this order. The inclusion of the term $\mathrm{O}(\mathrm{e}^{-1/\varepsilon})$ in the expansion valid for $x = \mathrm{O}(1)$ forces, *via* the matching principle, a term of this same order in the expansion for $x = \mathrm{O}(\varepsilon)$, and so the pattern continues. (The appearance of all these terms, in both expansions, can be seen by expanding the exact solution, (1.22), appropriately.)

E2.13 A nonlinear boundary-layer problem

We consider the problem

$$\varepsilon y'' - y' + \varepsilon x y^2 = 2x, \quad 0 \le x \le 1, \tag{2.77}$$

with $y(0;\varepsilon) = 2$, $y(1;\varepsilon) = 2+\varepsilon$, for $\varepsilon \to 0^+$; we are given that the boundary layer is near $x = 1$. We seek a solution, for $1 - x = O(1)$, in the form

$$y(x;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n y_n(x)$$

and so

$$y_0' = -2x; \quad y_1' = y_0'' + x y_0^2,$$

etc., with $y_0(0) = 2$, $y_n(0) = 0$ for $n \geq 1$. Thus we obtain

$$y_0(x) = 2 - x^2, \quad y_1(x) = \tfrac{1}{6}x^6 - x^4 + 2x^2 - 2x$$

which leads to

$$y(x;\varepsilon) \sim 2 - x^2 + \varepsilon\left\{\tfrac{1}{6}x^6 - x^4 + 2x^2 - 2x\right\}, \tag{2.78}$$

but this solution does not satisfy the boundary condition on $x = 1$.

For the solution near $x = 1$, we introduce $x = 1 - \delta X$ (with $X \geq 0$) and $y \equiv Y(X;\varepsilon)$, so that $\delta = \varepsilon$ i.e.

$$Y'' + Y' + \varepsilon^2(1 - \varepsilon X)Y^2 = 2\varepsilon(1 - \varepsilon X).$$

The solution in the form

$$Y(X;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n Y_n(X)$$

gives

$$Y_0'' + Y_0' = 0, \quad Y_1'' + Y_1' = 2,$$

and so on; the available boundary condition requires that

$$Y_0(0) = 2; \quad Y_1(0) = 1; \quad Y_n(0) = 0, \quad n \geq 2.$$

Thus

$$Y_0(X) = A_0 + (2 - A_0)e^{-X}$$

and

$$Y_1(X) = 2X + A_1 + (1 - A_1)e^{-X},$$

where A_0 and A_1 are the arbitrary constants to be determined by matching; that is, we must match

$$Y(X;\varepsilon) \sim A_0 + (2 - A_0)e^{-X} + \varepsilon\left\{2X + A_1 + (1 - A_1)e^{-X}\right\} \tag{2.79}$$

with (2.78). From (2.79) we obtain

$$y \sim A_0 + 2(1 - x) + \varepsilon A_1 \quad \text{for } x = \mathrm{O}(1);$$

from (2.78) we have

$$Y \sim 1 + 2\varepsilon X - \tfrac{5}{6}\varepsilon \quad \text{for } X = \mathrm{O}(1),$$

and these match if we select $A_0 = 1$ and $A_1 = -5/6$; thus (2.79) becomes

$$Y(X;\varepsilon) \sim 1 + \mathrm{e}^{-X} + \varepsilon\left\{2X - \tfrac{5}{6} + \tfrac{11}{6}\mathrm{e}^{-X}\right\}.$$

The fundamental issue relating to boundary-layer-type problems, which we have avoided thus far, addresses the question of where the boundary layer might be located. In the examples discussed above, we allowed ourselves the advantage of knowing where this layer was situated—and the consistency of the resulting asymptotic solution confirmed that unique, well-defined solutions existed, so presumably we started with the correct information. We now examine this important aspect of boundary-layer problems.

2.7 WHERE IS THE BOUNDARY LAYER?

For this discussion, we consider the general second-order ordinary differential equation in the form

$$\varepsilon y'' + a(x)y' + f(x, y; \varepsilon) = 0, \quad x_0 \le x \le x_1, \tag{2.80}$$

with suitable boundary conditions and for $\varepsilon \to 0^+$; note that the coefficient of y'' must be ε. The coefficient $a(x)$ will satisfy either $a(x) > 0$ or $a(x) < 0$, for $\forall x \in D$; the term $f(x, y; \varepsilon) = \mathrm{O}(1)$ (or smaller) as $\varepsilon \to 0^+$, for $\forall x \in D$ and for all solutions $y(x; \varepsilon)$ that may be of interest. Of course, this describes only one class of such boundary-layer problems, but this does cover by far the most common ones encountered in mathematical modelling. (Some of these conditions, both explicitly written and implied, can be relaxed; we will offer a few generalisations later.) The guiding principle that we will adopt is to seek solutions of (2.80) which remain bounded as $\varepsilon \to 0^+$, for $\forall x \in D$.

The starting point is the construction of the asymptotic solution valid for suitable $x = \mathrm{O}(1)$, directly from the equation as written in (2.80)—but this will necessarily generate a sequence of first-order equations. It is therefore impossible to impose the *two* boundary conditions (as we have already demonstrated in our examples above); the inclusion of a boundary layer remedies this deficiency in the solution. We introduce the boundary layer in the most general way possible: define $X = g(x)/\delta$ for some $g(x)$

and some $\delta(\varepsilon)$, and write $y \equiv \Delta Y(X;\varepsilon)$. Then we have

$$\frac{dy}{dx} = \frac{\Delta}{\delta} g'(x)\frac{dY}{dX}, \quad \frac{d^2y}{dx^2} = \frac{\Delta}{\delta} g''(x)\frac{dY}{dX} + \Delta\left(\frac{g'}{\delta}\right)^2 \frac{d^2Y}{dX^2}$$

and so equation (2.80) becomes

$$\frac{\varepsilon}{\delta^2}\left\{(g')^2 \frac{d^2Y}{dX^2} + \delta g''\frac{dY}{dX}\right\} + \frac{a(x)}{\delta} g'\frac{dY}{dX} + F(X, Y; \varepsilon, \Delta) = 0 \tag{2.81}$$

where $F = f/\Delta$. We will further assume, whatever the choice of Δ (and in almost all problems that we encounter, $\Delta = 1$), that the terms in δ^{-1} and $\varepsilon\delta^{-2}$ dominate as $\delta \to 0$. Thus the 'classical' choice for the balance of terms, $\delta = \varepsilon$, applies here for general $g(x)$ (which we have yet to determine).

The differential equation valid in the boundary layer can now be written

$$(g')^2 \frac{d^2Y}{dX^2} + ag'\frac{dY}{dX} + \varepsilon\left\{g''\frac{dY}{dX} + F\right\} = 0, \tag{2.82}$$

which has the first term in an asymptotic expansion, $Y \sim Y_0(X)$, satisfying

$$(g')^2 \frac{d^2Y_0}{dX^2} + ag'\frac{dY_0}{dX} = 0. \tag{2.83}$$

At this stage, $g'(x)$ has yet to be determined; let us choose $g'(x) = a(x)$ (we may not choose $g' = 0$), then we are left with the simple, generic problem for $Y_0(X)$:

$$Y_0'' + Y_0' = 0, \tag{2.84}$$

which also solves the difficulty over the mixing of the x and X notations in (2.83). Thus all boundary-layer problems in this class have the same general solution, from (2.84),

$$Y_0(X) = A_0 + B_0 e^{-X}. \tag{2.85}$$

However, we are no nearer finding the position of the boundary layer itself; this we now do by examining the available solutions for $g(x)$.

The general form for $g(x)$ is

$$g(x) = \int_C^x a(x')dx', \tag{2.86}$$

where C is an arbitrary constant (and this proves to be the most convenient way of including the constant of integration). First, we suppose that $a(x) > 0$, and examine

(2.85) when expressed in terms of x (as will be necessary for any matching); this gives

$$Y \sim A_0 + B_0 \exp\left\{-\frac{1}{\varepsilon}\int_C^x a(x')\,\mathrm{d}x'\right\}. \tag{2.87}$$

But we are seeking solutions that remain bounded as $\varepsilon \to 0^+$, and this is possible only if the exponent in (2.87) is non-positive for $\forall x \in D$. Thus we require

$$\int_C^x a(x')\,\mathrm{d}x' \geq 0 \quad \text{for } x_0 \leq x \leq x_1,$$

and this means that $x \geq C$ for the same xs i.e. $C = x_0$ (for otherwise $x < C$ can occur in the domain, and the integral will change sign). Then, for x sufficiently close to x_0, we have

$$\frac{1}{\varepsilon}\int_{x_0}^x a(x')\,\mathrm{d}x' = X = \mathrm{O}(1)$$

and we have a choice for the boundary-layer variable. Hence, for $a(x) > 0$, the boundary layer must sit near the left-hand edge of the domain. Conversely, the same argument in the case $a(x) < 0$ requires that the boundary layer be situated near $x = x_1$ (the right-hand edge of the domain). When we apply this rule to equation (2.63): $\varepsilon y'' + (1+\varepsilon)y' + y = 0$, $0 \leq x \leq 1$, we see that $a(x) = 1 + \varepsilon > 0$ and so the boundary layer is in the neighbourhood of $x = 0$ (and we introduced $x = \varepsilon X$ for this example). Similarly, equation (2.77): $\varepsilon y'' - y' + \varepsilon x y^2 = 2x$, $0 \leq x \leq 1$, has $a(x) = -1 < 0$, and so the boundary layer is now near $x = 1$ (and we used $x = 1 - \varepsilon X$).

It is rarely necessary to incorporate the formal definition of $g(x)$ to generate the appropriate variable that is to be used to represent the boundary layer (although it will always produce the simplest form of the solution). For example, the equation of this class: $\varepsilon y'' - y'/(2+x) + \cdots = 0$, $-1 \leq x \leq 2$, has a boundary layer near $x = 2$ (because $a(x) = -1/(2+x) < 0$ for $\forall x \in D$). Now an appropriate scaled variable is simply $x = 2 - \varepsilon X$, giving

$$\frac{1}{\varepsilon}\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2} + \frac{1}{\varepsilon}(4 - \varepsilon X)^{-1}\frac{\mathrm{d}Y}{\mathrm{d}X} + \cdots = 0,$$

and this choice will suffice, even though higher-order terms will require the expansion of $(4 - \varepsilon X)^{-1}$ (but this is usually a small price to pay—and we already know that this asymptotic expansion will breakdown for $\varepsilon X = \mathrm{O}(1)$, so retaining εX in the coefficient has no unforeseen complications). It should also be noted that, exceptionally, a boundary-layer-type problem may not require a boundary layer at all, in order to accommodate the given boundary value (to leading order or, possibly, to all orders). This is evident for the equation given in (2.63) (and see also the exact solution, (1.22)); in this example, if the boundary values satisfy the special condition $\alpha = \beta \mathrm{e}$, then no boundary layer whatsoever is required. Note, however, that if $\alpha - \beta\mathrm{e} = \mathrm{O}(\varepsilon)$, then the boundary layer is present, but only to correct the boundary value at $O(\varepsilon)$—the leading term (for $x = O(1)$) is uniformly valid. We call upon all these ideas in the next example.

