Chapter 2

ELEMENTS OF CONVEX ANALYSIS, DUALITY THEORY

The a posteriori error estimates presented in this work are derived based on the duality theory of convex analysis. The first research monograph specifically devoted to the topic of convex analysis is [136], emphasizing the finite-dimensional case. Convex analysis and duality theory in general normed spaces, mostly infinite dimensional ones, are thoroughly discussed in the well-known reference [49]. Another comprehensive treatment of the topic is [159]. Duality theory has been also extended for nonconvex systems, see, e.g. [59, 60] where the mathematical theory is motived by duality in natural phenomena with particular emphasis on mechanics.

In this chapter, we review some basic notions and results on convex sets, convex functions and their properties as well as the duality theory. Detailed discussions and proofs of the stated results can be found in [49] or [159].

In the theory of convex analysis, it is convenient to consider functions that take on values on the extended real line $\overline{\mathbb{R}}$. Recall that a functional $f : V \to \overline{\mathbb{R}}$ is said to be proper if $f(v) > -\infty \ \forall\, v \in V$ and $f(u) < \infty$ for some $u \in V$.

2.1. CONVEX SETS AND CONVEX FUNCTIONS

Let V be a linear space.

DEFINITION 2.1 *A subset $K \subset V$ is said to be* convex *if it has the property*

$$u, v \in K \Longrightarrow (1-t)\, u + t\, v \in K \ \forall\, t \in [0,1].$$

We see that if K is convex and $u, v \in K$, then the line segment connecting u and v, i.e. the set $\{(1-t)\, u + t\, v : t \in [0,1]\}$, is contained in K. By an induction argument, for any $u_1, \dots, u_n \in K$ and any nonnegative numbers $t_1, \dots, t_n$ with $\sum_{i=1}^n t_i = 1$, we have $\sum_{i=1}^n t_i u_i \in K$. The expression $\sum_{i=1}^n t_i u_i$ with nonnegative numbers $t_1, \ldots, t_n$ satisfying $\sum_{i=1}^n t_i = 1$ is called a *convex combination* of the elements $u_1, \ldots, u_n$.

DEFINITION 2.2 *A function $f : V \to \overline{\mathbb{R}}$ is* convex *if*

$$f((1-t)\,u + t\,v) \le (1-t)\,f(u) + t\,f(v) \tag{2.1}$$

for any $u, v \in V$ and $t \in (0,1)$ for which the right hand side is meaningful, i.e., $f(u)$ and $f(v)$ are not simultaneously infinite with opposite signs.

DEFINITION 2.3 *Let K be a convex set in V and $f : K \to \mathbb{R}$. If*

$$\tilde{f}(v) = \begin{cases} f(v), & v \in K, \\ +\infty, & v \notin K \end{cases}$$

is convex, then we say f is convex on K. The function f is strictly convex *on K if the strict inequality in (2.1) holds for any $u, v \in K$, $u \ne v$ and $t \in (0,1)$.*

In the future, for a function f defined on a subset $K \subset V$, we identify it with its extension $\tilde{f}$ introduced in Definition 2.3. In other words, we will use the same symbol f for both the function defined on K and its extension by ∞ to the complement of K in the space V. Thus, we will say that a function is convex over a subset $K \subset V$ to mean that the extension of the function is convex in the space V.

By an induction argument, if f is convex over a convex set K, then we have

$$f\Big(\sum_{i=1}^n t_i u_i\Big) \le \sum_{i=1}^n t_i f(u_i)$$

for any $u_1, \ldots, u_n \in K$ and any nonnegative numbers $t_1, \ldots, t_n$ with

$$\sum_{i=1}^n t_i = 1.$$

The next result follows easily from the definition of a convex function.

PROPOSITION 2.4 *Let V be a linear space, Λ be an index set. Assume $f, g, f_\alpha\,(\alpha \in \Lambda) : V \to \overline{\mathbb{R}}$ are convex. Then the functions $f + g$, $t\,f$ $(t \in (0,\infty))$, $\sup\{f, g\}$ and $\sup_{\alpha\in\Lambda} f_\alpha$ are all convex. Here we let $f(v)+g(v) = \infty$ if $f(v) = -g(v) = \pm\infty$.*

In the study of convex functions, it is convenient to use the notions of the effective domain and the epigraph.

DEFINITION 2.5 *Given a function $f : V \to \overline{\mathbb{R}}$, we define its* effective domain

$$dom\,(f) = \{v \in V : f(v) < \infty\}$$

and its epigraph

$$epi\,(f) = \{(v, a) \in V \times \mathbb{R} : f(v) \le a\}.$$

It is easy to show that for a convex function the effective domain is a convex set in V and the epigraph is a convex set in $V \times \mathbb{R}$.

From now on, we assume V is a normed space.

DEFINITION 2.6 *A function* $f : V \to \overline{\mathbb{R}}$ *is said to be* lower semicontinuous *(l.s.c.) if for any sequence* $\{u_n\} \subset V$ *with* $u_n \to u$ *in* V,

$$f(u) \le \liminf_{n \to \infty} f(u_n).$$

There is a useful characterization of the lower semicontinuity that provides an alternative definition of l.s.c. in some references.

PROPOSITION 2.7 *The function* $f : V \to \overline{\mathbb{R}}$ *is l.s.c. iff for any* $r \in \mathbb{R}$ *the set* $\{v \in V : f(v) \le r\}$ *is closed.*

Later on, we will also need the notion of weak l.s.c.

DEFINITION 2.8 *A function* $f : V \to \overline{\mathbb{R}}$ *is said to be* weakly lower semicontinuous *(w.l.s.c.) if for any sequence* $\{u_n\} \subset V$ *with* $u_n \rightharpoonup u$ *in* V,

$$f(u) \le \liminf_{n \to \infty} f(u_n).$$

EXAMPLE 2.9 Let $K \subset V$. The indicator function of the set K is defined by

$$I_K(v) = \begin{cases} 0 & v \in K, \\ +\infty & v \in V \backslash K. \end{cases}$$

Then it can be verified that K is closed iff I_K is l.s.c., whereas K is a convex set iff I_K is a convex function. ■

For a set K in the normed space V, we use $\text{int}\, K = \text{int}(K) = \overset{\circ}{K}$ to denote its interior, i.e. the set of the points in K such that each point is contained in an open ball that in turn lies in K. Roughly speaking, $\text{int}\, K$ is the subset of K excluding the boundary points. Some of the boundary points may not belong to K, unless K is a closed set. We use $\overline{K}$ to denote the closure of K, i.e., the union of the set K and its boundary.

