Chapter 2

LINEAR PIEZOELECTRICITY FOR INFINITESIMAL FIELDS

In this chapter we specialize the nonlinear equations in Chapter 1 to the case of infinitesimal deformations and fields, which results in the linear theory of piezoelectricity. A few theoretical aspects of the linear theory are also discussed.

1. LINEARIZATION

In this section we reduce the nonlinear electroelastic equations in the previous chapter to the linear theory of piezoelectricity for infinitesimal deformation and fields. We consider small amplitude motions of an electroelastic body around its reference state due to small mechanical and electrical loads. It is assumed that the displacement gradient is infinitesimal in the following sense that

$$\| u_{i,K} \| << 1, \tag{2.1-1}$$

under some norm, e.g., $\| u_{i,K} \| = \max | u_{i,K} |$. It is also assumed that the electric potential gradient $\phi_{,K}$ is infinitesimal. We neglect powers of $u_{i,M}$ and $\phi_{,K}$ higher than the first as well as their products in all expressions. The linear terms themselves are also dropped in comparison with any finite quantity such the Kronecker delta or 1. Under (2.1-1),

$$\begin{aligned} \frac{\partial u_i}{\partial X_K} &= \frac{\partial u_i}{\partial y_k} y_{k,K} = \frac{\partial u_i}{\partial y_k}(\delta_{kK} + u_{k,K}) \cong \frac{\partial u_i}{\partial y_k}\delta_{kK}, \\ \phi_{,K} &= \phi_{,i} y_{i,K} \cong \phi_{,i}\delta_{iK}, \end{aligned} \tag{2.1-2}$$

which implies that, to the first order of approximation, the displacement and potential gradients calculated from the material and spatial coordinates are numerically equal. Therefore, within the linear theory, there is no need to distinguish capital and lowercase indices. Only lowercase indices will be used in the linear theory. The material time derivative of an infinitesimal field variable *f*(**y**,*t*) is simply the partial derivative with respect to *t*:

$$\frac{Df}{Dt} = \left.\frac{\partial f}{\partial t}\right|_{\mathbf{X}\ \text{fixed}} = \left.\frac{\partial f}{\partial t}\right|_{\mathbf{y}\ \text{fixed}} + \left.\frac{\partial f}{\partial y_i}\right|_{t\ \text{fixed}} \left.\frac{\partial y_i}{\partial t}\right|_{\mathbf{X}\ \text{fixed}}$$
$$= \left.\frac{\partial f}{\partial t}\right|_{\mathbf{y}\ \text{fixed}} + v_i \frac{\partial f}{\partial y_i} \cong \left.\frac{\partial f}{\partial t}\right|_{\mathbf{y}\ \text{fixed}}. \tag{2.1-3}$$

For the finite strain tensor

$$S_{KL} = \frac{1}{2}(u_{L,K} + u_{K,L} + u_{M,K}u_{M,L}) \cong \frac{1}{2}(u_{L,K} + u_{K,L}). \tag{2.1-4}$$

In the linear theory, the infinitesimal strain tensor will be denoted by

$$S_{kl} = \frac{1}{2}(u_{l,k} + u_{k,l}). \tag{2.1-5}$$

The material electric field becomes

$$\mathcal{E}_K = E_i y_{i,K} \cong E_i \delta_{iK} \to E_k. \tag{2.1-6}$$

Similarly,

$$\sigma_{ij}^E \cong 0, \quad \sigma_{ij}^M \cong 0, \quad \sigma_{ij} \cong \sigma_{ij}^S \cong \tau_{ij},$$
$$M_{Lj} \cong 0, \quad K_{Lj} \cong F_{Lj} \cong \delta_{Ki}\sigma_{ij}, \quad T_{KL}^S \cong \delta_{Ki}\delta_{Lj}\sigma_{ij}, \tag{2.1-7}$$
$$\mathcal{P}_K \to P_k, \quad \mathcal{D}_K \to D_k.$$

Since the various stress tensors are either approximately zero (quadratic in the infinitesimal gradients) or about the same, we will use T_{ij} to denote the stress tensor that is linear in the infinitesimal gradients. This is according to the IEEE Standard on Piezoelectricity [11]. Our notation for the rest of the linear theory will also follow the IEEE Standard. Then

$$\sigma_{ij} \cong \sigma_{ij}^S \cong \tau_{ij} \to T_{ij},$$
$$K_{Lj} \cong F_{Lj} \cong \delta_{Li}\sigma_{ij} \to T_{lj}, \tag{2.1-8}$$
$$T_{KL}^S \cong \delta_{Ki}\delta_{Lj}\sigma_{ij} \to T_{kl}.$$

For small fields the total free energy can be approximated by

$$\rho_0\hat{\psi}(S_{KL}, \mathcal{E}_K) = \rho_0\psi(S_{KL}, \mathcal{E}_K) - \frac{1}{2}\varepsilon_0 J E_k E_k$$
$$\cong \frac{1}{2} c_{2\,ABCD} S_{AB} S_{CD} - e_{ABC}\mathcal{E}_A S_{BC} - \frac{1}{2}\chi_{2\,AB}\,\mathcal{E}_A\mathcal{E}_B - \frac{1}{2}\varepsilon_0 J E_k E_k \tag{2.1-9}$$
$$\to \frac{1}{2} c_{ijkl}^E S_{ij} S_{kl} - e_{ijk} E_i S_{jk} - \frac{1}{2}\varepsilon_{ij}^S E_i E_j = H(S_{kl}, E_k),$$

where

$$\varepsilon^S_{ij} = \chi_{\substack{\\ 2\ ij}} + \varepsilon_0 \delta_{ij}. \tag{2.1-10}$$

The superscript E in c^E_{ijkl} indicates that the independent electric constitutive variable is the electric field $\mathbf{E}$. The superscript S in ε^S_{ij} indicates that the mechanical constitutive variable is the strain tensor $\mathbf{S}$. We have also denoted the total free energy of the linear theory by H which is usually called the electric enthalpy. The constitutive relations generated by H are

$$\begin{aligned} T_{ij} &= \frac{\partial H}{\partial S_{ij}} = c^E_{ijkl} S_{kl} - e_{kij} E_k, \\ D_i &= -\frac{\partial H}{\partial E_i} = e_{ikl} S_{kl} + \varepsilon^S_{ik} E_k. \end{aligned} \tag{2.1-11}$$

Hence $\mathbf{T}$, $\mathbf{D}$ and $\mathbf{P}$ are also infinitesimal. The material constants in Equation (2.1-11) have the following symmetries:

$$\begin{aligned} &c^E_{ijkl} = c^E_{jikl} = c^E_{klij}, \\ &e_{kij} = e_{kji}, \\ &\varepsilon^S_{ij} = \varepsilon^S_{ji}. \end{aligned} \tag{2.1-12}$$

We also assume that the elastic and dielectric material tensors are positive-definite in the following sense:

$$\begin{aligned} &c^E_{ijkl} S_{ij} S_{kl} \geq 0, \quad \text{for any} \quad S_{ij} = S_{ji}, \\ &\quad \text{and} \quad c^E_{ijkl} S_{ij} S_{kl} = 0 \quad \Rightarrow \quad S_{ij} = 0, \\ &\varepsilon^S_{ij} E_i E_j \geq 0, \quad \text{for any} \quad E_i, \\ &\quad \text{and} \quad \varepsilon^S_{ij} E_i E_j = 0 \quad \Rightarrow \quad E_i = 0. \end{aligned} \tag{2.1-13}$$

The total internal energy density per unit volume can be obtained from H by a Legendre transform, given as

$$U(\mathbf{S}, \mathbf{D}) = H(\mathbf{S}, \mathbf{E}(\mathbf{S}, \mathbf{D})) + \mathbf{E}(\mathbf{S}, \mathbf{D}) \cdot \mathbf{D}. \tag{2.1-14}$$

Constitutive relations in the following form then follow:

$$\mathbf{T} = \frac{\partial U}{\partial \mathbf{S}}, \quad \mathbf{E} = \frac{\partial U}{\partial \mathbf{D}}, \tag{2.1-15}$$

or

$$\begin{aligned} T_{ij} &= c^D_{ijkl} S_{kl} - h_{kij} D_k, \\ E_i &= -h_{ikl} S_{kl} + \beta^S_{ik} D_k. \end{aligned} \tag{2.1-16}$$

It can be shown that U is positive-definite:

$$
\begin{aligned}
U &= H + E_i D_i \\
&= \frac{1}{2} c_{ijkl}^E S_{ij} S_{kl} - e_{ijk} E_i S_{jk} - \frac{1}{2} \varepsilon_{ij}^S E_i E_j + E_i (e_{ikl} S_{kl} + \varepsilon_{ik}^S E_k) \quad (2.1\text{-}17) \\
&= \frac{1}{2} c_{ijkl}^E S_{ij} S_{kl} + \frac{1}{2} \varepsilon_{ij}^S E_i E_j \geq 0.
\end{aligned}
$$

