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The Steiner Ratio of Banach-Minkowski spaces - A Survey

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1 Introduction

Starting with the famous book "What is Mathematics" by Courant and Robbins the following problem has been popularized under the name of Steiner:

> For a given finite set of points in a metric space find a network which connects all points of the set with minimal length.

Such a network must be a tree, which is called a Steiner Minimal Tree (SMT). It may contain vertices other than the points which are to be connected. Such points are called Steiner points.[1]
Given a set of points, it is a priori unclear how many Steiner points one has to add in order to construct an SMT, but one can prove that we need not more than $n-2$, whereby n is the number of given points.
A classical survey of Steiner's Problem in the Euclidean plane was presented by Gilbert and Pollak in 1968 [28] and christened "Steiner Minimal Tree" for the shortest interconnecting network and " Steiner points" for the additional vertices.
Without loss of generality, the following is true for any SMT for a finite set N of points in the Euclidean plane:

1. The degree of each vertex is at most three;

2. The degree of each Steiner point equals three; and two edges which are incident to a Steiner point meet at as angle of 120°;

3. There are at most $|N|-2$ Steiner points.

Moreover, in the paper by Gilbert and Pollak, there are a lot of interesting conjectures, stimulating the research in this field in the next years.

It is well-known that solutions of Steiner's problem depend essentially on the way in which the distances in space are determined. In recent years it turned out that in engineering design it is interesting to consider Steiner's Problem and similar problems in several two-dimensional Banach spaces and some specific higher-dimesnional cases. Over the years Steiner's Problem

[1]The history of Steiner's Problem started with P.Fermat [22] early in the 17th century and C.F.Gauß [27] in 1836. At first perhaps with the famous book *What is Mathematics* by R.Courant and H.Robbins in 1941, this problem became popularized under the name of Steiner.

has taken on an increasingly important role, it is one of the most famous combinatorial-geometrical problems. Consequently, in the last three decades the investigations and, naturally, the publications about Steiner's Problem have increased rapidly. Surveys are given by Cieslik [10], Hwang, Richards, Winter [32] and Ivanov, Tuzhilin [33]. However, all investigations showed the great complexity of the problem, as well in the sense of structural as in the sense of computational complexity. In other terms:

Observation I.

In general, methods to find an SMT are hard in the sense of computational complexity or still unknown. In any case we need a subtile description of the geometry of the space.

On the other hand, a Minimum Spanning Tree[2] (MST) can be found easily by simple and general applicable methods.

Observation II.

It is easy to find an MST by an algorithm which is simple to realize and running fast in all metric space.

Hence, it is of interest to know what the error is if we construct an MST instead of an SMT. In this sense, we define the Steiner ratio for a space to be the infimum over all finite sets of points of the length of an SMT divided by the length of an MST:

$$m := \inf \left\{ \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \text{ a finite set in the space} \right\}.$$

This quantity is a parameter of the considered space and describes the performance ratio of the the approximation for Steiner's Problem by a Minimum Spanning Tree.

This present paper concentrates on investigating the Steiner ratio. The goal is to determine or at least to estimate the Steiner ratio for many different spaces.

2 Banach-Minkowski Spaces

Obviously, Steiner's Problem depends essentially on the way how the distances in the plane are determined. In the present paper we consider finite-dimensional Banach spaces. These are defined in the following way: A_d

[2]This is a shortest tree interconnecting a finite set of points without Steiner points.

denotes the d-dimensional affine space with origin o. That means; A_d is a set of points and these points act over a d-dimensional linear space. We identify each point with its vector with respect to the origin. In other words, elements of A_d will be called either points when considerations have a geometrical character, or vectors when algebraic operations are applied. In this sense the zero-element o of the linear space is the origin of the affine space. The dimension of an affine space is given by the dimension of its linear space. A two-dimensional affine space is called a plane. A non-empty subset of a affine space which is itself an affine space is called an affine subspace.

The idea of normed spaces is based on the assumption that to each vector of a space can be assigned its "length" or norm, which satisfies some "natural" conditions.
A convex and compact body B of the d-dimensional affine space A_d centered in the origin o is called a unit ball, and induces a norm $||.|| = ||.||_B$ in the corresponding linear space by the so-called Minkowski functional:

$||v||_B = \inf\{t > 0 : v \in tB\}$ for any v in $A_d \setminus \{o\}$, and

$||o||_B = 0.$

On the other hand, let $||.||$ be a norm in A_d, which means:
$||.|| : A_d \to \mathbb{R}$ is a real-valued function satisfying

(i) positivity: $||v|| \geq 0$ for any v in A_d;

(ii) identity: $||v|| = 0$ if and only if $v = o$;

(iii) homogenity: $||tv|| = |t| \cdot ||v||$ for any v in A_d and any real t;

and

(iv) triangle inequality: $||v + v'|| \leq ||v|| + ||v'||$ for any v, v' in A_d.

Then $B = \{v \in A_d : ||v|| \leq 1\}$ is a unit ball in the above sense. It is not hard to see that the correspondences between unit balls B and norms $||.||$ are unique. That means that a norm is completely determined by its unit ball and vice versa. Consequently, a Banach-Minkowski space is uniquely defined by an affine space A_d and a unit ball B. This Banach-Minkowski space is abbreviated as $M_d(B)$. In each case we also have the induced norm $||.||_B$ in the space.

