ON THE COMPLETENESS OF SETS OF q-BESSEL FUNCTIONS $J_\nu^{(3)}(x;q)$

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Abstract We study completeness of systems of third Jackson q-Bessel functions by two quite different methods. The first uses a Dalzell-type criterion and relies on orthogonality and the evaluation of certain q-integrals. The second uses classical entire function theory.

1. Introduction

For $0 < q < 1$ define the q-integral on the interval $(0, a)$ by

$$\int_0^a f(x) d_q x = (1-q) \sum_{n=0}^{\infty} f(aq^n)\, aq^n. \tag{1.1}$$

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†Joaquín Bustoz (1939–2003) passed away in August 2003 as a consequence of a car accident. He will be missed both as a mathematician and for his work on teaching mathematics, in particular on getting students from minorities into higher education.

$L_q^2(0,1)$ will denote the Hilbert space associated with the inner product

$$\langle f,g\rangle = \int_0^1 f(x)g(x)d_qx.$$

It is a well known fact that the third Jackson q-Bessel function $J_\nu^{(3)}(z;q)$, defined as

$$J_\nu^{(3)}(z;q) = z^\nu \frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty} \sum_{k=0}^{\infty}(-1)^k \frac{q^{\frac{k(k+1)}{2}}}{\left(q^{\nu+1};q\right)_k (q;q)_k} z^{2k} \tag{1.2}$$

satisfies the orthogonality relation

$$\begin{aligned} &\int_0^1 xJ_\nu\left(j_{n\nu}qx;q^2\right) J_\nu\left(j_{m\nu}qx;q^2\right) d_qx \\ &= \frac{q-1}{2q^2} J_{\nu+1}\left(j_{n\nu};q^2\right) J_\nu\left(j_{n\nu};q^2\right) \delta_{n,m} \end{aligned} \tag{1.3}$$

where $j_{1\nu} < j_{2\nu} < \cdots$ are the zeros of $J_\nu^{(3)}\left(z;q^2\right)$ arranged in ascending order. Important information on the zeros of $J_\nu^{(3)}\left(z;q^2\right)$ has been given recently (Ismail, 2003; Koelink and Swarttouw, 1994; Koelink, 1999; Abreu et al., 2003). The orthogonality relation (1.3) is a consequence of the second order difference equation of Sturm-Liouville type satisfied by the functions $J_\nu^{(3)}\left(z;q^2\right)$ (Swartouw, 1992; Koelink and Swarttouw, 1994). In this paper we consider completeness properties of the third q-Bessel function in the spaces $L_q(0,1)$ and $L_q^2(0,1)$. We will approach the problem from two substantially different directions. In one case we will apply a q-version of the Dalzell Criterion (Higgins, 1977) to prove completeness of the system $\left\{J_\nu^3\left(j_{n\nu}qx;q^2\right)\right\}$ in $L_q^2(0,1)$. In another case we will use the machinery of entire functions and the Phragmén-Lindelöf principle to prove completeness of the system $\left\{J_\mu^3\left(j_{n\nu}qx;q^2\right)\right\}$, $\mu,\nu>0$ in $L_q^1(0,1)$. This theorem is in the spirit of classical results on Bessel functions (Boas and Pollard, 1947) that state the completeness of systems $\{J_\nu(\lambda_n(z))\}$ where the numbers λ_n are allowed a certain freedom. Although the entire function argument is more general, there is reason to present the Dalzell Criterion approach as well because it relies solely on techniques of q-integration and on properties of orthogonal expansions in a Hilbert space. Also, this approach requires the calculation of some q-integrals of q-Bessel functions that parallel results for classical Bessel functions. Thus this method of proof extends the q-theory of orthogonal functions.