E2.14 A nonlinear, variable coefficient boundary-layer problem

We consider

$$\varepsilon y'' - \frac{y'}{1+2x} - \frac{1}{y} = 0, \quad 0 \leq x \leq 1, \tag{2.88}$$

with $y(0;\varepsilon) = y(1;\varepsilon) = 3$, for $\varepsilon \to 0^+$. Because the coefficient of y' is negative for $\forall x \in D$, the boundary layer will be situated at the right-hand edge of the domain i.e. near $x = 1$. Away from $x = 1$, we seek a solution

$$y(x;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n y_n(x)$$

and so

$$-\frac{y_0'}{1+2x} - \frac{1}{y_0} = 0; \quad y_0'' - \frac{y_1'}{1+2x} + \frac{y_1}{y_0^2} = 0, \tag{2.89}$$

etc., and we may use the boundary condition on $x = 0$:

$$y_0(0) = 3, \quad y_n(0) = 0 \quad \text{for } n \geq 1.$$

Thus we obtain

$$y_0^2(x) = A_0 - 2(x + x^2)$$

and then the application of the boundary condition yields the solution

$$y_0(x) = \sqrt{9 - 2(x + x^2)}; \tag{2.90}$$

we note that y_0 remains real and positive as $x \to 1$. (The second term can also be found, but it is a slightly tiresome exercise and its inclusion teaches us little about the solution.) Clearly, $y_0 \to \sqrt{5}$ as the right-hand boundary is approached, which does not satisfy the given boundary condition $y(1;\varepsilon) = 3$, and so a boundary layer is required. (Of course, if $y(1;\varepsilon) = \sqrt{5}$ then (2.90) would be a uniformly valid 1-term asymptotic solution.)

We introduce $x = 1 - \varepsilon X$ (so that $X \geq 0$) and write $y \equiv Y(X;\varepsilon) = O(1)$; equation (2.88) then becomes

$$Y'' + \frac{Y'}{3 - 2\varepsilon X} - \frac{\varepsilon}{Y} = 0 \tag{2.91}$$

which gives, with $Y(X;\varepsilon) \sim Y_0(X)$,

$$Y_0'' + \tfrac{1}{3}Y_0' = 0. \tag{2.92}$$

The solution of (2.92), which satisfies the boundary condition $Y_0(0) = 3$, is

$$Y_0(X) = B_0 + (3 - B_0)\mathrm{e}^{-X/3} \tag{2.93}$$

and B_0 is to be determined by matching. From (2.90) we have directly that $Y \sim \sqrt{5}$, and from (2.93) we see that $y \sim B_0$; thus matching requires $B_0 = \sqrt{5}$ and the first term in the boundary-layer solution is

$$Y(X;\varepsilon) \sim \sqrt{5} + (3 - \sqrt{5})\mathrm{e}^{-X/3}$$

and then a composite expansion can be written down, if that is required.

The fundamental ideas that underpin the notion of boundary-layer-type solutions, in second-order ordinary differential equations, have been developed, but many variants of this simple idea exist; see also Q2.17–2.20. These lead to adjustments in the formulation, or to generalisations, or to a rather different structure (with corresponding interpretation). We now describe a few of these possibilities, but what we present is far from providing a comprehensive list; rather, we present some examples which emphasise the application of the basic technique of scaling to find thin layers where rapid changes occur. In the next section, we briefly describe a number of different scenarios, and present an example of each type.

2.8 BOUNDARY LAYERS AND TRANSITION LAYERS

Our first development from the simple notion of a boundary layer is afforded by an extension of our discussion of the position of this layer, *via* equation (2.80); here, we consider the case where $a(\alpha) = 0$, $x_0 < \alpha < x_1$. Such a point is analogous to a *turning point* (see Q2.24) and the solution valid near $x = \alpha$ takes the form of a *transition layer.* (The terminology 'turning point' is used to denote where the character of the solution changes or 'turns', typically from oscillatory to exponential, in equations such as $y'' + a(x)y = 0$.) The general approach is to seek a scaling–just as for a boundary layer–but now at this interior point. Let us suppose that $a(x) = \lambda(x - \alpha)^n$, for given constants λ and n, then equation (2.80) becomes

$$\varepsilon y'' + \lambda(x - \alpha)^n y' + f(x, y; \varepsilon) = 0$$

and we introduce

$$x = \alpha + \delta X, \quad y \equiv \Delta Y(X;\varepsilon)$$

to give

$$\frac{\varepsilon}{\delta^2}Y'' + \lambda\delta^{n-1}X^n Y' + F = 0,$$

where we have used the same notation as in (2.81). The balance that we seek is given by the choice $\delta^{n+1} = \varepsilon$ or $\delta = \varepsilon^{1/(1+n)}$, provided that $n < 1$ (in order that this balance does indeed produce the dominant terms, with $F = O(1)$ or smaller). The procedure then unfolds as for the boundary-layer problems, although we will now have solutions in $x_0 \le x < x_2$ and in $x_2 < x \le x_1$, where $(x_2 - \alpha) = O(1)$, together with a matched solution where $(x - \alpha) = O(\varepsilon^{1/(1+n)})$. If $n \ge 1$, then the balance of terms requires $\delta = \sqrt{\varepsilon}$, and then either ($n = 1$) all *three* terms in the equation contribute to leading order, or ($n > 1$) the balance is between Y'' and F. (This description assumes that $F = O(1)$; note also that the chosen behaviour of $a(x)$ used here need only apply *near* $x = \alpha$ for this approach to be relevant.)

E2.15 An equation with a transition layer

Consider the equation

$$\varepsilon y'' + x^{1/3}y' + y^2 = 0, \quad -1 \le x \le 1, \tag{2.94}$$

for $\varepsilon \to 0^+$, where, for real solutions, we interpret $(-x)^{1/3} = -x^{1/3}$; the boundary conditions are

$$y(-1;\varepsilon) = 2/9 \quad \text{and} \quad y(1;\varepsilon) = 1/3. \tag{2.95a,b}$$

We will find the first terms only in the asymptotic expansions valid away from $x = 0$ (where the coefficient $x^{1/3}$ is zero), in $-1 \le x < 0$, and then in $0 < x \le 1$, and finally valid near $x = 0$. For $x = O(1)$, we write $y(x;\varepsilon) \sim y_0(x)$ and so, from (2.94), we obtain

$$x^{1/3}y_0' + y_0^2 = 0 \quad \text{or} \quad y_0(x) = \frac{2}{3}(A_0 + x^{2/3})^{-1};$$

we determine the arbitrary constant by imposing the boundary conditions (2.95a,b), thereby producing solutions valid either on one side, or the other, of $x = 0$:

$$y_0(x) = \frac{2}{3\,(2 + x^{2/3})}, \quad -1 \le x < -|x_1|; \quad y_0(x) = \frac{2}{3\,(1 + x^{2/3})}, \quad |x_1| < x \le 1. \tag{2.96a,b}$$

Note that these solutions do not hold in the neighbourhood of $x = 0$ i.e. we may allow $x_1 = o(1)$ but with $\varepsilon/x_1 = o(1)$; this solution *would* be valid for $-1 \le x \le 1$, to leading order, if the function given in (2.96) were continuous at $x = 0$, and then no transition layer would be needed (to this order, at least).

Near $x = 0$, we write $x = \delta X$ and, in order to match, we require $y = O(1)$, so we write $y \equiv Y(X; \varepsilon)$; equation (2.94) then gives

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{X^{1/3}}{\delta^{2/3}} Y' + Y^2 = 0$$

and hence we select $\delta = \varepsilon^{3/4}$. Thus in the transition layer we have the equation

$$Y'' + X^{1/3} Y' + \sqrt{\varepsilon}\, Y^2 = 0,$$

and we seek a solution $Y(X; \varepsilon) \sim Y_0(X)$, where

$$Y_0'' + X^{1/3} Y_0' = 0 \quad \text{or} \quad Y_0(X) = B_0 + C_0 \int_0^X \exp\left(-\frac{3}{4}\hat{X}^{4/3}\right) \mathrm{d}\hat{X}. \tag{2.97}$$

(The lower limit in the integral here is simply a convenience; with this choice, the arbitrary constant following the second integration is B_0. Note that this integral exists for $-\infty < X < \infty$.) These arbitrary constants are determined by matching; from (2.96a,b) we obtain directly that

$$Y \sim \frac{1}{3} \quad \text{as } x \to 0^-; \quad Y \sim \frac{2}{3} \quad \text{as } x \to 0^+. \tag{2.98}$$

From (2.97), we first write

$$Y_0 = B_0 + C_0 \int_0^{\varepsilon^{-3/4} x} \exp\left(-\frac{3}{4}\hat{X}^{4/3}\right) \mathrm{d}\hat{X}; \tag{2.99}$$

now let us introduce the constant

$$k = \int_0^\infty \exp\left(-\frac{3}{4}X^{4/3}\right) \mathrm{d}X \quad \left[= \left(\frac{3}{4}\right)^{1/4} \Gamma(3/4)\right]$$

(where $\Gamma(z)$ is the *gamma function*), then

$$y \sim B_0 + C_0 k, \quad x > 0; \quad y \sim B_0 - C_0 k, \quad x < 0, \quad \text{as } \varepsilon \to 0^+ \tag{2.100}$$

and matching (2.98) and (2.100) requires $B_0 = 1/2$ and $C_0 = 1/(6k)$. The first term of the asymptotic expansion valid in the transition layer (around $x = 0$) is therefore

$$Y(X; \varepsilon) \sim \frac{1}{2} + \frac{1}{6k} \int_0^X \exp\left(-\frac{3}{4}\hat{X}^{4/3}\right) \mathrm{d}\hat{X}.$$

This example has demonstrated how boundary-layer techniques are equally applicable to interior (transition) layers and, further, they need not be restricted to single layers. Some problems exhibit multiple layers; for example

$$\varepsilon y'' + \left(x^2 - \tfrac{1}{4}\right)y' + f = 0, \quad -1 \le x \le 1,$$

has transition layers both near $x = 1/2$ and near $x = -1/2$, and the solution away from these layers is now in three parts.

