CHAPTER 2

BLASCHKE TYPE PRODUCTS

We shall consider a family of Blaschke type products in the lower half-plane $G^- = \{w : \operatorname{Im} w < 0\}$ and investigate some of the properties of these products. The factors of our products depend on a parameter $-1 < \alpha < +\infty$ and are chosen to be of the form

$$b_\alpha(w,\zeta) = \exp\left\{ \int_0^{|\eta|} \left(\big[\tau + i(w-\zeta)\big]^{-1-\alpha} + \big[i(w-\overline{\zeta}) - \tau\big]^{-1-\alpha} \right) \tau^\alpha d\tau \right\} \quad (1)$$

For $\alpha = 0$ this expression becomes $\frac{w-\zeta}{w-\overline{\zeta}}$, and the inversion $z = w^{-1}$ ($z_k = w_k^{-1}$, $k = 1, 2, \ldots$) transfers the properties of (1) into ones for several products in the upper half-plane $G^+ = \{z : \operatorname{Im} z > 0\}$, which for $\alpha = 0$ coincide with the widely known Blaschke product

$$\widetilde{B}(z, \{z_k\}) = \prod_k \frac{1 - z/z_k}{1 - z/\overline{z}_k}, \quad \{z_k\} \subset G^+, \quad \sum_k \left| \operatorname{Im} \frac{1}{z_k} \right| < +\infty. \qquad (2)$$

1. ELEMENTARY PROPERTIES OF FACTORS

1.1. We start by another representation of $b_\alpha(w,\zeta)$. Assuming $\zeta = \xi + i\eta \in G^-$ a fixed point, for any $\alpha \in (-1, +\infty)$ we introduce the function

$$\Omega_\alpha(w,\zeta) = \int_0^{|\eta|} \frac{\tau^\alpha d\tau}{[\tau + i(w-\zeta)]^{1+\alpha}} + \int_0^{|\eta|} \frac{\tau^\alpha d\tau}{\big[i(w-\overline{\zeta}) - \tau\big]^{1+\alpha}}. \qquad (1.1)$$

The changes of variables $t = -(\tau + \eta)$ and $t = \tau + \eta$ in the above integrals give

$$\Omega_\alpha(w,\zeta) = \int_{-|\eta|}^{|\eta|} \frac{(|\eta| - |t|)^\alpha dt}{[i(w-\xi) - t]^{1+\alpha}}, \quad -1 < \alpha < +\infty. \qquad (1.1')$$

By (1) and (1.1)

$$b_\alpha(w,\zeta) = \exp\{-\Omega_\alpha(w,\zeta)\}, \quad -1 < \alpha < +\infty. \qquad (1.2)$$

Sometimes we shall write $\Omega_\alpha(w,\zeta)$ in the form

$$\Omega_\alpha(w,\zeta)=U_\alpha(w,\zeta)+V_\alpha(w,\zeta),\quad -1<\alpha<+\infty, \tag{1.3}$$

where

$$U_\alpha(w,\zeta)=\int_{-|\eta|}^{0}\frac{(|\eta|+t)^\alpha dt}{[i(w-\xi)-t]^{1+\alpha}},\quad V_\alpha(w,\zeta)=\int_0^{|\eta|}\frac{(|\eta|-t)^\alpha dt}{[i(w-\xi)-t]^{1+\alpha}}. \tag{1.3'}$$

Theorem 1.1. *The function $b_\alpha(w,\zeta)$ $(-1<\alpha<+\infty)$ is holomorphic in the region $\mathbb{C}\backslash\{\xi+ih : 0\le h<+\infty\}$ and it vanishes only at the point $\zeta\in G^-$, which is a first order zero.*

Proof. By (1.2), (1.3), (1.4)

$$b_\alpha(w,\zeta)=\frac{w-\zeta}{w-\bar{\zeta}}\exp\Big\{U_0(w,\zeta)-U_\alpha(w,\zeta)+V_0(w,\zeta)-V_\alpha(w,\zeta)\Big\}.$$

The functions $U_\alpha(w,\zeta)$ and $U_0(w,\zeta)$ obviously are holomorphic in $\mathbb{C}\backslash\{\xi+ih : 0\le h<+\infty\}$. Thus, it suffices to show that

$$\begin{aligned}F_\alpha(w,\zeta)\equiv& V_0(w,\zeta)-V_\alpha(w,\zeta)\\ =&-\int_0^{|\eta|}\left\{\frac{(|\eta|-t)^\alpha}{[i(w-\xi)-t]^{1+\alpha}}-\frac{1}{i(w-\xi)-t}\right\}dt\end{aligned}$$

is holomorphic in the same domain. To this end, we write

$$\begin{aligned}F_\alpha(w,\zeta)&=\left(\int_{-\infty}^{0}-\int_{-\infty}^{|\eta|}\right)\left\{\frac{(|\eta|-t)^\alpha}{[i(w-\xi)-t]^{1+\alpha}}-\frac{1}{i(w-\xi)-t}\right\}dt\\ &\equiv I_\alpha^{(1)}(w,\zeta)-I_\alpha^{(2)}(w,\zeta)\end{aligned}$$

and separately prove the holomorphity of $I_\alpha^{(1)}$ and $I_\alpha^{(2)}$. First observe that

$$\varphi_\alpha(w,\zeta,t)\equiv\frac{(|\eta|-t)^\alpha}{[i(w-\xi)-t]^{1+\alpha}}-\frac{1}{i(w-\xi)-t}\quad(-\infty<t<0)$$

is holomorphic in $\mathbb{C}\backslash\{\xi+ih : 0\le h<+\infty\}$, and for $-\infty<t<|\eta|$ the same function is holomorphic in the smaller domain $\mathbb{C}\backslash\{\xi+ih : -|\eta|\le h<+\infty\}$. Next, for any compact $\mathbf{K}\subset\mathbb{C}$

$$\varphi_\alpha(w,\zeta,t)=O(t^{-2})\quad\text{as}\quad t\to-\infty$$

uniformly in respect to $w\in\mathbf{K}$. Therefore, by uniform convergence, the integrals $I_\alpha^{(1)}(w,\zeta)$ and $I_\alpha^{(2)}(w,\zeta)$ represent holomorphic functions in the domains

$\mathbb{C}\backslash\{\xi + ih : 0 \leq h < +\infty\}$ and $\mathbb{C}\backslash\{\xi + ih : -|\eta| \leq h < +\infty\}$ respectively. But $I_\alpha^{(2)}(w, \zeta)$ is a constant. Indeed, considering this function on the ray $w = \zeta - ih$ $(0 < h < +\infty)$ and taking $t = -\sigma h + |\eta|$ we get

$$I_\alpha^{(2)}(\zeta - ih, \zeta) = \int_0^{+\infty} \left\{ \frac{\sigma^\alpha}{(1+\sigma)^{1+\alpha}} - \frac{1}{1+\sigma} \right\} d\sigma \equiv \text{const} \neq \infty.$$

1.2. The below recurrent formulas for $U_\alpha(w, \zeta)$ and $V_\alpha(w, \zeta)$ lead to some representations of $\Omega_\alpha(w, \zeta)$. We shall assume that $\alpha > 0$ and $p \geq 1$ is the natural number deduced from $p - 1 < \alpha \leq p$. First observe that integration by parts in (1.3′) gives

$$U_\alpha(w, \zeta) = \frac{1}{\alpha} \int_{t=-|\eta|}^{0} (|\eta| + t)^\alpha d[i(w - \zeta) - t]^{-\alpha} = \frac{1}{\alpha} \left(\frac{i\eta}{w - \xi} \right)^\alpha - U_{\alpha-1}(w, \zeta).$$

Hence

$$U_\alpha(w, \zeta) = \sum_{j=1}^{p} \frac{(-1)^{p-j}}{\alpha - p + j} \left(\frac{i\eta}{w - \xi} \right)^{\alpha - p + j} + (-1)^p U_{\alpha-p}(w, \zeta). \tag{1.4}$$

Similar to this,

$$V_\alpha(w, \zeta) = -\sum_{j=1}^{p} \frac{1}{\alpha - p + j} \left(\frac{i\eta}{w - \xi} \right)^{\alpha - p + j} + V_{\alpha-p}(w, \zeta). \tag{1.4′}$$

By these recurrent formulas and (1.2), (1.3), (1.4)

$$\begin{aligned} b_\alpha(w, \zeta) = &\exp\Big\{(-1)^{p+1} U_{\alpha-p}(w, \zeta) - V_{\alpha-p}(w, \zeta)\Big\} \\ &\times \exp\left\{ \sum_{j=1}^{p} \frac{1 - (-1)^{p-j}}{\alpha - p + j} \left(\frac{i\eta}{w - \xi} \right)^{\alpha - p + j} \right\}. \end{aligned} \tag{1.5}$$

1.3. **Lemma 1.1.** *If $\alpha \in (-1, +\infty)$ and $\zeta = \xi + i\eta \in G^-$ are arbitrary, and $w = u + iv$ is an arbitrary point from the cut complex plane $\mathbb{C}\backslash\{\zeta + ih : 0 \leq h < +\infty\}$, such that $|w - \xi| > |\eta|$, then:*

$$|\Omega_\alpha(w, \zeta)| \leq \frac{2}{(|w - \xi| - |\eta|)^{1+\alpha}} \frac{|\eta|^{1+\alpha}}{1 + \alpha}, \tag{1.6}$$

$$\left| \frac{\partial}{\partial v} \Omega_\alpha(w, \zeta) \right| \leq \frac{2}{(|w - \xi| - |\eta|)^{2+\alpha}} |\eta|^{1+\alpha}. \tag{1.6′}$$

Proof. Evidently $|i(w - \xi) - t|^{1+\alpha} \geq (|w - \xi| - |\eta|)^{1+\alpha}$ for any $t \in [-|\eta|, |\eta|]$. Hence by (1.1′) we come to (1.6). For proving (1.6′), one have to repeat the same argument after a differentiation of (1.1′) by $v = \operatorname{Im} w$.

Lemma 1.2. *If $\alpha \in (-1, +\infty)$, then for any compact $\mathbf{K} \subset G^-$*

$$\Omega_\alpha(w, \xi + i\eta) = \frac{2e^{-i\frac{\pi}{2}(1+\alpha)}}{(w-\xi)^{1+\alpha}} \frac{|\eta|^{1+\alpha}}{1+\alpha} + O(|\eta|^{2+\alpha}) \quad as \quad \eta \to -0 \tag{1.7}$$

uniformly in respect to $w \in \mathbf{K}$ and $\xi \in (-\infty, +\infty)$.