For small fields, the total internal energy density U per unit volume and the internal energy density e per unit mass in the previous chapter are related by

$$
\begin{aligned}
U &= H + E_i D_i = \rho_0 \hat{\psi} + E_i D_i \\
&= \rho_0 \psi - \pi + E_i D_i = \rho_0 \psi - \frac{1}{2} \varepsilon_0 E_i E_i + E_i D_i \\
&= \rho_0 e - E_i P_i - \frac{1}{2} \varepsilon_0 E_i E_i + E_i (\varepsilon_0 E_i + P_i) \\
&= \rho_0 e + \frac{1}{2} \varepsilon_0 E_i E_i = \rho_0 e + \pi.
\end{aligned} \tag{2.1-18}
$$

Similar to (2.1-11) and (2.1-16), linear constitutive relations can also be written as [11]

$$
\begin{aligned}
S_{ij} &= s_{ijkl}^E T_{kl} + d_{kij} E_k, \\
D_i &= d_{ikl} T_{kl} + \varepsilon_{ik}^T E_k,
\end{aligned} \tag{2.1-19}
$$

and

$$
\begin{aligned}
S_{ij} &= s_{ijkl}^D T_{kl} + g_{kij} D_k, \\
E_i &= -g_{ikl} T_{kl} + \beta_{ik}^T D_k.
\end{aligned} \tag{2.1-20}
$$

The equations of motion and the charge equation become

$$
\begin{aligned}
T_{ji,j} + \rho f_i &= \rho \ddot{u}_i, \\
D_{i,i} &= \rho_e,
\end{aligned} \tag{2.1-21}
$$

in which the difference between the reference and present mass and charge densities can be ignored. The body force **f** and charge ρ_e are infinitesimal.

Within linear theory, the conservation of mass and the relation between the reference and present charge densities take the following form:

$$
\begin{aligned}
\rho_0 &\cong \rho(1 + u_{k,k}), \\
\rho_E &\cong \rho_e.
\end{aligned} \tag{2.1-22}
$$

The surface loads are also infinitesimal. Hence

$$\bar{T}_j \cong \bar{t}_j, \quad \bar{\sigma}_E \cong \bar{\sigma}_e, \tag{2.1-23}$$

and

$$K_{Lj} N_L \cong T_{ij} n_i, \quad \Delta_L N_L \cong D_i n_i. \tag{2.1-24}$$

Problem

2.1-1. Show (2.1-15).

2. BOUNDARY-VALUE PROBLEM

2.1 Displacement-Potential Formulation

In summary, the linear theory of piezoelectricity consists of the equations of motion and charge

$$T_{ji,j} + \rho f_i = \rho \ddot{u}_i, \quad D_{i,i} = \rho_e \ , \tag{2.2-1}$$

constitutive relations

$$T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \quad D_i = e_{ijk} S_{jk} + \varepsilon_{ij} E_j \ , \tag{2.2-2}$$

and the strain-displacement and electric field-potential relations

$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i} \ , \tag{2.2-3}$$

where **u** is the mechanical displacement vector, **T** is the stress tensor, **S** is the strain tensor, **E** is the electric field, **D** is the electric displacement, ϕ is the electric potential, ρ is the known reference mass density (or ρ_0 in the previous chapter), ρ_e is the body free charge density, and **f** is the body force per unit mass. The coefficients c_{ijkl}, e_{kij} and ε_{ij} are the elastic, piezoelectric and dielectric constants. We have neglected the superscripts in the material constants. With successive substitutions from Equations (2.2-2) and (2.2-3), Equation (2.2-1) can be written as four equations for **u** and ϕ

$$\begin{aligned} &c_{ijkl} u_{k,lj} + e_{kij} \phi_{,kj} + \rho f_i = \rho \ddot{u}_i, \\ &e_{ikl} u_{k,li} - \varepsilon_{ij} \phi_{,ij} = \rho_e. \end{aligned} \tag{2.2-4}$$

2.2 Boundary-Value Problem

Let the region occupied by the piezoelectric body be V and its boundary surface be S as shown in Figure 2.2-1. For linear piezoelectricity we use **x** as the independent spatial coordinates. Let the unit outward normal of S be **n**.

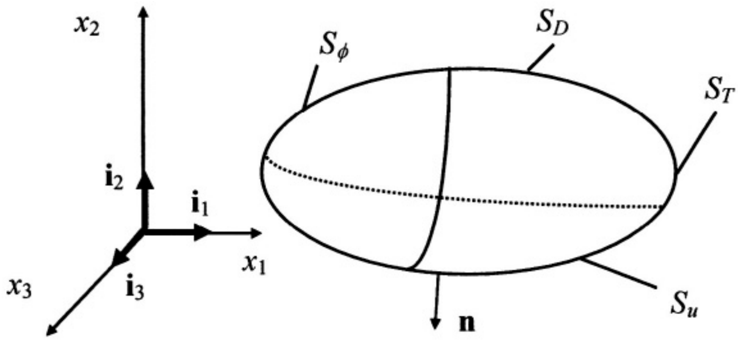

Figure 2.2-1. A piezoelectric body and partitions of its surface.

For boundary conditions we consider the following partitions of S:

$$\begin{aligned} S_u \cup S_T &= S_\phi \cup S_D = S, \\ S_u \cap S_T &= S_\phi \cap S_D = 0, \end{aligned} \tag{2.2-5}$$

where S_u is the part of S on which the mechanical displacement is prescribed, and S_T is the part of S where the traction vector is prescribed. S_ϕ represents the part of S which is electroded where the electric potential is no more than a function of time, and S_D is the unelectroded part. For mechanical boundary conditions we have prescribed displacement $\overline{u}_i$

$$u_i = \overline{u}_i, \quad \text{on} \quad S_u, \tag{2.2-6}$$

and prescribed traction $\bar{t}_j$

$$T_{ij} n_i = \bar{t}_j, \quad \text{on} \quad S_T. \tag{2.2-7}$$

Electrically, on the electroded portion of S,

$$\phi = \overline{\phi}, \quad \text{on} \quad S_\phi, \tag{2.2-8}$$

where $\overline{\phi}$ does not vary spatially. On the unelectroded part of S, the charge condition can be written as

$$D_j n_j = -\overline{\sigma}_e, \quad \text{on} \quad S_D, \tag{2.2-9}$$

where $\overline{\sigma_e}$ is free charge density per unit surface area. In the above formulation, the mechanical effect of the electrode is neglected because we assume very thin electrodes.

On an electrode S_ϕ, the total free electric charge Q_e can be represented by

$$Q_e = \int_{S_\phi} -n_i D_i dS\,. \tag{2.2-10}$$

The electric current flowing out of the electrode is given by

$$i = -\dot{Q}_e\,. \tag{2.2-11}$$

Sometimes there are two (or more) electrodes on a body which are connected to an electric circuit. In this case, circuit equation(s) will need to be considered.

2.3 Principle of Superposition

The linearity of Equation (2.2-4) allows the superposition of solutions. Suppose the solutions under two different sets of loads of $\{\mathbf{f}^{(1)},\rho_e^{(1)}\}$ and $\{\mathbf{f}^{(2)},\rho_e^{(2)}\}$ are $\{\mathbf{u}^{(1)},\phi^{(1)}\}$ and $\{\mathbf{u}^{(2)},\phi^{(2)}\}$, respectively. Then under the combined load of $\{\mathbf{f}^{(1)}+\mathbf{f}^{(2)},\rho_e^{(1)}+\rho_e^{(2)}\}$, the solution to (2.2-4) is $\{\mathbf{u}^{(1)}+\mathbf{u}^{(2)},\phi^{(1)}+\phi^{(2)}\}$. This is called the principle of superposition and can be shown as

$$\begin{aligned}
& c_{ijkl}(u_k^{(1)}+u_k^{(2)})_{,lj}+e_{kij}(\phi^{(1)}+\phi^{(2)})_{,kj} \\
&\quad +\rho(f_i^{(1)}+f_i^{(2)})-\rho\frac{\partial^2}{\partial t^2}(u_i^{(1)}+u_i^{(2)}) \\
&= c_{ijkl}u_{k,lj}^{(1)}+c_{ijkl}u_{k,lj}^{(2)}+e_{kij}\phi_{,kj}^{(1)}+\phi_{,kj}^{(2)} \\
&\quad +\rho f_i^{(1)}+\rho f_i^{(2)}-\rho\ddot{u}_i^{(1)}-\rho\ddot{u}_i^{(2)} \\
&= (c_{ijkl}u_{k,lj}^{(1)}+e_{kij}\phi_{,kj}^{(1)}+\rho f_i^{(1)}-\rho\ddot{u}_i^{(1)}) \\
&\quad +(c_{ijkl}u_{k,lj}^{(2)}+\phi_{,kj}^{(2)}+\rho f_i^{(2)}-\rho\ddot{u}_i^{(2)}) \\
&= 0+0=0,
\end{aligned} \tag{2.2-12}$$

and

$$\begin{aligned}
& e_{ikl}(u_k^{(1)} + u_k^{(2)})_{,li} - \varepsilon_{ij}(\phi^{(1)} + \phi^{(2)})_{,ij} - (\rho_e^{(1)} + \rho_e^{(2)}) \\
& = e_{ikl}u_{k,li}^{(1)} + e_{ikl}u_{k,li}^{(2)} - \varepsilon_{ij}\phi_{,ij}^{(1)} - \varepsilon_{ij}\phi_{,ij}^{(2)} - \rho_e^{(1)} - \rho_e^{(2)} \\
& = (e_{ikl}u_{k,li}^{(1)} - \varepsilon_{ij}\phi_{,ij}^{(1)} - \rho_e^{(1)}) + (e_{ikl}u_{k,li}^{(2)} - \varepsilon_{ij}\phi_{,ij}^{(2)} - \rho_e^{(2)}) \\
& = 0 + 0 = 0.
\end{aligned} \tag{2.2-13}$$

The principle of superposition can be generalized to include boundary loads.