A type of problem which contains elements of both boundary and transition layers occurs if the coefficient, *a(x),* of y', is zero at one (or both) boundaries. Because $a(x) \neq 0$ at an *internal* point, it is not a transition layer, but the fact that $a(x) \to 0$ at the end-point affects the scaling—it is no longer $O(\varepsilon)$ in general—and this must be determined directly (as we did for the transition layer).

E2.16 A boundary-layer problem with a new scaling

We consider the equation

$$\varepsilon y'' + \sqrt{x}y' + y^2 = 0, \quad 0 \le x \le 1, \tag{2.101}$$

for $\varepsilon \to 0^+$, with $y(0;\varepsilon) = 2$ and $y(1;\varepsilon) = 1/3$; note that $\sqrt{x} \ge 0$ and so we must expect a boundary layer near $x = 0$. Away from $x = 0$, we seek a solution

$$y(x;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n y_n(x)$$

which gives
$$\sqrt{x}y_0' + y_0^2 = 0, \quad \sqrt{x}y_1' + 2y_0y_1 + y_0'' = 0,$$

and so on, with $y_0(1) = 1/3$ and $y_n(1) = 0$ for $n \ge 1$. Thus we obtain

$$y_0 = \frac{1}{A_0 + 2\sqrt{x}}, \quad \text{and then} \quad y_0(x) = \frac{1}{1 + 2\sqrt{x}} \tag{2.102}$$

in order to satisfy the boundary condition (and as $x \to 0$ this does not approach $2 = y(0;\varepsilon)$, and so a boundary layer is certainly required). The equation for $y_1(x)$ is therefore

$$\sqrt{x}y_1' + \left(\frac{2}{1 + 2\sqrt{x}}\right) y_1 = \frac{1 + 6\sqrt{x}}{2x\sqrt{x}\left(1 + 2\sqrt{x}\right)^3}$$

or
$$\left[\left(1 + 2\sqrt{x}\right)^2 y_1\right]' = \frac{1 + 6\sqrt{x}}{2x^2\left(1 + 2\sqrt{x}\right)}$$

which has the general solution

$$y_1(x) = \frac{8\ln\left[\left(1+2\sqrt{x}\right)/\sqrt{x}\right] - 4x^{-1/2} - \frac{1}{2}x^{-1} + A_1}{\left(1+2\sqrt{x}\right)^2}$$

and $y_1(1) = 0$ requires $A_1 = \frac{9}{2} - 8\ln 3$. (It should be noted that $y_1(x) \sim -1/(2x)$ as $x \to 0$ which, alone, indicates a breakdown where $x = \mathrm{O}(\varepsilon)$—but this, as we shall see below, is irrelevant).

For the boundary layer, we scale $x = \delta X$ with $y \equiv Y(X;\varepsilon)$, then the equation, (2.101), becomes

$$\frac{\varepsilon}{\delta^2}Y'' + \delta^{-1/2}\sqrt{X}Y' + Y^2 = 0$$

and so we must select $\delta = \varepsilon^{2/3}$, which leads to

$$Y'' + \sqrt{X}Y' + \varepsilon^{1/3}Y^2 = 0.$$

(The apparent scaling, $x = \varepsilon X$, is therefore redundant—it is smaller than that required in the boundary layer.) We seek a solution $Y(X;\varepsilon) \sim Y_0(X)$, with

$$Y_0'' + \sqrt{X}Y_0' = 0 \quad \text{and so} \quad Y_0(X) = B_0 + C_0\int_0^X \exp\left(-\frac{2}{3}\hat{X}^{3/2}\right)\mathrm{d}\hat{X}$$

and the available boundary condition, $Y_0(0) = 2$, then gives

$$Y_0(X) = 2 + C_0\int_0^X \exp\left(-\frac{2}{3}\hat{X}^{3/2}\right)\mathrm{d}\hat{X}, \quad X \geq 0. \tag{2.103}$$

Finally, we determine C_0 by invoking the matching principle; from (2.102) we obtain $Y \sim 1$, and from (2.103) we see that

$$y \sim 2 + C_0 k \quad \text{where } k = \int_0^\infty \exp\left(-\frac{2}{3}X^{3/2}\right)\mathrm{d}X \quad \left[= \left(\frac{2}{3}\right)^{1/3}\Gamma(2/3)\right], \tag{2.104}$$

which match if $C_0 = -1/k$; the boundary-layer solution is therefore

$$Y(X;\varepsilon) \sim 2 - \frac{1}{k}\int_0^X \exp\left(-\frac{2}{3}\hat{X}^{3/2}\right)\mathrm{d}\hat{X},$$

where k is given in (2.104).

Boundary layers also arise even in the absence of the first-derivative term; indeed, equations of the form

$$\varepsilon y'' - a(x)y = f(x), \quad \varepsilon \to 0^+,$$

with $a(x) \geq 0$ for $\forall x \in D$, can have a boundary layer at each end of the domain. (If $a(x) \leq 0$, then the relevant part of the solution is oscillatory and boundary layers are not present. If $a(x) = 0$ at an interior point, then we have a classical turning-point problem and near this point we will require a transition layer.) The solution away from these layers is simply given, to leading order, by $y(x) = -f(x)/a(x)$. To see the nature of this problem, consider the case $a(x) = 1$. The equation that controls the solution in the boundary layers is then $Y'' - Y = 0$ and so $Y = Ae^X + Be^{-X}$, and exponentially decaying solutions—ensuring bounded solutions as $\varepsilon \to 0^+$—arise for $A = 0$ or $B = 0$, appropriately chosen, either on the left boundary or on the right boundary.

E2.17 Two boundary layers

Consider the equation

$$\varepsilon y'' - (1 + 3x^2)y = x, \quad 0 \leq x \leq 1, \tag{2.105}$$

for $\varepsilon \to 0^+$, with $y(0;\varepsilon) = y(1;\varepsilon) = 1$; the solution for suitable $x = O(1)$ is written as $y(x;\varepsilon) \sim y_0(x)$, to leading order, where

$$y_0(x) = -x/(1 + 3x^2). \tag{2.106}$$

This solution, (2.106), clearly does not satisfy the boundary conditions as $x \to 0$ nor as $x \to 1$. The boundary layer near $x = 1$ is expressed in terms of $x = 1 - \delta X$ $(X \geq 0)$ with $y \equiv Y(X;\varepsilon)$ $(= O(1))$; equation (2.105) then becomes

$$\frac{\varepsilon}{\delta^2}Y'' - (4 - 6\delta X + 3\delta^2 X^2)Y = 1 - \delta X$$

which leads to the choice $\delta = \sqrt{\varepsilon}$. Seeking a solution $Y(X;\varepsilon) \sim Y_0(X)$, then

$$Y_0'' - 4Y_0 = 1 \quad \text{or} \quad Y_0 = A_0 e^{2X} + B_0 e^{-2X} - \frac{1}{4}$$

and for bounded (i.e. matchable) solutions as $X \to \infty$ we must have $A_0 = 0$. The boundary condition, $Y_0(0) = 1$, gives $B_0 = 5/4$ and so

$$Y(X;\varepsilon) \sim \frac{1}{4}\left(5e^{-2X} - 1\right)$$

and this is completely determined, as is (2.106), so matching is used only to confirm the correctness of these results (and this is left as an exercise).

The boundary layer near $x = 0$ is written in terms of $x = \delta\chi$ ($\chi \geq 0$) with $y \equiv z(\chi; \varepsilon)$ ($= O(1)$ again); equation (2.105) now becomes

$$\frac{\varepsilon}{\delta^2} z'' - (1 + 3\delta^2\chi^2)z = \delta\chi.$$

The boundary layer is, not surprisingly, the same size at this end of the domain: $\delta = \sqrt{\varepsilon}$, and then with $z(\chi; \varepsilon) \sim z_0(\chi)$ we obtain

$$z_0'' - z_0 = 0 \quad \text{or} \quad z_0 = C_0 e^{\chi} + D_0 e^{-\chi}.$$

A bounded solution, as $\chi \to \infty$, requires the choice $C_0 = 0$, and $D_0 = 1$ will ensure that the boundary condition ($z_0(0) = 1$) is satisfied i.e. $z_0(\chi) = e^{-\chi}$. The next term in the asymptotic expansion $z(\chi; \varepsilon) \sim \sum_{n=0}^{N} \varepsilon^{n/2} z_n(\chi)$ satisfies

$$z_1'' - z_1 = \chi \quad \text{which gives} \quad z_1 = C_1 e^{\chi} + D_1 e^{-\chi} - \chi$$

with $C_1 = 0$ (for boundedness) and $D_1 = 0$ (because $z_1(0) = 0$) i.e.

$$z(\chi; \varepsilon) \sim e^{-\chi} - \sqrt{\varepsilon}\chi;$$

it is left as another exercise to confirm that this matches with (2.106).

In our examples so far (and see also Q2.21, 2.22), the character and position of the boundary layer (or its interior counterpart, the transition layer) have been controlled by the known function $a(x)$, as in $a(x)y'$ (or $a(x)y$ in our most recent example). We will now investigate how the same approach can be adopted when the relevant coefficient is a function of y. In this situation, we do not know, *a priori,* the sign of y—and this is usually critical. Typically, we make an appropriate assumption, seek a solution and then test the assumption. For example, if the two boundary values for y—we are thinking here of two-point boundary value problems—have the same sign, then we may reasonably suppose that y retains this sign throughout the domain. On the other hand, if the boundary values are of opposite sign, then the solution must have at least one zero somewhere on the domain (and this indicates the existence of a transition layer).

E2.18 A problem which exhibits either a boundary layer or a transition layer: I

(An example similar to this one is discussed in great detail in Kevorkian & Cole, 1981 & 1996; see Q2.23.)