The following results are not difficult to prove.

PROPOSITION 2.10 *Let* $f : V \to \overline{\mathbb{R}}$. *Then*
(a) f *is convex iff* $epi\,(f)$ *is convex;*
(b) f *is l.s.c. iff* $epi\,(f)$ *is closed;*
(c) f *is continuous at* u *and* $f(u) \ne \pm\infty \Longrightarrow int\, epi\,(f) \ne \emptyset$;
(d) $f \not\equiv +\infty \Longrightarrow epi\,(f) \ne \emptyset$;
(e) f *is convex* $\Longrightarrow dom\,(f)$ *is convex.*

2.2. HAHN–BANACH THEOREM AND SEPARATION OF CONVEX SETS

The Hahn–Banach theorem and its corollaries are of central importance in functional analysis (cf. e.g. [48]). In this work, we only need Corollary 2.17 given at the end of the section. For completeness, we state some related results and show how they lead to a proof of Corollary 2.17.

DEFINITION 2.11 *A function $p : V \to \mathbb{R}$ is* sublinear *if*

$$p(t\,v) = t\,p(v) \quad \forall\, v \in V,\ t \geq 0,$$
$$p(u + v) \leq p(u) + p(v) \quad \forall\, u, v \in V.$$

We observe that $p : V \to \mathbb{R}$ is a seminorm if it is sublinear and $p(t\,v) = |t|\,p(v)$ for any $v \in V$ and any $t \in \mathbb{R}$. The analytic form of a general Hahn–Banach Theorem is the following.

THEOREM 2.12 *(Hahn–Banach Theorem) Let V be a real linear space, $K \subset V$ a subspace. Assume $f : K \to \mathbb{R}$ is linear and $f(v) \leq p(v)$ for any $v \in K$, with some sublinear functional $p : V \to \mathbb{R}$. Then f can be extended to a linear functional $f : V \to \mathbb{R}$ such that $f(v) \leq p(v)$ for any $v \in V$.*

Taking the function $p(\cdot)$ to be a constant multiple of the norm, we immediately get the usual form of the Hahn–Banach Theorem.

COROLLARY 2.13 *Let V be a real Banach space, $K \subset V$ be a subspace. Assume $f : K \to \mathbb{R}$ is a linear functional satisfying*

$$|f(v)| \leq c_0 \|v\| \quad \forall\, v \in K.$$

Then f can be extended to a continuous linear functional on V with

$$|f(v)| \leq c_0 \|v\| \quad \forall\, v \in V.$$

There is a related geometric form of the Hahn–Banach theorem on separation of convex sets, Proposition 2.15. For this purpose, we introduce the following definition.

DEFINITION 2.14 *Let V be a real normed space, $A, B \subset V$ be non-empty. The sets A and B are* separated *if there exist $\ell \in V^*$, $\ell \neq 0$ and $a \in \mathbb{R}$ such that*

$$\ell(u) \leq a \leq \ell(v) \quad \forall\, u \in A,\ v \in B.$$

The separation is strict *if the inequalities can be replaced by strict inequalities*

$$\ell(u) < a < \ell(v) \quad \forall\, u \in A,\ v \in B.$$

PROPOSITION 2.15 *(Separation of convex sets) Let V be a real normed space, $A, B \subset V$ be non-empty and convex.*
(a) *If $int(A) \cap B = \emptyset$ and $int(A) \neq \emptyset$, then A and B can be separated; furthermore, $f(u) < a \; \forall\, u \in int(A)$.*
(b) *If $A \cap B = \emptyset$ and either A and B are open or A is closed and B is compact, then A and B can be strictly separated.*

LEMMA 2.16 *Let V be a real normed space, $f : V \to \overline{\mathbb{R}}$ be convex and l.s.c. Suppose*

$$-\infty < a < f(u)$$

for some $u \in \overline{dom(f)}$ (hence it is possible $f(u) = \infty$). Then $\exists\,(u^*, \alpha) \in V^* \times \mathbb{R}$ such that

$$\langle u^*, u\rangle - a > \alpha > \langle u^*, v\rangle - f(v) \quad \forall\, v \in V,\ f(v) > -\infty.$$

In particular, if $f(u) \neq \pm\infty$, then

$$f(v) \geq a + \langle u^*, v - u\rangle \quad \forall\, v \in V,\ f(v) > -\infty.$$

Proof. Every $z^* \in (V \times \mathbb{R})^*$ has the form

$$\langle z^*, (v, b)\rangle = \langle w^*, v\rangle + a^* b \quad \forall\,(v, b) \in V \times \mathbb{R},$$

where $w^* \in V^*$, $a^* \in \mathbb{R}$. If $f \equiv \infty$, then we choose $u^* = 0$. Now assume $f \not\equiv \infty$. Then $\mathrm{epi}(f)$ is convex, closed and non-empty. We have $(u, a) \notin \mathrm{epi}(f)$, and the set $\{(u, a)\}$ is convex and compact. By Proposition 2.15 (b), the sets $\{(u, a)\}$ and $\mathrm{epi}(f)$ can be strictly separated in $V \times \mathbb{R}$. So $\exists\, z^* = (w^*, a^*) \in (V \times \mathbb{R})^*$ and $\beta \in \mathbb{R}$ such that

$$\langle w^*, u\rangle + a^* a > \beta > \langle w^*, v\rangle + a^* b \quad \forall\,(v, b) \in \mathrm{epi}(f).$$

Now for $v \in \mathrm{dom}(f)$ with $f(v) > -\infty$, $(v, f(v)) \in \mathrm{epi}(f)$. Hence

$$\langle w^*, u\rangle + a^* a > \beta > \langle w^*, v\rangle + a^* f(v) \quad \forall\, v \in \mathrm{dom}(f),\ f(v) > -\infty.$$

Suppose $a^* \geq 0$. Then since $u \in \overline{\mathrm{dom}(f)}$, there is a sequence $\{v_n\} \subset \mathrm{dom}(f)$ with $v_n \to u$ as $n \to \infty$. Then

$$\langle w^*, u\rangle + a^* a > \beta \geq \langle w^*, v\rangle + a^* f(u),$$

contradicting $a < f(u)$. Thus we must have $a^* < 0$ and therefore

$$\left\langle -\frac{w^*}{a^*}, u\right\rangle - a > \left\langle -\frac{w^*}{a^*}, v\right\rangle - f(v) \quad \forall\, v \in \mathrm{dom}(f),\ f(v) > -\infty.$$

Then the inequality is valid for any $v \in V$ with $f(v) > -\infty$. ■

A consequence of Lemma 2.16 is the following.