2.4 Compatibility

Since the six strain components are derived from three displacement components, it is natural to expect some relations among the strain components whether they are linear or nonlinear. The following can be verified by direct substitution:

$$\begin{aligned}
S_{11,23} &= (S_{31,2} + S_{12,3} - S_{23,1})_{,1}, \\
S_{22,31} &= (S_{12,3} + S_{23,1} - S_{31,2})_{,2}, \\
S_{33,12} &= (S_{23,1} + S_{31,2} - S_{12,3})_{,3}, \\
2S_{23,23} &= S_{22,33} + S_{33,22}, \\
2S_{31,31} &= S_{33,11} + S_{11,33}, \\
2S_{12,12} &= S_{11,22} + S_{22,11}.
\end{aligned} \tag{2.2-14}$$

Equations (2.2-14) are called compatibility conditions. The compatibility conditions are necessary conditions for the six strain components derived from three displacement components. They are also sufficient in the sense that for six strain components satisfying these compatibility conditions, there exist three displacement components from which the six strain components are derivable. The sufficiency of (2.2-14) is true over a simply-connected domain only. For a multiply-connected domain, some additional conditions are needed. The compatibility conditions are useful when solving a problem using stress components rather than displacement components as the primary unknowns. In more compact form, Equation (2.2-14) can be written as

$$S_{ij,kl} + S_{kl,ij} - S_{ik,jl} - S_{jl,ik} = 0, \tag{2.2-15}$$

of which the six independent relations are (2.1-25), or

$$\varepsilon_{ijk}\varepsilon_{lmn}S_{il,jm} = 0. \tag{2.2-16}$$

3. VARIATIONAL PRINCIPLES

3.1 Hamilton's Principle

The equations and boundary conditions of linear piezoelectricity can be derived from a variational principle. Consider [4]

$$\Pi(\mathbf{u},\phi) = \int_{t_0}^{t_1} dt \int_V \left[\frac{1}{2}\rho\dot{u}_i\dot{u}_i - H(\mathbf{S},\mathbf{E}) + \rho f_i u_i - \rho_e\phi\right] dV + \int_{t_0}^{t_1} dt \int_{S_T} \bar{t}_i u_i dS - \int_{t_0}^{t_1} dt \int_{S_D} \bar{\sigma}_e \phi dS, \tag{2.3-1}$$

where

$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}. \tag{2.3-2}$$

$\mathbf{u}$ and ϕ are variationally admissible if they are smooth enough and satisfy

$$\begin{aligned} &\delta u_i\,|_{t_0} = \delta u_i\,|_{t_1} = 0, \quad \text{in} \quad V, \\ &u_i = \bar{u}_i, \quad \text{on} \quad S_u, \quad t_0 < t < t_1, \\ &\phi = \bar{\phi}, \quad \text{on} \quad S_\phi, \quad t_0 < t < t_1. \end{aligned} \tag{2.3-3}$$

The first variation of Π is

$$\delta\Pi = \int_{t_0}^{t_1} dt \int_V \left[(T_{ji,j} + \rho f_i - \rho\ddot{u}_i)\delta u_i + (D_{i,i} - \rho_e)\delta\phi\right] dV - \int_{t_0}^{t_1} dt \int_{S_T} (T_{ji}n_j - \bar{t}_i)\delta u_i dS - \int_{t_0}^{t_1} dt \int_{S_D} (D_i n_i + \bar{\sigma}_e)\delta\phi dS, \tag{2.3-4}$$

where we have denoted

$$\mathbf{T} = \frac{\partial H}{\partial \mathbf{S}}, \quad \mathbf{D} = -\frac{\partial H}{\partial \mathbf{E}}. \tag{2.3-5}$$

Therefore the stationary condition of Π is

$$\begin{aligned} &T_{ji,j} + \rho f_i = \rho\ddot{u}_i, \quad \text{in} \quad V, \quad t_0 < t < t_1, \\ &D_{i,i} = \rho_e, \quad \text{in} \quad V, \quad t_0 < t < t_1, \\ &T_{ji}n_j = \bar{t}_i, \quad \text{on} \quad S_T, \quad t_0 < t < t_1, \\ &D_i n_i = -\bar{\sigma}_e, \quad \text{on} \quad S_D, \quad t_0 < t < t_1. \end{aligned} \tag{2.3-6}$$

Hamilton's principle can be stated as: Among all the admissible $\{\mathbf{u}, \phi\}$, the one that also satisfies (2.3-6) makes Π stationary.

3.2 Mixed Variational Principles

If the functional in Equation (2.3-1) is viewed to be dependent on $\mathbf{u}$, ϕ, $\mathbf{S}$ and $\mathbf{E}$, then Equation (2.3-2) should be considered as constraints among the independent variables. These constraints, along with the boundary data in Equations $(2.3\text{-}3)_{2,3}$, can be removed by the method of Lagrange multipliers. Then the following variational functional will result [12]:

$$\begin{aligned}
&\Pi(\mathbf{u},\phi,\mathbf{S},\mathbf{T},\mathbf{E},\mathbf{D}) \\
&= \int_{t_0}^{t_1} dt \int_V \left\{ \frac{1}{2}\rho \dot{u}_i \dot{u}_i - H(\mathbf{S},\mathbf{E}) + T_{ij}\left[S_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i}) \right] \right. \\
&\qquad \left. - D_i(E_i + \phi_{,i}) + \rho f_i u_i - \rho_e \phi \right\} dV \\
&\quad + \int_{t_0}^{t_1} dt \int_{S_u} T_{ji} n_j (u_i - \bar{u}_i) dS \\
&\quad + \int_{t_0}^{t_1} dt \int_{S_\phi} D_i n_i (\phi - \bar{\phi}) dS \\
&\quad + \int_{t_0}^{t_1} dt \int_{S_T} \bar{t}_i u_i dS - \int_{t_0}^{t_1} dt \int_{S_D} \bar{\sigma}_e \phi dS.
\end{aligned} \tag{2.3-7}$$

$\mathbf{u}$, ϕ, $\mathbf{S}$, $\mathbf{T}$, $\mathbf{E}$ and $\mathbf{D}$ are admissible if they are smooth enough and satisfy

$$\delta u_i \,|_{t_0} = \delta u_i \,|_{t_1} = 0, \quad \text{in} \quad V. \tag{2.3-8}$$

The first variation of Π is

$$\begin{aligned}
\delta\Pi &= \int_{t_0}^{t_1} dt \int_V \left\{ (T_{ji,j} + \rho f_i - \rho \ddot{u}_i)\delta u_i + (D_{i,i} - \rho_e)\delta\phi \right. \\
&\quad + \left[S_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i}) \right]\delta T_{ij} - (E_i + \phi_{,i})\delta D_i \\
&\quad \left. + \left(T_{ij} - \frac{\partial H}{\partial S_{ij}} \right)\delta S_{ij} - \left(D_i + \frac{\partial H}{\partial E_i} \right)\delta E_i \right\} dV \\
&\quad + \int_{t_0}^{t_1} dt \int_{S_u} (u_i - \bar{u}_i)\delta T_{ji} n_j dS + \int_{t_0}^{t_1} dt \int_{S_\phi} (\phi - \bar{\phi})\delta D_i n_i dS \\
&\quad - \int_{t_0}^{t_1} dt \int_{S_T} (T_{ji} n_j - \bar{t}_i)\delta u_i dS - \int_{t_0}^{t_1} dt \int_{S_D} (D_i n_i + \bar{\sigma}_e)\delta\phi dS.
\end{aligned} \tag{2.3-9}$$

Therefore the stationary condition of Π is

$$
\begin{aligned}
&T_{ji,j} + \rho f_i = \rho \ddot{u}_i, \quad D_{i,i} = \rho_e, \quad \text{in} \quad V, \quad t_0 < t < t_1, \\
&S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad \text{in} \quad V, \quad t_0 < t < t_1, \\
&T_{ij} = \frac{\partial H}{\partial S_{ij}}, \quad D_i = -\frac{\partial H}{\partial E_i}, \quad \text{in} \quad V, \quad t_0 < t < t_1, \\
&u_i = \overline{u}_i, \quad \text{on} \quad S_u, \quad t_0 < t < t_1, \\
&T_{ji} n_j = \bar{t}_i, \quad \text{on} \quad S_T, \quad t_0 < t < t_1, \\
&\phi = \overline{\phi}_e, \quad \text{on} \quad S_\phi, \quad t_0 < t < t_1. \\
&D_i n_i = -\overline{\sigma}_e, \quad \text{on} \quad S_D, \quad t_0 < t < t_1.
\end{aligned}
\tag{2.3-10}
$$

Hence, among all the admissible {**u**, ϕ , **S, T, E, D**}, the one that also satisfies Equation (2.3-10) makes Π(**u,** ϕ , **S, T, E, D)** stationary. The functional in Equation (2.3-7) has all of the fields as independent variables. Its stationary condition yields all the equations and boundary conditions. Variational principles like this are called mixed or generalized variational principles.