We consider the equation

$$\tfrac{1}{2}\varepsilon y'' + yy' - 2xy = 0, \quad 0 \le x \le 1, \tag{2.107}$$

for $\varepsilon \to 0^+$, with $y(0;\varepsilon) = \alpha$ and $y(1;\varepsilon) = \beta$ (where α and β are independent of ε); for suitable $x = O(1)$, we seek a solution in the usual form

$$y(x;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n y_n(x),$$

which gives

$$y_0 y_0' - 2xy_0 = 0; \quad (y_0 y_1)' - 2xy_1 + \tfrac{1}{2} y_0'' = 0, \tag{2.108a,b}$$

and so on. The general solution to equation (2.108a) is

$$y_0(x) = A_0 + x^2; \tag{2.109}$$

we exclude the solution $y_0(x) \equiv 0$ because we will consider problems for which $\alpha \neq 0$ and $\beta \neq 0$. (Any special solutions which may need to make use of the zero solution are easily incorporated if required.) The next term in this asymptotic expansion is obtained from (2.108b) i.e.

$$[(A_0 + x^2)y_1]' - 2xy_1 + 1 = 0$$

which yields

$$y_1(x) = A_1 - \frac{1}{\sqrt{A_0}} \arctan\left(\frac{x}{\sqrt{A_0}}\right), \tag{2.110}$$

where A_1 is the second arbitrary constant (and we have taken $A_0 > 0$).

It is clear, however, that it is impossible to proceed without more information about the boundary values, α and β. Let us examine, first, the problem for which both values are positive; we therefore assume that a solution, $y > 0$, exists and hence that any boundary layer must be situated in the neighbourhood of $x = 0$ (indicated by the term yy' with $y > 0$). With this in mind, we may use the one available boundary condition away from $x = 0$, i.e. $y(1;\varepsilon) = \beta$; thus we obtain

$$y(x;\varepsilon) \sim \beta - 1 + x^2. \tag{2.111}$$

Correspondingly, with $y_1(1) = 0$, we see that $A_1 = 0$; we will assume hereafter that $\beta - 1 > 0$ (but we clearly have an interesting case if $-1 < \beta - 1 \le 0$, for then the solution *does* have a zero near $x = \sqrt{1-\beta}$, even with $\alpha > 0$–a possibility not pursued here).

Now, as $x \to 0$, we see that $y_0(x) \to \beta - 1$ (> 0), and if this does not equal α then a boundary layer is required near $x = 0$. For this layer, we write $x = \delta X$, with $y \equiv Y(X; \varepsilon)$ ($= \mathrm{O}(1)$), and then (2.107) becomes

$$\frac{\varepsilon}{2\delta^2} Y'' + \frac{1}{\delta} YY' - 2\delta XY = 0 \quad \text{or} \quad \tfrac{1}{2} Y'' + YY' - 2\varepsilon^2 XY = 0 \tag{2.112}$$

with the choice $\delta = \varepsilon$. We seek a solution $Y(X; \varepsilon) \sim Y_0(X)$, where

$$\tfrac{1}{2} Y_0'' + Y_0 Y_0' = 0 \quad \text{or} \quad Y_0' = B_0^2 - Y_0^2 \tag{2.113}$$

and we have chosen to write the arbitrary constant of integration as B_0^2 (> 0); any other choice produces a solution which cannot be matched—an investigation that is left as an exercise. The next integral of (2.113) gives the general solution

$$Y_0(X) = B_0 \left(\frac{C_0 \mathrm{e}^{2B_0 X} - 1}{C_0 \mathrm{e}^{2B_0 X} + 1} \right), \quad B_0 > 0,$$

and to satisfy $Y_0(0) = \alpha$ this can be written as

$$Y_0(X) = B_0 \left\{ \frac{\alpha + B_0 + (\alpha - B_0)\mathrm{e}^{-2B_0 X}}{\alpha + B_0 - (\alpha - B_0)\mathrm{e}^{-2B_0 X}} \right\}. \tag{2.114}$$

The value of the remaining arbitrary constant, B_0, is now determined by matching (2.114) and (2.111); from (2.111) we obtain

$$Y \sim \beta - 1 \quad \text{for } X = \mathrm{O}(1)$$

and from (2.114) we see that

$$y \sim B_0 \quad \text{for } x = \mathrm{O}(1),$$

which requires that $B_0 = \beta - 1$. Thus we have successfully completed the initial calculations in the construction of asymptotic expansions valid for $x = \mathrm{O}(1)$ and for $x = \mathrm{O}(\varepsilon)$; these demonstrate that, in the case $\alpha > 0$ and $\beta > 1$, we have a solution which satisfies $y > 0$ for $\forall x \in D$.

E2.19 A problem which exhibits either a boundary layer or a transition layer: II

We repeat example E2.18, but now with the boundary values $\beta > 1$ (chosen to avoid the difficulties already noted) and $\alpha < 0$, and thus the solution must change sign (at least once) in order to accommodate these boundary values. This indicates the need for a transition layer at some $x = x_0$, $0 < x_0 < 1$, and determining x_0 becomes an essential element in the construction of the solution.

For $x = \mathrm{O}(1)$ we have, as before (see (2.111)),

$$y(x;\varepsilon) \sim \beta - 1 + x^2 \quad \text{as } \varepsilon \to 0^+, \tag{2.115}$$

but this can hold only for $x_0 + a < x \leq 1$ ($a \to 0$ but $\delta(\varepsilon)/a \to 0$, where $\delta(\varepsilon)$ is introduced below); for $0 \leq x < x_0 - a$ we have the corresponding solution

$$y(x;\varepsilon) \sim \alpha + x^2 \quad \text{as} \quad \varepsilon \to 0^+. \tag{2.116}$$

Near $x = x_0$ we write $x = x_0 + \delta X$, with $y \equiv Y(X;\varepsilon)$ ($= \mathrm{O}(1)$), which essentially repeats (2.112) i.e. $\delta = \varepsilon$ and so

$$\tfrac{1}{2}Y'' + YY' - 2\varepsilon(x_0 + \varepsilon X)Y = 0,$$

and this gives the same general solution, to leading order, as before (see (2.113), *et seq.*):

$$Y_0(X) = B_0\left(\frac{C_0 e^{2B_0X} - 1}{C_0 e^{2B_0X} + 1}\right). \tag{2.117}$$

However, for a transition layer, we do not have *any* boundary conditions; here, we must match (2.117) to both (2.115) and (2.116).

From (2.115) and (2.116) we obtain

$$\beta - 1 + x^2 \sim \beta - 1 + x_0^2 \quad \text{and} \quad \alpha + x^2 \sim \alpha + x_0^2, \tag{2.118}$$

respectively, both for $X = \mathrm{O}(1)$; from (2.117), with $X = (x - x_0)/\varepsilon$, we have

$$B_0\left(\frac{C_0 e^{2B_0X} - 1}{C_0 e^{2B_0X} + 1}\right) \sim \begin{cases} B_0 & \text{for } x > x_0 \\ -B_0 & \text{for } x < x_0 \end{cases} \tag{2.119}$$

as $\varepsilon \to 0^+$. We observe, immediately, that a property of this transition layer is to admit only a change in value across it from $-B_0$ to $+B_0$ (which will fix the value of x_0) and that the matching excludes C_0 (so this cannot be determined at this stage). Now (2.119) does match with (2.118) when we choose

$$B_0 = \beta - 1 + x_0^2 \quad \text{and} \quad -B_0 = \alpha + x_0^2$$

which requires that

$$x_0^2 = \tfrac{1}{2}(1 - \beta - \alpha)$$

and hence a transition layer exists at $x = x_0$ $(0 < x_0 < 1)$ provided

$$0 < 1 - \beta - \alpha < 2. \tag{2.120}$$

If this condition is not satisfied, for given α and β, then the adjustment to the given boundary value must be through a boundary layer near $x = 0$. Thus, for example, with $\beta = 2$ (> 1) and $\alpha = -2$ (< 0), there is a transition layer at $x_0 = 1/\sqrt{2}$ and the jump across the layer is between $\pm 3/2$ (to leading order). On the other hand, the problem with $\beta = 2$ and $\alpha = -4$ does not admit a transition layer; the boundary layer near $x = 0$ is used to accommodate the change in value from $\beta - 1 = 1$ (to leading order) to $\alpha = -4$. The dominant solution in the transition layer is given by (2.117) with $B_0 = \frac{1}{2}(\beta - 1 - \alpha) > 0$, although C_0 is unknown at this stage. (The role of C_0 is to determine the position of the transition layer, correct at $O(\varepsilon)$.)

These two examples, E2.18 and E2.19 (and see also Q2.23), demonstrate the complexity and richness of solutions that are available for this type of problem, depending on the particular boundary values that are prescribed. All this can be traced to the nonlinearity associated with the y' term; if this term were simply $a(x)y'$, then we would have a fixed boundary layer, or fixed transition layers, independent of the specific boundary values (as we have seen in our earlier examples). We conclude this section with an example which shows how these ideas can be extended, fairly straightforwardly, to higher-order equations. (The following example is based on the type of problem that can arise when examining the displacement of a loaded beam.)

E2.20 A problem with two boundary layers

We consider

$$\varepsilon^2 y'''' - \frac{y''}{1 - \frac{3}{4}x} = 8, \quad 0 \le x \le 1, \tag{2.121}$$

for $\varepsilon \to 0^+$ (and the use of ε^2 here is merely an algebraic convenience), with

$$y(0;\varepsilon) = y'(0;\varepsilon) = y(1;\varepsilon) = y'(1;\varepsilon) = 0. \tag{2.122}$$

Before we begin the detailed analysis of this problem, a couple of points should be noted. First, the variable coefficient, $1/(1 - \frac{3}{4}x)$, in (1.121), is positive for $\forall x \in D$ and so, second, this implies that we have available (locally) two exponential solutions. These arise from, approximately,

$$\varepsilon^2 y'''' - \lambda^2 y'' = 0 \quad \text{i.e.} \quad y \propto \exp(\pm\lambda x/\varepsilon)$$

and so we may select one exponential near $x = 0$ and the other near $x = 1$, ensuring exponential decay as we move away from the boundaries. Thus we must anticipate boundary layers near both $x = 0$ and $x = 1$; we will assume that they exist—we can always ignore them if they are not required (because the boundary values are automatically satisfied).