A Banach-Minkowski space $M_d(B)$ is a complete metric linear space if we define the metric by

$$\rho(v, v') = ||v - v'||_B. \tag{1}$$

Usually, a (finitely- or infinitely-dimensional) linear space which is complete with regard to its given norm is called a Banach space. Essentially, every Banach-Minkowski space is a finite-dimensional Banach space and vice versa.

All norms in a d-dimensional affine space induce the same topology, the well-known topology with coordinate-wise convergence.[3] In other words: On a finite dimensional linear space all norms are topologically equivalent, i.e. there are positive constants c_1 and c_2 such that

$$c_1 \cdot ||.|| \leq |||.||| \leq c_2 \cdot ||.|| \tag{2}$$

for the two norms $||.||$ and $|||.|||$.
Conversely, there is exactly one topology that generates a finite-dimensional linear space to a metric linear space satisfying the separating property by Hausdorff.

Let $M_d(B)$ and $M_d(B')$ be Banach-Minkowski spaces.
$M_d(B)$ is said to be isometric to $M_d(B')$ if there is a mapping $\Phi : A_d \to A_d$ (called an isometry) which preserves the distances:

$$||\Phi(v) - \Phi(v')||_{B'} = ||v - v'||_B \tag{3}$$

for all v, v' in A_d.
A well-known fact given by Mazur and Ulam says that each isometry mapping a Banach-Minkowski space onto another, such that it maps o on o, is a linear operator. Hence, $M_d(B)$ is isometric to $M_d(B')$ if and only if there is an affine map $\Phi : A_d \to A_d$ with $\Phi B = B'$. Also the affine map Φ is the isometry itself.

Steiner's Problem looks for a shortest network and in particular for a shortest length of a curve C joining two points. For our purpose, we regard a geodesic curve as any curve of shortest length.
If we parametrize the curve C by a differentiable map $\gamma : [0, 1] \to I\!R^d$ we define

$$\text{length of } C = \int_0^1 ||\dot{\gamma}||\, dt. \tag{4}$$

[3]This is the topology derived from the Euclidean metric.

It is not hard to see that among all differentiable curves $\mathcal{C}$ from the point v to the point v' the segment

$$\underline{vv'} = \{tv + (1-t)v' : 0 \leq t \leq 1\} \tag{5}$$

minimizes the length of $\mathcal{C}$.

A unit ball B in an affine space is called strictly convex if one of the following pairwise equivalent properties is fulfilled:

- For any two different points v and v' on the boundary of B, each point $w = tv + (1-t)v', 0 < t < 1$, lies in int$B$.
- No segment is a subset of bd$B$.
- $||v + v'||_B = ||v||_B + ||v'||_B$ for two vectors v and v' implies that v and v' are linearly dependent.

One property more we have in

Lemma 2.1 *All segments in a Banach-Minkowski space are shortest curves (in the sense of inner geometry). They are the unique shortest curves if and only if the unit ball is strictly convex.*

Hence, we can define the metric in a Banach-Minkowski space $M_d(B)$ by

$$\rho(v, v') = \frac{2 \cdot ||v - v'||_{B^e}}{||w - w'||_{B^e}}, \tag{6}$$

where $\underline{ww'}$ is the Euclidean diameter of B parallel to the line through v and v' and $||.||_{B^e}$ denotes the Euclidean norm.

A function F defined on a convex subset of the affine space is called a convex function if for any two points v and v' and each real number t with $0 \leq t \leq 1$, the following is true

$$F(tv + (1-t)v') \leq tF(v) + (1-t)F(v'). \tag{7}$$

A function F is called a strictly convex function, if the following is true for any two different points v and v' and each real number t with $0 < t < 1$:

$$F(tv + (1-t)v') < tF(v) + (1-t)F(v'). \tag{8}$$

A norm is a convex function. Moreover, the unit ball of a strictly convex norm is a strictly convex set.

Lemma 2.2 *For a norm* $||.||$ *in a finite-dimensional affine space the following holds:*

(a) *A norm* $||.||$ *in a finite-dimensional affine space is a convex and thus a continuous function.*

(b) *A norm* $||.||$ *is a strictly convex function if and only if its unit ball* $B = \{v \in A_d : ||v|| \leq 1\}$ *is a strictly convex set.*

The dual norm $||\cdot||_{DB}$ of the norm $||\cdot||_B$ is defined as

$$||v||_{DB} = \max_{w \neq o} \frac{(v, w)}{||w||_B} \tag{9}$$

and has the unit ball *DB*, called the dual unit ball, which can be described as

$$DB = \{w : (v, w) \leq 1 \text{ for all } v \in B\}.$$

(Here, (., .) denotes the standard inner product.) Immediately, we have that for any two vectors v and w the inequality

$$(v, w) \leq ||v||_{DB} \cdot ||w||_B; \tag{10}$$

is true and it is not hard to see that $B \subseteq B'$ holds if and only if $DB' \subseteq DB$. An example of non-Euclidean norms dual to each other is

$$||(t_1, \ldots, t_d)||_B = \max\{|t_1|, \ldots, |t_d|\} \tag{11}$$

and

$$||(t_1, \ldots, t_d)||_{DB} = |t_1| + \ldots + |t_d|, \tag{12}$$

whereby *B* is a hypercube and *DB* is a cross-polytope.