3.3 Conservation Laws

From Noether's theorem on variational principles invariant under infinitesimal transformations, the following relations can be shown [13]:

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\rho \dot{u}_j u_{j,i}) + \frac{\partial}{\partial x_k}(\Sigma \delta_{ik} - u_{j,i} T_{jk} - \phi_{,i} D_k) = 0, \\
&\frac{\partial}{\partial t}\varepsilon_{ijk} \rho (x_j \dot{u}_m u_{m,k} + \dot{u}_j u_k) \\
&\quad + \frac{\partial}{\partial x_k}\varepsilon_{imj}(x_m \Sigma \delta_{jk} - x_m u_{l,j} T_{lk} - x_m \varphi_{,j} D_k + T_{jk} u_m) = 0, \\
&\frac{\partial}{\partial t}[\rho \dot{u}_j (u_j + x_m u_{j,m} + t\dot{u}_j) + t\Sigma] \\
&\quad + \frac{\partial}{\partial x_k}[x_k \Sigma - T_{jk}(u_j + x_m u_{j,m} + t\dot{u}_j) - D_k(\phi + x_m \phi_{,m} + t\dot{\phi})] = 0,
\end{aligned}
\tag{2.3-11}
$$

where

$$\Sigma = H(\mathbf{S},\mathbf{E}) - T_{ij}\left[S_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})\right] + D_i(E_i + \phi_{,i}) - \frac{1}{2}\rho\dot{u}_i\dot{u}_i. \tag{2.3-12}$$

Equation (2.3-11)$_{1,2,3}$ are obtained by the invariance of the functional in Equation (2.3-7) under translations, rotations, and scale changes, respectively. They can be verified by direct differentiation. The relations in Equation (2.3-11) are in divergence-free form and are called conservation laws. They can be transformed to path-independent integrals by the divergence theorem.

Problems

2.3-1. Show (2.3-4).
2.3-2. Show (2.3-9)
2.3-3. Study the conservation laws for linear, static piezoelectricity [13].

4. UNIQUENESS

4.1 Poynting's Theorem

We begin with the rate of change of the total internal energy density, given as

$$\begin{aligned}\dot{U} &= \frac{\partial U}{\partial S_{ij}}\dot{S}_{ij} + \frac{\partial U}{\partial D_i}\dot{D}_i = T_{ij}\dot{S}_{ij} + E_i\dot{D}_i \\ &= T_{ij}\dot{S}_{ij} + E_i\dot{D}_i = T_{ij}\dot{u}_{i,j} - \phi_{,i}\dot{D}_i \\ &= (T_{ij}\dot{u}_i)_{,j} - T_{ij,j}\dot{u}_i - (\phi\ \dot{D}_i)_{,i} + \phi\dot{D}_{i,i} \\ &= (T_{ij}\dot{u}_i)_{,j} - (\rho\ddot{u}_i - \rho f_i)\dot{u}_i - (\phi\ \dot{D}_i)_{,i} + \phi\dot{\rho}_e \\ &= (T_{ji}\dot{u}_j)_{,i} - \frac{\partial}{\partial t}\left(\frac{1}{2}\rho\dot{u}_i\dot{u}_i\right) + \rho f_i\dot{u}_i - (\phi\ \dot{D}_i)_{,i} + \phi\dot{\rho}_e,\end{aligned} \tag{2.4-1}$$

where (2.2-1) has been used. Therefore,

$$\frac{\partial}{\partial t}(T + U) = \rho f_i\dot{u}_i + \phi\dot{\rho}_e - (\phi\ \dot{D}_i - T_{ji}\dot{u}_j)_{,i}, \tag{2.4-2}$$

where

$$T = \frac{1}{2}\rho\dot{u}_i\dot{u}_i \tag{2.4-3}$$

is the kinetic energy density, and $\phi\dot{D}_i$ is the quasistatic Poynting vector. Equation (2.4-2) may be considered as a generalized version of Poynting's theorem in electromagnetics.

4.2 Energy Integral

Integration of (2.4-2) over V gives

$$\begin{aligned}\frac{\partial}{\partial t}\int_V (T+U)dV &= \int_V \rho(f_i\dot{u}_i + \phi\dot{\rho}_e)dV \\ &+ \int_{S_u} T_{ji}n_j\dot{\bar{u}}_i dS + \int_{S_T} \bar{t}_i\dot{u}_i dS \\ &- \int_{S_\phi} \dot{D}_i n_i\bar{\phi}dS + \int_{S_D} \dot{\bar{\sigma}}_e\phi dS,\end{aligned} \tag{2.4-4}$$

where Equations (2.2-6) through (2.2-9) have been used. Integrating Equation (2.4-4) from t_0 to t, we obtain

$$\begin{aligned}\int_V (T+U)\big|_t dV &= \int_V (T+U)\big|_{t_0} dV \\ &+ \int_{t_0}^{t} dt\int_V \rho(f_i\dot{u}_i + \phi\dot{\rho}_e)dV \\ &+ \int_{t_0}^{t} dt\int_{S_u} T_{ji}n_j\dot{\bar{u}}_i dS + \int_{t_0}^{t} dt\int_{S_T} \bar{t}_i\dot{u}_i dS \\ &- \int_{t_0}^{t} dt\int_{S_\phi} \dot{D}_i n_i\bar{\phi}dS + \int_{t_0}^{t} dt\int_{S_D} \dot{\bar{\sigma}}_e\phi dS.\end{aligned} \tag{2.4-5}$$

Equation (2.4-5) is called the energy integral which states that the energy at time t is the energy at time t_0 plus the work done to the body from t_0 to t.

4.3 Uniqueness

Consider two solutions to the following initial-boundary value problem:

$$
\begin{aligned}
&T_{ji,j} + \rho f_i = \rho \ddot{u}_i, \quad \text{in} \quad V, \quad t > t_0, \\
&D_{i,i} = \rho_e, \quad \text{in} \quad V, \quad t > t_0, \\
&T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \quad \text{in} \quad V, \quad t > t_0, \\
&D_i = e_{ijk} S_{jk} + \varepsilon_{ij} E_j, \quad \text{in} \quad V, \quad t > t_0, \\
&S_{ij} = (u_{i,j} + u_{j,i})/2, \quad \text{in} \quad V, \quad t > t_0, \\
&E_i = -\phi_{,i}, \quad \text{in} \quad V, \quad t > t_0, \\
&u_i = \bar{u}_i, \quad \text{on} \quad S_u, \quad t > t_0, \\
&T_{ji} n_j = \bar{t}_i, \quad \text{on} \quad S_T, \quad t > t_0, \\
&\phi = \bar{\phi}, \quad \text{on} \quad S_\phi, \quad t > t_0, \\
&D_i n_i = -\bar{\sigma}_e, \quad \text{on} \quad S_D, \quad t > t_0, \\
&u_i = u_i^0, \quad \text{in} \quad V, \quad t = t_0, \\
&\dot{u}_i = v_i^0, \quad \text{in} \quad V, \quad t = t_0.
\end{aligned}
\tag{2.4-6}
$$

From the principle of superposition, the difference of the two solutions satisfies the homogeneous version of (2.4-6). Let $\mathbf{u}^*$, ϕ^*, $\mathbf{S}^*$, $\mathbf{T}^*$, $\mathbf{E}^*$, and $\mathbf{D}^*$ denote the difference of the corresponding fields and apply (2.4-5) to it. The initial energy and the external work for the difference are zero. Then the energy integral (2.4-5) implies that, for the difference, at any $t > t_0$

$$\int_V \left(T^* + U^*\right)\Big|_t dV = 0, \quad t > t_0. \tag{2.4-7}$$

Since both T and U are nonnegative,

$$U^* = 0, \quad T^* = 0, \quad \text{in} \quad V, \quad t > t_0. \tag{2.4-8}$$

From the positive-definiteness of T and U,

$$\mathbf{S}^* = 0, \quad \mathbf{E}^* = 0, \quad \dot{\mathbf{u}}^* = 0, \quad \text{in} \quad V, \quad t > t_0. \tag{2.4-9}$$

Hence the two solutions are identical to within a static rigid body displacement and a constant potential.