Hence, for $x = O(1)$ away from $x = 0$ and $x = 1$, we seek a solution with $y(x;\varepsilon) \sim y_0(x)$, where

$$-\frac{y_0''}{1-\frac{3}{4}x} = 8 \quad \text{or} \quad y_0(x) = A_0 + B_0 + x^3 - 4x^2, \tag{2.123}$$

but no boundary conditions are available (by virtue of the assumed existence of boundary layers). For the boundary layer near $x = 0$, it is clear that we require $x = \varepsilon X$, $y \equiv \Delta Y(X;\varepsilon)$ (but we do not know Δ at this stage; since $y(0;\varepsilon) = 0$, *presumably* $y_0(x) \to 0$ as $x \to 0$ although the *size* of y_0 in this limit is unknown). Thus equation (2.121) becomes

$$Y'''' - \frac{Y''}{1-\frac{3}{4}\varepsilon X} = \frac{8\varepsilon^2}{\Delta}.$$

We seek a solution $Y(X;\varepsilon) \sim Y_0(X)$, where $Y_0(X)$ satisfies

$$Y_0'''' - Y_0'' = 0 \quad \text{or} \quad Y_0(X) = C_0 + D_0X + E_0\mathrm{e}^{X} + F_0\mathrm{e}^{-X}$$

provided $\varepsilon^2/\Delta = o(1)$ as $\varepsilon \to 0$; we will assume that this condition applies, and we will check it shortly. The boundary conditions are $Y_0(0) = Y_0'(0) = 0$ and we must not allow the term which grows exponentially away from $X = 0$, so $E_0 = 0$, and then

$$Y_0(X) = C_0(1 - X - \mathrm{e}^{-X}).$$

Immediately we observe that the term $C_0X = C_0x/\varepsilon$ is unmatchable to $y_0(x)$ $(= O(1))$ unless we select $\Delta = \varepsilon$ (and then $\varepsilon^2/\Delta = \varepsilon$); this we do, so that

$$Y(X;\varepsilon) \sim \varepsilon C_0(1 - X - \mathrm{e}^{-X}). \tag{2.124}$$

Correspondingly, for the boundary layer near $x = 1$, we write $x = 1 - \varepsilon\chi$ $(\chi \geq 0)$ and now choose $y \equiv \varepsilon\hat{Y}(\chi;\varepsilon)$ to give

$$\hat{Y}'''' - \frac{\hat{Y}''}{\frac{1}{4}+\frac{3}{4}\varepsilon\chi} = 8\varepsilon.$$

Thus the first term in the expansion, $\hat{Y}(\chi;\varepsilon) \sim \hat{Y}_0(\chi)$, satisfies

$$\hat{Y}_0'''' - 4\hat{Y}_0'' = 0 \quad \text{or} \quad \hat{Y}_0(\chi) = G_0 + H_0\chi + J_0\mathrm{e}^{2\chi} + K_0\mathrm{e}^{-2\chi}$$

and then (exactly as described for Y_0) we obtain

$$\hat{Y}(\chi;\varepsilon) \sim \varepsilon G_0(1 - 2\chi - \mathrm{e}^{-2\chi}). \tag{2.125}$$

We now determine the constants A_0, B_0, C_0 and G_0 by matching (2.123) with, in turn, (2.124) and (2.125). First, from (2.123) with $x = \varepsilon X$ and $y = \varepsilon Y$, we obtain

$$\varepsilon Y \sim \varepsilon A_0 X + B_0 \quad \text{for } X = \mathrm{O}(1);$$

from (2.124) we have

$$y \sim -\varepsilon C_0 x \quad \text{for } x = \mathrm{O}(1)$$

which requires $B_0 = 0$ and then $C_0 = -A_0$. Again, from (2.123), but now with $x = 1 - \varepsilon\chi$ and $y = \varepsilon\hat{Y}$, we obtain

$$\varepsilon\hat{Y} \sim A_0 - 3 + (5 - A_0)\varepsilon\chi \quad \text{for } \chi = \mathrm{O}(1),$$

and, finally, (2.125) gives

$$y \sim -2\varepsilon G_0(1 - x) \quad \text{for } x = \mathrm{O}(1).$$

Now we require $A_0 = 3$ and $5 - A_0 = -2G_0$; thus, collecting all these results, we see that

$$A_0 = 3, \quad B_0 = 0, \quad C_0 = -3, \quad G_0 = -1$$

and hence, to leading order, we have

$$y(x;\varepsilon) \sim 3x + x^3 - 4x^2$$

with $$Y(X;\varepsilon) \sim 3\varepsilon(X - 1 + \mathrm{e}^{-X}); \quad \hat{Y}(\chi;\varepsilon) \sim \varepsilon(2\chi - 1 + \mathrm{e}^{-2\chi}).$$

So, indeed, boundary layers are required at each end in order to accommodate the boundary conditions there (although we may note that the solution for $y(x;\varepsilon)$ does satisfy $y(0;\varepsilon) = y(1;\varepsilon) = 0$, but not the derivative conditions).

Some further examples of higher-order equations that exhibit boundary-layer behaviour are offered in Q2.26.

This chapter has been devoted to a presentation of some of the fairly routine applications of singular perturbation theory to various types of mathematical problem.

Although we have touched on methods for finding roots of equations, and on integration, the main thrust has been to develop basic techniques for solving differential equations—the most important use, by far, of these methods. We shall devote the rest of the text to extending and developing the methods for solving differential equations, both ordinary and partial, and their applications to many practical problems that are encountered in various branches of mathematical modelling. Many of the examples and exercises in this chapter are, perforce, invented to make a point or to test ideas; however, a few of the later exercises that are included at the end of this chapter (see Q2.27–2.35) begin to employ the techniques in physically relevant problems. In the next chapter, we will show how these ideas can be applied to a broader class of problems and, in particular, begin our discussion of partial differential equations. This will allow us, in turn, to begin to extend the applications of singular perturbation theory to more problems which arise within a physically relevant context.

FURTHER READING

A few of the existing texts include a discussion of the methods for finding roots of equations, and for evaluating integrals of functions which contain a small parameter; in particular, the interested reader is directed to Holmes (1995) and Hinch (1991). Differential equations that give rise to regular problems are given little consideration—they are quite rare, after all—but some can be found in Holmes (1995) and in Georgescu (1995). We have already mentioned those texts that present a more formal approach to perturbation theory (Eckhaus, 1979; Smith, 1985; O'Malley, 1991), but some further developments along these lines are also given in Chang & Howes (1984).

The whole arena of scaling with respect to a parameter, and we should include here the construction of non-dimensional variables, is fairly routine but very powerful. These ideas play a rôle, not only in the identification of asymptotic regions (as we have seen), but also in providing more general pointers to the construction of solutions. A very thorough introduction to these ideas, and their connection with asymptotics, can be found in Barenblatt (1996). A discussion of the applications of group theory to the study of differential equations is likely to be available in any good, relevant text; one such, which emphasises precisely the application to differential equations, is Dresner (1999).

The nature of a boundary layer (which is, for our current interest, limited to a property of certain ordinary differential equations) is described at length, and carefully, in most available texts on singular perturbation theory. We can mention, as examples of the extent and depth of what is discussed, the excellent presentations on this subject given by Smith (1985) and Holmes (1995). The determination of the position of a boundary layer is also covered in most existing texts, although O'Malley (1991) probably provides the most detailed analysis. (This work also includes a number of relevant references which the interested reader may wish to investigate.) An excellent discussion of the interplay between boundary layers and transition layers (for nonlinear equations) is given in Kevorkian & Cole (1981, 1996). (Those readers who wish to examine techniques applicable to turning points, at this stage, are encouraged to study Wasow (1965) and Holmes (1995); we will touch on these ideas in Chapter 4.)

Finally, some examples of higher-order equations, which exhibit boundary-layer-type solutions, are discussed in Smith (1985) and O'Malley (1991).

EXERCISES

Q2.1 *Quadratic equations.* Write down the exact roots of these quadratic equations, where ε is a positive parameter.

(a) $x^2 + 2\varepsilon x - 1 = 0$;   (b) $\varepsilon x^2 + 4x - 4 = 0$;   (c) $x^2 - (2 + \varepsilon)x + 1 = 0$.

Now, in each case, use the binomial expansion to obtain power-series representations of these roots, valid for $\varepsilon \to 0^+$, writing down the first three terms for each root. (You may wish to investigate how these same expansions can be derived directly from the original quadratic equations.)

Finally, obtain the corresponding power series which are valid for $\varepsilon \to +\infty$.

Q2.2 *Equations I.* Find the first two terms in the asymptotic expansions of all the real roots of these equations, for $\varepsilon \to 0^+$.

(a) $\varepsilon x^3 - x^2 + x + \frac{3}{4} = 0$;   (b) $\varepsilon x^3 - x^2 + 2x - 1 = 0$;   (c) $\varepsilon x^3 + \sqrt{x} - 1 = 0$;
(d) $\varepsilon x^3 - x + 2\sqrt{x} - 1 = 0$;   (e) $\varepsilon x + \varepsilon^2 - \tanh x = 0$;
(f) $\varepsilon x^3 - x^2 + 1 + 2\varepsilon - \mathrm{e}^{-x/\varepsilon} = 0$;   (g) $\varepsilon^5 x^3 - \varepsilon x - \varepsilon + 1 - \mathrm{e}^{-x} = 0$;
(h) $3\varepsilon x^4 - x^3 + 6\sqrt{x} - 5 = 0$;   (i) $1 + \varepsilon - \mathrm{e}^{-x/\varepsilon} - \varepsilon \sin x = 0$;
(j) $\varepsilon x^3 + x^2 - 1 - \varepsilon \sinh(\varepsilon x) = 0$.

Q2.3 *Equations II.* Repeat Q2.2 for these slightly more involved equations.

(a) $1 + \varepsilon - 2\sqrt{\varepsilon} + (x\sqrt{\varepsilon} - 1)^3 - \sqrt{\varepsilon}(x + \mathrm{e}^{-x/\varepsilon}) = 0$;

(b) $2\varepsilon x + \dfrac{2(1-\varepsilon)}{2 + \varepsilon x} - e^{-|x|/\varepsilon} = 0$;   (c) $\varepsilon x^2 - x + \varepsilon(\ln \varepsilon)^2 - \dfrac{x^2}{x^2 + \varepsilon^2 \mathrm{e}^{-x/\varepsilon}} = 0$;

(d) $\mathrm{e}^x - 2 + \dfrac{\varepsilon}{1 + x^2} = 0$;   (e) $\varepsilon^3 x^3 - \varepsilon^4 x^2 - \varepsilon - \tanh x = 0$.

Q2.4 *Kepler's equation.* A routine problem in celestial mechanics is to find the *eccentric anomaly, u,* given both the *eccentricity e* $(0 \le e < 1)$ and the *mean anomaly nt* (where t is time measured from where $u = 0$, and $n = 2\pi/P$, where P is the period); u is then the solution of *Kepler's equation*

$$nt = u - e \sin u$$

(see e.g. Boccaletti & Pucacco, 1996). For many orbits (for example, most planets in our solar system), the eccentricity is very small; find the first three terms in the asymptotic solution for u as $e \to 0^+$. Confirm that your 3-term expansion is uniformly valid for all *nt.*

Q2.5 *Complex roots.* Find the first two terms in the asymptotic expansions of all the roots of these equations, for $\varepsilon \to 0^+$.