Particularly, we consider finite-dimensional spaces with p-norm, defined in the following way: Let A_d be the d-dimensional affine space. For the point $v = (x_1, ..., x_d)$ we define the norm by

$$||v|| = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}$$

where $1 \leq p < \infty$ is a real number. If p runs to infinity we get the so-called Maximum norm

$$||v||_\infty = \max\{|x_i| : 0 \leq i \leq d\}$$

In each case we obtain a Banach-Minkowski space written by $\mathcal{L}_p^d$.
$\mathcal{L}_1^d$ and $\mathcal{L}_\infty^d$ normed by a cross-polytope and a cube, respectively. For $1 < p < \infty$ the space $\mathcal{L}_p^d$ is strictly convex. The spaces $\mathcal{L}_p^d$ and $\mathcal{L}_q^d$ with $1/p + 1/q = 1$ are dual.

3 Steiner's Problem and the Steiner ratio

A (finite) graph $G = (V, E)$ with the set V of vertices and the set E of edges is embedded in the Banach-Minkowski space $M_d(B)$ in the sense that

- V is a finite set of points in the space;
- Each edge $\underline{vv'} \in E$ is a segment $\{tv + (1-t)v' : 0 \leq t \leq 1\}$, $v, v' \in V$; and
- The length of G is defined by

$$L(G) = L_B(G) = \sum_{\underline{vv'} \in E} ||v - v'||_B.$$

Now, **Steiner's Problem of Minimal Trees** is the following:

Given: A finite set N of points in the Banach-Minkowski space $M_d(B)$.

Find: A connected graph $G = (V, E)$ embedded in the space such that
- $N \subseteq V$ and
- $L_B(G)$ is minimal as possible.

A solution of Steiner's Problem is called a Steiner Minimal Tree (SMT) for N in the space $M_d(B)$.[4] The vertices in the set $V \setminus N$ are called Steiner points. We may assume that for any SMT $T = (V, E)$ for N the following holds: The degree of each Steiner point is at least three and

$$|V \setminus N| \leq |N| - 2. \tag{13}$$

If we don't allow Steiner points, that is if we connect certain pairs of given points only, then we refer to a Minimum Spanning Tree (MST). Starting with Boruvka in 1926 and Kruskal in 1956, Minimum Spanning Trees have a well-documented history [29] and effective constructions [3].

A minimum spanning tree in a graph $G = (N, E)$ with a positive length-function $f : E \to \mathbb{R}$, can be found with the help of Kruskal's [34] well-known method:

1. Start with the forest $T = (N, \emptyset)$;

[4] That for any finite set of points there an SMT always exists is not obvious. Particularly, it is proved in [10].

2. Sequentially choose the shortest edge that does not form a circle with already chosen edges;

3. Stop when all vertices are connected, that is when $|N| - 1$ edges have been chosen.

Then an MST for a finite set *N* of points in $M_d(B)$ can be found obtaining the graph $G = (N, \binom{N}{2})$ with the length-function $f(\underline{vv'}) = ||v - v'||_B$.

Let *N* be a finite set of points in $M_d(B)$. We saw that it is easy to find an MST for *N;* this is valid in the sense of the combinatorial structure as well as in the sense of computational complexity. On the other hand, methods to find an SMT for *N* are still unknown or at least hard in the sense of computational complexity. More exactly:

space	complexity	source
Euclidean plane	$\mathcal{NP}$-hard	[24]
Rectilinear plane $\mathcal{L}_1^2$	$\mathcal{NP}$-hard	[25]
$\mathcal{L}_p$-planes	algorithm needs exponential time	[13]
Banach plane	algorithm needs exponential time	[9]

For higher-dimensional spaces the problems are not easier than in the planes. For a complete discussion of these difficulties see [10] and [32].
Moreover, to solve Steiner's Problem we need facts about the geometry of the space. On the other hand, for an MST we only use the mutual distances between the points.

Consequently, we are interested in the value

$$m_d(B) := \inf\left\{\frac{L_B(\text{SMT for } N)}{L_B(\text{MST for } N)} : N \subseteq M_d(B) \text{ is a finite set}\right\}, \qquad (14)$$

which is called the Steiner ratio of the space $M_d(B)$.
The quantity $m_d(B) \cdot L(\text{MST for } N)$ would be a convenient lower bound for the length of an SMT for *N* in the space $M_d(B)$; that means, roughly speaking, $m_d(B)$ says how much the total length of an MST can be decreased by allowing Steiner points.

For the space $\mathcal{L}_p^d$ the Steiner ratio will be briefly written by $m(d, p)$.

4 Basic properties for the Steiner ratio

It is obvious that $0 < m_d(B) \leq 1$ for the Steiner ratio $m_d(B)$ of each Banach-Minkowski space $M_d(B)$. Of course, if $d = 1$ then the MST and the SMT are identical, and it is $m_1(B) = 1$. Moreover,

Theorem 4.1 *(E.F.Moore in [28]) For the Steiner ratio of every Banach-Minkowski Space*

$$m_d(B) \geq \frac{1}{2} = 0.5$$

holds.

In the d-dimensional affine space A_d, the unit ball $B(1)$ is the convex hull of

$$N = \{\pm(0, ..., 0, 1, 0, ..., 0) : \text{the i'th component is equal to } 1, i = 1, \ldots, d\}. \tag{15}$$

The set N contains $2d$ points. The rectilinear distance of any two different points in N equals 2. Hence, an MST for N has the length $2(2d - 1)$. Conversely, an SMT[5] for N with the Steiner point o = (0,...,0) has the length $2d$. This implies the first fact of

Theorem 4.2 *For the Steiner ratio of spaces with rectilinear norm the following are true.*

(a) *In the case of* $\mathbf{d}$ *dimensions we have*

$$m(d, 1) \leq \frac{d}{2d - 1}. \tag{16}$$

(b) *(Hwang [31]) In two dimensions in (16) equality holds:*

$$m(2, 1) = \frac{2}{3}. \tag{17}$$

Graham and Hwang [30] conjectured that in (16) always equality holds, which is true in the planar case, (17), but the methods by Hwang do not seem to be applicable to proving the conjecture in the higher dimensional case.
Comparing the last two theorems, we observe the following:

[5] that this tree is indeed an SMT is not simple to see!