5. OTHER FORMULATIONS

5.1 Four-Vector Formulation

Let us define the four-space coordinate system [14]

$$x_p = \{x_i, t\}, \tag{2.5-1}$$

and the four-vector

$$U_p = \{u_i, \phi\}, \tag{2.5-2}$$

where subscripts p, q, r, s will be assumed to run 1 to 4. Also define the second-rank four-tensor

$$\rho_{pq} = \begin{cases} \rho\delta_{pq}, & p,q = 1,2,3, \\ 0, & p,q = 4, \end{cases} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.5-3}$$

and the fourth-rank four-tensor M_{pqrs}, where

$$\begin{gathered} M_{ijkl} = c_{ijkl}, \quad M_{4jkl} = e_{jkl}, \quad M_{ijk4} = e_{kij}, \\ M_{4jk4} = -\varepsilon_{jk}, \quad M_{p44s} = -\rho_{ps}, \end{gathered} \tag{2.5-4}$$

and all other components of $M_{pqrs} = 0$. Then

$$\begin{aligned} (U_{p,q}M_{pqrl})_{,r} &= (U_{i,j}M_{ijrl} + U_{4,j}M_{4jrl} + U_{i,4}M_{i4rl} + U_{4,4}M_{44rl})_{,r} \\ &= (U_{i,j}M_{ijkl} + U_{4,j}M_{4jkl} + U_{i,4}M_{i4kl} + U_{4,4}M_{44kl})_{,k} \\ &\quad + (U_{i,j}M_{ij4l} + U_{4,j}M_{4j4l} + U_{i,4}M_{i44l} + U_{4,4}M_{444l})_{,4} \\ &= (u_{i,j}c_{ijkl} + \phi_{,j}e_{jkl})_{,k} + (-\dot{u}_i\rho_{il})_{,4} \\ &= c_{ijkl}u_{i,jk} + e_{jkl}\phi_{,jk} - \rho\ddot{u}_l, \end{aligned} \tag{2.5-5}$$

and

$$\begin{aligned} &(U_{p,q}M_{pqr4})_{,r} \\ &= (U_{i,j}M_{ijr4} + U_{4,j}M_{4jr4} + U_{i,4}M_{i4r4} + U_{4,4}M_{44r4})_{,r} \\ &= (U_{i,j}M_{ijk4} + U_{4,j}M_{4jk4} + U_{i,4}M_{i4k4} + U_{4,4}M_{44k4})_{,k} \\ &\quad + (U_{i,j}M_{ij44} + U_{4,j}M_{4j44} + U_{i,4}M_{i444} + U_{4,4}M_{4444})_{,4} \\ &= (u_{i,j}e_{kij} - \phi_{,j}\varepsilon_{jk})_{,k} \\ &= u_{i,jk}e_{kij} - \phi_{,jk}\varepsilon_{jk}. \end{aligned} \tag{2.5-6}$$

Therefore,

$$(U_{p,q}M_{pqrs})_{,r} = 0 \tag{2.5-7}$$

yields the homogeneous equation of motion and the charge equation.

5.2 Vector Potential Formulation

Consider the case when there is no body charge. Since the divergence of **D** vanishes, we can introduce a vector potential ψ_i by

$$D_i = \frac{1}{2}\varepsilon_{ijk}\psi_{k,j}, \tag{2.5-8}$$

which satisfies the divergence-free condition on **D**. Corresponding to vector **D**, we introduce an anti-symmetric tensor by [15]

$$D_i = \frac{1}{2}\varepsilon_{ijk}\mathsf{D}_{jk}, \tag{2.5-9}$$

which, when substituted into (2.5-8), yields

$$\mathsf{D}_{ij} = \frac{1}{2}(\psi_{j,i} - \psi_{i,j}). \tag{2.5-10}$$

Similarly, for the electric field **E**, we introduce an anti-symmetric tensor by

$$E_i = \frac{1}{2}\varepsilon_{ijk}\mathsf{E}_{jk}. \tag{2.5-11}$$

Then the curl-free condition on **E** takes the following form:

$$\mathsf{E}_{ij,i} = 0. \tag{2.5-12}$$

In summary, the equations for this formulation are

$$\begin{gathered} T_{ij,i} = \rho\ddot{u}_j, \quad \mathsf{E}_{ij,i} = 0, \\ T_{ij} = \frac{\partial U}{\partial S_{ij}}, \quad \mathsf{E}_{ij} = \frac{\partial U}{\partial \mathsf{D}_{ij}}, \\ S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \mathsf{D}_{ij} = \frac{1}{2}(\psi_{j,i} - \psi_{i,j}). \end{gathered} \tag{2.5-13}$$

Note that in this formulation the internal energy U is used, which is positive definite.

6. CURVILINEAR COORDINATES

Cylindrical and spherical shapes are often used in piezoelectric devices. To analyze these devices, it is usually convenient to use cylindrical or spherical coordinates.

6.1 Cylindrical Coordinates

The cylindrical coordinates (r, θ, z) are defined by

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z. \tag{2.6-1}$$

In cylindrical coordinates we have the strain-displacement relation

$$S_{rr} = u_{r,r}, \quad S_{\theta\theta} = \frac{1}{r}u_{\theta,\theta} + \frac{u_r}{r}, \quad S_{zz} = u_{z,z},$$

$$2S_{r\theta} = u_{\theta,r} + \frac{1}{r}u_{r,\theta} - \frac{u_\theta}{r}, \quad 2S_{\theta z} = \frac{1}{r}u_{z,\theta} + u_{\theta,z}, \tag{2.6-2}$$

$$2S_{zr} = u_{r,z} + u_{z,r}.$$

The electric field-potential relation is given by

$$E_r = -\phi_{,r}, \quad E_\theta = -\frac{1}{r}\phi_{,\theta}, \quad E_z = -\phi_{,z}. \tag{2.6-3}$$

The equations of motion are

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{\partial T_{zr}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r} + \rho f_r = \rho \ddot{u}_r,$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{2}{r}T_{r\theta} + \rho f_\theta = \rho \ddot{u}_\theta, \tag{2.6-4}$$

$$\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{rz} + \rho f_z = \rho \ddot{u}_z.$$

The electrostatic charge equation is

$$\frac{1}{r}(rD_r)_{,r} + \frac{1}{r}D_{\theta,\theta} + D_{z,z} = \rho_e. \tag{2.6-5}$$

6.2 Spherical Coordinates

The spherical coordinates (r,θ,φ) are defined by

$$x_1 = r\sin\theta\cos\varphi, \quad x_2 = r\sin\theta\sin\varphi, \quad x_3 = r\cos\theta. \tag{2.6-6}$$

In spherical coordinates we have the strain-displacement relation

$$S_{rr} = \frac{\partial u_r}{\partial r}, \quad S_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r},$$

$$S_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta,$$

$$2S_{r\theta} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}, \tag{2.6-7}$$

$$2S_{\theta\varphi} = \frac{1}{r}\frac{\partial u_\varphi}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{r}\cot\theta,$$

$$2S_{\varphi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}.$$

The electric field-potential relation is

$$E_r = -\frac{\partial \phi}{\partial r}, \quad E_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}, \quad E_\varphi = -\frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \varphi}. \tag{2.6-8}$$

The equations of motion are

$$\begin{aligned}
&\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi r}}{\partial \varphi} \\
&\quad + \frac{1}{r}(2T_{rr} - T_{\theta\theta} - T_{\varphi\varphi} + T_{\theta r}\cot\theta) + \rho f_r = \rho \ddot{u}_r, \\
&\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi\theta}}{\partial \varphi} \\
&\quad + \frac{1}{r}[3T_{r\theta} + (T_{\theta\theta} - T_{\varphi\varphi})\cot\theta] + \rho f_\theta = \rho \ddot{u}_\theta, \\
&\frac{\partial T_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\varphi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi\varphi}}{\partial \varphi} \\
&\quad + \frac{1}{r}(3T_{r\varphi} + 2T_{\theta\varphi}\cot\theta) + \rho f_z = \rho \ddot{u}_\varphi.
\end{aligned} \tag{2.6-9}$$

The electrostatic charge equation is

$$r^2\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(D_\theta \sin\varphi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}D_\varphi = \rho_e. \tag{2.6-10}$$

7. COMPACT MATRIX NOTATION

We now introduce a compact matrix notation [11]. This notation consists of replacing pairs of indices *ij* or *kl* by single indices *p* or *q,* where *i, j, k* and *l* take the values of 1, 2, and 3, and *p* and *q* take the values 1, 2, 3, 4, 5, and 6 according to

$$\begin{array}{lcccccc} ij \text{ or } kl: & 11 & 22 & 33 & 23 \text{ or } 32 & 31 \text{ or } 13 & 12 \text{ or } 21 \\ p \text{ or } q: & 1 & 2 & 3 & 4 & 5 & 6 \end{array}. \tag{2.7-1}$$

Thus

$$c_{ijkl} \to c_{pq}, \quad e_{ikl} \to e_{ip}, \quad T_{ij} \to T_p. \tag{2.7-2}$$

For the strain tensor, we introduce S_p such that

$$\begin{aligned} &S_1 = S_{11}, \quad S_2 = S_{22}, \quad S_3 = S_{33}, \\ &S_4 = 2S_{23}, \quad S_5 = 2S_{31}, \quad S_6 = 2S_{12}. \end{aligned} \tag{2.7-3}$$

The constitutive relations in (2.1-11) can then be written as

$$
\begin{aligned}
T_p &= c_{pq}^E S_q - e_{kp} E_k, \\
D_i &= e_{iq} S_q + \varepsilon_{ik}^S E_k.
\end{aligned}
\tag{2.7-4}
$$