(a) $\varepsilon z^3 + z^2 + z + 1 = 0$;   (b) $\varepsilon z^4 - z^2 + 2z - 2 = 0$;   (c) $\varepsilon z^4 + z^2 - 2z + 2 = 0$;

(d) $z^4 + z^3 + z^2 + \varepsilon = 0$;   (e) $\mathrm{e}^z = 1 + \varepsilon\, |z|^2$;   (f) $\mathrm{e}^{z/\varepsilon} = \mathrm{e}^2 + \dfrac{\varepsilon}{1 + |z|}$.

Q2.6 *Simple integrals.* Obtain estimates for these integrals, for $\varepsilon \to 0^+$, by first finding asymptotic expansions of the integrand for each relevant size of x, retaining the first two terms in each case. (These integrals can be evaluated exactly, so you may wish to check your results against the expansions of the exact values.)

(a) $\displaystyle\int_0^1\left(x^2 + \frac{\varepsilon}{\sqrt{x+\varepsilon}} + \mathrm{e}^{-x/\varepsilon}\right)\mathrm{d}x$; (b) $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(1+x)(x+\varepsilon)}$;

(c) $\displaystyle\int_0^1\left(2x + \varepsilon + \frac{\varepsilon}{\sqrt{x}} + 3\varepsilon^2x^2 + \mathrm{e}^{-x/\varepsilon}\right)\mathrm{d}x$; (d) $\displaystyle\int_0^\infty \frac{(2\varepsilon - 2\varepsilon^2 - 1)x - (1+3\varepsilon)}{\varepsilon x^2(x+1-\varepsilon) + (x+1)^2 - \varepsilon^2}\mathrm{d}x$.

Q2.7 *More integrals.* See Q2.6; repeat for these integrals (but here you are not expected to have available the exact values).

(a) $\displaystyle\int_0^1 \frac{1 + \varepsilon x + \mathrm{e}^{-x/\varepsilon}}{1+x^2}\mathrm{d}x$; (b) $\displaystyle\int_0^1 \frac{x\mathrm{d}x}{\sqrt{(1-x)(x^2+\varepsilon^2)}}$;

(c) $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(1+x)(x+\varepsilon)^n}$ for $n = \frac{1}{2}$; (d) repeat (c) for $n = 2$;

(e) $\displaystyle\int_0^\infty \frac{x\mathrm{d}x}{(1+x)(1+\varepsilon x)\sqrt{x^2+\varepsilon^2}}$; (f) $\displaystyle\int_0^1 \frac{x + \varepsilon + \sqrt{1+\varepsilon\mathrm{e}^{-x/\varepsilon}}}{\sqrt{1 + x - \varepsilon + 2\varepsilon\mathrm{e}^{-x/\varepsilon}}}\mathrm{d}x$.

Q2.8 *An integral from thin aerofoil theory.* An integral of the type that can appear in the study of thin aerofoil theory (for the velocity components in the flow field) is

$$I(x;\varepsilon) = \varepsilon\int_{-1}^1 \frac{\xi + \varepsilon/\sqrt{1+\xi}}{(\xi - x)^2 + \varepsilon^2}\mathrm{d}\xi, \quad -1 \le x \le 1;$$

obtain the first terms in the asymptotic expansions of the integrand (for $\varepsilon \to 0^+$), with x away from the end-points, for each of: (a) ξ away from x and away from $\xi = -1$; (b) $\xi = x + \varepsilon\varsigma$; (c) $\xi = -1 + \varepsilon^2 z$. Hence find an estimate for $I(x;\varepsilon)$. Repeat the calculations with $x = 1 - \varepsilon X$, and then with $x = -1 + \varepsilon X$, for ξ away from the end-points, and then with $\xi = 1 - \varepsilon\varsigma$, $\xi = -1 + \varepsilon^2 z$; $\xi = 1 - \varepsilon\varsigma$, $\xi = -1 + \varepsilon\varsigma$, $\xi = -1 + \varepsilon^2 z$, respectively. Again, find estimates for $I(1 - \varepsilon X;\varepsilon)$ and for $I(-1 + \varepsilon X;\varepsilon)$; show that your asymptotic approximations for $I(x;\varepsilon)$ satisfy the matching principle.

Q2.9 *Regular expansions for differential equations.* Find the first two terms in the asymptotic expansions of the solutions of these equations, satisfying the given boundary conditions. In each case you should use the asymptotic sequence $\{\varepsilon^n\}$, and you should confirm that your 2-term expansions are uniformly valid. (You may wish to examine the nature of the general term, and hence produce an argument that shows the uniform validity of the expansion to all orders in ε.)

(a) $y' = 2x + \varepsilon y^2$, $0 \le x \le 1$, $y(0;\varepsilon) = 0$;

(b) $y'' + y' + \varepsilon y^3 = 0$, $0 \le x \le 1$, $y(0;\varepsilon) = 1$, $y(1;\varepsilon) = \mathrm{e}^{-1}$;

(c) $y'' + y + \varepsilon y^3 = 0$, $0 \le x \le \pi/2$, $y(0;\varepsilon) = 1$, $y(\pi/2;\varepsilon) = 0$;

(d) $x^2y'' - (4x + \varepsilon x^3\mathrm{e}^x)y' + 6y = 0$, $-1 \le x \le 1$, $y(-1;\varepsilon) = 2$, $y(1;\varepsilon) = 0$;

(e) $\ddot{x} + (1 - \varepsilon t\mathrm{e}^{-t})x = 0$, $t \ge 0$, $x(0;\varepsilon) = 0$, $\dot{x}(0;\varepsilon) = 1$.

Q2.10 *Eigenvalue problems.* A standard problem in many branches of applied mathematics and physics is to find the eigenvalues (and eigenfunctions) of appropriate problems based on ordinary differential equations. In these examples, find the first two terms in the asymptotic expansions of both the eigenvalues (λ) and the eigenfunctions; for each use the asymptotic sequence $\{\varepsilon^n\}$.

(a) $y'' + \lambda(1+\varepsilon x)y = 0$, $0 \le x \le 1$, $y(0;\varepsilon) = y(1;\varepsilon) = 0$;

(b) $y'' + \lambda(1+\varepsilon \sin x)y = 0$, $0 \le x \le 1$, $y(0;\varepsilon) = y(1;\varepsilon) = 0$;

(c) $y'' + \lambda y = \varepsilon y^2$, $0 \le x \le 1$, $y(0;\varepsilon) = y(1;\varepsilon) = 0$.

Q2.11 *Breakdown of asymptotic solutions of differential equations.* These ordinary differential equations define solutions on the domain $0 \le x \le 1$, with conditions given on $x = 1$. In each case, find the first two terms in an asymptotic solution valid for $x = \mathrm{O}(1)$ as $\varepsilon \to 0^+$, which allows the application of the given boundary condition(s). Show, in each case, that the resulting expansion is not uniformly valid as $x \to 0$; find the breakdown, rescale and hence find the first term in an asymptotic expansion valid near $x = 0$, matching as necessary. Finally find, for each problem, the dominant asymptotic behaviour of $y(0;\varepsilon)$ as $\varepsilon \to 0^+$.

(a) $(x^2 + \varepsilon y)y' + 2xy = \dfrac{3\varepsilon}{2y}$, $y(1;\varepsilon) = 1$;

(b) $(x^2 + \varepsilon y^2)y' + xy = \varepsilon y^2$, $y(1;\varepsilon) = 1$;

(c) $yy'' - \frac{1}{2}(y')^2 - 2\varepsilon = 2\varepsilon y^{3/2}$, $y(1;\varepsilon) = 1 + 2\varepsilon$, $y'(1;\varepsilon) = 2 + 3\varepsilon$;

(d) $(x^2 + \varepsilon y)y' + \dfrac{2xy}{1+x} = 3x^2(1+x)^2$, $y(1;\varepsilon) = 8$;

(e) $yy'' - 2(y')^2 = \varepsilon\left[2(y')^3 + 6y'\right]$, $y(1;\varepsilon) = 1 - \varepsilon$, $y'(1;\varepsilon) = -1 + \varepsilon$.

Q2.12 *Another breakdown problem.* See Q2.11; repeat for the problem

$$(x^3 + \varepsilon y)y' - \frac{x^4}{y} = \varepsilon\left(3x^3 y + \frac{x}{y}\right), \quad y(1;\varepsilon) = 1 + \varepsilon, \quad x \le 1,$$

but show that, for a real solution to exist, the domain is $x_0(\varepsilon) \le x \le 1$, where $x_0(\varepsilon) \sim (2\varepsilon)^{1/3}$, and then find the dominant asymptotic behaviour of $y(2\varepsilon^{1/3};\varepsilon)$ as $\varepsilon \to 0^+$.

Q2.13 *Breakdown as $x \to \infty$: I.* Find the first two terms in an asymptotic solution, valid for $x = \mathrm{O}(1)$ as $\varepsilon \to 0^+$, of

$$y''' + y'' + \varepsilon[(y')^2 + y] = \varepsilon \mathrm{e}^{-2x}, \quad x \ge 0,$$

with $y(0;\varepsilon) = 2$, $y'(0;\varepsilon) = -1$, $y''(0;\varepsilon) = 1$. Now show that this expansion is not uniformly valid as $x \to \infty$, find the breakdown, rescale and find the first two terms in an expansion valid for large x, matching as necessary. Show, also, that this 2-term expansion breaks down for even larger x, but do not take the analysis further.

Q2.14 *Breakdown as $x \to \infty$: II.* See Q2.11 (a) and (c); for these equations and boundary conditions, and the asymptotic solutions already found for $x = O(1)$, take the domain now to be $x \geq 1$. Hence show that the expansions are not uniformly valid as $x \to \infty$, find the breakdown, rescale and then find the first terms in the expansions valid for large x, matching as necessary.

Q2.15 *Problem E 2.7 reconsidered.* Find the first two terms in an asymptotic expansion, valid for $x = O(1)$ as $\varepsilon \to 0$, of

$$y'' + y' + \varepsilon y^2 = 0, \quad 0 \leq x \leq 1,$$

with $y(0; \varepsilon) = \varepsilon$, $y(1; \varepsilon) = 1 - e^{-1} - \frac{1}{2}\varepsilon e^{-2}$. Show that, formally, this requires two matched expansions, but that the asymptotic solution obtained for $x = O(1)$ correctly recovers the solution for $\forall x \in D$ i.e. it is uniformly valid. (Note the *balance* of terms, when scaled near $x = 0$!)