Corollary 4.3 *The lower bound* 1/2 *is the best possible for the Steiner ratio over the class of all Banach-Minkowski spaces.*

Let $M_d(B)$ be a d-dimensional Banach-Minkowski space, and let $A_{d'}$ be a d'-dimensional affine subspace $(d' \leq d)$ with $o \in A_{d'}$. Clearly, the intersection $B \cap A_{d'}$ can be considered as the unit ball of the space $A_{d'}$. This means that $M_{d'}(B \cap A_{d'})$ is a (Banach-Minkowski) subspace of $M_d(B)$. Let v and v' be two different points in $A_{d'}$. Then the line through v and v' lies completely in $A_{d'}$, and in view of 2.1 and (6) we see that the distance between the points v and v' is preserved:

$$||v - v'||_B = ||v - v'||_{B \cap A_{d'}}. \tag{18}$$

Then we have

Theorem 4.4 *Let* $M_{d'}(B')$ *be a (Banach-Minkowski) subspace of* $M_d(B)$. *Then* $m_{d'}(B') \geq m_d(B)$.

An interesting problem, but which seems as very difficult, is to determine the range of the Steiner ratio for d-dimensional Banach-Minkowski spaces, depending on the value d. More exactly, determine the best possible reals c_d and C_d such that

$$c_d \leq m_d(B) \leq C_d, \tag{19}$$

for all unit balls B of A_d.
The quantity C_d is defined as the upper bound of all numbers $m_d(B)$ ranging over all unit balls B of A_d:

$$C_d = \sup\{m_d(B) : B \in \underline{B}_d\}. \tag{20}$$

The sequence $\{C_d\}_{d=1,2,\ldots}$, starting with $C_1 = 1$ is a decreasing and bounded, consequently a convergent one. Is it true that $C_d = m(d, 2)$ for $d = 2, 3, \ldots$? On the other hand,

$$c_d = \inf\{m_d(B) : B \in \underline{B}_d\}. \tag{21}$$

is of interest. Does the equality $c_d = m(d, 1)$ for $d = 2, 3, \ldots$ hold?

5 The Steiner ratio of the Euclidean plane

Original, Steiner's Problem was considered in the Euclidean plane. Even here, we find that the complexity of computing an SMT is $\mathcal{NP}$-hard. The complexity of computing SMT's in higher-dimensional spaces is demonstrably even more difficult, since here is no inherent combinatorial structure present in the problem, compare [40].

A long-standing conjecture, given by Gilbert and Pollak in 1968, said that $m(2,2) = \sqrt{3}/2$. Many persons have tried to show this; successively establishing that $\sqrt{3}/2$ does indeed hold for sets with a small number of points: Pollak [36] and Du, Yao, Hwang [15] have shown that the conjecture is valid for sets *N* consisting of $n = 4$ points; Du, Hwang, Yao [17] stated this result to the case $n = 5$, and Rubinstein, Thomas [37] have done the same for the case $n = 6$.

On the other hand, many attempts have been made to estimate the Steiner ratio for the Euclidean plane from below:

$m(2,2) \geq 1/\sqrt{3}$	$= 0.57735\ldots$	Graham, Hwang [30]
$m(2,2) \geq \sqrt{2\sqrt{3}+2-(7+2\sqrt{3})}$	$= 0.74309\ldots$	Chung, Hwang [5]
$m(2,2) \geq 4/5$	$= 0.8$	Du, Hwang [16]
$m(2,2)$	$\geq 0.82416\ldots$	Chung, Graham [6]

Finally, Du and Hwang created a lot of new methods and succeeded in proving the Gilbert-Pollak conjecture completely:

Theorem 5.1 *(Du, Hwang [18], [19]) The Steiner Ratio of the Euclidean plane equals*

$$m(2,2) = \frac{\sqrt{3}}{2} = 0.86602\ldots.$$

Now, we are interested in the sets of points which achieve the Steiner ratio. Clearly, when *N* contains the nodes of an equilateral triangle we have

$$\frac{L(\text{SMT for } N)}{L(\text{MST for } N)} = \frac{\sqrt{3}}{2} = m(2,2). \tag{22}$$

In a first view, it seems that no other finite set of points has this property. Probably, this is true, but Du and Smith [21] had an surprising idea: Let's look at some special set configurations created by joining equilateral triangles at a common side. This is actually called a 2-sausage, more formally:

1. Start with a unit circle;
2. Successively add unit circles so that the n'th circle you add is always touching the $\min\{2, n-1\}$ most recently added circles.