COROLLARY 2.17 *Let V be a normed space, $j : V \to \overline{\mathbb{R}}$ be proper, convex and l.s.c. Assume $u \in \mathrm{dom}\,(j)$. Then there exist $u^* \in V^*$ and $a \in \mathbb{R}$ such that*

$$j(v) \geq a + \langle u^*, v - u\rangle \quad \forall\, v \in V.$$

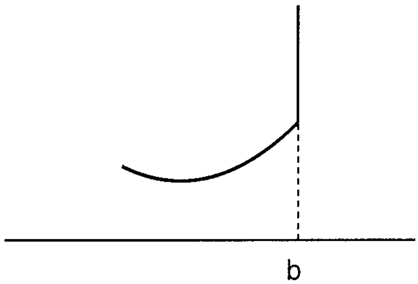

Figure 2.1. Continuity of a convex function

2.3. CONTINUITY AND DIFFERENTIABILITY

The purpose of this section is to list a few results on the continuity and differentiability of convex functions so that readers with little background on convex analysis can get familiar with basic properties of convex functions.

Continuity. The basic result concerning the continuity of convex functions is the following.

PROPOSITION 2.18 *Let V be a real normed space, $f : V \to \overline{\mathbb{R}}$ be convex.*
(a) *Assume $f(u) \in \mathbb{R}$. Then f is continuous at u iff f is bounded from above in a neighborhood of u.*
(b) *If f if finite on an open set $M \subset V$ and is continuous at some point of M, then f is continuous on M.*

Figure 2.1 shows a convex function f that is continuous in the interior of its effective domain, and is not continuous at b ($f(x) = \infty$ for $x \geq b$), a boundary point of the effective domain. Note that f is not bounded from above to the right of b.

The next two results can be deduced from Proposition 2.18.

COROLLARY 2.19 *Let $M \subset \mathbb{R}^d$ be an open convex set. Then every convex function $f : M \to \mathbb{R}$ is continuous.*

COROLLARY 2.20 *Let V be a real Banach space, $M \subset V$ be closed and convex. Let $f : M \to \mathbb{R}$ be convex and l.s.c. Then f if continuous on* $\text{int}(M)$.

Subdifferential. The notion of subdifferential is useful in describing various mechanical laws arising in contact problems, plasticity, etc. Although in later chapters, we do not explicitly use the notion of subdifferential in deriving a posteriori error estimates, it is an important concept in convex analysis. Now we introduce the definition of subdifferential and subgradient.

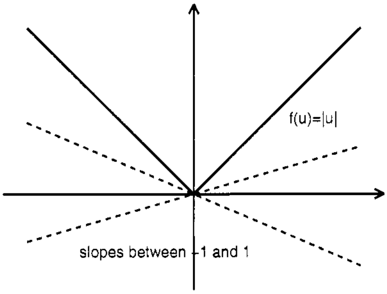

Figure 2.2. Subdifferential of the absolute value function

DEFINITION 2.21 *Let V be a real normed space with the dual V^*, and $f : V \to \overline{\mathbb{R}}$. Let $u \in V$ be such that $f(u) \neq \pm\infty$. Then the* subdifferential *of f at u is defined to be the set*

$$\partial f(u) = \{u^* \in V^* : f(v) \geq f(u) + \langle u^*, v - u\rangle \quad \forall\, v \in V\}.$$

Any $u^* \in \partial f(u)$ is called a subgradient *of f at u.*

We see that if $\partial f(u) \neq \emptyset$, then $f(v) > -\infty$ for any $v \in V$.

EXAMPLE 2.22 For a real-valued real-variable function $f : \mathbb{R} \to \mathbb{R}$, its subdifferential at $u \in \mathbb{R}$ is the set of the slopes of straight lines passing through the point $(u, f(u))$ and lying below the curve of f. For example, the absolute value function $f(u) = |u|$ is not differentiable at $u = 0$, but is subdifferentiable there, and $\partial f(0) = [-1, 1]$ (Figure 2.2).

On the other hand, differentiable functions may not be subdifferentiable. For instance, the smooth function $f(u) = u^3$ is not subdifferentiable at $u = 0$. The notion of the subdifferential is most suitable for convex functions. ■

EXAMPLE 2.23 (Support functional) Let V be a real normed space, $K \subset V$ be a convex set. Consider the subdifferential of the indicator function

$$I_K(v) = \begin{cases} 0 & \text{if } v \in K, \\ +\infty & \text{if } v \notin K. \end{cases}$$

If $u \notin K$, then $\partial I_K(u) = \emptyset$. Assume $u \in K$. Then $u^* \in \partial I_K(u)$ iff

$$I_K(v) \geq \langle u^*, v - u\rangle \quad \forall\, v \in V,$$

i.e.,

$$\langle u^*, v - u\rangle \leq 0 \quad \forall\, v \in K.$$

Thus we have the characterization

$$\partial I_K(u) = \{u^* \in V^* : \langle u^*, v - u\rangle \leq 0 \;\, \forall\, v \in K\}.$$

Any subgradient $u^* \in \partial I_K(u)$ is called a support functional to K at u. We always have $0 \in \partial I_K(u)$ for $u \in K$. It is easily seen that if $u \in \text{int}(K)$, then $\partial I_K(u) = \{0\}$. For a boundary point $u \in \partial K$ and the case $\text{int}(K) \neq \emptyset$, by separating u and $\text{int}(K)$, we can show the existence of a nonzero subgradient $u^* \in \partial I_K(u)$. If K is a subspace, then

$$\partial I_K(u) = \{u^* \in V^* : \langle u^*, v\rangle = 0 \;\; \forall\, v \in K\}$$

which can be viewed as the orthogonal complement of K. ■

The following important result plays a central role in the duality theory.

THEOREM 2.24 *Assume V is a reflexive Banach space, $f : (-\infty, \infty]$ is convex and l.s.c. Then $v^* \in \partial f(v)$ iff $v \in \partial f^*(v^*)$.*

As is commented in Example 2.22, the notion of the subdifferential is mainly applied to convex functions. This is supported by the next result on the existence of subgradients.