The third Jackson q-Bessel function was also studied by Exton (Exton, 1983) and sometimes appears in the literature as *The Hahn-Exton q-Bessel Function.* There are other two analogues of the Bessel function introduced by Jackson (Jackson, 1904). The notation of Ismail (Ismail, 1982; Ismail, 2003), denoting all three analogues by $J_\nu^{(k)}(z;q)$, $k = 1, 2, 3$ has become common and we adhere to it here. However, because the present work will deal exclusively with $J_\nu^{(3)}\left(z;q^2\right)$, to simplify notation we write from now on

$$J_\nu(z) = J_\nu^{(3)}\left(z;q^2\right).$$

It is critical to keep in mind that in definition (1.2) the q-Bessel function is defined with base q, whereas in defining $J_\nu(z)$ we have changed the base to q^2. Thus the series definition for $J_\nu(z)$ is

$$J_\nu(z) = z^\nu \frac{\left(q^{2\nu+2};q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)}}{\left(q^{2\nu+2};q^2\right)_\infty \left(q^2;q^2\right)_\infty} z^{2k}.$$

Let $z_{n\nu}$, $n = 1, 2, \ldots$ denote the positive roots of $J_\nu^{(3)}(z;q)$ arranged in increasing order. From (Kvitsinsky, 1995) we have that

$$\sum_{n=1}^{\infty} \left(z_{n\nu}\right)^{-2} = \frac{q}{(1-q)\left(1-q^{\nu+1}\right)}. \tag{1.4}$$

Replacing q by q^2, we find for the roots $j_{n\nu}$ of $J_\nu(z)$ that

$$\sum_{n=1}^{\infty} \left(j_{n\nu}\right)^{-2} = \frac{q}{\left(1-q^2\right)\left(1-q^{2\nu+2}\right)}. \tag{1.5}$$

Expression (1.5) will be used in Section 2.

2. Completeness: A Dalzell type criterion

It is easy to verify (Higgins, 1977) that if $\{\Phi_n\}$ and $\{\Psi_n\}$ are two sequences in a Hilbert space H, with Ψ_n complete in H and Φ_n complete in Ψ_n and orthogonal in H, then Φ_n is also complete in H. Then, if Ψ_n is complete in H, a necessary and sufficient condition for the orthogonal sequence Φ_n to be complete in H is that it satisfies the Parseval relation

$$\sum_n \left|\langle \Phi_n, \Psi_k\rangle\right|^2 = \|\Psi_k\|, \text{ for every } \Psi_k,\ k = 0, 1, \ldots. \tag{2.1}$$

This fact was used by Dalzell to derive a completeness criterion and apply it to several sequences of special functions (Dalzell, 1945). In this section we will derive a similar criterion suitable to be used in $L_q^2(0,1)$. Then, we use it to prove completeness in $L_q^2(0,1)$ of the orthonomal set of functions

$$\Phi_n(x) = \frac{x^{\frac{1}{2}} J_\nu\left(j_{n\nu} q x\right)}{\left\| x^{\frac{1}{2}} J_\nu\left(j_{n\nu} q x\right)\right\|}.$$

To do so, we will evaluate explicitly some q-integrals using the results from the preceding section. We start by stating and proving the following lemma:

Lemma 2.1. *Let $g \in L_q^2(0,1)$ such that $g\left(q^n\right) > 0$, $n = 0,1,2\ldots$. Define $\chi_n(x) = 1$ if $x \in [0, q^n]$ and $\chi_n(x) = 0$ otherwise. Then $\{g\chi_n\}$ is complete in $L_q^2(0,1)$.*