This procedure uniquely[6] defines an infinite sequence of interior-disjoint

[6] up to congruence

numbered circles. The centers of these circles form a discrete point set, which is called the (infinity) 2-sausage. The first n points of the 2-sausage will be called the "n-point 2-sausage" $N(n,2)$ or "flat-sausage" for simplicity.
What is remarkable about the sausages? At first their regularity and then the fact that the ratio between the length of an SMT and the length of an MST for $N(2n,2)$ decreases with increasing number n. Moreover,

Theorem 5.2 *(Du, Smith [21])*

$$\lim_{n\to\infty} \frac{L(SMT\ for\ N(2n,2))}{L(MST\ for\ N(2n,n))} = \frac{\sqrt{3}}{2} = m(2,2).$$

6 The Steiner ratio of $\mathcal{L}_p$-planes

If we have an analytic formula, which decribes the norm, we have also the possibility to estimate the Steiner ratio with direct calculations. In this section we will determine upper and lower bounds for the Steiner ratio $m(2,p)$ of two-dimensional $\mathcal{L}_p$-spaces.

Du and Liu determined an upper bound for the Steiner ratio using direct calculations of the ratio between the length of SMT's and of MST's for sets with three elements:

Theorem 6.1 *(Du, Liu [35]) The following is true for the Steiner ratio of the $\mathcal{L}_p$-planes $M_2(B(p))$:*

$$m(2,p) \leq \frac{(2^p-1)^{1/p} + (2^q-1)^{1/q}}{4}, \tag{23}$$

where q is the conjugated number to p; that means $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 6.2 *For $1 < p < \infty$ it holds*

$$m(2,p) \leq m(2,2) = \frac{\sqrt{3}}{2} = 0.866025\ldots.$$

Furthermore, equality holds if and only if $p = 2$.

The proof of 6.1 uses a specific triangle. Now, we will use a triangle which has a side parallel to the line $\{(x,x) : x \in \mathbb{R}\}$. Let $1 < p < \infty$

and $u = (0,1)$, $v = (1,0)$ and $w = (x_p, x_p)$. We want that the triangle spanned by u,v and w is equilateral and, additionally, x_p lies between 1 and 2. Considering this triangle Albrecht [1] gives the following new upper bounds for $m(2,p)$ for specific values p:

p	q	Theorem 6.1	new with p	new with q
1.1	11	0.782399...	0.775933...	0.775933...
1.2	6	0.809264...	0.797975...	0.797975...
1.3	4.3...	0.829043...	0.816708...	0.816708...
1.4	3.5	0.842759...	0.832320...	0.832320...
1.5	3	0.852049...	0.844625...	0.844625...
1.6	2.6...	0.858207...	0.853640...	0.853640...
1.7	2.428571...	0.862145...	0.859755...	0.859755...
1.8	2.25	0.864491...	0.863518...	0.863518...
1.9	2.1...	0.865681...	0.865460...	0.865460...
2.0	2	0.866025...	0.866025...	0.866025...

It is not hard to see, that considering sets with three points only creates estimates for the Steiner ratio which are at least 3/4, compare [28]. Using sets with four points gives

Theorem 6.3 *(Albrecht [1]) The Steiner ratio of $\mathcal{L}_p^2$ is essentially less than $\frac{3}{4}$ if $p \le 1.2$ or if $p \ge 6$.*

How can we find a lower estimate for the Steiner ratio of the planes? Here, we use two facts:

1. The values for $\mathcal{L}_1^2$ and $\mathcal{L}_2^2$ are exactly known: $m(2,1) = 2/3$ and $m(2,2) = \sqrt{3}/2$.

2. We introduce a distance function between classes of Banach-Minkowski spaces in the following way: Let $\underline{B}_d$ denote the class of all unit balls in A_d, and let $[\underline{B}_d]$ be the space of classes of isometrics for $\underline{B}_d$. Then the Banach-Mazur distance abst is a metric on $[\underline{B}_d]$ defined as

$$\begin{aligned} abst([B],[B']) &= \ln \inf\{h \ge 1 : \textit{there is an isometry } A \textit{ such that} \\ &\qquad B \subseteq AB \subseteq hB\} \end{aligned}$$

for $[B]$, $[B']$ in $[\underline{B}_d]$. The space $([\underline{B}_2], \text{abst})$ is a compact metric space, with diameter $= \ln \frac{3}{2}$, compare [42].

With these facts in mind we find

Theorem 6.4 *(C. [8], [10]) The following are true for the Steiner ratio of the $\mathcal{L}_p$-planes:*

$$m(2,p) \geq \begin{cases} \frac{2^{1/p}}{3} & \text{if } 1 \leq p \leq \frac{\ln 16}{\ln 13.5} \\ \frac{\sqrt{6}}{2} \cdot 2^{-1/p} & \text{if } \frac{\ln 16}{\ln 13.5} \leq p \leq 2. \end{cases}$$

We can find bounds for $m(2,p)$, $p \geq 2$, if we replace p by $p/(p-1)$ on the right side.

7 The Steiner ratio of Banach-Minkowski planes

For many specific planes the Steiner ratio is known or well estimated. In the section before we find the values for $\mathcal{L}_p$-planes, including the exact value for the Euclidean plane and the plane with rectilinear norm. Additionally, it is an interesting question to consider planes which are normed by a regular polygon with an even number of corners.
We defined the so-called $\lambda-$geometry $M_2(B^{(\lambda)})$ in the following way: The unit ball $B^{(\lambda)}$ is a regular 2λ-gon with the x-axis being a diagonal direction. Particularly, it holds

$$M_2(B^{(2)}) = \mathcal{L}_1^2 \quad (24)$$
$$M_2(B^{(\infty)}) = \mathcal{L}_2^2. \quad (25)$$