THEOREM 2.25 *Let V be a real normed space, $f : V \to \overline{\mathbb{R}}$ be convex.*
(a) *For any $v \in V$, $\partial f(v)$ is convex and weak* closed.*
(b) *If f is finite and continuous at v, then $\partial f(v) \neq \emptyset$.*

Ordinary differentiation rules in calculus carry over to subdifferentials, either straightforwardly or with some additional assumptions. For example, it is easy to verify the following relations from the definition of the subdifferential.

$$\partial(\lambda f(u)) = \lambda\, \partial f(u) \quad \forall\, \lambda > 0,$$
$$\partial(f_1 + f_2)(u) \supset \partial(f_1)(u) + \partial(f_2)(u).$$

A natural question is when the equality holds for the summation rule.

PROPOSITION 2.26 *Let V be a real normed space, $f_i : V \to (-\infty, \infty]$ be convex for $i = 1, \dots, n$. Assume there is a $u_0 \in V$ such that $f_i(u_0) \in \mathbb{R}$, $1 \le i \le n$, and f_i, $1 \le i \le n-1$, are continuous at u_0. Then*

$$\partial(f_1 + \cdots + f_n)(v) = \partial(f_1)(v) + \cdots + \partial(f_n)(v) \quad \forall\, v \in V.$$

A proof of this result and that of the next Chain rule can be found in [49].

PROPOSITION 2.27 *Let V and W be two real normed spaces, $L : V \to W$ be linear and continuous, $f : W \to \mathbb{R}$ be convex and l.s.c. If f is finite and continuous at some point, then*

$$\partial(f \circ L)(v) = L^* \partial f(Lv) \quad \forall\, v \in V.$$

Relationship between subgradient and Gâteaux derivative. First, we recall the definitions of the directional derivative and Gâteaux derivative.

DEFINITION 2.28 *Let $f : V \to \overline{\mathbb{R}}$ and $f(u) \in \mathbb{R}$. For a $v \in V$, if*

$$\lim_{t \to 0+} \frac{f(u + t\,v) - f(u)}{t}$$

exists, we call it the directional derivative *of f at u in the direction v, and denote it by $f'(u; v)$. If there exists $u^* \in V^*$ such that*

$$\lim_{t \to 0} \frac{f(u + t\,v) - f(u)}{t} = \langle u^*, v \rangle \quad \forall\, v \in V,$$

then f is said to be Gâteaux differentiable *at u. The element u^* is called the Gâteaux derivative of f at u and is denoted by $f'(u)$.*

Higher order derivatives are defined recursively. For instance, the second-order Gâteaux derivative is defined to be the Gâteaux derivative of the Gâteaux derivative.

PROPOSITION 2.29 *Let V be a real normed space, $f : V \to \overline{\mathbb{R}}$ be convex. Assume $f(u) \in \mathbb{R}$.*
(a) *If $f'(u)$ exists as a Gâteaux derivative, then $\partial f(u) = \{f'(u)\}$.*
(b) *If f is continuous at u and $\partial f(u)$ contains exactly one element, then $f'(u)$ exists as a Gâteaux derivative.*

Proof. For any $v \in V$, we define a function $\phi(t) = f(u + t\,(v - u))$, $t \in \mathbb{R}$. Then ϕ is a convex function of the real variable t. By the Mean Value Theorem,

$$\phi(1) - \phi(0) = \phi'(\theta) \quad \text{for some } \theta \in (0, 1).$$

Since ϕ is convex, $\phi'(\theta) \ge \phi'(0)$. Then

$$\phi(1) - \phi(0) \ge \phi'(0),$$

i.e.,

$$f(v) - f(u) \ge \langle f'(u), v - u \rangle \quad \forall\, v \in V.$$

Let $u^* \in \partial f(u)$. Then

$$f(v) - f(u) \ge \langle u^*, v - u \rangle \quad \forall\, v \in V.$$

Let $v = u + t\,h$, $h \in V$, and let $t \to 0+$ to obtain

$$\langle f'(u) - u^*, h \rangle \ge 0 \quad \forall\, h \in V.$$

Therefore, $f'(u) = u^*$. This completes a proof of part (a).

A proof of part (b) can be found in [49]. ■

Characterization of convex functions. We can use the Gâteaux derivative to characterize the convexity of a function. Let V be a normed space and $f : V \to \mathbb{R}$ be Gâteaux differentiable. Then the following three statements are equivalent.
(a) f is convex.
(b) $f(v) \geq f(u) + \langle f'(u), v - u\rangle \ \forall u, v \in V$.
(c) $\langle f'(v) - f'(u), v - u\rangle \geq 0 \ \forall u, v \in V$.

2.4. CONVEX OPTIMIZATION

Given a space V, its subset K, and a functional $f : K \to \mathbb{R}$, we consider the problem

$$\inf_{v\in K} f(v). \tag{2.2}$$

When K is unbounded, we say the function f is *coercive* on K if

$$f(v) \to \infty \quad \text{as } \|v\| \to \infty,\ v \in K.$$

We have a standard general result on the existence of a minimizer to the problem (2.2).

THEOREM 2.30 *Assume V is a reflexive Banach space, $K \subset V$ convex and closed, and $f : K \to \mathbb{R}$ is convex and l.s.c. If either*
(a) *K is bounded*
or
(b) *f is coercive on K,*
then the minimization problem (2.2) has a solution. Moreover, if f is strictly convex on K, then a solution of the minimization problem (2.2) is unique.

This theorem will be applied later to show the existence and uniqueness of weak solutions to some nonlinear boundary value problems that are equivalent certain convex minimization problems.

From the definition of subdifferential, immediately we get an extremal principle.

PROPOSITION 2.31 *Let V be a real normed space, $f : V \to \overline{\mathbb{R}}$ be proper. Then u is a solution of $\inf_{v\in V} f(v)$ iff $0 \in \partial f(u)$.*

THEOREM 2.32 *Suppose V is a real normed space, $K \subset V$ is a non-empty convex set. Assume $f : K \to \mathbb{R}$ is convex. Then for $u \in K$ to be a solution of the problem*

$$\inf_{v\in K} f(v)$$

a necessary and sufficient condition is that $u \in V$ is a solution of the unconstrained optimization problem

$$\inf_{v \in V} \left[f(v) + I_K(v) \right],$$

or $u \in V$ satisfies the relation

$$0 \in \partial f(u) + \partial I_K(u),$$

or

$$\exists\, u^* \in V^* \text{ with } \langle u^*, v - u \rangle \geq 0 \ \ \forall\, v \in K, \text{ such that}$$
$$f(v) \geq f(u) + \langle u^*, v - u \rangle \quad \forall\, v \in V.$$

The solution set is convex. If f is l.s.c. and K is closed, then the solution set is closed. Every local minimum of f is also a global minimum. A minimizer of f is unique if f is strictly convex.