In matrix form, Equation (2.7-4) becomes

$$
\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{pmatrix} c_{11}^E & c_{12}^E & c_{13}^E & c_{14}^E & c_{15}^E & c_{16}^E \\ c_{21}^E & c_{22}^E & c_{23}^E & c_{24}^E & c_{25}^E & c_{26}^E \\ c_{31}^E & c_{32}^E & c_{33}^E & c_{34}^E & c_{35}^E & c_{36}^E \\ c_{41}^E & c_{42}^E & c_{43}^E & c_{44}^E & c_{45}^E & c_{46}^E \\ c_{51}^E & c_{52}^E & c_{53}^E & c_{54}^E & c_{55}^E & c_{56}^E \\ c_{61}^E & c_{62}^E & c_{63}^E & c_{64}^E & c_{65}^E & c_{66}^E \end{pmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} - \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \\ e_{14} & e_{24} & e_{34} \\ e_{15} & e_{25} & e_{35} \\ e_{16} & e_{26} & e_{36} \end{pmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix},
$$

$$
\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} + \begin{pmatrix} \varepsilon_{11}^S & \varepsilon_{12}^S & \varepsilon_{13}^S \\ \varepsilon_{21}^S & \varepsilon_{22}^S & \varepsilon_{22}^S \\ \varepsilon_{31}^S & \varepsilon_{32}^S & \varepsilon_{33}^S \end{pmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}.
\tag{2.7-5}
$$

Similarly, Equations (2.1-16), (2.1-19) and (2.1-20) can also be written in matrix form. The matrices of the material constants in various expressions are related by [11]

$$
\begin{aligned}
& c_{pr}^E s_{qr}^E = \delta_{pq}, \quad c_{pr}^D s_{qr}^D = \delta_{pq}, \\
& \beta_{ik}^S \varepsilon_{jk}^S = \delta_{ijq}, \quad \beta_{ik}^T \varepsilon_{jk}^T = \delta_{ijq}, \\
& c_{pq}^D = c_{pq}^E + e_{kp} h_{kq}, \quad s_{pq}^D = s_{pq}^E - d_{kp} g_{kq}, \\
& \varepsilon_{ij}^T = \varepsilon_{ij}^S + d_{iq} e_{jq}, \quad \beta_{ij}^T = \beta_{ij}^S - g_{iq} h_{jq}, \\
& e_{ip} = d_{iq} c_{pq}^E, \quad d_{ip} = \varepsilon_{ik}^T g_{kp}, \\
& g_{ip} = \beta_{ik}^T d_{kp}, \quad h_{ip} = g_{iq} c_{qp}^D.
\end{aligned}
\tag{2.7-6}
$$

As an example, some of the relations in (2.7-6) are shown below. In matrix-vector notation (2.7-4) can be written as

$$
\begin{aligned}
\{T\} &= [c^E]\{S\} - [e]^T \{E\}, \\
\{D\} &= [e]\{S\} + [\varepsilon^S]\{E\}.
\end{aligned}
\tag{2.7-7}
$$

From $(2.7\text{-}7)_1$,

$$[c^E]\{S\} = \{T\} + [e]^T\{E\}. \tag{2.7-8}$$

Multiplication of both sides of (2.7-8) by the inverse of $[c^E]$ yields

$$\{S\} = [c^E]^{-1}\{T\} + [c^E]^{-1}[e]^T\{E\}. \tag{2.7-9}$$

Substituting Equation (2.7-9) into $(2.7\text{-}7)_2$ gives

$$\begin{aligned}\{D\} &= [e]([c^E]^{-1}\{T\} + [c^E]^{-1}[e]^T\{E\}) + [\varepsilon^S]\{E\}\\ &= [e][c^E]^{-1}\{T\} + ([e][c^E]^{-1}[e]^T + [\varepsilon^S])\{E\}.\end{aligned} \tag{2.7-10}$$

Compare Equations (2.7-9) and (2.7-10) with (2.1-19) which is rewritten in matrix form below:

$$\begin{aligned}\{S\} &= [s^E]\{T\} + [d]^T\{E\},\\ \{D\} &= [d]\{T\} + [\varepsilon^T]\{E\},\end{aligned} \tag{2.7-11}$$

we identify

$$\begin{aligned}&[s^E] = [c^E]^{-1}, \quad [d] = [e][c^E]^{-1},\\ &[\varepsilon^T] = [\varepsilon^S] + [e][c^E]^{-1}[e]^T.\end{aligned} \tag{2.7-12}$$

8. POLARIZED CERAMICS

Polarized ceramics are transversely isotropic. Let **a**, a constant unit vector, represent the direction of the axis of rotational symmetry or the poling direction of the ceramics. For linear constitutive relations we need a quadratic electric enthalpy function H. For transversely isotropic materials, a quadratic H is a function of the following invariants of degrees one and two [16] (higher degree invariants are not included):

$$\begin{aligned}&I_1 = \mathbf{a}\cdot\mathbf{S}\cdot\mathbf{a}, \quad I_2 = \mathrm{tr}\mathbf{S}, \quad I_3 = \mathbf{a}\cdot\mathbf{E},\\ &II_1 = \mathbf{a}\cdot\mathbf{S}^2\cdot\mathbf{a}, \quad II_2 = \mathrm{tr}\mathbf{S}^2,\\ &II_3 = \mathbf{E}\cdot\mathbf{E}, \quad II_4 = \mathbf{a}\cdot\mathbf{S}\cdot\mathbf{E} + \mathbf{E}\cdot\mathbf{S}\cdot\mathbf{a}.\end{aligned} \tag{2.8-1}$$

A complete quadratic function of the above seven invariants can be written as [16]

$$\begin{aligned}H &= c_1 I_1^2 + c_2 I_2^2 + c_3 I_1 I_2 + c_4 II_1 + c_5 II_2\\ &+ \varepsilon_1 I_3^2 + \varepsilon_2 II_3\\ &+ e_1 I_1 I_3 + e_2 I_2 I_3 + e_3 I_4,\end{aligned} \tag{2.8-2}$$

where c_1, c_2, c_3, c_4, and c_5 are elastic constants, ε_1 and ε_2 are dielectric constants, and e_1, e_2, and e_3 are piezoelectric constants. Differentiation of Equation (2.8-2) yields

$$\begin{aligned}\mathbf{T} &= \frac{\partial H}{\partial \mathbf{S}} = \frac{\partial H}{\partial I_1}\mathbf{a}\otimes\mathbf{a} + \frac{\partial H}{\partial I_2}\mathbf{1} + \frac{\partial H}{\partial II_1}(\mathbf{a}\otimes\mathbf{S}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{S}\otimes\mathbf{a}) \\ &\quad + 2\frac{\partial H}{\partial II_2}\mathbf{S} + \frac{\partial H}{\partial II_4}(\mathbf{a}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{a}) \\ &= (2c_1 I_1 + c_3 I_2 + e_1 I_3)\mathbf{a}\otimes\mathbf{a} + (2c_2 I_2 + c_3 I_1 + e_2 I_3)\mathbf{1} \\ &\quad + c_4(\mathbf{a}\otimes\mathbf{S}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{S}\otimes\mathbf{a}) + 2c_5\mathbf{S} + e_3(\mathbf{a}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{a}),\end{aligned} \tag{2.8-3}$$

and

$$\begin{aligned}\mathbf{D} &= -\frac{\partial H}{\partial \mathbf{E}} = -\frac{\partial H}{\partial I_3}\mathbf{a} - 2\frac{\partial H}{\partial II_3}\mathbf{E} - 2\frac{\partial H}{\partial II_4}\mathbf{S}\cdot\mathbf{a} \\ &= -(2\varepsilon_1 I_3 + e_1 I_1 + e_2 I_2)\mathbf{a} - 2\varepsilon_2\mathbf{E} - 2e_3\mathbf{S}\cdot\mathbf{a}.\end{aligned} \tag{2.8-4}$$

Let $\mathbf{a} = \mathbf{i}_3$, and rearrange (2.8-3) and (2.8-4) in the form of (2.7-5). The following matrices will result:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{31} & c_{31} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}, \tag{2.8-5}$$

where $c_{66} = (c_{11} - c_{12})/2$. The matrices in Equation (2.8-5) have the same structures as those of crystals class C_{6v} (or 6mm). The elements of the matrices in (2.8-5) are related to the material constants in (2.8-2) by

$$
\begin{aligned}
&c_1 = c_{11} - 2c_{13} + c_{33} - 4c_{44}, \quad c_2 = c_{12}/2, \\
&c_3 = c_{13} - c_{12}, \quad c_4 = -c_{11} + c_{12} + 2c_{44}, \quad c_5 = (c_{11} - c_{12})/2, \\
&\varepsilon_1 = (\varepsilon_{11} - \varepsilon_{22})/2, \quad \varepsilon_2 = -\varepsilon_{11}/2, \\
&e_1 = e_{31} + 2e_{15} - e_{33}, \quad e_2 = -e_{31}, \quad e_3 = -e_{15}.
\end{aligned}
\tag{2.8-6}
$$