Proof. Integration by parts in (1.3′) gives

$$U_\alpha(w, \zeta) = \frac{1}{1+\alpha}\left(\frac{i\eta}{w-\xi}\right)^{1+\alpha} - U_{1+\alpha}(w, \zeta),$$

$$V_\alpha(w, \zeta) = \frac{1}{1+\alpha}\left(\frac{i\eta}{w-\xi}\right)^{1+\alpha} + V_{1+\alpha}(w, \zeta).$$

Hence by (1.3)

$$\Omega_\alpha(w, \xi + i\eta) = \frac{2e^{-i\frac{\pi}{2}(1+\alpha)}}{(w-\xi)^{1+\alpha}} \frac{|\eta|^{1+\alpha}}{1+\alpha} + R_\alpha(w, \zeta),$$

where $R_\alpha(w, \zeta) = -U_{1+\alpha}(w, \zeta) + V_{1+\alpha}(w, \zeta)$. We shall prove that there exists a constant C depending on α and on the compact $\mathbf{K}$, for which

$$|R_\alpha(w, \zeta)| \leq C|\eta|^{2+\alpha} \quad \text{as} \quad \eta \to -0$$

uniformly in respect to $w \in \mathbf{K}$ and $\xi \in (-\infty, +\infty)$. To this end, we denote $\rho = \min_{w \in \mathbf{K}} |\operatorname{Im} w|$ and observe that if $|\eta| < \rho/2$, then $|w - \xi| - |\eta| \geq \rho/2$ for any $w \in \mathbf{K}$ and $\xi \in (-\infty, +\infty)$. The desired estimate for R_α follows from (1.3′).

Remark 1.1. If $|w| \geq R_0 \geq 4|\zeta|$, then $|w - \xi| - |\eta| \geq |w| - 2|\zeta| \geq |w|/2$ since $2|\zeta| > |\xi| + |\eta|$. Hence by (1.2), (1.6) and (1.6′)

$$|\log|b_\alpha(w, \zeta)|| \leq |\Omega_\alpha(w, \zeta)| \leq \frac{2^{2+\alpha}}{1+\alpha}|\eta|^{1+\alpha}|w|^{-(1+\alpha)}, \tag{1.8}$$

$$\left|\frac{\partial}{\partial v}\log|b_\alpha(w, \zeta)|\right| \leq \left|\frac{\partial}{\partial v}\Omega_\alpha(w, \zeta)\right| \leq 2^{3+\alpha}|\eta|^{1+\alpha}|w|^{-(2+\alpha)} \tag{1.8′}$$

for any $w \in \mathbb{C} \backslash \{\zeta + ih : 0 \leq h < +\infty\}$ ($|w| \geq R_0$). More simple representations than (1.5) are true for $b_\alpha(w, \zeta)$ for $\alpha = p$, where p is an even or odd number (see [54], formulas (1.6)–(1.8′)).

1.4. Assuming that $s = \delta + i\lambda$ is an arbitrary fixed point from G^+ and $\zeta = \xi + i\eta \in G^- = s^{-1}$ ($\in G^-$), consider the ray

$$\Gamma^*[\xi, \infty] = \{w = \xi + ih : 0 \leq h \leq +\infty\} \tag{1.9}$$

and the arc

$$L^*[\xi^{-1},0] = \{\Gamma^*[\xi,\infty)\}^{-1} = \{z = w^{-1} : w \in \Gamma^*[\xi,\infty)\}. \tag{1.10}$$

One can see that $L^*[\xi^{-1},0]$ is contained in the closed half-plane $\overline{G^-} = \{w : \text{Im } w \le 0\}$ and for $\xi \ne 0$ it becomes a semi-circle connecting $\xi^{-1} = |s|^2/\delta$ with the origin, and the center is on the real axis. Now introduce the functions

$$\widetilde{b}_\alpha(z,s) \equiv b_\alpha(w,\zeta)\Big|_{w=z^{-1},\,\zeta=s^{-1}}, \quad -1 < \alpha < +\infty, \tag{1.11}$$

and observe that their properties are simple restatement of those of $b_\alpha(w,\zeta)$. Particularly,

$$\widetilde{b}_\alpha(z,s) = \frac{1 - z/s}{1 - z/\overline{s}}. \tag{1.12}$$

Further, in view of (1.2), (1.2′) and (1.5)

$$\widetilde{b}_\alpha(z,s) = \exp\left\{-\int_{-|\text{Im } s^{-1}|}^{|\text{Im } s^{-1}|} \frac{\left(|\text{Im } s^{-1}| - |t|\right)^\alpha dt}{\left[i(z^{-1} - \text{Re } s^{-1}) - t\right]^{1+\alpha}}\right\} \tag{1.13}$$

for $-1 < \alpha < +\infty$, and the following similarity of Theorem 1.1 is true.

Theorem 1.1*. *The function $\widetilde{b}_\alpha(z,s)$ $(-1 < \alpha < +\infty)$ is holomorphic in $\mathbb{C}\backslash L^*\left[|s|^2/\delta, 0\right]$ and vanishes only at the point $s \in G^+$, which is a zero of first order.*

2. INTEGRO-DIFFERENTIAL PROPERTIES OF FACTORS

Assuming $\zeta = \xi + i\eta$ a fixed point in G^-, we shall apply Liouville's integro-differential operator $W^{-\alpha}$ (formulas (1.13)-(1.15) in Ch. 1) to $\log b_\alpha(w,\zeta) \equiv -\Omega_\alpha(w,z)$ and consider the properties of the function

$$W^{-\alpha} \log|b_\alpha(w,\zeta)| = -W^{-\alpha}\text{Re } \Omega_\alpha(w,\zeta), \quad -1 < \alpha < +\infty.$$

2.1. Introduce the following line intercepts:

$$\begin{aligned}(\zeta,\overline{\zeta}) &= \{\xi + ih : -|\eta| < h < |\eta|\},\\ [\zeta,\overline{\zeta}] &= \{\xi + ih : -|\eta| \le h \le |\eta|\},\\ (\zeta,\xi) &= \{\xi + ih : -|\eta| < h < 0\}.\end{aligned} \tag{2.1}$$

Lemma 2.1. *The function $W^{-\alpha}\log b_\alpha(w,\zeta)$ $(-1 < \alpha < +\infty)$ is holomorphic in the domain $\mathbb{C}\backslash[\zeta,\overline{\zeta}]$, where*

$$W^{-\alpha}\log b_\alpha(w,\zeta) = -W^{-\alpha}\Omega_\alpha(w,\zeta) = \frac{1}{\Gamma(1+\alpha)}\int_{-|\eta|}^{|\eta|} \frac{(|\eta| - |t|)^\alpha dt}{t - i(w - \xi)}. \tag{2.2}$$

Proof. As W^0 is identical, for $\alpha = 0$ our assertion is trivial by (1.1′) and (1.2). The cases $\alpha > 0$ and $\alpha < 0$ have to be considered separately.

(a) $0 < \alpha < +\infty$. By (1.1′), (1.2) and (1.13) of Ch. 1

$$W^{-\alpha}\log b_\alpha(w,\zeta) \equiv -W^{-\alpha}\Omega_\alpha(w,\zeta)$$
$$= -\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\sigma^{\alpha-1}d\sigma\int_{-|\eta|}^{|\eta|}\frac{(|\eta|-|t|)^\alpha dt}{[i(w-\xi)-t+\sigma]^{1+\alpha}}$$

for $w \in \mathbb{C}\backslash\{\zeta + ih;\ 0 \le h < +\infty\}$. It is obvious that the right-hand side integral in this equality is absolutely convergent. Therefore, using the well-known formula

$$\int_0^{+\infty}\frac{\sigma^{\alpha-1}d\sigma}{(z+\sigma)^{1+\alpha}} = \frac{1}{\alpha z}\left(\frac{\sigma}{z+\sigma}\right)^\alpha\Bigg|_{\alpha=0}^{+\infty} = \frac{1}{\alpha z}, \quad \alpha > 0, \tag{2.3}$$

we come to (2.2) for $w \in \mathbb{C}\backslash\{\zeta + ih : 0 \le h < +\infty\}$. From (2.2) it follows that $W^{-\alpha}\Omega_\alpha(w,\zeta)$ permits a holomorphic continuation onto $\mathbb{C}\backslash[\zeta,\bar{\zeta}]$. Therefore, (2.2) is true in whole $\mathbb{C}\backslash[\zeta,\bar{\zeta}]$.

(b) $-1 < \alpha < 0$. By (1.1′), (1.2) and (1.15) of Ch. 1

$$W^{-\alpha}\log b_\alpha(w,\zeta) \equiv -W^{-\alpha}\Omega_\alpha(w,\zeta)$$
$$= -\frac{1+\alpha}{\Gamma(1+\alpha)}\int_0^{+\infty}\sigma^\alpha d\sigma\int_{-|\eta|}^{|\eta|}\frac{(|\eta|-|t|)^\alpha dt}{[i(w-\xi)-t+\sigma]^{2+\alpha}}$$

for $w \in \mathbb{C}\backslash\{\zeta + ih : 0 \le h < +\infty\}$. Hence we again come to (2.2).

Remark 2.1. From (2.2) it follows that

$$\left|W^{-\alpha}\log|b_\alpha(w,\zeta)|\right| \le \frac{2}{|w-\xi|-|\eta|}\frac{|\eta|^{1+\alpha}}{\Gamma(2+\alpha)}, \quad -1 < \alpha < +\infty, \tag{2.4}$$

for any $w \in \mathbb{C}$ such that $|w-\xi| > |\eta|$. The proof of this estimate is similar to that of (1.6).