We have the

Theorem 7.1 *(Sarrafzadeh, Wong [38]) For the Steiner ratio of the planes with λ-geometry it holds that*

$$m_2(B^{(\lambda)}) \geq \frac{\sqrt{3}}{2}\cos\frac{\pi}{2\lambda}. \quad (26)$$

It follows from the last theorem that $m_2(B^{(3)}) \geq 3/4$. $B^{(3)}$ is an affinely regular hexagon. In view of an isometry, we may assume that

$$B^{(3)} = \mathrm{conv}\{(1,1),(-1,-1),(1,0),(-1,0),(0,1),(0,-1)\}, \quad (27)$$

which implies that

$$||(x_1,x_2)||_{B^{(3)}} = \max\{|x_1|,|x_2|,|x_1-x_2|\}. \quad (28)$$

Then it is easy to see that the set $N = \{(1,1), (-1,0), (0,-1)\}$ has an MST of the length 4 and an SMT of the length at most 3. Hence, the Steiner Ratio is not greater than 3/4, and we have

Theorem 7.2 *(Du et.al. [20]) Let B be an affinely regular hexagon in the plane. Then*

$$m_2(B) = \frac{3}{4} = 0.75.$$

What is known about the Steiner ratio of two-dimensional Banach spaces in general? A sharp lower bound for the Steiner ratio of any Banach-Minkowski plane we have in

Theorem 7.3 *(Gao, Du, Graham [23]) For the Steiner ratio of Banach-Minkowski planes the following is true:*

$$m_2(B) \geq \frac{2}{3}.$$

If there is a natural number n such that the bound 2/3 is adopted by a set of n points, then $n = 4$, and B is a parallelogram.

This bound is the best possible one, since the plane with rectilinear norm has Steiner ratio 2/3. Such a nice result for an upper bound is unknown, but we have

Theorem 7.4 *(Du et.al. [20]) For any unit ball B in the plane the following is true:*

$$m_2(B) \leq \frac{\sqrt{13}-1}{3} = 0.8685\ldots. \tag{29}$$

Is it true that

$$m_2(B) \leq \frac{\sqrt{3}}{2} = 0.86602\ldots, \tag{30}$$

which is the Steiner ratio of the Euclidean plane?

8 The Steiner ratio of $\mathcal{L}_p^3$

In this section we will determine upper bounds for the Steiner ratio $m(3,p)$ of three-dimensional $\mathcal{L}_p$-spaces. Our goal is to estimate the quantities $m(3,p)$ with help of investigations for configurations of points in a regular situation in the space $\mathcal{L}_p^3$. To do this we start with the consideration of tetrahedrons: Let $1 < p < \infty$ and consider the four points

$$\begin{aligned} v_1 &= (1,0,0), \\ v_2 &= (0,1,0), \\ v_3 &= (0,0,1) \quad \text{and} \\ v_4 &= (1,1,1) \end{aligned}$$

which build an equilateral set. Hence,

Theorem 8.1 *(Albrecht [1]) Let $1 < p < \infty$ and let q be conjugated to p. Then we have for the Steiner ratio of $\mathcal{L}_p^3$*

$$m(3,p) \le \begin{cases} \frac{1}{3}\left(2^{-1/p} + (2^q - 1)^{1/q}\right) & : \quad \textit{if} \quad 1 < p \le \frac{\log 3}{\log 3 - \log 2} \\ \left(\frac{2}{3}\right)^{1/q} & : \quad \textit{otherwise} \end{cases}$$

Another way: We consider a cross-polytope created by the set of nodes $N = \{v_1, \ldots, v_6\}$ whereby

$$\begin{aligned} v_1 &= (1,0,0), \\ v_2 &= (0,1,0), \\ v_3 &= (0,0,1), \\ v_4 &= (-1,0,0), \\ v_5 &= (0,-1,0) \quad \text{and} \\ v_6 &= (0,0,-1). \end{aligned}$$

The points v_i and v_j, $i \neq j$, have the distance

$$\rho(v_i, v_j) = \begin{cases} 2 & : \text{ if } \quad |i - j| = 3 \\ 2^{1/p} \le 2 & : \text{ otherwise} \end{cases}$$

and consequently an MST for N has the length

$$L_p(\text{MST for } N) = 5 \cdot 2^{1/p}. \tag{31}$$

On the other hand, using the fact that it holds $\rho(v_i, o) = 1$ for $i = 1, \ldots, 6$, there is a tree for $N \cup \{o\}$ of the length 6. Hence,

Theorem 8.2 *For the Steiner ratio of a three-dimensional $\mathcal{L}_p$-space it holds*

$$m(3,p) \leq \frac{6}{5} \cdot \left(\frac{1}{2}\right)^{1/p}.$$

Now, we will consider a cross-polytope in another way which gives better bounds for the Steiner ratio if $p > 2$.[7]
At first we assume that $p \neq \infty$ and consider the set $N = \{v_1, \ldots, v_6\}$ of given points with

$$\begin{aligned}
v_1 &= (x_0, x_0 - 1, 1 - x_0), \\
v_2 &= (x_0, x_0, 2 - x_0), \\
v_3 &= (1, 0, 1), \\
v_4 &= (0, 0, 0), \\
v_5 &= (0, 1, 1) \quad \text{and} \\
v_6 &= (x_0 - 1, x_0, 1 - x_0),
\end{aligned}$$

whereby x_0 denotes the unique zero of the function f with

$$f(x) = x^p + 2(x-1)^p - 2 \tag{32}$$

over the range (1,2). Here,

Theorem 8.3 *(Albrecht [1]) Let $1 < p < \infty$, $1/p + 1/q = 1$ and let x_0 be the unique determined zero of the function f defined in (32) over the range* (1,2)*. Then the Steiner ratio of $\mathcal{L}_p^3$ can be estimated by*

$$m(3,p) \leq \begin{cases} \frac{1}{5}\left((2^q - 1)^{1/q} + \left(\frac{1}{2}\right)^{1/p} + \left(\frac{3}{2}\right)^{1/p} x_0\right) & : \quad 1 < p \leq \frac{\log 3}{\log 3 - \log 2} \\ \frac{1}{5}\left(\frac{3}{2}\right)^{1/p}(x_0 + 2) & : \quad \frac{\log 3}{\log 3 - \log 2} < p < \infty \end{cases}$$