With Equation (2.8-5), the constitutive relations of ceramics poled in the x_3 direction take the following form:

$$
\begin{aligned}
T_{11} &= c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + e_{31}\phi_{,3}, \\
T_{22} &= c_{12}u_{1,1} + c_{22}u_{2,2} + c_{13}u_{3,3} + e_{31}\phi_{,3}, \\
T_{33} &= c_{13}u_{1,1} + c_{13}u_{2,2} + c_{33}u_{3,3} + e_{33}\phi_{,3}, \\
T_{23} &= c_{44}(u_{2,3} + u_{3,2}) + e_{15}\phi_{,2}, \\
T_{31} &= c_{44}(u_{3,1} + u_{1,3}) + e_{15}\phi_{,1}, \\
T_{12} &= c_{66}(u_{1,2} + u_{2,1}),
\end{aligned}
\tag{2.8-7}
$$

and

$$
\begin{aligned}
D_1 &= e_{15}(u_{3,1} + u_{1,3}) - \varepsilon_{11}\phi_{,1}, \\
D_2 &= e_{15}(u_{2,3} + u_{3,2}) - \varepsilon_{11}\phi_{,2}, \\
D_3 &= e_{31}(u_{1,1} + u_{2,2}) + e_{33}u_{3,3} - \varepsilon_{33}\phi_{,3}.
\end{aligned}
\tag{2.8-8}
$$

The equations of motion and charge are

$$
\begin{aligned}
&c_{11}u_{1,11} + (c_{12} + c_{66})u_{2,12} + (c_{13} + c_{44})u_{3,13} + c_{66}u_{1,22} \\
&\quad + c_{44}u_{1,33} + (e_{31} + e_{15})\phi_{,13} = \rho\ddot{u}_1, \\
&c_{66}u_{2,11} + (c_{12} + c_{66})u_{1,12} + c_{11}u_{2,22} + (c_{13} + c_{44})u_{3,23} \\
&\quad + c_{44}u_{2,33} + (e_{31} + e_{15})\phi_{,23} = \rho\ddot{u}_2, \\
&c_{44}u_{3,11} + (c_{44} + c_{13})u_{1,31} + c_{44}u_{3,22} + (c_{13} + c_{44})u_{2,23} \\
&\quad + c_{33}u_{3,33} + e_{15}(\phi_{,11} + \phi_{,22}) + e_{33}\phi_{,33} = \rho\ddot{u}_3, \\
&e_{15}u_{3,11} + (e_{15} + e_{31})u_{1,13} + e_{15}u_{3,22} + (e_{15} + e_{31})u_{2,32} \\
&\quad + e_{31}u_{3,33} - \varepsilon_{11}(\phi_{,11} + \phi_{,22}) - \varepsilon_{33}\phi_{,33} = 0.
\end{aligned}
\tag{2.8-9}
$$

Sometimes a piezoelectric device is heterogeneous with ceramics poled in different directions in different parts. In this case it is not possible to orient the x_3 axis along different poling directions unless a few local coordinate systems are introduced. Therefore, material matrices of ceramics poled along other axes are useful. They can be obtained from the matrices in (2.8-5) by rotating rows and columns properly. For ceramics poled in the x_1 direction, we have

$$\begin{pmatrix} c_{33} & c_{13} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{13} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{pmatrix},$$

$$\begin{pmatrix} e_{33} & e_{31} & e_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{15} \\ 0 & 0 & 0 & 0 & e_{15} & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{33} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{11} \end{pmatrix}. \qquad (2.8\text{-}10)$$

For ceramics poled in the x_2 direction, we obtain

$$\begin{pmatrix} c_{11} & c_{13} & c_{12} & 0 & 0 & 0 \\ c_{13} & c_{33} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{13} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & e_{15} \\ e_{31} & e_{33} & e_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{33} & 0 \\ 0 & 0 & \varepsilon_{11} \end{pmatrix}. \qquad (2.8\text{-}11)$$

9. QUARTZ AND LANGASITE

Quartz is probably the most widely used piezoelectric crystal. It belongs to crystal class 32 (or D_3). Langasite and some of its isomorphs (langanite and langatate) are emerging piezoelectric crystals which have stronger piezoelectric coupling than quartz and also belong to crystal class 32. For such a crystal with x_3 a trigonal axis and x_1 a diagonal axis, the material matrices are

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{21} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & 0 & c_{14} & c_{66} \end{pmatrix},$$

$$\begin{pmatrix} e_{11} & -e_{11} & 0 & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{14} & -e_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}. \tag{2.9-1}$$

The independent material constants are 6 + 2 + 2=10.

Quartz plates are often used to make devices. Plates taken from a bulk crystal at different orientations are referred to as plates of different cuts. A particular cut is specified by two angles, φ and θ, with respect to the crystal axes (X,Y,Z).

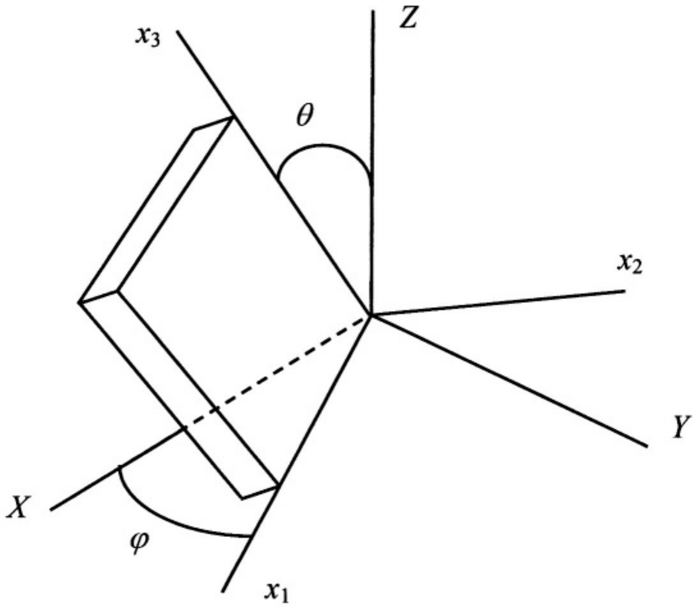

Figure 2.9-1. A quartz plate cut from a bulk crystal.

Plates of different cuts have different material matrices with respect to coordinate systems in and normal to the plane of the plates. One class of cuts of quartz plates, called rotated Y-cuts, has $\varphi = 0$ and is particularly useful in device applications. Rotated Y-cut quartz exhibits monoclinic symmetry of class 2 (or C_2) in a coordinate system (x_1,x_2) in and normal to the plane of the plate. Therefore we list the equations for monoclinic crystals below which are useful for studying quartz devices. For monoclinic crystals, with the diagonal axis along the x_1 axis,

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & c_{34} & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{65} & c_{66} \end{pmatrix},$$

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{25} & e_{26} \\ 0 & 0 & 0 & 0 & e_{35} & e_{36} \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{23} \\ 0 & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}. \tag{2.9-2}$$

The constitutive relations are

$$\begin{aligned} T_{11} &= c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + c_{14}(u_{2,3} + u_{3,2}) + e_{11}\phi_{,1}, \\ T_{22} &= c_{12}u_{1,1} + c_{22}u_{2,2} + c_{23}u_{3,3} + c_{24}(u_{2,3} + u_{3,2}) + e_{12}\phi_{,1}, \\ T_{33} &= c_{13}u_{1,1} + c_{23}u_{2,2} + c_{33}u_{3,3} + c_{34}(u_{2,3} + u_{3,2}) + e_{13}\phi_{,1}, \\ T_{23} &= c_{14}u_{1,1} + c_{24}u_{2,2} + c_{34}u_{3,3} + c_{44}(u_{2,3} + u_{3,2}) + e_{14}\phi_{,1}, \\ T_{31} &= c_{55}(u_{3,1} + u_{1,3}) + c_{56}(u_{1,2} + u_{2,1}) + e_{25}\phi_{,2} + e_{35}\phi_{,3}, \\ T_{12} &= c_{56}(u_{3,1} + u_{1,3}) + c_{66}(u_{1,2} + u_{2,1}) + e_{26}\phi_{,2} + e_{36}\phi_{,3}, \end{aligned} \tag{2.9-3}$$

and

$$\begin{aligned} D_1 &= e_{11}u_{1,1} + e_{12}u_{2,2} + e_{13}u_{3,3} + e_{14}(u_{2,3} + u_{3,2}) - \varepsilon_{11}\phi_{,1}, \\ D_2 &= e_{25}(u_{3,1} + u_{1,3}) + e_{26}(u_{1,2} + u_{2,1}) - \varepsilon_{22}\phi_{,2} - \varepsilon_{23}\phi_{,3}, \\ D_3 &= e_{35}(u_{3,1} + u_{1,3}) + e_{36}(u_{1,2} + u_{2,1}) - \varepsilon_{23}\phi_{,2} - \varepsilon_{33}\phi_{,3}. \end{aligned} \tag{2.9-4}$$