Lemma 2.2. 1°. *For any* $\alpha \in (-1,+\infty)$ *and any* $w \in \mathbb{C}\backslash[\zeta,\bar{\zeta}]$

$$W^{-\alpha}\log|b_\alpha(w,\zeta)| = -\mathrm{Re}\ W^{-\alpha}\Omega_\alpha(w,\zeta)$$
$$= \frac{2\mathrm{Im}\ w}{\Gamma(1+\alpha)}\int_0^{|\eta|}\frac{|w-\xi|^2-t^2}{|(w-\xi)^2+t^2|^2}(|\eta|-t)^\alpha dt. \tag{2.5}$$

2°. *For any* $\alpha \in (0,+\infty)$ *and any* $w \in \mathbb{C}$

$$W^{-\alpha}\log|b_\alpha(w,\zeta)| = -\mathrm{Re}\ W^{-\alpha}\Omega_\alpha(w,\zeta)$$
$$= \frac{1}{\Gamma(\alpha)}\int_0^{|\eta|}\log\left|\frac{\xi-it-w}{\xi+it-w}\right|(|\eta|-t)^\alpha dt$$
$$= \frac{1}{\Gamma(\alpha)}\iint_{G^-}\log\left|\frac{w-\bar{s}}{w-s}\right||\xi-s|^{\alpha-1}\mathfrak{X}_\zeta(s)d\sigma(s), \tag{2.6}$$

where $\mathfrak{X}_\zeta(s)$ is the characteristic function of the interval (ζ, ξ) and $d\sigma(s)$ is the area element.

Proof. 1°. Denoting $z = i(w - \xi)$, from (2.2) we obtain

$$W^{-\alpha} \log |b_\alpha(w, \zeta)| = \frac{1}{\Gamma(1+\alpha)} \int_0^{|\eta|} \mathrm{Re} \left\{ \frac{1}{t-z} - \frac{1}{t+z} \right\} (|\eta| - t)^\alpha dt.$$

Hence (2.5) follows.

2°. Observe that $\mathrm{Re}\,\{dt/[t - i(w - \xi)]\} = d_t \log |t - i(w - \xi)|$ for $w \in \mathbb{C}$ and $t \in (-\infty, +\infty)$. Therefore, integration by parts in (2.2) gives

$$\begin{aligned} W^{-\alpha} \log |b_\alpha(w, \zeta)| &= \frac{1}{\Gamma(1+\alpha)} \int_{-|\eta|}^{|\eta|} \log \frac{1}{|t - i(w - \xi)|} d(|\eta| - t)^\alpha \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{|\eta|} \log \left| \frac{\xi - it - w}{\xi + it - w} \right| (|\eta| - t)^\alpha dt \\ &= \frac{1}{\Gamma(\alpha)} \iint_{G^-} \log \left| \frac{w - \bar{s}}{w - s} \right| |\xi - s|^{\alpha - 1} \mathfrak{X}_\zeta(s) d\sigma(s), \end{aligned}$$

and the proof is complete.

From (2.5) it follows that $W^{-\alpha} \log |b_\alpha(w, \zeta)|$ is harmonic outside the intercept $[\zeta, \bar{\zeta}]$, and also that this function vanishes on the real axis, i.e.

$$W^{-\alpha} \log |b_\alpha(u, \zeta)| = 0, \quad -\infty < u < +\infty, \quad u \neq \xi \quad \alpha > -1. \tag{2.7}$$

Further, for $|w - \xi| \geq |\eta|$ the integrand in (2.5) is nonnegative. Hence, for $|w - \xi| > |\eta|$

$$W^{-\alpha} \log |b_\alpha(w, \zeta)| \begin{cases} < 0, & \text{if} \quad w \in G^-, \\ > 0, & \text{if} \quad w \in G^+, \end{cases} \qquad -1 < \alpha < +\infty. \tag{2.8}$$

2.2. For further analysis of $W^{-\alpha} \log |b_\alpha(w, \zeta)|$ we use some well-known properties of Cauchy type integrals. From (2.2) it follows that for $z \notin \mathbb{C} \backslash [-|\eta|, |\eta|]$

$$W^{-\alpha} \log |b_\alpha(-iz + \xi, \zeta)| = -\frac{2}{\Gamma(1+\alpha)} \mathrm{Im}\ \Phi_\alpha(z), \quad -1 < \alpha < +\infty, \tag{2.9}$$

where

$$\Phi_\alpha(z) = \frac{1}{2\pi} \int_{-|\eta|}^{|\eta|} \frac{(|\eta| - |t|)^\alpha}{t - z} dt. \tag{2.9'}$$

In view of these formulas

$$W^{-\alpha} \log |b_\alpha(-iz + \xi, \zeta)| \equiv -W^{-\alpha} \log |b_\alpha(iz + \xi, \zeta)|, \quad z \in \mathbb{C}. \tag{2.10}$$

The function $\Phi_\alpha(z)$ is holomorphic in $\mathbb{C}\backslash[-|\eta|,|\eta|]$. Besides, $\varphi_\alpha(t)=(|\eta|-|t|)^\alpha$ satisfies the Lipschitz condition (for $\alpha=0$ and $\alpha\geq 1$ of order 1 on $[-|\eta|,|\eta|]$, for $0<\alpha<1$ of order α on $[-|\eta|,|\eta|]$ and for $-1<\alpha<0$ of order 1 on $(-|\eta|,|\eta|)$). Therefore, the below statements are true by the well-known theory of Cauchy type integrals (see, for instance, [34], Ch. 1).

1°. The Cauchy type integral

$$\Phi_\alpha(x)=\frac{1}{2\pi}\int_{-|\eta|}^{|\eta|}\frac{(|\eta|-|t|)^\alpha}{t-x}dt,\quad -1<\alpha<+\infty,$$

is a continuous function of Lipλ in $(-|\eta|,|\eta|)$ for some $\lambda\in(0,1)$.

2°. For any $\alpha\in(-1,+\infty)$ and any $x\in(-|\eta|,|\eta|)$ the limits

$$\lim_{z\to x,\, z\in G^+}\Phi_\alpha(z)=\Phi_\alpha^+(x),\qquad \lim_{z\to x,\, z\in G^-}\Phi_\alpha(z)=\Phi_\alpha^-(x)$$

exist, are finite and connected by the following formulas:

$$\Phi_\alpha^+(x)-\Phi_\alpha^-(x)=(|\eta|-|t|)^\alpha,\quad \Phi_\alpha^+(x)+\Phi_\alpha^-(x)=2\Phi_\alpha(x).$$

3°. For $\alpha>-1$ these limits are continuous functions of Lip λ on $(-|\eta|,|\eta|)$ with some $\lambda\in(0,1)$.

4°. If $\alpha>0$, then $\Phi_\alpha(z)$ is continuous at the points $-|\eta|$ and $|\eta|$, as a function of complex variable.

5°. If $-1<\alpha<0$, then the following representations are true in enough small neighborhoods of the points $-|\eta|$ and $|\eta|$:

$$\Phi_\alpha(z)=-\frac{e^{-i\pi\alpha}}{2i\sin\pi\alpha}(z+|\eta|)^\alpha+\psi_\alpha^0(z),\;\; \Phi_\alpha(z)=\frac{e^{-i\pi\alpha}}{2i\sin\pi\alpha}(-z+|\eta|)^\alpha+\psi_\alpha^1(z),$$

where $\psi_\alpha^0(z)$ and $\psi_\alpha^1(z)$ are holomorphic in the same neighborhoods.

In view of (2.9) and (2.9′) we come to the following

Lemma 2.3. *The function $W^{-\alpha}\log|b_\alpha(w,\zeta)|$ $(-1<\alpha<+\infty)$, which is harmonic in $\mathbb{C}\backslash[\zeta,\overline{\zeta}]$, is continuous on $[\zeta,\overline{\zeta}]$ for $\alpha>0$, and is continuous on $(\zeta,\overline{\zeta})$ for $-1<\alpha<0$. Besides, for $-1<\alpha<0$*

$$W^{-\alpha}\log|b_\alpha(w,\zeta)| \\ =\begin{cases}\dfrac{\Gamma(1-\alpha)}{\alpha}|w-\zeta|^\alpha\cos\left[\alpha\arg i(w-\zeta)\right]+u_\alpha^0(w),\\[2mm] \dfrac{\Gamma(1-\alpha)}{\alpha}|w-\overline{\zeta}|^\alpha\cos\left[\alpha\arg i(\overline{\zeta}-w)\right]+u_\alpha^1(w),\end{cases}\tag{2.11}$$

in enough small neighborhoods of $\zeta \in G^-$ *and* $\overline{\zeta} \in G^+$, *where* $u_\alpha^1(w)$ *and* $u_\alpha^1(w)$ *are harmonic.*

In contrast to the properties of $\log|b_0(w,\zeta)|$, it appears that the entire intercept $[\zeta,\overline{\zeta}]$ is the support of the mass of $W^{-\alpha}\log|b_\alpha(w,\zeta)|$.

Theorem 2.1. 1°. *For* $\alpha \in (0,+\infty)$ *the function* $W^{-\alpha}\log|b_\alpha(w,\zeta)|$ *is continuous in the closed* w*-plane, harmonic out of* $[\zeta,\overline{\zeta}]$, *subharmonic in* G^- *and superharmonic in* G^+. *Besides,*

$$W^{-\alpha}\log|b_\alpha(w,\zeta)| \begin{cases} <0 & \text{for} \quad w \in G^-, \\ >0 & \text{for} \quad w \in G^+, \end{cases} \quad 0<\alpha<+\infty. \tag{2.12}$$

2°. *For* $\alpha \in (-1,0)$ *the function* $W^{-\alpha}\log|b_\alpha(w,\zeta)|$ *is continuous in the closed* w*-plane, except the points* $\zeta \in G^-$ *and* $\overline{\zeta} \in G^+$, *is harmonic out of* $[\zeta,\overline{\zeta}]$, *superharmonic in* $G^-\backslash\zeta$ *and subharmonic in* $G^+\backslash\overline{\zeta}$. *Besides, for* $-1<\alpha<0$.

$$W^{-\alpha}\log|b_\alpha(w,\zeta)| \begin{cases} <0 & \text{for} \quad w \in G^-, \quad |w-\xi|>|\eta|, \\ >0 & \text{for} \quad w \in G^+, \quad |w-\xi|>|\eta|. \end{cases} \tag{2.13}$$

3°. *For any* $\alpha \in (-1,+\infty)$

$$W^{-\alpha}\log|b_\alpha(u,\zeta)| = 0, \quad -\infty<u<+\infty. \tag{2.14}$$

Proof. Whatever be $\alpha \in (-1,+\infty)$, by (2.2) $W^{-\alpha}\log|b_\alpha(w,\zeta)|$ is continuous in the closure of w-plane (by (2.2) $W^{-\alpha}\log b_\alpha(\infty,\zeta)=0$), with possible exception of points ζ and $\overline{\zeta}$. Therefore, 3° follows from (2.7).