Q2.16 *Scaling of equations.* See Q2.11 and Q2.14; use the dominant terms only, valid for $x = O(1)$, together with appropriate scalings associated with the relevant balance of terms, to analyse these equations. Compare your results with the scalings obtained from the breakdown of the asymptotic expansions.

Q2.17 *Boundary-layer problems I.* Find the first two terms in asymptotic expansions, valid for $x = O(1)$ (away from the boundary layer) as $\varepsilon \to 0^+$, for each of these equations, with the given boundary conditions. Then, for each, find the first term in the boundary-layer solution, matching as necessary. (You may wish to use your expansions to construct composite expansions valid for $\forall x \in D$, to this order.)

(a) $\varepsilon y'' + (1 + x)^2 y' - y^2 = 0, 0 \leq x \leq 1, y(0; \varepsilon) = 2, y(1; \varepsilon) = 2 + 4\varepsilon;$

(b) $\varepsilon y'' + (1 + \varepsilon x) y' - 2\sqrt{y} = -2\varepsilon x, 0 \leq x \leq 1, y(0; \varepsilon) = 2, y(1; \varepsilon) = 4;$

(c) $\varepsilon y'' + (1 + \varepsilon x) y' + (1 + 2x) e^y = 0, 0 \leq x \leq 1, y(0; \varepsilon) = 1, y(1; \varepsilon) = 2 \ln(2/3);$

(d) $\varepsilon y' = \dfrac{1 + y}{(1 + x)^2} \left(\dfrac{1 + y}{2 + x} - y \right), 0 \leq x < \infty, y(0; \varepsilon) = \frac{1}{2};$

(e) $\varepsilon y'' + \sqrt{x}(y' - y) = 0, 0 \leq x \leq 1, y(0; \varepsilon) = 0, y(1; \varepsilon) = 1.$

Q2.18 *Boundary-layer problems II.* See Q2.17; repeat for these more involved equations.

(a) $\varepsilon[y'' + (y')^2] + y' + y^2 = \dfrac{2\varepsilon}{1 + x}, 0 \leq x \leq 1, y(0; \varepsilon) = 1 + \ln 2, y(1; \varepsilon) = \frac{1}{2};$

(b) $\varepsilon y y'' + y^2 y' - y = \varepsilon x, 0 \leq x \leq 1, y(0; \varepsilon) = 2\sqrt{2}, y(1; \varepsilon) = 2;$

(c) $\varepsilon y'' - y y' - y^2 = \varepsilon(e^{-x} + e^{-2x}), 0 \leq x \leq 1, y(0; \varepsilon) = 1, y(1; \varepsilon) = \frac{1}{2} e^{-1};$

(d) $\varepsilon y'' + (x + y)[y' - x - \varepsilon(x + y)] = \varepsilon, 0 \leq x \leq 1, y(0; \varepsilon) = 0, y'(0; \varepsilon) = 2/\varepsilon;$

(e) $\varepsilon y' y'' + y(y')^2 - 4y^2 = 4\varepsilon y^2 \left[(1 + x)^{-3} + 2(1 + x)\right], 0 \leq x \leq 1, y(0; \varepsilon) = \frac{1}{3},$
$y(1; \varepsilon) = 4(1 + 2\varepsilon);$

(f) $\varepsilon y''' + y'' + \varepsilon y'(y' + 2) + y = 1, 0 \leq x \leq \pi/2, y(0; \varepsilon) = 0, y(\pi/2; \varepsilon) = 1 - \frac{1}{3}\varepsilon,$
$y'(\pi/2; \varepsilon) = -1 + (\varepsilon\pi)/4.$

Q2.19 *Two boundary layers.* The function $y(x;\varepsilon)$ is defined by the problem

$$\varepsilon^2 y'' + 4\varepsilon x^2 y' - 5y = x^2 - 2, \quad 0 \le x \le 1,$$

with $y(0;\varepsilon) = 1$, $y(1;\varepsilon) = 2$. Find the first two terms in an asymptotic expansion valid for $x = \mathrm{O}(1)$, as $\varepsilon \to 0^+$, away from $x = 0$ and $x = 1$. Hence show that, in this problem, boundary layers exist both near $x = 0$ and near $x = 1$, and find the first term in each boundary-layer solution, matching as necessary.

Q2.20 *A boundary layer within a thin layer.* Consider the equation

$$\varepsilon^2 y'' + y' - \frac{\varepsilon y^2}{x^2 + \varepsilon^2} = 2x, \quad 0 \le x \le 1,$$

for $\varepsilon \to 0^+$, with $y(0;\varepsilon) = 1$, $y(1;\varepsilon) = 2$. Find the first terms in each of three regions, two of which are near $x = 0$, matching as necessary. (Here, only the inner-most region is a true boundary layer; the other is simply a scaled-thin-connecting region.)

Q2.21 *Boundary layers or transition layers?* Decide if these equations, on the given domains, possess solutions which may include boundary layers or transition layers; give reasons for your conclusions.

(a) $\varepsilon y'' + (x-2)y' + y = 0, 0 \le x \le 1;$

(b) $\varepsilon y'' + (x - x^2)y' + y^2 = 0, 0 \le x \le 1;$

(c) $\varepsilon y'' + (2x^2 - x)y' + y^2 = 3x, 0 \le x \le 1;$

(d) $\varepsilon y'' + (x - x^2)y' + y^2 = 0, -1 \le x \le 1;$

(e) $\varepsilon y'' + (x + y)y' + y = 0, 1 \le x \le 2.$

Q2.22 *Transition layer near a fixed point.* In these problems, a transition layer exists at a fixed point, independent of the boundary values. Find, for (a), the first two terms, and for (b) the first term only, in an asymptotic expansion (as $\varepsilon \to 0^+$) valid away from the transition layer and the first term only of an expansion valid in this layer; match your expansions as necessary.

(a) $\varepsilon y'' + 2(x-1)y' = 8(x-1)^3, 0 \le x \le 2, y(0;\varepsilon) = 1, y(2;\varepsilon) = 2;$

(b) $\varepsilon y'' + x^{1/5}y' + \frac{5}{6}y^2 = 0, -1 \le x \le 1, y(-1;\varepsilon) = \frac{1}{2}, y(1;\varepsilon) = \frac{1}{3}.$

Q2.23 *Boundary layer or transition layer?* (This example is based on the one which is discussed carefully and extensively in Kevorkian & Cole, 1981 & 1996.) The equation is

$$\tfrac{1}{2}\varepsilon y'' + yy' + y = 0, \quad 0 \le x \le 1,$$

for $\varepsilon \to 0^+$ and given $y(0;\varepsilon) = \alpha$ $(\neq 0)$, $y(1;\varepsilon) = \beta$ $(\neq 0)$. Suppose that a transition layer exists near $x = x_0$, $0 < x_0 < 1$; find the leading terms in the asymptotic expansions valid outside the transition layer, and in the transition layer. Hence deduce that a transition layer is required if α and β are of opposite sign and $0 < \alpha + \beta + 1 < 2$, and then find the leading terms in all the relevant regions for:

(a) $\alpha = 1, \beta = -1$; (b) $\alpha = 1, \beta = 2$; (c) $\alpha = -1, \beta = 3$.

Q2.24 *Transition layers and turning points.* Consider the equation

$$\varepsilon y'' + a(x)y' + f(x, y; \varepsilon) = 0;$$

introduce $y(x;\varepsilon) = u(x;\varepsilon)v(x;\varepsilon)$ and find a choice of $u(x;\varepsilon)$ which produces an equation for $v(x;\varepsilon)$ in the form

$$\varepsilon^2 v'' + F(x, v; \varepsilon) = 0$$

and identify F. If F changes sign on the domain of the solution, then the point where this occurs is called a *turning point;* find the equation that defines the turning points in the case $f = \varepsilon^{-1} b(x;\varepsilon)y$.

Q2.25 *Transition layer at a turning point.* Consider the equation

$$\varepsilon^3 y'' + (x - 2x^2)y = 0, \quad 0 \le x \le 1;$$

find the position of the turning point and scale in the neighbourhood of the transition layer. Write down the general solution, to leading order, valid in the transition layer, as $\varepsilon \to 0^+$. (This solution is best written in terms of *Airy* functions. A uniformly valid solution is usually expressed using the *WKB* method; see Chapter 4.)

Q2.26 *Higher-order equations.* For these problems, find the first terms only in asymptotic expansions valid in each region of the solution, for $\varepsilon \to 0^+$.

(a) $\varepsilon^2 y'''' - (1 + x)^2 y'' = 1, 0 \le x \le 1, y(0;\varepsilon) = y'(0;\varepsilon) = y(1;\varepsilon) = y'(1;\varepsilon) = 0$;

(b) $\varepsilon^2 y''' - xy' + y = 2x^3, 1 \le x \le 4, y(1;\varepsilon) = 1, y'(1;\varepsilon) = 0, y(4;\varepsilon) = 0$;

(c) $\varepsilon^2 y''' - (1 + x)^2 y' + y^2 = 0, 0 \le x \le 1, y(0;\varepsilon) = 1, y(1;\varepsilon) = \frac{2}{3}, y'(1;\varepsilon) = 0$.

Q2.27 *Vertical motion under gravity.* Consider an object that is projected vertically upwards from the surface of a planetary body (or, rather, for example, from our moon, because we will assume no atmosphere in this model). The height above the surface is $z(t)$, where t is time, and this function is a solution of

$$\ddot{z} = -\frac{gR^2}{(R + z)^2}, \quad t \ge 0,$$

where R is the distance from the centre of mass of the body to the point of projection, and g is the appropriate (constant) acceleration of gravity. (For our moon, $R \approx 1735$ km, $g \approx 1{\cdot}63$ m/s^2.) The initial conditions are $z(0) = 0$, $\dot{z}(0) = V (> 0)$; find the relevant solution in the form $t = t(z)$ (and you may assume that $V^2 < 2gR$).

(a) Write your solution, and the differential equation, in terms of the non-dimensional variables (Z, τ): $t = (V/g)\tau$, $z = (V^2/g)Z$, and introduce the parameter $\varepsilon = V^2/(Rg)$. Suppose that the limit of interest is $\varepsilon \to 0^+$ (which you may care to interpret); find the first two terms of an asymptotic expansion, valid for $\tau = O(1)$ (in the form $Z = Z(\tau; \varepsilon)$), directly from the governing equation. (You should compare this with the expansion of the exact solution.) From your results, find approximations to the time to reach the maximum height, the value of this height and the time to return to the point of projection.