9 The Steiner ratio of Euclidean spaces

In the d-dimensional Euclidean space, we consider the set N of $d+1$ nodes of a regular simplex with exclusively edges of unit length. Then an MST for N has the length d. It is easy to compute that the sphere that circumscribes N has the radius $R(N) = \sqrt{d/(2d+2)}$. With the center of this sphere as Steiner point, we find a tree T interconnecting N with the length $L_{B(2)}(T) = (d+1)R(N)$. Hence, we find the following nontrivial upper bound:

[7]But between $p = 2$ and $p \approx 2.0619508$ the bound will be greater than the bounds given before.

Theorem 9.1 *The Steiner ratio of the d-dimensional Euclidean space can be bounded as follows:*

$$m(d,2) \le \sqrt{\frac{1}{2}+\frac{1}{2d}}. \tag{33}$$

In the proof we used a Steiner point of degree $d+1$, but it is well-known that all Steiner points in an SMT in Euclidean space are of degree 3. Consequently, the tree T described above is not an SMT for N, if $d > 2$. Better estimates for $m(d,2)$, we find in

Theorem 9.2 *(Chung, Gilbert [4], Smith [39] and Du, Smith [21]) The Steiner ratio of the d-dimensional Euclidean space is bounded as follows:*

dimension	*upper bound by Chung, Gilbert*	*upper bound by Smith*	*upper bound by Du, Smith*
= 2	0.86602...		
= 3	0.81305...	0.81119...	0.78419...
= 4	0.78374...	0.76871...	0.74398...
= 5	0.76456...	0.74574...	0.72181...
= 6	0.75142...	0.73199...	0.70853...
= 7	0.74126...	0.72247...	0.70012...
= 8	0.73376...	0.71550...	0.69455...
= 9	0.72743...	0.71112...	0.69076...
= 10	0.72250...		0.68812...
= 11	0.71811...		0.68624...
= 20	0.69839...		
= 40	0.68499...		
= 80	0.67775...		
= 160	0.67392...		
$\to \infty$	0.66984...		

The first column was computed by Chung and Gilbert considering regular simplices. Here, Du and Smith [21] showed that the regular d-simplex cannot achieve the Steiner ratio if $d > 2$. That means that these bounds cannot be the Steiner ratio of the space when $d > 2$.
The second column given by Smith investigates regular octahedra, respectively cross polytopes. Note, that it is not easy to compute an SMT for the nodes of an octahedra.
The third column used the ratio of sausages, whereby a sausage is constructed by

1. Start with a ball (of unit diameter) in $\mathcal{L}_2^d$;
2. Successively add balls so that the n'th ball you add is always touching the $\min\{d, n-1\}$ most recently added balls.

This procedure uniquely[8] defines an infinite sequence of interior-disjoint numbered balls. The centers of these balls form a discrete point set, which is called the (infinity) d-sausage $N(\infty, d)$. The first n points of the d-sausage will be called the "n-point d-sausage" $N(n, d)$. Note, that $N(d+1, d)$ is a d-simplex if $d \geq 3$.
Du and Smith [21] present many properties of the d-sausage, in particular, that

$$u(d) := \frac{L(\text{SMT for } N(\infty, d))}{L(\text{MST for } N(\infty, d))} \tag{34}$$

is a strictly decreasing function of the dimension d.[9] Hence, $u(d)$, $d = 2, 3, \ldots$ is a convergent sequence, but the limit is still unknown.
It seems that probably there does not exist a finite set of points in the d-dimensional Euclidean space, $d \geq 3$, which achieves the Steiner ratio $m(d, 2)$. But, if such set in spite of it exists, then it must contain exponentially many points. More exactly: Smith and McGregor Smith [41] investigate sausages in the three-dimensional Euclidean space to determine the Steiner Ratio and following they conjectured that for the Steiner Ratio of the three-dimensional Euclidean space

$$\begin{aligned} m(3,2) &= \sqrt{\frac{283}{700} - \frac{3\sqrt{21}}{700} + \frac{9\sqrt{11-\sqrt{21}}\sqrt{2}}{140}} \\ &= 0.78419\ldots \end{aligned}$$

holds.

Moreover, Du and Smith used the theory of packings to get the following

[8] up to congruence

[9] Here, we use a generalization of Steiner's Problem to sets of infinetly many points. This is simple to understand. For a finite number of points it is shown that

$$\frac{L(\text{SMT for } N(2d+1, d))}{L(\text{MST for } N(2d+1, d))} \leq \frac{L(\text{SMT for } N(d+1, d))}{L(\text{MST for } N(d+1, d))},$$

which is a finite version of

$$\frac{L(\text{SMT for } N(\infty, d))}{L(\text{MST for } N(\infty, d))} \leq \frac{L(\text{SMT for } N(d+1, d))}{L(\text{MST for } N(d+1, d))},$$

for $d > 1$.