2.5. CONJUGATE FUNCTIONALS

The idea of the duality theory can be described as follows: Let f be a given function on a normed space V. For a minimization problem

$$\inf_{v \in V} f(v), \tag{2.3}$$

we look for a maximization problem

$$\sup_{q \in Q} g(q) \tag{2.4}$$

such that

$$\inf_{v \in V} f(v) = \sup_{q \in Q} g(q).$$

The problem (2.3) is called the *primal problem*, and (2.4) is called the *dual problem*. Then we have the following two-sided bounds for the optimal value:

$$g(q) \leq \inf_V f \leq f(v) \quad \forall\, v \in V,\ q \in Q.$$

This is the basis for deriving most of the a posteriori error estimates in this book.

The space Q and the function g in the dual problem (2.4) are to be constructed from the primal problem (2.3). In particular, the construction of g is related to the concept of conjugate functionals.

DEFINITION 2.33 *Assume V is a normed space, and let $f : V \to \overline{\mathbb{R}}$. The conjugate functional $f^* : V^* \to \overline{\mathbb{R}}$ is defined by the formula*

$$f^*(v^*) = \sup_{v \in V} [\langle v^*, v \rangle - f(v)] \quad \forall\, v^* \in V^*.$$

Using the effective domain of the functional, we can also write

$$f^*(v^*) = \sup_{v \in \mathrm{dom}(f)} \left[\langle v^*, v \rangle - f(v) \right] \quad \forall\, v^* \in V^*.$$

It follows from the definition that the conjugate functional f^* is convex and l.s.c. on V^*.

We have the following generalized Young inequality:

$$\begin{aligned} &f^*(v^*) + f(v) \geq \langle v^*, v \rangle, \\ &f^*(v^*) + f(v) = \langle v^*, v \rangle \quad \text{iff} \quad v^* \in \partial f(v) \end{aligned}$$

for all $v \in V$, $v^* \in V^*$ as long as the expression $f^*(v^*) + f(v)$ is meaningful, i.e., not of the form $\infty - \infty$. This inequality is a generalization of the usual Young inequality:

$$\frac{|v|^p}{p} + \frac{|v^*|^{p^*}}{p^*} \geq v^* v \quad \forall\, v, v^* \in \mathbb{R},$$

where $p > 1$ and p^* is the conjugate exponent, defined through the equality $1/p^* + 1/p = 1$.

We will frequently need to calculate the conjugate functional for a functional defined by an integral of the form

$$G(\boldsymbol{q}) = \int_\Omega g\left(\boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x})\right)\, dx.$$

Before stating a theorem on how to calculate its conjugate function, we introduce the following notion.

DEFINITION 2.34 *Let Ω be an open set of $\mathbb{R}^d$, $g : \Omega \times \mathbb{R}^l \to \mathbb{R}$. We say g is a* Carathéodory function *if*
(a) $\forall\, \boldsymbol{\xi} \in \mathbb{R}^l$, $\boldsymbol{x} \mapsto g(\boldsymbol{x}, \boldsymbol{\xi})$ *is a measurable function;*
(b) *for a.e.* $\boldsymbol{x} \in \Omega$, $\boldsymbol{\xi} \mapsto g(\boldsymbol{x}, \boldsymbol{\xi})$ *is a continuous function.*

Let there be given $m_i \in (1, \infty)$, $i = 1, \ldots, l$. We have the following theorem which will be applied repeatedly in calculating conjugate functionals.

THEOREM 2.35 *Assume $g : \Omega \times \mathbb{R}^l \to \mathbb{R}$ is a Carathéodory function. For any $\boldsymbol{q} \in Q = \Pi_{i=1}^l L^{m_i}(\Omega)$, define*

$$G(\boldsymbol{q}) = \int_\Omega g\left(\boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x})\right) dx.$$

Then for the conjugate function of G, we have the formula

$$G^*(\boldsymbol{q}^*) = \int_\Omega g^*\left(\boldsymbol{x}, \boldsymbol{q}^*(\boldsymbol{x})\right) dx \quad \forall\, \boldsymbol{q}^* \in V^*,$$

where

$$g^*(\boldsymbol{x}, \boldsymbol{y}) = \sup_{\boldsymbol{\xi} \in \mathbb{R}^l} [\, \boldsymbol{y} \cdot \boldsymbol{\xi} - g(\boldsymbol{x}, \boldsymbol{\xi}) \,] .$$

EXAMPLE 2.36 Let Ω be a domain in $\mathbb{R}^d$, $Q = \left(L^2(\Omega)\right)^d$. We equate the dual space Q^* with Q. Let us compute the conjugate of the functional

$$G(\boldsymbol{q}) = \int_\Omega \frac{1}{2} \, |\boldsymbol{q}(\boldsymbol{x})|^2 dx, \quad \boldsymbol{q} \in Q.$$

By definition,

$$\begin{aligned} G^*(\boldsymbol{q}^*) &= \sup_{\boldsymbol{q} \in Q} [\langle \boldsymbol{q}^*, \boldsymbol{q} \rangle - G(\boldsymbol{q})] \\ &= \sup_{\boldsymbol{q} \in Q} \int_\Omega \left(\boldsymbol{q}^* \cdot \boldsymbol{q} - \frac{1}{2} \, |\boldsymbol{q}|^2 \right) dx. \end{aligned}$$

Applying Theorem 2.35, we have

$$\begin{aligned} G^*(\boldsymbol{q}^*) &= \int_\Omega \sup_{\boldsymbol{\xi} \in \mathbb{R}^d} \left(\boldsymbol{q}^* \cdot \boldsymbol{\xi} - \frac{1}{2} \, |\boldsymbol{\xi}|^2 \right) dx \\ &= \int_\Omega \sup_{t \geq 0} \left(t \, |\boldsymbol{q}^*| - \frac{1}{2} \, t^2 \right) dx \\ &= \int_\Omega \frac{1}{2} \, |\boldsymbol{q}^*|^2 dx. \end{aligned}$$

In later chapters, we will follow the above procedure to compute conjugate functionals of similar kind, and we will not always state explicitly the application of Theorem 2.35. ■

2.6. DUALITY THEORY

We now introduce some basic results in the duality theory; detailed discussion and proofs of these results can be found in [49].