1°. Let $\alpha>0$. Then in view of (2.9) and (2.9′) $W^{-\alpha}\log|b_\alpha(w,\zeta)|$ is harmonic in w-plane, except $[\zeta,\overline{\zeta}]$. Because of continuity of this function, it suffices to prove that for any $s \in [\zeta,\overline{\zeta}]$ and enough small $\rho>0$

$$\frac{1}{2\pi}\int_0^{2\pi} W^{-\alpha}\log|b_\alpha(s+\rho e^{i\vartheta},\zeta)|d\vartheta$$

$$\begin{cases} >W^{-\alpha}\log|b_\alpha(s,\zeta)|, & s \in [\zeta,\xi), \\ <W^{-\alpha}\log|b_\alpha(s,\zeta)|, & s \in (\xi,\overline{\zeta}]. \end{cases} \tag{2.15}$$

Hence our assertion will follow by the maximum and the minimum principles of subharmonic and superharmonic functions. Before proving (2.15), observe that by (2.10) and (2.14)

$$\frac{1}{2\pi}\int_0^{2\pi} W^{-\alpha}\log|b_\alpha(\xi+\rho e^{i\vartheta},\zeta)|d\vartheta = W^{-\alpha}\log|b_\alpha(\xi,\zeta)| = 0 \tag{2.16}$$

for any $\alpha \in (-1,+\infty)$ and $\rho>0$. Now assume that $0<h<|\eta|$ and $0<\rho<\max\{h,|\eta|-h\}$. Then by (2.2)

$$\frac{1}{2\pi}\int_0^{2\pi} W^{-\alpha}\log|b_\alpha(\xi-ih+\rho e^{i\vartheta},\zeta)|d\vartheta$$

$$= \frac{1}{\Gamma(1+\alpha)}\int_{-|\eta|}^{|\eta|}(|\eta|-|t|)^\alpha \mathrm{Re}\left\{\frac{1}{2\pi}\int_{|s|=\rho}\frac{ds}{s(t-h-is)}\right\}dt$$

for any $1 < \alpha < +\infty$. The last inner integral vanishes for $|t-h| < \rho$ and equals $(t-h)^{-1}$ for $|t-h| > \rho$. Hence

$$\frac{1}{2\pi}\int_0^{2\pi} W^{-\alpha}\log|b_\alpha(\xi - ih + \rho e^{i\vartheta},\zeta)|d\vartheta$$
$$= \frac{1}{\Gamma(1+\alpha)}\left(\int_{-|\eta|}^{h-\rho} + \int_{h+\rho}^{|\eta|}\right)\frac{(|\eta|-|t|)^\alpha}{t-h}dt.$$

Consequently, by (2.2) we obtain that

$$\frac{1}{2\pi}\int_0^{2\pi} W^{-\alpha}\log|b_\alpha(\xi - ih + \rho e^{i\vartheta},\zeta)|d\vartheta - W^{-\alpha}\log|b_\alpha(\xi - ih,\zeta)|$$
$$= -\frac{1}{\Gamma(1+\alpha)}\int_{h-\rho}^{h+\rho}\frac{(|\eta|-|t|)^\alpha}{t-h}dt \qquad (2.17)$$

for $\alpha \in (-1,+\infty)$, $0 < h < |\eta|$ and $0 < \rho < \max\{h, |\eta| - h\}$. One can be convinced that also

$$\frac{1}{2\pi}\int_0^{2\pi} W^{-\alpha}\log|b_\alpha(\zeta + \rho e^{i\vartheta},\zeta)|d\vartheta - W^{-\alpha}\log|b_\alpha(\zeta,\zeta)|$$
$$= \frac{1}{\Gamma(1+\alpha)}\frac{\rho^\alpha}{\alpha} > 0. \qquad (2.17')$$

for any $\alpha > 0$ and $\rho \in (0,|\eta|)$. Now observe that

$$\int_{h-\rho}^{h+\rho}\frac{(|\eta|-|t|)^\alpha}{t-h}dt = \int_0^\rho \frac{(|\eta|-x-h)^\alpha - (|\eta|+x-h)^\alpha}{x}dx.$$

Consequently,

$$\frac{1}{2\pi}\int_0^{2\pi} W^{-\alpha}\log|b_\alpha(\xi - ih + \rho e^{i\vartheta},\zeta)|d\vartheta - W^{-\alpha}\log|b_\alpha(\xi - ih,\zeta)|$$
$$= \frac{1}{\Gamma(1+\alpha)}\int_0^\rho \frac{(|\eta|+x-h)^\alpha - (|\eta|-x-h)^\alpha}{x}dx$$
$$\begin{cases} > 0, & 0 < \alpha < +\infty \\ < 0, & -1 < \alpha < 0 \end{cases} \qquad (2.18)$$

for any $\alpha \in (-1,+\infty)$, $0 < h < |\eta|$ and $0 < \rho < \max\{h, |\eta| - h\}$. Hence the estimates (2.15) follow by (2.17′) and (2.10).

2°. As we have shown, the function $W^{-\alpha}\log|b_\alpha(w,\zeta)|$ $(-1 < \alpha < 0)$ is continuous in the closure of w-plane, except the points ζ and $\overline{\zeta}$. Besides, this function is harmonic out of the intercept $[\zeta,\overline{\zeta}]$ and the estimates (2.13) (see (2.8)) are true. Therefore, it suffices to see that by the second inequality in (2.18) and (2.10) $W^{-\alpha}\log|b_\alpha(w,\zeta)|$ is superharmonic in $G^-\backslash\zeta$ and subharmonic in $G^+\backslash\bar{\zeta}$.

2.3. **Lemma 2.4.** *For any* $\alpha\in(-1,+\infty)$

$$\lim_{v\to 0}\int_{-\infty}^{+\infty}\left|W^{-\alpha}\log|b_\alpha(u+iv,\zeta)|\right|du=0. \tag{2.19}$$

Proof. By (2.2)

$$W^{-\alpha}\log|b_\alpha(u+iv,\zeta)|=\frac{1}{\Gamma(1+\alpha)}\int_{-|\eta|}^{|\eta|}\frac{t+v}{(u-\xi)^2+(t+v)^2}(|\eta|-|t|)^\alpha dt.$$

for any u, $v\in(-\infty,+\infty)$. Hence, changing the order of integration we get

$$\int_{-\infty}^{+\infty}W^{-\alpha}\log|b_\alpha(u+iv,\zeta)|du$$
$$=\frac{\pi}{\Gamma(1+\alpha)}\int_{-|\eta|}^{|\eta|}(|\eta|-|t|)^\alpha\text{sign }(t+v)dt\equiv I_\alpha(|\eta|,v).$$

For calculation of the last integral, observe that sign $(t+v)=$ sign v for $|v|\geq|\eta|$ and $-|\eta|<t<|\eta|$. Hence, for $|v|\geq|\eta|$

$$I_\alpha(|\eta|,v)=(\text{sign }v)\frac{\pi}{\Gamma(1+\alpha)}\int_{-|\eta|}^{|\eta|}(|\eta|-|t|)^\alpha dt=(\text{sign }v)\frac{2\pi}{\Gamma(2+\alpha)}|\eta|^{1+\alpha}.$$

On the other hand,

$$I_\alpha(|\eta|,v)=(\text{sign }v)\frac{2\pi}{\Gamma(2+\alpha)}\left[|\eta|^{1+\alpha}-(|\eta|-|v|)^{1+\alpha}\right]$$

for $|v|<|\eta|$. Consequently, for any $v\in(-\infty,+\infty)$

$$\int_{-\infty}^{+\infty}W^{-\alpha}\log|b_\alpha(u+iv,\zeta)|du$$
$$=\begin{cases}(\text{sign }v)\dfrac{2\pi}{\Gamma(2+\alpha)}|\eta|^{1+\alpha} & \text{if}\quad |v|\geq|\eta|,\\[2ex] (\text{sign }v)\dfrac{2\pi}{\Gamma(2+\alpha)}\left[|\eta|^{1+\alpha}-(|\eta|-|v|)^{1+\alpha}\right] & \text{if}\quad |v|<|\eta|.\end{cases} \tag{2.20}$$

As the function $W^{-\alpha}\log|b_\alpha(w,\zeta)|$ $(\alpha\geq 0)$ keeps its sign in G^- and G^+, (2.20) implies (2.19). For proving (2.19) with $\alpha\in(-1,0)$, note that in view of (2.10) it suffices to prove (2.19) as $v\to -0$. To this end, suppose $v<0$ and denote by S_v^+ the part of the line $\{w = u + iv : -\infty < u < +\infty\}$, where $W^{-\alpha} \log |b_\alpha(w, \zeta)| \geq 0$, and by S_v^- the remaining part of the same line, where $W^{-\alpha} \log |b_\alpha(w, \zeta)| < 0$. Then

$$\int_{-\infty}^{+\infty} \left|W^{-\alpha} \log |b_\alpha(u + iv, \zeta)|\right| du = \left(\int_{S_v^+} - \int_{S_v^-}\right) W^{-\alpha} \log |b_\alpha(u + iv, \zeta)| du$$
$$= \left(2 \int_{S_v^+} - \int_{-\infty}^{+\infty}\right) W^{-\alpha} \log |b_\alpha(u + iv, \zeta)| du. \tag{2.21}$$

But the last integral in the right-hand side of this formula vanishes as $v \to -0$. Therefore, it remains to show that

$$\lim_{v \to -0} \int_{S_v^+} W^{-\alpha} \log |b_\alpha(u + iv, \zeta)| du = 0.$$

To this end, observe that in virtue of first inequality in (2.13) S_v^+ is contained in the semidisc $\{w : |w - \xi| \leq |\eta|,\ w \in G^-\}$ for $|v| < |\eta|$ $(v < 0)$. Hence

$$0 \leq \int_{S_v^+} W^{-\alpha} \log |b_\alpha(u+iv, \zeta)| du < \int_{\xi - |\eta|}^{\xi + |\eta|} \left|W^{-\alpha} \log |b_\alpha(u + iv, \zeta)|\right| du. \tag{2.22}$$

Further, by Theorem 2.1 $W^{-\alpha} \log |b_\alpha(w, \zeta)|$ is continuous in the closed quadrat $\{w = u + iv : |u - \xi| \leq |\eta|,\ \eta/2 \leq v \leq 0\}$, and therefore is bounded there. Thus, by Fatou's lemma and (2.14)

$$\limsup_{v \to -0} \int_{\xi - |\eta|}^{\xi + |\eta|} \left|W^{-\alpha} \log |b_\alpha(u + iv, \zeta)|\right| du$$
$$\leq \int_{\xi - |\eta|}^{\xi + |\eta|} \lim_{v \to -0} \left|W^{-\alpha} \log |b_\alpha(u + iv, \zeta)|\right| du = 0.$$