The equations of motion and charge are

$$
\begin{aligned}
& c_{11}u_{1,11} + (c_{12} + c_{66})u_{2,12} + (c_{13} + c_{55})u_{3,13} + (c_{14} + c_{56})u_{2,13} \\
& \quad + (c_{14} + c_{56})u_{3,12} + 2c_{56}u_{1,23} + c_{66}u_{1,22} + c_{55}u_{1,33} \\
& \quad + e_{11}\phi_{,11} + e_{26}\phi_{,22} + (e_{36} + e_{25})\phi_{,23} + e_{35}\phi_{,33} = \rho\ddot{u}_1, \\
& c_{56}u_{3,11} + (c_{56} + c_{14})u_{1,13} + (c_{66} + c_{12})u_{1,12} + c_{66}u_{2,11} \\
& \quad + c_{22}u_{2,22} + (c_{23} + c_{44})u_{3,23} + 2c_{24}u_{2,23} + c_{24}u_{3,22} \\
& \quad + c_{34}u_{3,33} + c_{44}u_{2,33} + (e_{26} + e_{12})\phi_{,12} + (e_{36} + e_{14})\phi_{,13} = \rho\ddot{u}_2, \\
& c_{55}u_{3,11} + (c_{55} + c_{13})u_{1,13} + (c_{56} + c_{14})u_{1,12} + c_{56}u_{2,11} \\
& \quad + c_{24}u_{2,22} + 2c_{34}u_{3,23} + (c_{44} + c_{23})u_{2,23} + c_{44}u_{3,22} \\
& \quad + c_{33}u_{3,33} + c_{34}u_{2,33} + (e_{25} + e_{14})\phi_{,12} + (e_{35} + e_{13})\phi_{,13} = \rho\ddot{u}_3, \\
& e_{11}u_{1,11} + (e_{12} + e_{26})u_{2,12} + (e_{13} + e_{35})u_{3,13} + (e_{14} + e_{36})u_{2,13} \\
& \quad + (e_{14} + e_{25})u_{3,12} + (e_{25} + e_{36})u_{1,23} + e_{26}u_{1,22} \\
& \quad + e_{35}u_{1,33} - \varepsilon_{11}\phi_{,11} - \varepsilon_{22}\phi_{,22} - 2\varepsilon_{23}\phi_{,23} - \varepsilon_{33}\phi_{,33} = 0.
\end{aligned}
\tag{2.9-5}
$$

For rotated Y-cut quartz, motions with only one displacement component, u_1, are particularly useful in device applications. Consider

$$
\begin{aligned}
& u_1 = u_1(x_2, x_3, t), \quad u_2 = u_3 = 0, \\
& \phi = \phi(x_2, x_3, t).
\end{aligned}
\tag{2.9-6}
$$

Equation (2.9-6) yields the following non-vanishing components of strain, electric field, stress, and electric displacement:

$$
\begin{aligned}
& S_5 = u_{1,3}, \quad S_6 = u_{1,2}, \\
& E_2 = -\phi_{,2}, \quad E_3 = -\phi_{,3}, \\
& T_{31} = c_{55}u_{1,3} + c_{56}u_{1,2} + e_{25}\phi_{,2} + e_{35}\phi_{,3}, \\
& T_{21} = c_{56}u_{1,3} + c_{66}u_{1,2} + e_{26}\phi_{,2} + e_{36}\phi_{,3}, \\
& D_2 = e_{25}u_{1,3} + e_{26}u_{1,2} - \varepsilon_{22}\phi_{,2} - \varepsilon_{23}\phi_{,3}, \\
& D_3 = e_{35}u_{1,3} + e_{36}u_{1,2} - \varepsilon_{23}\phi_{,2} - \varepsilon_{33}\phi_{,3}.
\end{aligned}
\tag{2.9-7}
$$

The equations left to be satisfied by u_1 and ϕ are

$$\begin{aligned}&c_{66}u_{1,22}+c_{55}u_{1,33}+2c_{56}u_{1,23}\\&\quad+e_{26}\phi_{,22}+e_{35}\phi_{,33}+(e_{25}+e_{36})\phi_{,23}=\rho\ddot{u}_1,\\&e_{26}u_{1,22}+e_{35}u_{1,33}+(e_{25}+e_{36})u_{1,23}\\&\quad-\varepsilon_{22}\phi_{,22}-\varepsilon_{33}\phi_{,33}-2\varepsilon_{23}\phi_{,23}=0.\end{aligned}\tag{2.9-8}$$

10. LITHIUM NIOBATE AND LITHIUM TANTALATE

Lithium niobate and lithium tantalate have stronger piezoelectric coupling than quartz. For these two crystals the crystal class is $C_{3v} = 3m$. The material matrices are

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{21} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & 0 & c_{14} & c_{66} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & -e_{22} \\ -e_{22} & e_{22} & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}. \tag{2.10-1}$$

When a rotated Y-cut is formed, the material apparently has m-monoclinic symmetry with the following matrices

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & c_{34} & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{65} & c_{66} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & e_{16} \\ e_{21} & e_{22} & e_{23} & e_{24} & 0 & 0 \\ e_{31} & e_{32} & e_{33} & e_{34} & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{23} \\ 0 & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}. \tag{2.10-2}$$

The constitutive relations are

$$T_{11} = c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + c_{14}(u_{2,3} + u_{3,2}) + e_{21}\phi_{,2} + e_{31}\phi_{,3},$$
$$T_{22} = c_{12}u_{1,1} + c_{22}u_{2,2} + c_{23}u_{3,3} + c_{24}(u_{2,3} + u_{3,2}) + e_{22}\phi_{,2} + e_{32}\phi_{,3},$$
$$T_{33} = c_{13}u_{1,1} + c_{23}u_{2,2} + c_{33}u_{3,3} + c_{34}(u_{2,3} + u_{3,2}) + e_{23}\phi_{,2} + e_{33}\phi_{,3},$$
$$T_{23} = c_{14}u_{1,1} + c_{24}u_{2,2} + c_{34}u_{3,3} + c_{44}(u_{2,3} + u_{3,2}) + e_{24}\phi_{,2} + e_{34}\phi_{,3},$$
$$T_{31} = c_{55}(u_{3,1} + u_{1,3}) + c_{56}(u_{1,2} + u_{2,1}) + e_{15}\phi_{,1},$$
$$T_{12} = c_{56}(u_{3,1} + u_{1,3}) + c_{66}(u_{1,2} + u_{2,1}) + e_{16}\phi_{,1},$$

(2.10-3)

and

$$D_1 = e_{15}(u_{1,3} + u_{3,1}) + e_{16}(u_{1,2} + u_{2,1}) - \varepsilon_{11}\phi_{,1},$$
$$D_2 = e_{21}u_{1,1} + e_{22}u_{2,2} + e_{23}u_{3,3} + e_{24}(u_{2,3} + u_{3,2}) - \varepsilon_{22}\phi_{,2} - \varepsilon_{23}\phi_{,3},$$
$$D_3 = e_{31}u_{1,1} + e_{32}u_{2,2} + e_{33}u_{3,3} + e_{34}(u_{2,3} + u_{3,2}) - \varepsilon_{23}\phi_{,2} - \varepsilon_{33}\phi_{,3}.$$

(2.10-4)

The equations of motion and charge are

$$\begin{aligned}
&c_{11}u_{1,11} + (c_{12} + c_{66})u_{2,12} + (c_{13} + c_{55})u_{3,13} + (c_{14} + c_{56})u_{2,13} \\
&\quad + (c_{14} + c_{56})u_{3,12} + 2c_{56}u_{1,23} + c_{66}u_{1,22} + c_{55}u_{1,33} \\
&\quad + (e_{21} + e_{16})\phi_{,12} + (e_{31} + e_{15})\phi_{,13} = \rho\ddot{u}_1, \\
&c_{56}u_{3,11} + (c_{56} + c_{14})u_{1,13} + (c_{66} + c_{12})u_{1,12} + c_{66}u_{2,11} + c_{22}u_{2,22} \\
&\quad + (c_{23} + c_{44})u_{3,23} + 2c_{24}u_{2,23} + c_{24}u_{3,22} + c_{34}u_{3,33} + c_{44}u_{2,33} \\
&\quad + e_{16}\phi_{,11} + e_{22}\phi_{,22} + (e_{32} + e_{24})\phi_{,23} + e_{34}\phi_{,33} = \rho\ddot{u}_2, \\
&c_{55}u_{3,11} + (c_{55} + c_{13})u_{1,13} + (c_{56} + c_{14})u_{1,12} + c_{56}u_{2,11} + c_{24}u_{2,22} \\
&\quad + 2c_{34}u_{3,23} + (c_{44} + c_{23})u_{2,23} + c_{44}u_{3,22} + c_{33}u_{3,33} + c_{34}u_{2,33} \\
&\quad + e_{15}\phi_{,11} + e_{24}\phi_{,22} + (e_{34} + e_{23})\phi_{,23} + e_{33}\phi_{,33} = \rho\ddot{u}_3, \\
&(e_{15} + e_{31})u_{1,31} + e_{15}u_{3,11} + (e_{16} + e_{21})u_{1,12} + e_{16}u_{2,11} + e_{22}u_{2,22} \\
&\quad + (e_{23} + e_{34})u_{3,23} + (e_{24} + e_{32})u_{2,23} + e_{24}u_{3,22} + e_{33}u_{3,33} + e_{34}u_{2,33} \\
&\quad - \varepsilon_{11}\phi_{,11} - \varepsilon_{22}\phi_{,22} - 2\varepsilon_{23}\phi_{,23} - \varepsilon_{33}\phi_{,33} = 0.
\end{aligned}$$

(2.10-5)

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