Let V and Q be two normed spaces, V^* and Q^* denote their dual spaces. The duality pairings in both V, V^* and Q, Q^* will be denoted by $\langle \cdot, \cdot \rangle$. Assume there exists a linear continuous operator $\Lambda \in \mathcal{L}(V, Q)$. The transpose $\Lambda^* \in \mathcal{L}(Q^*, V^*)$ of the operator Λ is defined through the relation

$$\langle \Lambda^* q^*, v \rangle = \langle q^*, \Lambda v \rangle \quad \forall \, v \in V, \, q^* \in Q^*.$$

Let J be a function mapping $V \times Q$ into $\overline{\mathbb{R}}$. We consider the minimization problem (the primal problem)

$$\inf_{v \in V} J(v, \Lambda v). \tag{2.5}$$

Define its dual problem by

$$\sup_{q^* \in Q^*} \left[-J^*(\Lambda^* q^*, -q^*) \right], \tag{2.6}$$

where $J^* : V^* \times Q^* \to \overline{\mathbb{R}}$ is the conjugate function of J:

$$J^*(v^*, q^*) = \sup_{\substack{v \in V \\ q \in Q}} \left[\langle v^*, v \rangle + \langle q^*, q \rangle - J(v, q) \right], \quad v^* \in V^*,\ q^* \in Q^*. \tag{2.7}$$

For the relation between problems (2.5) and (2.6), we have the following duality theorem.

THEOREM 2.37 *Assume the following conditions:*

(1) *V is a reflexive Banach space and Q is a normed space; $\Lambda \in \mathcal{L}(V, Q)$.*

(2) *$J : V \times Q \to \overline{\mathbb{R}}$ is proper, lower semi-continuous and convex.*

(3) *There exists $u_0 \in V$ such that $J(u_0, \Lambda u_0) < \infty$ and the mapping $q \mapsto J(u_0, q)$ from Q to $\overline{\mathbb{R}}$ is continuous at Λu_0.*

(4) *$J(v, \Lambda v) \to +\infty$ as $\|v\| \to \infty$, $v \in V$.*

Then the problem (2.5) has a solution $u \in V$, the problem (2.6) has a solution $p^* \in Q^*$, and

$$J(u, \Lambda u) = -J^*(\Lambda^* p^*, -p^*). \tag{2.8}$$

Furthermore, if $J(v, \Lambda v)$ is strictly convex in its effective domain, then a solution u of the problem (2.5) is unique.

It is possible to weaken the assumptions of Theorem 2.37, then a weaker conclusion holds.

THEOREM 2.38 *Assume:*

(1) *V and Q are normed spaces; $\Lambda \in \mathcal{L}(V, Q)$.*

(2) *$J : V \times Q \to \overline{\mathbb{R}}$ is convex.*

(3) *There exists $u_0 \in V$ such that $J(u_0, \Lambda u_0) < \infty$ and the mapping $q \mapsto J(u_0, q)$ from Q to $\overline{\mathbb{R}}$ is continuous at Λu_0.*

(4) *$\inf_{v \in V} J(v, \Lambda v)$ is finite.*

Then the problem (2.6) has a solution $p^* \in Q^*$ and

$$\inf_{v \in V} J(v, \Lambda v) = -J^*(\Lambda^* p^*, -p^*). \tag{2.9}$$

Furthermore, if $J(v, \Lambda v)$ is strictly convex in its effective domain, then a solution u (if it exists) of the problem (2.5) is unique.

This theorem is of special interest where the primal minimization problem does not have a solution; one can study the primal problem through the dual problem. The two problems are connected by the equality (2.9) and note that the dual problem does have a solution. In the rest of this work, though, we do not need Theorem 2.38.

We will often encounter the situation where the function J is of a separated form, i.e.,

$$J(v,q) = F(v) + G(q), \quad v \in V,\ q \in Q. \tag{2.10}$$

It is then usually more convenient to compute its conjugate as follows:

$$J^*(v^*, q^*) = F^*(v^*) + G^*(q^*),$$

where F^* and G^* are the conjugate functions of F and G, respectively. This follows from the definition of the conjugate functional. Specializing Theorem 2.37 to this case, we obtain the next result.

THEOREM 2.39 *Assume:*

(1) *V is a reflexive Banach space and Q is a normed space; $\Lambda \in \mathcal{L}(V, Q)$.*

(2) *$F : V \to \overline{\mathbb{R}}$, $G : Q \to \overline{\mathbb{R}}$ are proper, lower semi-continuous, convex functions.*

(3) *There exists $u_0 \in V$ such that $F(u_0) < \infty$, $G(\Lambda u_0) < \infty$ and the mapping $q \mapsto G(q)$ is continuous at Λu_0.*

(4) *$F(v) + G(\Lambda v) \to +\infty$ as $\|v\| \to \infty$, $v \in V$.*

Denote $J(v,q) = F(v) + G(q)$, then $J^*(v^*, q^*) = F^*(v^*) + G^*(q^*)$. There is a solution $u \in V$ to the problem (2.5), a solution $p^* \in Q^*$ to the problem (2.6), and (2.8) holds. Moreover, if $J(v, \Lambda v)$ is strictly convex on its effective domain, then a solution u of the problem (2.5) is unique.

2.7. APPLICATIONS OF DUALITY THEORY IN A POSTERIORI ERROR ANALYSIS

Let $u \in V$ be a solution of the minimization problem (2.5). For any $v \in V$, we define the energy difference

$$ED(u,v) = J(v, \Lambda v) - J(u, \Lambda u). \tag{2.11}$$

Let $v \in V$ be any element with $J(v, \Lambda v) < \infty$. If the directional derivative $J'((u, \Lambda u); (v-u, \Lambda v - \Lambda u))$ exists, then we further define the quantity

$$D(u,v) = J(v, \Lambda v) - J(u, \Lambda u) - J'((u, \Lambda u); (v-u, \Lambda v - \Lambda u)). \tag{2.12}$$

THEOREM 2.40 *We make the same assumptions as in Theorem 2.37. Then*

$$ED(u,v) \le J(v, \Lambda v) + J^*(\Lambda^* q^*, -q^*) \quad \forall\, v \in V,\ q^* \in Q^*. \tag{2.13}$$

Assume the directional derivative $J'((u, \Lambda u); (v-u, \Lambda v - \Lambda u))$ exists, then

$$D(u,v) \le ED(u,v) \tag{2.14}$$

and

$$D(u,v) \le J(v,\Lambda v) + J^*(\Lambda^* q^*, -q^*) \quad \forall\, q^* \in Q^*. \tag{2.15}$$

Proof. The inequality (2.13) follows from the definition (2.11), the equality (2.8), and the definition of the dual problem (2.6): with p^* a solution of the dual problem,

$$\begin{aligned} ED(u,v) &= J(v,\Lambda v) + J^*(\Lambda^* p^*, -p^*) \\ &\le J(v,\Lambda v) + J^*(\Lambda^* q^*, -q^*) \end{aligned}$$

for any $q^* \in Q^*$.

