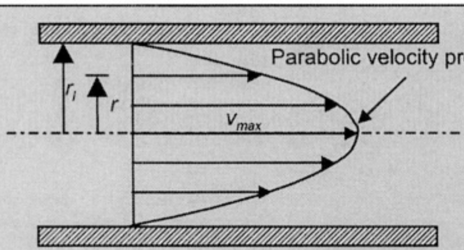
Chapter 2 LAW OF POISEUILLE



$v(r) = (\Delta P/l)\cdot(r_i^2 - r^2)/4\eta$

$v_{max} = (\Delta P/l)\cdot r_i^2/4\eta$

$v(r) = v_{max}\cdot(1 - r^2/r_i^2)$

$Q = (\Delta P/l)\cdot(\pi/8\eta)\cdot r_i^4$

$v_{mean} = Q/\pi r_i^2 = (\Delta P/l)\cdot r_i^2/8\eta$

$\tau = 4\eta Q/(\pi r_i^3)$

THE LAW OF POISEUILLE describes the relation between pressure drop, ΔP, and blood flow, Q, under steady flow conditions. This figure shows a tube with circular cross-section where the blood flows in a laminar fashion, i.e., each fluid layer stays at the same constant distance from the center. The velocity profile, v(r), is parabolic (top formula). The flow depends strongly on the radius of the tube (fourth power), the pressure drop over the tube length ($\Delta P/l$) and the viscosity of the blood (η). The wall shear stress, τ, acting on the intimal layer (endothelium) equals $4\eta\, Q/(\pi r_i^3)$.

Description

With laminar and steady flow through a uniform tube of radius r_i the velocity profile over the cross-section is a parabola. The formula that describes the velocity (v) as a function of the radius, r is:

$$v_r = \frac{\Delta P\cdot(r_i^2 - r^2)}{4\cdot\eta\cdot l}$$

ΔP is the pressure drop over the tube of length (l), and η is blood viscosity. At the axis ($r = 0$), velocity is maximal, v_{max}, while at the wall ($r = r_i$) the velocity is zero. Mean velocity is:

$$v_{mean} = \frac{\Delta P\cdot r_i^2}{8\cdot\eta\cdot l}$$

and is found at $r \approx 0.7\, r_i$.

Blood flow (Q) is mean velocity times the cross-sectional area of the tube, πr_i^2, giving:

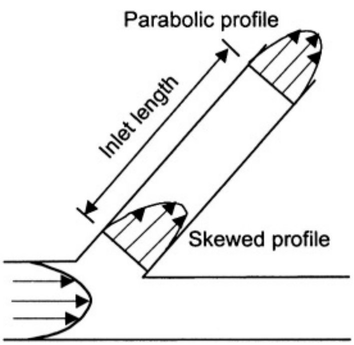
$$Q = \frac{\Delta P\cdot\pi\cdot r_i^4}{8\cdot\eta\cdot l}$$

This is Poiseuille's law relating the pressure difference, ΔP, and the steady flow, Q, through a uniform (constant radius) and stiff blood vessel. Hagen, in 1860, theoretically derived the law and therefore it is sometimes called the law of Hagen-Poiseuille. The law can be derived from very basic physics (Newton's law) or the general Navier-Stokes equations.

The major assumptions for Poiseuille's law to hold are:

- The tube is stiff, straight, and uniform

- Blood is Newtonian, i.e., viscosity is constant
- The flow is laminar and steady, not pulsatile, and the velocity at the wall is zero (no slip at the wall).



INLET LENGTH. Flow entering a side branch results in skewed profile. It takes a certain inlet length before the velocity develops into a parabolic profile again.