Theorem 9.3 *(Du, Smith [21]) Let N be a finite set of* n *points in the* d*-dimensional Euclidean space* $M_d(B(2))$, $d \geq 3$, *which achieves the Steiner ratio* $m_d(B(2))$ *of the space. Then*

$$n \geq \left\lceil \frac{1}{2} \cdot \sqrt{f\left(\frac{\pi}{3}, d\right)} \right\rceil + 1,$$

where

$$f(\theta, d) = \frac{2I_{d-2}(\pi/2)}{I_{d-2}(\theta)}$$

and

$$I_m(x) = \int_0^x (\sin u)^m \, du.$$

9.3 implies that the number n grows at least exponentially in the dimension d. Some numbers are computed:

$d =$	n is at least
49	49
50	53
100	2218
200	3481911
500	10^{16}
1000	$5 \cdot 10^{31}$

10 The Steiner ratio of $\mathcal{L}_p^d$

For our investigations we have the following facts: Let $1 < p < \infty$ and $d \geq 3$. Then there are in $\mathcal{L}_p^d$ at least $d+1$ equidistant points. This can be seen with the following considerations: For $i = 1, \ldots, d$ let

$$v_i = (x_{i,1}, \ldots, x_{i,d})$$

with

$$x_{i,j} = \begin{cases} 1 & : \text{ if } i = j \\ 0 & : \text{ otherwise.} \end{cases}$$

These are d points with $||v_i - v_j|| = 2^{1/p}$ for all $1 \leq i < j \leq d$.
For the point $v = (x, \ldots, x)$ it holds $||v - v_i|| = ||v - v_j||$ for all $1 \leq i, j \leq d$.
To create $||v - v_i|| = 2^{1/p}$ the value x has to fulfill the equation

$$((d-1)|x|^p + |1 - x|^p)^{1/p} = 2^{1/p}.$$

This we can realize by the fact that the function $f:[0,1]\to \mathbb{R}$ with

$$f(x)=((d-1)x^p+(1-x)^p)^{1/p}-2^{1/p}$$

has exactly one zero in [0,1].

Theorem 10.1 *(Albrecht [1], [2]) Let* $1<p<\infty$ *and* $d\geq 3$. *Then*

$$m(d,p)\leq \frac{d+1}{2d}\cdot\left(\frac{d}{2}\right)^{1/p}.$$

Moreover, when we use an idea by Liu and Du [35] for the planar case extending to an approach using equilateral sets and a "center" we find:

Theorem 10.2 *(Albrecht [1], [2]) Let* $1<p<\infty$. *Then*

$$m(d,p)<\frac{d+1}{d}\cdot\left(\frac{1}{2}\right)^{1/p}.$$

This bound is not sharp, since the estimation of the distance of the points to the center is to inefficient, at least for small dimensions. On the other hand, we only use one additional point, and it is to assume that more than one of such points decrease the length.

Now, we compare the bounds given in 10.1 and 10.2. Obviously,

$$\frac{d+1}{2d}\cdot\left(\frac{d}{2}\right)^{1/p} \quad\leq\quad \frac{d+1}{d}\cdot\left(\frac{1}{2}\right)^{1/p}$$

holds if and only if

$$d \quad\leq\quad 2^p.$$

Hence,

Corollary 10.3 *Looking for the Steiner ratio of high dimensional* $\mathcal{L}_p$*-spaces we have only to consider the bound given in 10.2.*

11 When the dimension runs to infinity

What do we know of the development of the Steiner ratio of Banach-Minkowski spaces if the dimension d runs to infinity? First, we consider the spaces $\mathcal{L}_p^d$, namely

$$\underline{m}(p)=\lim_{d\to\infty} m(d,p). \tag{35}$$

We know the following values and estimates for $\underline{m}(p)$:

$p=$	lower bound	exact value	upper bound	source
1		$\frac{1}{2}=0.5$		4.2
2	$\frac{1}{\sqrt{3}}=0,57735\ldots$		$0.66984\ldots$	9.2
∞		$\frac{1}{2}=0.5$		calculation
arbitrary	$\frac{1}{2}=0.5$		$\left(\frac{1}{2}\right)^{1/p}$	10.2

Moreover, combining all facts, we have

Theorem 11.1 *Let* $\underline{m}(p)=\lim_{d\to\infty} m(d,p)$. *Then it holds*

$$\frac{1}{2}\leq \underline{m}(p)\leq \min\left\{\left(\frac{1}{2}\right)^{1/p}, \underline{m}(2)\right\}$$

for any real $1<p<\infty$, *and*

$$\underline{m}(1)=\underline{m}(\infty)=\frac{1}{2}.$$

Above we discussed the limiting process for the class of all Banach-Minkowski spaces considering the sequence $\{C_d\}_{d=1,2,\ldots}$ whereby

$$C_d=\sup\{m_d(B): B\in \underline{B}_d\}.$$

Moreover, we find that our conjectures from the end of section 4 are true if we go to infinity.

Theorem 11.2 *(C. [12])*

$$\lim_{d\to\infty} C_d=\lim_{d\to\infty} m(d,2),$$

whereby

$$0.57735\ldots\leq \lim_{d\to\infty} m(d,2)\leq 066984\ldots;$$

and

$$\lim_{d\to\infty} c_d=\lim_{d\to\infty} m(d,1)=0.5.$$

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