Now assume the directional derivative $J'((u,\Lambda u);(v-u,\Lambda v - \Lambda u))$ exists. What remains to be proved is the inequality (2.14). Since u is a solution of the minimization problem (2.5), for any $v \in V$ we have the inequality

$$J\big(u + t\,(v-u), \Lambda u + t\,(\Lambda v - \Lambda u)\big) \ge J(u,\Lambda u) \quad \forall\, t \in (0,1).$$

Thus,

$$\frac{1}{t}\,\big[J\big(u + t\,(v-u), \Lambda u + t\,(\Lambda v - \Lambda u)\big) - J(u,\Lambda u)\big] \ge 0 \quad \forall\, t \in (0,1).$$

Taking the limit $t \to 0+$, we obtain

$$J'((u,\Lambda u);(v-u,\Lambda v - \Lambda u)) \ge 0.$$

Hence, (2.14) holds. ■

In the case $J(\cdot,\Lambda\cdot)$ is Gâteaux-differentiable at u, we can replace (2.12) by

$$D(u,v) = J(v,\Lambda v) - J(u,\Lambda u) - \langle J'(u,\Lambda u), (v-u,\Lambda v - \Lambda u)\rangle. \tag{2.16}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $(V \times Q)^*$ and $V \times Q$.

In most of the applications of Theorem 2.40 later in this work, $J(v,q)$ is of the separated form (2.10), $F(v)$ is linear over its effective domain $\mathrm{dom}\,(F) \subset V$, and $G : Q \to \mathbb{R}$ is real-valued Gâteaux-differentiable over Q. Obviously,

$$\mathrm{dom}\,(F) = \mathrm{dom}\, J(\cdot,\Lambda\cdot).$$

Suppose $v \in \mathrm{dom}\,(F)$. Since $F(\cdot)$ is linear over $\mathrm{dom}\,(F)$, it is easy to see that

$$D(u,v) = G(\Lambda v) - G(\Lambda u) - \langle G'(\Lambda u), \Lambda v - \Lambda u\rangle, \tag{2.17}$$

where G' denotes the Gâteaux derivative, and $\langle\cdot,\cdot\rangle$ is the duality pairing between Q^* and Q.

We now consider further the quantity $D(u,v)$, starting with the expression (2.17). For this purpose, we recall two elementary formulas from Taylor's theorem. Consider a real valued function $f(t)$, continuously differentiable for $t \in [0, 1]$. Then

$$f(1) - f(0) = \int_0^1 f'(t)\, dt. \tag{2.18}$$

If $f(t)$ is twice continuously differentiable, then we can apply the above formula to $f'(t)$:

$$\begin{aligned}\int_0^1 f'(t)\, dt &= \int_0^1 \left[f'(0) + \int_0^t f''(s)\, ds \right] dt \\ &= f'(0) + \int_0^1 \int_0^t f''(s)\, ds\, dt \\ &= f'(0) + \int_0^1 (1-s)\, f''(s)\, ds\end{aligned}$$

and get

$$f(1) - f(0) - f'(0) = \int_0^1 (1-t)\, f''(t)\, dt. \tag{2.19}$$

Assume $G(q)$ is continuously or twice continuously Gâteaux differentiable. For $p, q \in Q$, we apply (2.18) and (2.19) to the real variable function $G(p+t\,(q-p))$ to get

$$G(q) - G(p) = \int_0^1 \langle G'(p + t\,(q-p)), q - p \rangle\, dt$$

and

$$\begin{aligned}&G(q) - G(p) - \langle G'(p), q - p\rangle \\ &\quad = \int_0^1 (1-t)\, \langle G''(p + t\,(q-p))(q-p), q-p \rangle\, dt,\end{aligned}$$

respectively. Here, $G''(p + t\,(q-p))(q-p)$ is a mapping from Q to $\mathbb{R}$, since the second order Gâteaux derivative $G''(p + t\,(q-p))$ is a mapping from Q to Q^*. Then we have the formulas

$$D(u, v) = \int_0^1 \langle G'(\Lambda u + t\,(\Lambda v - \Lambda u)) - G'(\Lambda u), \Lambda v - \Lambda u \rangle\, dt, \tag{2.20}$$

$$D(u, v) = \int_0^1 (1-t)\, \langle G''(\Lambda u + t\,(\Lambda v - \Lambda u))(\Lambda v - \Lambda u), \Lambda v - \Lambda u \rangle\, dt, \tag{2.21}$$

for any $u, v \in \operatorname{dom} J(\cdot, \Lambda\cdot)$.

We will mainly use the formula (2.21) later. In many situations, G is *strongly convex* in Q: for some constant $\alpha > 0$,

$$\langle G''(p)\, q, q \rangle \geq \alpha\, \|q\|_Q^2 \quad \forall p, q \in Q. \tag{2.22}$$

In this case, from (2.21), we can conclude that

$$D(u,v) \geq \frac{\alpha}{2}\,\|\Lambda v - \Lambda u\|_Q^2 \quad \forall\, u, v \in \operatorname{dom} J(\cdot, \Lambda\cdot). \tag{2.23}$$

EXAMPLE 2.41 We consider an example which will be useful for later chapters. Let Ω be a domain in $\mathbb{R}^d$. Suppose $A(\boldsymbol{x}) = (a_{ij}(\boldsymbol{x}))_{d\times d}$ is symmetric, bounded and uniformly positive definite:

$$a_{ij} = a_{ji} \in L^\infty(\Omega), \quad 1 \leq i, j \leq d,$$
$$a_{ij}(\boldsymbol{x})\xi_i\xi_j \geq \alpha\,|\boldsymbol{\xi}|^2 \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega,$$

for some constant $\alpha > 0$. Suppose Λ is a linear continuous operator from $H^1(\Omega)$ to $Q = \left(L^2(\Omega)\right)^d$. In such a situation, $\Lambda = \nabla$ will be the gradient operator in later applications. Define the functional

$$\begin{aligned} G(\boldsymbol{q}) &= \int_\Omega \frac{1}{2}\, a_{ij}(\boldsymbol{x})\, q_i(\boldsymbol{x})\, q_j(\boldsymbol{x})\, dx \\ &= \int_\Omega \frac{1}{2}\, \boldsymbol{q}(\boldsymbol{x})^T A(\boldsymbol{x})\, \boldsymbol{q}(\boldsymbol{x})\, dx \end{aligned}$$

for $\boldsymbol{q} \in \left(L^2(\Omega)\right)^d$. In the following, we will usually omit the variable $\boldsymbol{x}$ in integrands. Then for any $\boldsymbol{p}, \boldsymbol{q} \in \left(L^2(\Omega)\right)^d$,

$$\begin{aligned} \langle G'(\boldsymbol{p}), \boldsymbol{q}\rangle &= \int_\Omega \boldsymbol{q}^T A\,\boldsymbol{p}\, dx, \\ \langle G''(\boldsymbol{p})\,\boldsymbol{q}, \boldsymbol{q}\rangle &= \int_\Omega \boldsymbol{q}^T A\,\boldsymbol{q}\, dx. \end{aligned}$$