(b) A better model, for motion through an atmosphere, is represented by the equation

$$\ddot{z} = -\frac{gR^2}{(R+z)^2} - \frac{k\dot{z}}{R+z},$$

where k (> 0) is a constant. (This is only a rather crude model for air resistance, but it has the considerable advantage that it is valid for both $\dot{z} > 0$ and $\dot{z} < 0$.) Non-dimensionalise this equation as in (a), and then write $(kV)/(Rg) = (V^2/(Rg))(k/V) = \varepsilon\delta$ where $\delta = O(1)$ as $\varepsilon \to 0^+$. Repeat all the calculations in (a).

(c) Finally, in the special case $\varepsilon = 2$ (i.e. $V = \sqrt{2Rg}$, the *escape speed),* find the first term in an asymptotic expansion valid as $\delta \to 0^+$. Now find the equation for the second term and a particular integral of it. On the assumption that the rest of the solution contributes only an exponentially decaying solution, show that your expansion breaks down at large distances; rescale and write down—do not solve—the equation valid in this new region.

Q2.28 *Earth-moon-spaceship (1D).* In this simple model for the passage of a spaceship moving from the Earth to our moon, we assume that both these objects are fixed in our chosen coordinate system, and that the trajectory is along the straight line joining the two centres of mass. (More complete and accurate models will be discussed in later exercises.) We take $x(t)$ to be the distance measured along this line from the Earth, and then Newton's Law of Gravitation gives the equation of motion as

$$m\ddot{x} = \frac{GmM_m}{(d-x)^2} - \frac{GmM_e}{x^2},$$

where m is the mass of the spaceship, M_e and M_m the masses of the Earth and Moon, respectively, G is the universal gravitational constant and d the distance between the mass centres. Non-dimensionalise this equation, using d as the distance scale and $\sqrt{d^3/(G(M_e + M_m))}$ as the time scale, to give the non-dimensional version of the equation (x and t now non-dimensional) as

$$\ddot{x} = \frac{\varepsilon}{(1-x)^2} - \frac{1-\varepsilon}{x^2}$$

where $\varepsilon = M_m/(M_e + M_m)$. We will construct an asymptotic solution, for $x \in (0, 1)$, as $\varepsilon \to 0^+$. (The actual values, for the Earth and Moon, give $\varepsilon \approx 0.012$, and a trajectory from surface to surface requires $x \in [0.017, 0.996]$, approximately.) Write down a first integral of the equation.

(a) Find the first two terms in an asymptotic expansion valid for $x = O(1)$, by seeking $t = t(x; \varepsilon)$ (cf. Q2.27), and use the data $\dot{x}(\Delta; \varepsilon) = \alpha$, $t \to 0$ as $x \to 0$, and write

$$\frac{1}{2}\alpha^2 = -k + \frac{1 - \varepsilon - \Delta + 2\varepsilon\Delta}{\Delta(1 - \Delta)} \quad (k = O(1), 0 < k < 1).$$

(Here, α is the non-dimensional initial speed away from the Earth, Δ is small ($\Delta \approx 0{\cdot}017$), and the condition on k ensures that the spaceship reaches the Moon, but not at such a high speed that it can escape to infinity.) Show that this expansion breaks down as $x \to 1$.

(b) Seek a scaling of the governing equation in the neighbourhood of $x = 1$ by writing $x = 1 - \delta X$, $t = T_0 + \delta T(X; \varepsilon)$ (which is consistent with the solution obtained in (a), where the first term, $T_0 =$ constant, provides the dominant contribution at $x = 1$). Find the first term in an asymptotic expansion of $T(X; \varepsilon)$, match to your solution from (a) and hence determine $T_0(k)$. (Be warned that $\ln \varepsilon$ terms appear in this problem.)

Q2.29 *Eigenvalues for a vibrating beam.* The (linearised) problem of an elastic beam clamped at each end is

$$\varepsilon^2 y'''' - y'' = \lambda^2 y, \quad 0 \le x \le 1,$$

for $\varepsilon \to 0^+$, with $y(0; \varepsilon) = y'(0; \varepsilon) = y(1; \varepsilon) = y'(1; \varepsilon) = 0$, where λ is the eigenvalue (which arises from the time-dependence), and $\varepsilon^2 \propto E$, *Young's modulus.* Find the first term in an asymptotic expansion of the eigenvalues. (This problem can be solved exactly, and then the exponents expanded for $\varepsilon \to 0^+$; this is an alternative that could be explored.)

Q2.30 *Heat transfer in 1D.* An equation which describes heat transfer in the presence of a one-dimensional, steady flow (Hanks, 1971) is

$$\varepsilon T'' + xT' - xT = 0, \quad 0 \le x \le 1, \quad \varepsilon > 0,$$

with temperature conditions $T(0; \varepsilon) = T_0$, $T(1; \varepsilon) = T_1$. Find the first two terms in an asymptotic expansion, valid for $x = O(1)$ as $\varepsilon \to 0^+$, and the leading term valid in the boundary layer, matching as necessary.

Q2.31 *Self-gravitating annulus.* A particular model for the study of planetary rings is represented by the equation

$$\frac{d}{dr}\left[r\frac{d\rho}{dr}\right] + \alpha r\rho = \frac{1}{r^2} - \frac{\beta}{r^3}, \quad 1 - \varepsilon \le r \le 1 + \varepsilon,$$

where $\alpha > 0$, $\beta > 1$ and $0 < \varepsilon < 1$ are constants, with the density, ρ, satisfying $\rho(1-\varepsilon;\varepsilon) = \rho(1+\varepsilon;\varepsilon) = 0$. (This example is based on the more general equation given in Christodoulou & Narayan, 1992.) For $\varepsilon \to 0^+$ (the narrow annulus approximation), introduce $r = 1 + \varepsilon R$ $(-1 \le R \le 1)$ and then write the density as $\rho = \varepsilon^2 P(R;\varepsilon)$; find the first three terms in an asymptotic expansion for P. On the basis of this information, deduce that your expansion would appear to be uniformly valid for $\forall R \in [-1, 1]$.

Q2.32 *An elastic displacement problem.* A simplified version of an equation which describes the displacement of a (weakly) nonlinear string, in the presence of forcing, which rests on an elastic bed, is

$$[y'(1-\varepsilon y')]' - y = x, \quad 0 \le x \le 1,$$

where ε is a constant, with $y(0;\varepsilon) = y(1;\varepsilon) = 0$. Find the first two terms in an asymptotic expansion, for $\varepsilon \to 0^+$, and use this evidence to deduce that this expansion would appear to be uniformly valid for $\forall x \in [0, 1]$.

Q2.33 *Laminar flow through a channel.* A model for laminar flow through a channel which has porous walls, through which suction occurs, can be reduced to

$$\varepsilon y''' - yy'' + (y')^2 = A(\varepsilon), \quad 0 \le x \le 1,$$

where $A(\varepsilon)$ is an arbitrary constant of integration, with $y(0;\varepsilon) = k$ $(0 < k < 1)$, $y'(0;\varepsilon) = 0$, $y(1;\varepsilon) = 1$, $y'(1;\varepsilon) = 0$. (This is taken from Proudman, 1960; see also Terrill & Shrestha, 1965, and McLeod in Segur, *et al.,* 1991; here, the stream function is proportional to the function $y(x;\varepsilon)$ and $\varepsilon \propto$ 1/(Reynolds' Number).) Assume that $A(0)$ exists and is non-zero, and then find the first term in an asymptotic expansion for $y(x;\varepsilon)$, and for $A(\varepsilon)$, valid for $x = O(1)$, and then the first two terms valid in the boundary layer (the first being simply the boundary value there).

Q2.34 *Slider bearing.* The pressure, p, within the fluid film of a slider bearing, based on Reynolds' equation, can be reduced to the equation

$$\varepsilon(h^3 pp')' = (hp)', \quad 0 \le x \le 1,$$

written in non-dimensional form; here, $\varepsilon > 0$ is a constant and $h(x)$ (> 0) is the given (smooth) film thickness, with $p(0;\varepsilon) = p(1;\varepsilon) = 1$ (and $h(0) \neq h(1)$). Find the first two terms in an asymptotic expansion, for $\varepsilon \to 0^+$, valid for $x = O(1)$, and then the first term only in the boundary layer, matching as necessary. (The first term in the boundary layer can be written only in implicit form, but this is sufficient to allow matching.)

Q2.35 *An enzyme reaction.* The concentration, $c(r;\varepsilon)$, of oxygen in an enzyme reaction can be modelled by the equation

$$\varepsilon\left(c'' + \frac{2}{r}c'\right) = \frac{c}{c+k}, \quad 0 \le r \le 1,$$

with $\varepsilon \to 0^+$, where k (> 0) is a constant. The boundary conditions specify the concentration on $r = 1$ and that the flux of oxygen must be zero at $r = 0$; these are expressed as

$$c(1;\varepsilon) = 1 \quad \text{and} \quad c'(r;\varepsilon) \to 0 \quad \text{as } r \to 0.$$

(a) Find the size of the boundary layer near $r = 1$ (in the form $r = 1 - \delta R$, for suitable δ) and hence show that $c(1 - \delta R; \varepsilon) \sim C_0(R)$ satisfies

$$\frac{1}{\sqrt{2}} \int_{C_0(R)}^{1} [c - k\ln(1 + c/k)]^{-1/2} \mathrm{d}c = R,$$

where we have assumed that $C_0(R) \to 0$ and $C_0'(R) \to 0$ as $R \to \infty$ (which is consistent with the equation).

(b) From the result in (a), deduce that $C_0(R)$ is *exponentially* small as $R \to \infty$; for $1 - r = \mathrm{O}(1)$, seek a solution (which is exponentially small) in terms of the scaled variable $\rho = r/\sqrt{\varepsilon}$, and show that $c(r;\varepsilon) \sim A(\varepsilon)s(\rho)$, where $s(\rho)$ satisfies

$$s'' + \frac{2}{\rho}s' = \frac{s}{k}.$$

Solve this equation, apply the relevant boundary condition, match and hence show that

$$c(r;\varepsilon) \sim \frac{A_0}{r} \mathrm{e}^{-1/\sqrt{\varepsilon k}} \sinh\left(r/\sqrt{\varepsilon k}\right)$$

where A_0 is a constant (independent of ε) which should be identified.

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