Proof. Let $f \in L_q^2(0,1)$ be such that

$$\int_0^1 f(x) g(x) \chi_n(x) d_q x = 0,\ n = 0,1,2,\ldots.$$

Now, by (1.1) and using the fact that $\chi_n\left(q^k\right) = 0$ if $k < n$, we get:

$$A_n = \sum_{k=n}^{\infty} f\left(q^k\right) g\left(q^k\right) q^k = 0.$$

Then,

$$0 = A_n - A_{n+1} = f\left(q^n\right) g\left(q^n\right) q^n$$

because $g\left(q^n\right) > 0$ it follows that

$$f\left(q^n\right) = 0,\ n = 0,1,2,\ldots.$$

□

Theorem 2.2. *Let $g \in L_q^2(0,1)$ such that $g\left(q^n\right) > 0$, $n = 1,2\ldots$ and let $w(x)$ be such that $\int_0^1 w(x) d_q x$ exists and $w\left(q^n\right) > 0$, $n = 1,2\ldots$. Then an orthonormal sequence $\{\Phi_n\} \subset L_q^2(0,1)$ is complete in $L_q^2(0,1)$ if and only if*

$$\sum_n^{\infty} \int_0^1 \left| \int_0^r \Phi_n(x) g(x) d_q x \right|^2 w(r) d_q r = \int_0^1 \left[\int_0^r |g(x)|^2 d_q x \right] w(r) d_q r. \quad (2.2)$$

Proof. Writing $\Psi_k = g\chi_k$ in (2.1), by the preceding lemma, the sequence $\{\Phi_n\}$ is complete in $L_q^2(0,1)$ if and only if

$$\sum_n^\infty \left| \int_0^1 \Phi_n(x)g(x)\chi_k(x)d_qx \right|^2 = \int_0^1 |g(x)\chi_k(x)|^2 \, d_qx, \; k = 0, 1, \ldots .$$

that is

$$\sum_n^\infty \left| \int_0^r \Phi_n(x)g(x)d_qx \right|^2$$

$$= \int_0^r |g(x)|^2 \, d_qx \text{ for every } r \in \left\{ q^k, k = 0, 1, \ldots \right\}.$$

Integrating both sides of this relation after multiplying by $w(x)$, one gets the relation (2.2). On the other hand, if (2.2) holds, then define

$$F(r) = \int_0^r |g(x)|^2 \, d_qx - \sum_n^\infty \left| \int_0^r \Phi_n(x)g(x)d_qx \right|^2 .$$

From the hypothesis,

$$\int_0^1 F(r)w(r)d_qr = 0.$$

Observing that by the Bessel inequality, $F(r)$ is non-negative, we get

$$F\left(q^k\right) = 0, \; k = 1, 2, \ldots .$$

□

We proceed to evaluate two important q-integrals.

Lemma 2.3. *For every real number r,*

$$\int_0^r x^{\nu+1} J_\nu \left(j_{n\nu} qx\right) d_qx = \frac{1-q}{qj_{nv}} r^{\nu+1} J_{\nu+1} \left(j_{n\nu} qr\right).$$

Proof. Express

$$\int_0^r x^{\nu+1} J_\nu \left(j_{n\nu} qx\right) d_qx$$

using the power series expansion (1.2). Then interchange the q-integral with the sum and use the following fact:

$$\int_0^r x^{2\nu+2k+1} d_q x = (1-q) r^{2\nu+2k+2} \sum_{n=0}^{\infty} q^{n(2\nu+2k+2)}$$
$$= \frac{1-q}{1-q^{2\nu+2k+2}} r^{2\nu+2k+2}.$$

Rearranging terms the result follows in a straightforward manner. □

Lemma 2.4.

$$\int_0^1 x J_{\nu+1}^2 (j_{n\nu} q x)\, d_q x = \int_0^1 x J_\nu^2 (j_{n\nu} q x)\, d_q x.$$

Proof. Consider the following formula from (Koelink and Swarttouw, 1994):

$$\int_0^1 x J_{\nu+1}^2 (aqz) d_q x = \frac{(1-q)q^\nu}{-2a} \{ a J_{\nu+2}(aq) J'_{\nu+1}(a) \tag{2.3}$$
$$- J_{\nu+2}(aq) J_{\nu+1}(a) - aq J'_{\nu+2}(aq) J_{\nu+1}(a) \}$$

and

$$J_{\nu+1}(xq) = q^{-\nu-1} \left(\frac{1-q^{2\nu}}{x} J_\nu(x) - J_{\nu-1}(x) \right). \tag{2.4}$$