2.4. Note that by (2.20) and (2.21)

$$\int_{-\infty}^{+\infty} \left|W^{-\alpha} \log |b_\alpha(u + iv, \zeta)|\right| du \leq \frac{6\pi}{\Gamma(2 + \alpha)} |\eta|^{1+\alpha} \tag{2.23}$$

for any $-1 < \alpha < +\infty$ and $-\infty < v < +\infty$. Further, note that the results of this section can be inverted to similar results on properties of $\widetilde{W}^{-\alpha} \log |\widetilde{b}_\alpha(z, s)|$, where $\widetilde{W}^{-\alpha}$ is the integro-differential operator mentioned in Ch. 1 (formulas (1.21)–(1.23)) since by our notation (1.11)

$$\widetilde{W}^{-\alpha} \log |\widetilde{b}_\alpha(z, s)| \Big|_{z = w^{-1},\, s = \zeta^{-1}} \equiv W^{-\alpha} \log |b_\alpha(w, \zeta)|, \quad z \in \overline{\mathbb{C}}. \tag{2.24}$$

3. BLASCHKE TYPE PRODUCTS

Now we proceed to the convergence and main properties of the products

$$B_\alpha(w, \{w_k\}) \equiv \prod_k b_\alpha(w, w_k), \quad -1 < \alpha < +\infty, \tag{3.1}$$

with zeros $\{w_k\}$ in the lower half-plane G^-. These will result in similar properties of the product

$$\widetilde{B}_\alpha(z, \{z_k\}) \equiv \prod_k \widetilde{b}_\alpha(z, z_k), \quad -1 < \alpha < +\infty, \tag{3.2}$$

the zeros $z_k = w_k^{-1}$ $(k = 1, 2, \dots)$ of which lie in the upper half-plane G^+.

3.1. For $\rho < 0$ we set $G_\rho^- = \{w : \text{Im } w < \rho\}$ and $\overline{G_\rho^-} = \{w : \text{Im } w \le \rho\}$.

Theorem 3.1. $1°$. *Let $\{w_k\}_1^\infty \subset G^-$ be any sequence of complex numbers satisfying*

$$\sum_{k=1}^\infty |\text{Im } w_k|^{1+\alpha} < +\infty \tag{3.3}$$

for a given $\alpha \in (-1, +\infty)$. Then for any $\rho < 0$ the infinite product

$$B_\alpha(w, \{w_k\}_1^\infty) \equiv \prod_{k=1}^\infty b_\alpha(w, w_k) = \exp\left\{-\sum_{k=1}^\infty \Omega_\alpha(w, w_k)\right\} \tag{3.4}$$

is absolutely and uniformly convergent in the closed half-plane $\overline{G_\rho^-}$, and the function $B_\alpha(w, \{w_k\}_1^\infty)$ is holomorphic in G^-, where $\{w_k\}_1^\infty$ are its zeros.

$2°$. *If $\{w_k\}_1^\infty \subset G^-$ is a bounded sequence, and for a given $\alpha \in (-1, +\infty)$ the product (3.4) is absolutely and uniformly convergent inside G^-, then $\{w_k\}_1^\infty$ satisfies (3.3).*

Poof. $1°$. Let $\rho < 0$ be a fixed number. Since by (3.3) $\text{Im } w_k \to -0$ as $k \to \infty$, one can choose a natural number $N_\rho \ge 1$ such that $w_k \notin \overline{G_\rho^-}$ for $k \ge N_\rho + 1$. Besides, $w_k \in \overline{G_\rho^-}$, then $|w - \text{Re } w_k| \ge |\text{Im } w| \ge |\rho| > |\text{Im } w_k|$ for $k \ge N_\rho + 1$. Therefore, by (1.6)

$$|\Omega_\alpha(w, w_k)| \equiv |\log b_\alpha(w, w_k)| \le \frac{1}{(|\rho| - |\text{Im } w_k|)^{1+\alpha}} \frac{|\text{Im } w_k|^{1+\alpha}}{1+\alpha}$$

for any $w_k \in \overline{G_\rho^-}$ and $k \ge N_\rho + 1$. Together with (3.3), this implies the absolute and uniform convergence of the series

$$\sum_{k=N_\rho+1}^\infty \log b_\alpha(w, w_k) = -\sum_{k=N_\rho+1}^\infty \Omega_\alpha(w, w_k) \tag{3.5}$$

in $\overline{G_\rho^-}$. Hence, also the product (3.4) is absolutely and uniformly convergent in $\overline{G_\rho^-}$. Consequently, $B_\alpha(w, \{w_k\}_1^\infty)$ is holomorphic in G^- and $\{w_k\}_1^\infty$ are its zeros.

2°. The function $B_\alpha(w, \{w_k\}_1^\infty) \not\equiv 0$ is holomorphic in G^- since (3.4) is assumed uniform convergent inside G^-. Besides, $\text{Im } w_k \to 0$ as $k \to \infty$ since the sequence $\{w_k\}_1^\infty$ is bounded. Observe that the absolute convergence of the product (3.4) in any fixed point $a \in G^-$ is equivalent to the absolute convergence of the series (3.5) (where $\rho = \text{Im } a$ and N_ρ is chosen by ρ as above) in the same point. Besides, by our requirements $\sup_k |\text{Re } w_k| \le \sup_k |w_k| = M < +\infty$. Hence

$$\left| \frac{2e^{-i\frac{\pi}{2}(1+\alpha)}|\text{Im } w_k|^{1+\alpha}}{(a - \text{Re } w_k)^{1+\alpha}(1+\alpha)} \right| \ge \big(|a+M| + |a-M|\big)^{-1-\alpha} \frac{2|\text{Im } w_k|^{1+\alpha}}{1+\alpha}.$$

Since $\text{Im } w_k \to 0$ as $k \to \infty$, (1.7) and the last inequality lead to

$$|\Omega_\alpha(a, w_k)| \ge C|\text{Im } w_k|^{1+\alpha}, \quad k \ge k_0,$$

for enough great $k_0 \ge 1$, where $C \equiv C(\alpha, a, M) > 0$ is a constant independent of k. Therefore, the absolute convergence of (3.5) in $w = a$ implies (3.3), and the proof is complete.

Remark 3.1. Considering the convergence of $B_\alpha(w, \{w_k\}_1^\infty)$ on compacts $\mathbf{K} \subset \mathbb{C}$, one can show that under (3.3) $B_\alpha(w, \{w_k\}_1^\infty)$ $(\alpha > -1)$ admits holomorphic continuation into the strip $\{w : a < \text{Re } w < b,\ 0 < \text{Im } w < +\infty\} \subset G^+$, through any interval (a, b) disjoint from the points $u_k = \text{Re } w_k$ $(k = 1, 2, \dots)$.

Let $\{w_k\} \subset G^-$ be a bounded sequence (finite or infinite) satisfying (3.3) for a given $\alpha \in (-1, +\infty)$. Denote $\sup_k |w_k| = M$ $(< +\infty)$ and assume $w \in G^-$ a fixed point such that $|w| > 4M$. Then, in view of absolute convergence of (3.5) and (1.8)

$$\begin{aligned} \big|\log|B_\alpha(w, \{w_k\})|\big| &\le \sum_k \big|\log|b_\alpha(w, w_k)|\big| \\ &\le \left\{ \frac{2^{2+\alpha}}{1+\alpha} \sum_k |\text{Im } w_k|^{1+\alpha} \right\} |w|^{-(1+\alpha)}. \end{aligned} \tag{3.6}$$

3.2. The below lemmas are to be used in investigation of the properties of $W^{-\alpha} \log|B_\alpha(w, \{w_k\})|$. Let the sequence $\{w_k\}_1^\infty \subset G^-$ satisfy (3.3) for a given $\alpha \in (-1, +\infty)$. Then $\text{Im } w_k \to 0$ as $k \to \infty$ and, as it was mentioned above, for any closed half-plane $\overline{G_\rho^-} = \{w : \text{Im } w \le \rho < 0\}$ one can choose $N \equiv N_\rho + 1$ such that $w_k \notin \overline{G_\rho^-}$ for $k \ge N + 1$. Introduce the function

$$\begin{aligned} \psi_\alpha(w) &\equiv \log|B_\alpha(w, \{w_k\}_1^\infty)| - \sum_{k=1}^{N} \log|b_\alpha(w, w_k)| \\ &\equiv \sum_{k=N+1}^{\infty} \log|b_\alpha(w, w_k)|. \end{aligned} \tag{3.7}$$

For $k \geq N+1$ the summands in the last series are harmonic functions in $\overline{G_\rho^-}$. On the other hand, from the proof of Theorem 3.1 it follows that this series is absolutely and uniformly convergent in $\overline{G_\rho^-}$. Therefore $\psi_\alpha(w)$ is harmonic in G_ρ^-. Using the inequalities (1.6′) one can show that in $\overline{G_\rho^-}$ there is also another absolutely and uniformly convergent representation:

$$\frac{\partial}{\partial(\mathrm{Im}\ w)}\psi_\alpha(w) = \sum_{k=N+1}^{\infty} \frac{\partial}{\partial(\mathrm{Im}\ w)} \log|b_\alpha(w, w_k)|. \tag{3.7′}$$

Lemma 3.2. *If (3.3) is true for a given $\alpha \in (-1, +\infty)$, then the series*

$$\sum_{k=N+1}^{\infty} W^{-\alpha} \log|b_\alpha(w, w_k)|$$

is absolutely and uniformly convergent in the closed half-plane $\overline{G_\rho^-}$.

Proof. We have $w_k \notin \overline{G_\rho^-}$ for $k \geq N+1$ by the choice of $N \equiv N_\rho$. Besides, for $k \geq N+1$

$$\left|W^{-\alpha} \log|b_\alpha(w, w_k)|\right| \leq \frac{2}{\Gamma(2+\alpha)} \frac{|\mathrm{Im}\ w_k|^{1+\alpha}}{|\rho| - \delta}, \quad w_k \in \overline{G_\rho^-}, \tag{3.8}$$

where $\delta = \max_{k \geq N+1} |\mathrm{Im}\ w_k| < |\rho|$. Hence the desired assertion follows.