By (2.21), we have

$$D(u,v) = \frac{1}{2}\int_\Omega (\Lambda v - \Lambda u)^T A\,(\Lambda v - \Lambda u)\, dx.$$

Since the matrix $A(\boldsymbol{x})$ is uniformly positive definite in Ω, we have

$$\langle G''(\boldsymbol{p})\,\boldsymbol{q}, \boldsymbol{q}\rangle \geq \alpha\,\|\boldsymbol{q}\|^2_{(L^2(\Omega))^d} \quad \forall\, \boldsymbol{p}, \boldsymbol{q} \in \left(L^2(\Omega)\right)^d.$$

Therefore, the condition (2.22) is valid and we have the following lower bound from (2.23):

$$D(u,v) \geq \frac{\alpha}{2}\,\|\Lambda v - \Lambda u\|^2_{(L^2(\Omega))^d} \quad \forall\, u, v \in \operatorname{dom} J(\cdot, \Lambda\cdot).$$

In particular, for

$$G(\boldsymbol{q}) = \int_\Omega \frac{1}{2}\,|\boldsymbol{q}|^2 dx,$$

we have

$$\langle G''(\boldsymbol{p})\, \boldsymbol{q}, \boldsymbol{q}\rangle = \int_\Omega |\boldsymbol{q}|^2 dx$$

and then

$$D(u,v) = \frac{1}{2} \|\Lambda v - \Lambda u\|^2_{(L^2(\Omega))^d}$$

for any $u, v \in \operatorname{dom} J(\cdot, \Lambda\cdot)$. ■

EXAMPLE 2.42 In Section 4.5, we have the situation with

$$G(\boldsymbol{q}) = \int_\Omega \frac{1}{p} \left(1 + |\boldsymbol{q}|^2\right)^{p/2} dx, \quad \boldsymbol{q} \in (L^p(\Omega))^d,$$

where $p > 1$, and Ω is a domain in $\mathbb{R}^d$. Using the chain rule, we can find the Gâteaux derivatives:

$$\langle G'(\boldsymbol{p}), \boldsymbol{q}\rangle = \int_\Omega \left(1 + |\boldsymbol{p}|^2\right)^{p/2-1} \boldsymbol{p}\cdot\boldsymbol{q}\, dx,$$
$$\langle G''(\boldsymbol{p})\, \boldsymbol{q}, \boldsymbol{q}\rangle = \int_\Omega \left(1 + |\boldsymbol{p}|^2\right)^{p/2-2} \left[(p-2)\,(\boldsymbol{p}\cdot\boldsymbol{q})^2 + |\boldsymbol{p}|^2|\boldsymbol{q}|^2 + |\boldsymbol{q}|^2\right] dx.$$

By Cauchy-Schwarz inequality, $(\boldsymbol{p}\cdot\boldsymbol{q})^2 \le |\boldsymbol{p}|^2|\boldsymbol{q}|^2$. Hence,

$$\begin{aligned}\langle G''(\boldsymbol{p})\, \boldsymbol{q}, \boldsymbol{q}\rangle &\ge \int_\Omega \left(1 + |\boldsymbol{p}|^2\right)^{p/2-2} \left[(p-1)\,(\boldsymbol{p}\cdot\boldsymbol{q})^2 + |\boldsymbol{q}|^2\right] dx \\ &\ge \int_\Omega \left(1 + |\boldsymbol{p}|^2\right)^{p/2-2} |\boldsymbol{q}|^2 dx.\end{aligned}$$

And then, for $u, v \in \operatorname{dom} J(\cdot, \Lambda\cdot)$, a lower bound for $D(u,v)$ could be

$$\int_0^1 (1-t) \int_\Omega \left(1 + |\nabla u + t\,\nabla(v-u)|^2\right)^{p/2-2} |\nabla(v-u)|^2 dx\, dt. \tag{2.24}$$

Now if $p \ge 4$, then

$$\left(1 + |\nabla u + t\,\nabla(v-u)|^2\right)^{p/2-2} \ge 1$$

and a further lower bound for $D(u,v)$ could be

$$\frac{1}{2} \int_\Omega |\nabla(v-u)|^2 dx.$$

This lower bound is not of desirable form, since the natural space for the solution u of the corresponding weak formulation (cf. Section 4.5) is $W^{1,p}(\Omega)$. For $p < 4$, the expression (2.24) does not lead to a convenient form for a lower bound. In such a situation, we will use directly the energy difference (2.11) to measure the difference between u and v. ■

Armed with Theorem 2.40, the procedure of deriving an estimate for the difference between u and v is decomposed into two steps:

STEP 1. Find a suitable lower bound for $D(u, v)$ that measures the difference between u and v. Usually, this lower bound will be some quantity depending on $\|v - u\|$. We will use (2.23), (2.21) or (2.20) for this purpose. When it is not convenient to relate the lower bound with a norm-like quantity, we will directly use the energy difference $ED(u, v)$ to measure the difference between u and v.

STEP 2. Construct an appropriate dual variable q^* so that the bound from the right hand side of (2.13) or (2.15) is as accurate as possible. If q^* is chosen to be a solution p^* of the dual problem, then the right-hand side of the estimate attains its minimum. However, usually it is not easy to find p^*. So it is desirable to have a strategy on determining a q^* that is easy to get and that produces a good bound for the right-hand side of the estimate. The function q^* is called a *dual variable* since it is related to the dual problem; it will also be called an *auxiliary function.*

To use Theorem 2.40 for an error estimate, we will take u to be the solution of the original problem, $v = u_0$ the solution of an idealized problem or an approximate problem. We will construct suitable auxiliary functions q^* based on the information from the solution u_0 and the idealized or approximate problem to produce good estimates for the error $(u - u_0)$.

As will be evident from the rest of the work, it is amazing that the above approach can be used to derive a posteriori error estimates in so many contexts.

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