Shift $\nu \to \nu+1$ in (2.4) and set $x = j_{n\nu}$. This yields

$$J_{\nu+2}(j_{n\nu} q) = \frac{1-q^{2\nu+2}}{q^{\nu+2} j_{n\nu}} J_{\nu+1}(j_{n\nu}).$$

Taking derivatives in both members of (2.4), changing $\nu \to \nu+1$ and again setting $x = j_{n\nu}$ the result is

$$J'_{\nu+2}(j_{n\nu} q) = q^{-\nu-3} \Big\{ \left(1 - q^{2\nu+2}\right) \left[\frac{1}{j_{n\nu}} J'_{\nu+1}(j_{n\nu}) \right.$$
$$\left. - \frac{1}{j_{n\nu}^2} J_{\nu+1}(j_{n\nu}) \right] - J_\nu(j_{n\nu}) \Big\}.$$

Substituting this in (2.3) we get the simplification:

$$\int_0^1 xJ_{\nu+1}^2(j_{n\nu}qx)\,d_qx = \frac{(1-q)}{-2q^2}J'_\nu(j_{n\nu})\,J_{\nu+1}(j_{n\nu})$$

$$= -\frac{1}{2}(1-q)q^{\nu-2}J_{\nu+1}(j_{n\nu}q)\,J_\nu(j_{n\nu}) = \int_0^1 xJ_\nu^2(j_{n\nu}qx)\,d_qx$$

where (1.3) was used in the last identity. □

Theorem 2.5. *The orthonormal sequence* $\{\Phi_n\}$ *defined by*

$$\Phi_n(x) = \frac{x^{\frac{1}{2}}J_\nu(j_{n\nu}qx)}{\left\|x^{\frac{1}{2}}J_\nu(j_{n\nu}qx)\right\|}$$

is complete in $L_q^2(0,1)$.

Proof. In (2.2) take $\{\Phi_n\}$ defined as above, $g(x) = x^{\nu+\frac{1}{2}}$ and $w(r) = r^{-2\nu-1}$. We need thus to prove the identity

$$\sum_{n=1}^{\infty}\int_0^1\left|\int_0^r \frac{x^{\frac{1}{2}}J_\nu(j_{n\nu}qx)}{\left\|x^{\frac{1}{2}}J_\nu(j_{n\nu}qx)\right\|}x^{\nu+1}d_qx\right|^2 r^{-2\nu-1}d_qr$$

$$= \int_0^1\left[\int_0^r |g(x)|^2\,d_qx\right]w(r)d_qr.$$

Lemma 1 and Lemma 2 allow us to reduce the left hand member of above to:

$$\frac{(1-q)^2}{q^2}\sum_{n=1}^{\infty}\frac{1}{j_{n\nu}^2},$$

that is,

$$\frac{1-q}{(1+q)(1-q^{2\nu+2})}$$

by (1.5). It is straightforward to compute

$$\int_0^1\left[\int_0^r |g(x)|^2\,d_qx\right]w(r)d_qr = \frac{1-q}{(1+q)\,(1-q^{2\nu+2})},$$

and the Theorem is proved. □

3. Completeness: An entire function approach

From (1.2) we can write

$$J_\nu(w) = \frac{\left(q^{2\nu+2};q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} z^\nu F_\nu(w),$$

where

$$F_\nu(w) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} z^{2k}}{\left(q^{2\nu+2};q^2\right)_k \left(q^2;q^2\right)_k}.$$

The function $F_\nu(w)$ is entire and it is directly shown that $F_\nu(w)$ has order zero.

Set $G(w) = \int\limits_0^1 g(x) F_\mu(qwx) d_q x$, and $h(w) = \frac{G(w)}{F_\nu(w)}$.