Lemma 3.3. *If (3.3) is true for a given $\alpha \in (-1, +\infty)$, then*

$$W^{-\alpha}\psi_\alpha(w) \equiv \sum_{k=N+1}^{\infty} W^{-\alpha} \log|b_\alpha(w, w_k)|, \quad w \in \overline{G_\rho^-}, \tag{3.9}$$

where the series is absolutely and uniformly convergent in $\overline{G_\rho^-}$.

Proof. Setting $\delta = \max_{k \geq N+1} |\mathrm{Im}\ w_k|$, preliminarily we give two estimates which are consequences of (1.6) and (1.6 ′) and are true for $u \in (-\infty, +\infty)$, $t \in (-\infty, \rho)$ and $k \geq N+1$:

$$|\log|b_\alpha(u+it, w_k)|| \leq \frac{2}{(|t| - \delta)^{1+\alpha}} \frac{|\mathrm{Im}\ w_k|^{1+\alpha}}{1+\alpha}, \tag{3.10}$$

$$\left|\frac{\partial}{\partial t} \log|b_\alpha(u+it, w_k)|\right| \leq \frac{2}{(|t| - \delta)^{2+\alpha}} |\mathrm{Im}\ w_k|^{1+\alpha}. \tag{3.10′}$$

(a) Let $\alpha > 0$, then from (3.10) it follows that the integral

$$I(w) \equiv \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{v} (v-t)^{\alpha-1} \sum_{k=N+1}^{\infty} |\log|b_\alpha(u+it, w_k)||\, dt$$

is convergent for $w = u + iv \in \overline{G_\rho^-}$. Indeed, since $-\infty < t < v < \rho$

$$I(w) \le \frac{2}{1+\alpha}\frac{1}{\Gamma(\alpha)}\left(\sum_{k=N+1}^{\infty} |\mathrm{Im}\ w_k|^{1+\alpha}\right)\int_{-\infty}^{v} \frac{(v-t)^{\alpha-1}}{(|t|-\delta)^{1+\alpha}}dt < +\infty$$

by (3.10) and the convergence of the series (3.3). Hence, for $w = u + iv \in \overline{G_\rho^-}$

$$\begin{aligned} W^{-\alpha}\psi_\alpha(w) \equiv& \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{v}(v-t)^{\alpha-1}\sum_{k=N+1}^{\infty}\log|b_\alpha(u+it,w_k)|dt \\ =& \sum_{k=N+1}^{\infty}\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{v}(v-t)^{\alpha-1}\log|b_\alpha(u+it,w_k)|dt \\ =& \sum_{k=N+1}^{\infty} W^{-\alpha}\log|b_\alpha(w,w_k)| \end{aligned}$$

in view of the definition (3.7) of $\psi_\alpha(w)$ and the convergence of $I(w)$.

(b) Let $-1 < \alpha < 0$. Similar to previous case, (3.10′) implies the convergence of

$$J(w) \equiv \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{v}(v-t)^{\alpha}\sum_{k=N+1}^{\infty}\left|\frac{\partial}{\partial t}\log|b_\alpha(u+it,w_k)|\right|dt$$

for $w = u + iv \in \overline{G_\rho^-}$. Hence, at any $w = u + iv \in \overline{G_\rho^-}$

$$\begin{aligned} W^{-\alpha}\psi_\alpha(w) \equiv& \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{v}(v-t)^{\alpha}\sum_{k=N+1}^{\infty}\frac{\partial}{\partial t}\log|b_\alpha(u+it,w_k)|dt \\ =& \sum_{k=N+1}^{\infty}\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{v}(v-t)^{\alpha}\frac{\partial}{\partial t}\log|b_\alpha(u+it,w_k)|dt \\ =& \sum_{k=N+1}^{\infty} W^{-\alpha}\log|b_\alpha(w,w_k)|. \end{aligned}$$

3.3. The following theorem is a consequence of formula (3.7), Lemmas 3.2 and 3.3 and the properties of $W^{-\alpha}\log|b_\alpha(w,w_k)|$ stated in Theorem 2.1.

Theorem 3.2. 1°. *If a sequence $\{w_k\} \subset G^-$ satisfies (3.3) for a given $\alpha \in (0, +\infty)$, then the function $W^{-\alpha}\log|B_\alpha(w,\{w_k\})|$ is continuous and subharmonic in G^- and harmonic in the domain $G^- \setminus \cup_k [w_k, \mathrm{Re}\ w_k)$, and*

$$W^{-\alpha}\log|B_\alpha(w,\{w_k\})| \le 0, \quad w \in G^-.$$

2°. *If* $\{w_k\} \subset G^-$ *satisfies (3.3) for an* $\alpha \in (-1,0)$*, then the function* $W^{-\alpha} \log|B_\alpha(w,\{w_k\})|$ *is continuous and superharmonic in* $G^- \setminus \{w_k\}$ *and is harmonic in the domain* $G^- \setminus \cup_k [w_k, \mathrm{Re}\, w_k)$*. Besides,*

$$W^{-\alpha} \log|B_\alpha(w,\{w_k\})| \leq 0$$

for any $w \in G^-$ *such that* $|w - \mathrm{Re}\, w_k| > |\mathrm{Im}\, w_k|$ $(k = 1, 2, \ldots)$.

Remark 3.2. By a similar argument, one can also show that under (3.3) the series

$$W^{-\alpha} \log|B_\alpha(w,\{w_k\})| = \sum_k W^{-\alpha} \log|b_\alpha(w, w_k)|, \quad \alpha \in (-1, +\infty), \tag{3.11}$$

is absolutely and uniformly convergent in any compact $\mathbf{K} \subset \mathbb{C}$ disjoint from condensation points of the sequence $\{\mathrm{Re}\, w_k\}$. Therefore, by Theorem 2.1 $W^{-\alpha} \log|B_\alpha(w,\{w_k\})|$ permits continuous extension from G^- through any interval (a,b) containing not more than finite number of points $u_k = \mathrm{Re}\, w_k$. Besides,

$$W^{-\alpha} \log|B_\alpha(u,\{w_k\})| = 0, \quad a < u < b. \tag{3.11'}$$

Theorem 3.3. *If a sequence* $\{w_k\} \subset G^-$ *satisfies (3.3) for a given* $\alpha \in (-1, +\infty)$*, then*

$$\lim_{v \to -0} \int_{-\infty}^{+\infty} \left| W^{-\alpha} \log|B_\alpha(u+iv,\{w_k\})| \right| du = 0. \tag{3.12}$$

Proof. By (3.7) for $w \in G^- \setminus \{w_k\}$

$$\left| W^{-\alpha} \log|B_\alpha(w,\{w_k\})| \right| \leq \sum_k \left| W^{-\alpha} \log|b_\alpha(w, w_k)| \right|, \tag{3.13}$$

where the series is convergent. Therefore, by (2.23)

$$\int_{-\infty}^{+\infty} \left| W^{-\alpha} \log|B_\alpha(u+iv,\{w_k\})| \right| du \\ \leq \frac{6\pi}{\Gamma(2+\alpha)} \sum_k |\mathrm{Im}\, w_k|^{1+\alpha} < +\infty \tag{3.14}$$

for any $v < 0$. Hence, by (2.19) and Fatou's lemma

$$0 \leq \limsup_{v \to -0} \int_{-\infty}^{+\infty} \left| W^{-\alpha} \log|B_\alpha(u+iv,\{w_k\})| \right| du \\ \leq \sum_k \limsup_{v \to -0} \int_{-\infty}^{+\infty} \left| W^{-\alpha} \log|b_\alpha(u+iv,\{w_k\})| \right| du = 0.$$

Remark 3.3. Since $W^{-\alpha}\log|B_\alpha(w,\{w_k\})|$ ($\alpha \geq 0$) is subharmonic in G^- and satisfies (3.12), one can use the well-known Littlewood's theorem on boundary values of subharmonic functions (see, for instance, [105], Ch. IV, Sec. 10, Theorem IV.34) to prove that for almost all $u \in (-\infty, +\infty)$

$$\lim_{w \to u_0,\, w \in l(u_0)} W^{-\alpha} \log|B_\alpha(w,\{w_k\})| = 0, \tag{3.15}$$

where $l(u_0)$ is the image of a radius for any conformal mapping of $|z|<1$ onto G^-.

Lemma 3.4. *If $\{w_k\} \subset G^-$ is a bounded sequence (i.e. $\sup_k |w_k| = r_0 < +\infty$) satisfying (3.3) for a given $\alpha \in (-1,+\infty)$, then for $w \in G^-$ ($|w| > 4r_0$)*

$$\left|W^{-\alpha}\log|B_\alpha(w,\{w_k\})|\right| \leq \left\{\frac{4}{\Gamma(2+\alpha)}\sum_k |\mathrm{Im}\ w_k|^{1+\alpha}\right\}|w|^{-1}. \tag{3.16}$$

Proof. Assume $w \in G^-$ and $|w| > 4r_0$. Since $|w - \mathrm{Re}\ w_k| - |\mathrm{Im}\ w_k| \geq |w| - (|\mathrm{Re}\ w_k| + |\mathrm{Im}\ w_k|) \geq |w| - 2|w_k| > |w|/2 + 2(r_0 - |w_k|) \geq |w_k|/2$, (2.4) implies

$$\left|W^{-\alpha}\log|b_\alpha(w,\{w_k\})|\right| \leq \frac{4}{\Gamma(2+\alpha)}|\mathrm{Im}\ w_k|^{1+\alpha}|w|^{-1}.$$

Therefore, (3.16) follows from (3.11).

3.4. Now we pass to the properties of the products

$$\widetilde{B}_\alpha(z,\{z_k\}) = \prod_k \widetilde{b}_\alpha(z,z_k), \quad -1 < \alpha < +\infty, \tag{3.17}$$

with zeros in the upper half-plane G^+. Some of these properties are described by means of the integro-differential operator $\widetilde{W}^{-\alpha}$ (formulas (1.23)–(1.25) in Ch. 1). One can verify that the inversion $w = z^{-1}$ ($w_k = z_k^{-1}$, $k = 1,2,\ldots$) transforms Theorems 3.1, 3.2 and 3.3 into the following equivalent assertions.

Theorem 3.1*. 1°. *Let $\{z_k\}_1^\infty$ be a sequence of complex numbers in the upper half-plane G^+ and let*

$$\sum_{k=1}^\infty \left|\mathrm{Im}\ \frac{1}{z_k}\right|^{1+\alpha} < +\infty \tag{3.18}$$

for a given $\alpha \in (-1,+\infty)$. Then the infinite product

$$\widetilde{B}_\alpha(z,\{z_k\}_1^\infty) \equiv \prod_{k=1}^\infty \widetilde{b}_\alpha(z,z_k) \tag{3.19}$$

is absolutely and uniformly convergent inside G^+, and $\widetilde{B}_\alpha(z,\{z_k\})$ is a holomorphic function in G^+, with zeros $\{z_k\}$.