In curved vessels and distal to branching points the velocity profile is not parabolic and the blood flow profile needs some length of straight tube to develop, this length is called inlet length. The inlet length depends on the Reynolds number (Re, see Chapter 4) as:

$$l_{inlet}/D \approx 0.06\, Re$$

with D vessel diameter. For the aorta mean blood flow is about 6 l/min, and the diameter 3 cm, so that the mean velocity is ~ 15 cm/s. The Reynolds number is therefore ~ 1350. This means that l_{inlet}/D is ~ 80, and the inlet length ~240 cm, which is much longer than the length of the entire aorta. In the common iliac artery the Reynolds number is about 500 and diameter 0.6 cm giving an inlet length of ~18 cm. In other, more peripheral arteries the inlet length is much shorter but their length is shorter as well. Clearly, a parabolic flow profile is not even approximated in the arterial system. Nevertheless, the law of Poiseuille can be used as a concept relating pressure drop to flow.

A less detailed and thus more general form of Poiseuille's law is $Q = \Delta P/R$ with resistance R being:

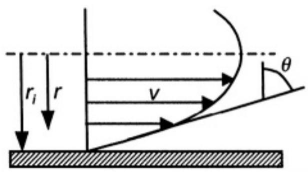
$$R = 8\eta \cdot l/\pi r_i^4$$

This law is used in analogy to Ohm's law of electricity, where resistance equals voltage drop/current. The analogy is that voltage difference is compared to pressure drop and current to volume flow. In hemodynamics we call also it Ohm's law. Thus:

$$\Delta P/Q = R$$

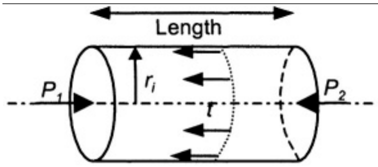
This means that resistance can be calculated from pressure and flow measurements.

Calculation of wall shear stress



THE SHEAR RATE at the wall of a blood vessel can be calculated from the 'rate of change of velocity' at the wall, as indicated by angle θ.

The wall shear rate can be calculated from the slope of the velocity profile near the wall (angle θ in the figure above), which relates to the velocity gradient, tan $\theta = \mathrm{d}v/\mathrm{d}r$, near the wall (see Chapter 1). The derivative of the velocity profile gives the shear rate $\gamma = (\Delta P/l) \cdot r/2\eta$. Shear stress is shear rate times viscosity $\tau = (\Delta P/l) \cdot r/2$. The shear rate at the vessel axis, $r = 0$, is zero, and at the wall, $r = r_i$, it is $\tau = (\Delta P/l) \cdot r_i/2$, so the blood cells encounter a range of shear stresses and shear rates over the vessel's cross-section.



SHEAR STRESS at the wall can also be calculated directly by the balance of pressure and frictional forces.

The shear stress at the wall can also be calculated from basic principles. For an arterial segment of length l, the force resulting from the pressure difference $(P_1 - P_2) = \Delta P$, times the cross-sectional area, πr_i^2, should equal the opposing force generated by friction. This frictional force on the wall equals the shear stress, τ, times the lateral surface, $2\pi r_i \cdot l$. Equating these forces gives $\Delta P \cdot \pi r_i^2 = \tau \cdot 2\pi r_i \cdot l$, and

$$\tau = (\Delta P/l) \cdot (r_i/2)$$

This formulation shows that with constant perfusion pressure an increase in viscosity does not affect wall shear stress.

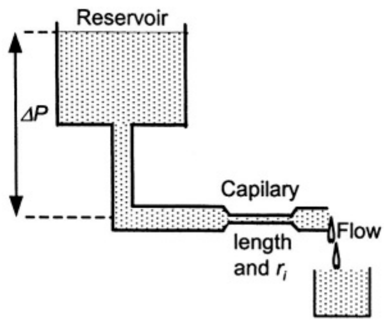
The wall shear stress may also be expressed as a function of volume flow using Poiseuille's law

$$\tau = 4\eta \cdot Q/\pi r_i^3$$

this is a more useful formula for estimating shear stress because flow and radius can be measured noninvasively using ultrasound or MRI, whereas pressure gradient cannot.