2°. *If an infinite sequence $\{z_k\}_1^\infty \subset G^+$ lies out of some neighborhood of the origin, and for an $\alpha \in (-1,+\infty)$ the product (3.4) is absolutely and uniformly convergent inside G^+, then $\{z_k\}_1^\infty$ satisfies (3.18).*

In the next theorem $L[s, \mathrm{Re}\, s^{-1})$ $(s \in G^+)$ means an arc with the endpoints z_k and $\mathrm{Re}\, {z_k}^{-1}$, which is the part of a tangential to the imaginary axis circle centered on the real axis.

Theorem 3.2*. 1°. *If a sequence $\{z_k\} \subset G^+$ satisfies (3.18) for a given $\alpha \in (0,+\infty)$, then $\widetilde{W}^{-\alpha} \log|\widetilde{B}_\alpha(z,\{z_k\})|$ is continuous and subharmonic in G^+ and this function is harmonic in $G^+ \setminus \cup_k L[z_k, \mathrm{Re}\, z_k)$. Besides*

$$\widetilde{W}^{-\alpha} \log|\widetilde{B}_\alpha(z,\{z_k\})| \le 0, \quad z \in G^+.$$

2°. *If a sequence $\{z_k\} \subset G^+$ satisfies (3.18) for some $\alpha \in (-1,0)$, then $\widetilde{W}^{-\alpha} \log|\widetilde{B}_\alpha(z,\{z_k\})|$ is continuous and superharmonic in $G^+ \setminus \{z_k\}$ and is harmonic in $G^+ \setminus \cup_k L[z_k, \mathrm{Re}\, z_k)$. Besides,*

$$\widetilde{W}^{-\alpha} \log|\widetilde{B}_\alpha(z,\{z_k\})| \le 0$$

for any $z \in G^+$ such that $|1/z - \mathrm{Re}\, 1/z_k| > |\mathrm{Im}\, 1/z_k|$ $(k = 1,2,\dots)$.

Theorem 3.3*. *If $\{z_k\} \subset G^+$ satisfies (3.18) for an $\alpha \in (-1,+\infty)$, then*

$$\sup_{0<R<+\infty} \int_\pi^\pi \left|\widetilde{W}^{-\alpha} \log|\widetilde{B}_\alpha(R\sin\vartheta e^{i\vartheta},\{z_k\})|\right| \frac{d\vartheta}{R\sin^2\vartheta} \le \frac{6\pi}{\Gamma(2+\alpha)} \sum_k \left|\mathrm{Im}\, \frac{1}{z_k}\right|^{1+\alpha} < +\infty, \tag{3.20}$$

$$\lim_{R\to+\infty} \int_{-\pi}^\pi \left|\widetilde{W}^{-\alpha} \log|\widetilde{B}_\alpha(R\sin\vartheta e^{i\vartheta},\{z_k\})|\right| \frac{d\vartheta}{R\sin^2\vartheta} = 0. \tag{3.21}$$

4. A REPRESENTATION OF THE PRODUCT

4.1. Below we shall prove a representation which will be used later, in Ch. 5, for investigation of boundary properties of our Blaschke type products.

Theorem 4.1. *Let $\alpha \in (-1,1)$ be any number, and let $\{w_k\} \subset G^-$ be a sequence of complex numbers such that simultaneously*

$$\sum_k |\mathrm{Im}\, w_k|^{1+\alpha} < +\infty \quad \textit{and} \quad \sum_k |\mathrm{Im}\, w_k| < +\infty. \tag{4.1}$$

Then for any $w \in G^-$

$$B_\alpha(w,\{w_k\}) = B_0(w,\{w_k\}) \times \exp\left\{\frac{\Gamma(1+\alpha)}{\pi} e^{-i\frac{\pi}{2}(1+\alpha)} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(w-t)^{1+\alpha}}\right\}, \tag{4.2}$$

where $\mu(t)$ is a nonincreasing, bounded function in $(-\infty,+\infty)$ for $-1<\alpha<0$, and for $0<\alpha<1$ this function is nondecreasing in $(-\infty,+\infty)$ but such that

$$\int_{-\infty}^{+\infty}\frac{d\mu(t)}{1+|t|^{\alpha+\varepsilon}}<+\infty \tag{4.3}$$

for any $\varepsilon>0$. Besides, whatever be $\alpha\in(-1,1)$ there exists a sequence $\delta_n\downarrow 0$ by which

$$\mu(t)=\lim_{n\to\infty}\int_0^t W^{-\alpha}\log\left|\frac{B_\alpha(u-i\delta_n,\{w_k\})}{B_0(u-i\delta_n,\{w_k\})}\right|du,\quad -\infty<t<+\infty. \tag{4.4}$$

4.2. Before establishing some necessary lemmas, note that for $\alpha=0$

$$\log b_\alpha(w,\zeta)\equiv-\Omega_\alpha(w,\zeta),\quad w,\zeta\in G^-,\quad \alpha>-1,$$

becomes the main branch of the logarithm.

Lemma 4.1. *Let $\alpha\in(-1,1)$ and $\zeta=\xi+i\eta\in G^-$ be any fixed numbers. Then the function*

$$\varphi_\alpha(w,\zeta)=W^{-\alpha}\log\frac{b_\alpha(w,\zeta)}{b_0(w,\zeta)} \tag{4.5}$$

is holomorphic in $\mathbb{C}\backslash\{\xi+ih:\ 0\le h<+\infty\}$ where

$$\begin{aligned}\Gamma(1+\alpha)\varphi_\alpha(w,\zeta)=&\int_{|\eta|}^{+\infty}\left\{[\sigma+i(w-\zeta)]^{-1}-\left[\sigma+i(w-\overline{\zeta})\right]^{-1}\right\}\sigma^\alpha d\sigma\\&+\int_0^{|\eta|}\left\{\left[\sigma-i(w-\overline{\zeta})\right]^{-1}-\left[\sigma+i(w-\overline{\zeta})\right]^{-1}\right\}\sigma^\alpha d\sigma.\end{aligned} \tag{4.6}$$

Proof. Preliminarily we shall show that

$$\begin{aligned}&W^{-\alpha}\log b_0(w,\zeta)\\&=-\frac{1}{\Gamma(1+\alpha)}\int_0^{+\infty}\left\{[\sigma+i(w-\zeta)]^{-1}-[\sigma+i(w-\zeta)]^{-1}\right\}\sigma^\alpha d\sigma\end{aligned} \tag{4.7}$$

for $w\notin\{\zeta+ih:\ 0\le h<+\infty\}$. Indeed, for $\alpha=0$ (4.7) is trivial since W^0 is identical operator. For $0<\alpha<1$ integration by parts gives

$$W^{-\alpha}\log b_0(w,\zeta)=\frac{1}{\Gamma(1+\alpha)}\int_0^{+\infty}\sigma^\alpha d\log\frac{\sigma+i(w-\overline{\zeta})}{\sigma+i(w-\zeta)}$$

which implies (4.7). If $-1<\alpha<0$, then

$$W^{-\alpha}\log b_0(w,\zeta)=W^{-(1+\alpha)}\left\{\left[i(w-\overline{\zeta})\right]^{-1}-[i(w-\zeta)]^{-1}\right\}.$$

Hence (4.7) follows. The representation (4.6) providing the holomorphity of $\varphi_\alpha(w, \zeta)$ in the required domain follows from (4.7) and (2.2).

Lemma 4.2. *Let $\alpha \in (-1, 1)$ and $\zeta = \xi + i\eta \in G^-$ be fixed. Then the function* Re $\varphi_\alpha(w, \zeta)$ *is harmonic in G^- and continuous in $\overline{G^-}$.*

Proof. By Lemma 4.1, it suffices to prove the continuity of Re $\varphi_\alpha(w, \zeta)$ at the point $w = \xi$ or, which is the same, the continuity of

$$\varphi_\alpha^*(w, \zeta) = \Gamma(1 + \alpha)\varphi_\alpha(w + \xi, \zeta) \tag{4.9}$$

at the origin. To this end, we subtract from (4.6) the similar formula for $\varphi_0(w, \zeta)$ $(\equiv 0)$ multiplied by $|\eta|^\alpha$. This gives

$$\begin{aligned}\varphi_\alpha^*(w, \zeta) = &\int_{|\eta|}^{+\infty} \left\{[\sigma - |\eta| + iw]^{-1} - [\sigma + |\eta| + iw]^{-1}\right\}(\sigma^\alpha - |\eta|^\alpha)d\sigma \\ &+ \int_0^{|\eta|} \left\{[\sigma - |\eta| - iw]^{-1} - [\sigma - |\eta| + iw]^{-1}\right\}(\sigma^\alpha - |\eta|^\alpha)d\sigma.\end{aligned}$$

But $|\sigma^\alpha - |\eta|^\alpha| < C_\alpha|\sigma - |\eta||$ $(0 < \sigma < +\infty)$, where C_α depends only on α and $|\eta|$. Therefore, estimating the real parts of integrands and using the Lebesgue theorem on dominated convergence, we conclude that Re $\varphi_\alpha^*(w, \zeta)$ is continuous in $\overline{G^-}$ and

$$\lim_{\substack{w \to 0 \\ \text{Im } w \le 0}} \text{Re } \varphi_\alpha^*(w, \zeta) = 2|\eta| \int_0^{+\infty} \frac{\sigma^\alpha - |\eta|^\alpha}{\sigma^2 - |\eta|^2} d\sigma.$$

4.3. **Lemma 4.3.** *For any $\zeta \in G^-$ and $w \in \overline{G^-}$*

$$\text{Re } \varphi_\alpha^*(w, \zeta) \begin{cases} < 0, & \textit{if} \quad -1 < \alpha < 0, \\ > 0, & \textit{if} \quad 0 < \alpha < 1. \end{cases} \tag{4.10}$$

Proof. We shall initially prove these estimates for $w = u \in (-\infty, +\infty)$. By (4.9) and (4.6)

$$\text{Re } \varphi_\alpha^*(u, \zeta) = \left(\int_0^{|\eta|} + \int_{|\eta|}^{+\infty}\right) \sigma^\alpha d \log \left|\frac{\sigma - |\eta| + iu}{\sigma + |\eta| + iu}\right| \equiv I_1(u) + I_2(u).$$