Example of the use of Poiseuille's law to obtain viscosity

A relatively simple way to obtain viscosity is to use a reservoir that empties through a capillary. Knowing the dimensions of the capillary and using Poiseuille's law viscosity can be calculated. Even simpler is the determination of viscosity relative to that of water. In that case only a beaker and stopwatch are required. The amounts of blood and water obtained for a chosen time are inversely proportional to their viscosities. The practical design based on this principle is the Ostwald viscometer.



A WIDE BORE RESERVOIR maintaining constant pressure, provides the blood flow through a capillary. The application of Poiseuille's law, or comparison with water, gives absolute or relative viscosity, respectively.

Murray's law

Murray' law (1926) was originally proposed by Hess in 1913 and assumes that the energy required for blood flow and the energy needed to maintain the vasculature is assumed minimal [1]. The first term equals pressure times flow and, using Poiseuille's law, this is $P \cdot Q = Q^2 \cdot 8 \cdot \eta l/\pi r_i^4$. The second term is proportional to vessel volume and thus equals $b \cdot \pi r_i^2 l$, with b a proportionality constant. The total energy, E_m, is:

$$E_m = Q^2 \cdot 8\eta \cdot l/\pi r_i^4 + b \cdot \pi r_i^2 l$$

The minimal value is found for $dE_m/dr = 0$ and this leads to:

$$Q = (\pi/4l) \cdot (b/\eta)^{0.5} \cdot r_i^3 = k \cdot r_i^3$$

For a bifurcation it holds that

$$Q_{mother} = Q_{daughter1} + Q_{daughter2}$$

and thus

$$r^3_{mother} = r^3_{daughter1} + r^3_{daughter2}$$

with two equal daughters it holds that:

$$r^3_{mother} = 2 \cdot r^3_{daughter}$$

and we find that

$$r_{daughter} = (½)^{1/3}\, r_{mother} \approx 0.79\, r_{mother}$$

The area of both daughters together is $2 \cdot 0.79^2 \approx 1.25$ the area of the mother vessel. This area ratio is close to the area ratio predicted by Womersley on the basis of the oscillatory flow theory, to obtain minimal reflection of waves at a bifurcation, namely between 1.15 and 1.33 [2]. Thus Murray's law suggests a minimal size of blood vessels and an optimum bifurcation [1].

Physiological and clinical relevance

The more general form of Poiseuille's law given above, i.e., $Q = \Delta P/R$ allows us to derive resistance, R, from mean pressure and mean flow measurement.

The wall shear stress, i.e., the shear force on the endothelial cells plays an important role in short term, second to minutes, and long term, weeks, months or years, effects. Short-term effects are vasomotor tone and flow mediated dilatation (FMD). Long-term effects are vascular remodeling, endothelial damage, changes in barrier function, and atherosclerosis.

It is still not possible to directly measure wall shear stress or shear rate *in vivo*. Shear rate is therefore derived from the velocity profile. Velocity profiles can be measured with MRI and Ultrasound Doppler. From the velocity profile the velocity gradient is often calculated. However, the calculations to obtain shear rate require extrapolation, because very near the wall velocity measurements are not possible. To calculate wall shear stress the blood viscosity near the wall has to be known as well, but viscosity close to the wall is not known because of plasma skimming. Plasma skimming refers to the relative absence of erythrocytes in the region near the wall. Also the diameter variation over the heartbeat is almost impossible to account for.

Wall shear stresses are ~ 10 - 20 dynes/cm^2, which is about 10,000 times less than the hoop stress (Chapter 9). Despite this enormous difference in magnitude, both stresses are equally important in the functional wall behavior in physiological and pathological conditions (see Chapters 27 and 28).

References

1. Weibel E. *Symmorphosis.* 2000, Cambridge MA, Harvard Univ Press.
2. Womersley JR. *The mathematical analysis of the arterial circulation in a state of oscillatory motion.* 1957, Wright Air Dev. Center, Tech Report WADC-TR-56-614.

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