Besides, if $0 < \sigma < |\eta|$, then

$$\frac{\partial}{\partial \sigma} \log \left|\frac{\sigma - |\eta| + iu}{\sigma + |\eta| + iu}\right| = -2|\eta| \frac{u^2 + \eta^2 - \sigma^2}{|(\sigma + iu)^2 - \eta^2|^2} < 0.$$

But $\sigma^\alpha < |\eta|^\alpha$ for $\alpha > 0$, and $\sigma^\alpha > |\eta|^\alpha$ for $\alpha < 0$. Consequently,

$$-|\eta|^\alpha \log \left|\frac{2|\eta| + iu}{u}\right| \begin{cases} > I_1(u) & \text{if} \quad -1 < \alpha < 0, \\ < I_1(u) & \text{if} \quad 0 < \alpha < 1, \end{cases} \qquad -\infty < u < +\infty.$$

On the other hand, integration by parts gives

$$I_2(u) = |\eta|^\alpha \log\left|\frac{2|\eta| + iu}{u}\right| - \alpha\int_{|\eta|}^{+\infty} \log\left|\frac{\sigma - |\eta| + iu}{\sigma + |\eta| + iu}\right| \sigma^{\alpha-1} d\sigma.$$

The above estimates of $I_1(u)$ imply (4.10) for Re $\varphi_\alpha^*(w,\zeta)$ (Im $w = 0$). For extending the inequalities (4.10) to all $w \in \overline{G^-}$, it suffices to use the Phragmén–Lindelöf principle and the estimate

$$|\varphi_\alpha^*(w,\zeta)| \le \frac{8}{3}\left[\frac{|\eta|}{|w|^{1-\alpha}} + \frac{1}{1+\alpha}\frac{|\eta|^{1+\alpha}}{|w|}\right], \quad w \in G^-, \ |w| > 2|\eta|,$$

which follows from the representation (4.7), (4.9) of $\varphi_\alpha^*(w,\zeta)$ by evaluation of modules of integrands.

Remark 4.1. The last estimate of $\varphi_\alpha^*(w,\zeta)$ implies that

$$|\varphi_\alpha(u+iv,\zeta)| \le \frac{8}{3\Gamma(1+\alpha)}\left[\frac{|\eta|}{|v|^{1-\alpha}} + \frac{1}{1+\alpha}\frac{|\eta|^{1+\alpha}}{|v|}\right], \ v < -2|\eta|. \tag{4.11}$$

On the other hand, using (4.6) and (4.10) one can verify that if $v < 0$ is enough small, then for any $\eta \in (-1,0)$

$$|\text{Re } \varphi_\alpha(iv, i\eta)| > \begin{cases} C_\alpha|\eta||v|^{-1+\alpha} & \text{if } \ 0 < \alpha < 1, \\ C'_\alpha|\eta|^{1+\alpha}|v|^{-1} & \text{if } \ -1 < \alpha < 0, \end{cases} \tag{4.12}$$

where C_α and C'_α are positive constants depending solely on α.

4.4. Assuming $\{w_k\} \subset G^-$ an arbitrary sequence satisfying the conditions (4.1) for an $\alpha \in (-1,1)$, consider the following function which is holomorphic in G^-:

$$\begin{aligned}\Phi_\alpha(w,\{w_k\}) \equiv& W^{-\alpha}\log\frac{B_\alpha(w,\{w_k\})}{B_0(w,\{w_k\})} \\ =& \sum_k W^{-\alpha}\log\frac{b_\alpha(w,w_k)}{b_0(w,w_k)} \equiv \sum_k \varphi_\alpha(w,w_k).\end{aligned} \tag{4.13}$$

Note that the last two series are absolutely and uniformly convergent inside G^- (this follows by (4.11) and by the convergence of the series of $W^{-\alpha}|\log b_\alpha|$). On the other hand, the first two equalities in (4.13) are true by Fubini's theorem (see (1.8) and (2.4)).

Lemma 4.4. *Let $\alpha \in (-1,1)$ be arbitrary, and let a sequence $\{w_k\} \subset G^-$ satisfy (4.1). Then*

$$\Phi_\alpha(w,\{w_k\}) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{w-t}, \quad w \in G^-, \tag{4.14}$$

where $\mu(t)$ is a nonincreasing and bounded function $(-\infty, +\infty)$ for $-1 < \alpha < 0$, and for $0 < \alpha < 1$ this function is nondecreasing in $(-\infty, +\infty)$ but satisfying (4.3) for any $\varepsilon > 0$. Besides, for any $\alpha \in (-1, 1)$ there exists a sequence $\delta_n \downarrow 0$ by which (4.4) holds.

Proof. By (4.13), (4.10) and the Herglotz-Riesz' theorem, in both cases $-1 < \alpha < 0$ and $0 < \alpha < 1$

$$\Phi_\alpha(w, \{w_k\}) = ipw + \frac{1}{\pi i} \int_{-\infty}^{+\infty} \left\{ \frac{1}{w-t} + \frac{t}{1+t^2} \right\} d\mu(t) + iC, \quad w \in G^-,$$

where p and C are some real numbers and the function $\mu(t)$ is nonincreasing for $-1 < \alpha < 0$, nondecreasing for $0 < \alpha < 1$ and such that

$$\int_{-\infty}^{+\infty} \frac{|d\mu(t)|}{1+t^2} < +\infty, \quad -1 < \alpha < +\infty.$$

Besides (see, for instance, [4], Ch. I, Sec. 4), there exists a sequence $\delta_n \downarrow 0$ by which (4.4) holds. Now observe that by (4.11) and (4.13)

$$|\Phi_\alpha(u + iv, \{w_k\})| \\ \leq \frac{8}{3\Gamma(1+\alpha)} \left[\frac{1}{|v|^{1-\alpha}} \sum_k |\text{Im } w_k| + \frac{1}{(1+\alpha)|v|} \sum_k |\text{Im } w_k|^{1+\alpha} \right]$$

for $v < -2 \max_k |\text{Im } w_k|$. Consequently, $\sup_{v<0} |v\Phi_\alpha(iv, \{w_k\})| < +\infty$ for $-1 < \alpha < 0$. Hence (4.14) and the boundedness of $\mu(t)$ follows (see, for instance, [4], Addendum I, Sec. 4). As to the case $0 < \alpha < 1$, it is obvious that

$$\lim_{v \to -\infty} \Phi_\alpha(iv, \{w_k\}) = 0 \quad \text{and} \quad \int_1^{+\infty} |\Phi_\alpha(-it, \{w_k\})| \frac{dt}{t^{\alpha+\varepsilon}} < +\infty$$

for any $\varepsilon > 0$ (we cannot get rid of ε in view of the below estimate for Re Φ_α, which follows from (4.12)). Hence we come to the representation (4.14) and to the relation (4.3) with obligatory $\varepsilon > 0$ (see [4], Addendum I, Sec. 3).

4.5. **Lemma 4.5.** *Under the conditions of Lemma* 4.4,

$$W^\alpha \Phi_\alpha(w, \{w_k\}) \equiv \log \frac{B_\alpha(w, \{w_k\})}{B_0(w, \{w_k\})}, \qquad \text{Im } w < -\max_k |\text{Im } w_k|.$$

Proof. We start by the case when the sequence $\{w_k\}$ consists of a single term $\zeta = \xi + i\eta \in G^-$ and use the formula

$$\int_0^{+\infty} \frac{\sigma^{-\gamma} d\sigma}{(z+\sigma)^{n+1}} = \frac{\Gamma(\gamma+n)\Gamma(1-\gamma)}{n!} \frac{1}{z^{\gamma+n}}, \quad z \notin (-\infty, 0], \qquad (4.15)$$

which particularly is true for any integer $n \geq 0$ and any $\gamma \in (-n, 1)$.

Let $-1 < \alpha < 0$ and Im $w < \eta$. Then by formulas (2.2), (1.15) of Ch. 1 and (1.1′)–(1.2)

$$W^{\alpha}W^{-\alpha}\log b_{\alpha}(w,\zeta)$$
$$= -\frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\int_{-|\eta|}^{|\eta|}(|\eta|-|t|)^{\alpha}dt\int_{0}^{+\infty}\frac{\sigma^{-\alpha}d\sigma}{[i(w-\xi)-t+\sigma]^{2}}$$
$$= \log b_{\alpha}(w,\zeta).$$

Besides,

$$W^{\alpha}W^{-\alpha}\log b_{0}(w,\zeta)$$
$$= -(1-\alpha)\int_{-|\eta|}^{|\eta|}dt\int_{0}^{+\infty}\frac{\sigma^{-\alpha}d\sigma}{[i(w-\xi)-t+\sigma]^{2-\alpha}} = \log b_{0}(w,\zeta)$$

since by formula (1.13) of Ch. 1 and (4.15)

$$W^{-\alpha}\log b_{0}(w,\zeta) = -\Gamma(1-\alpha)\int_{-|\eta|}^{|\eta|}\frac{dt}{[i(w-\xi)-t]^{1-\alpha}}.$$

For $0 < \alpha < 1$ and Im $w < \eta$ one can obtain the same equalities in a similar way. Thus

$$W^{\alpha}\varphi_{\alpha}(w,\zeta) = \log\frac{b_{\alpha}(w,\zeta)}{b_{0}(w,\zeta)}, \quad \text{Im } w < \eta, \quad \alpha \in (-1,1).$$

Hence the desired formula follows by (4.5) and the absolute and uniform convergence of the series

$$\log\frac{B_{\alpha}(w,\{w_k\})}{b_{0}(w,\{w_k\})} = \sum_{k}\log\frac{b_{\alpha}(w,w_k)}{b_{0}(w,w_k)}$$

for Im $w < -\max_k |\text{Im } w_k|$ (see. (3.5)).

Proof of Theorem 4.1. In view of Lemmas 4.4 and 4.5, it is sufficient to prove the equality

$$W^{\alpha}\left\{\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{w-t}\right\} = \frac{\Gamma(1+\alpha)}{\pi}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{[i(w-t)]^{1+\alpha}}, \quad w \in G^{-},$$

under the assumption that $\mu(t)$ is as required. One can directly verify this equality using (4.15).

http://www.springer.com/978-0-387-23625-4

Functions of a-Bounded Type in the Half-Plane
Jerbashian, A.M.
2005, XVI, 196 p., Hardcover
ISBN: 978-0-387-23625-4