Lemma 3.1. *If* $\mu > 0$, $\nu > 0$ *and* $g(x) \in L_q^1(0,1)$ *then* $h(w)$ *is entire of order* 0.

Proof. We first show that $G(w)$ is entire of order 0. From the definition of the q-integral we have

$$G(w) = (1-q) \sum_{k=0}^{\infty} g\left(q^k\right) F_\mu\left(wq^{k+1}q^k\right). \tag{3.1}$$

The series in (3.1) converges uniformly in any disk $|w| \le R$. Hence $G(w)$ is entire. Recall that the order $\rho(f)$ of an entire function $f(w)$ is given by

$$\rho(f) = \limsup_{r\to\infty} \frac{\ln\ln M(r;f)}{\ln r}$$

where

$$M(r,f) = \max_{|w|\le r} |f(w)|.$$

From (3.1)

$$M(r;G) \le M\left(r;F_\mu\right) \int\limits_0^1 |g(x)|\, d_q x.$$

Since $\rho\left(F_\mu\right) = 0$ we have that $\rho(G) = 0$.

Both the numerator and the denominator of $h(w)$ are entire functions of order 0. If we write $G(w)$ and $F_\nu(w)$ as canonical products, each factor of $F_\nu(w)$ divides out with a factor of $G(w)$ by the hypothesis of Theorem 3.3. $h(w)$ is thus entire of order 0. □

Lemma 3.2. *If $\mu > 0$, $\nu > 0$, and $0 < q < 1$ then the quotient $\frac{F_\mu(q^m w)}{F_\nu(w)}$ is bounded on the imaginary w axis.*

Proof. We will make use of the simple inequality

$$(q^\alpha; q)_\infty < (q^\alpha; q)_k < 1, \quad \alpha > 0, \quad 0 < q < 1.$$

Using this inequality we get for $w = iy$, y real,

$$F_\mu(q^m iy) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)} q^{2mn} y^{2n}}{(q^{2\mu+2}; q^2)_n (q^2; q^2)_n} < \frac{1}{(q^{2\mu+2}; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)} y^{2n}}{(q^2; q^2)_n},$$

$$F_\nu(iy) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)} y^{2n}}{(q^{2\nu+2}; q^2)_n (q^2; q^2)_n} > \sum_{n=0}^{\infty} \frac{q^{n(n+1)} y^{2n}}{(q^2; q^2)_n}.$$

Thus we have

$$0 \leq \frac{F_\mu(q^m; y)}{F_\nu(iy)} < \frac{1}{(q^{2\mu+2}; q^2)_\infty}.$$

□

Theorem 3.3. *Let $\mu > 0$, $\nu > 0$ and $g(x) \in L^1_q(0,1)$. If*

$$\int_0^1 g(x) J_\mu(q j_{n\nu} x)\, d_q x = 0,$$

$n = 1, 2, \ldots$ then $g(x) = 0$ for $x = q^m$, $m = 0, 1, \ldots$.

Proof. Lemma 3.2 implies that $h(iy)$ is bounded. Since $h(w)$ is entire of order 0, we can apply one of the versions of the Phragmén-Lindelöf theorem (Levin, 1980, p. 49) and Lemma 3.2 and conclude that $h(w)$ is bounded in the entire w-plane. Next by Liouville's theorem we conclude that $h(w)$ is constant. Say that $h(w) \equiv C$. We will prove that $C = 0$. We have

$$G(w) - CF_\nu(w) \equiv 0.$$

In infinite series form this equality produces an identity of the form

$$\sum_{k=0}^{\infty} A_k w^{2k} \equiv 0.$$

From the identity theorem for analytic functions we conclude that $A_k = 0$. Calculating A_k we find

$$\frac{q^{k(k+1)+2k}(1-q)(-1)^k}{(q^{2\mu+2}; q^2)_k (q^2; q^2)_k} \sum_{j=0}^{\infty} g(q^j)\, q^{(2k+1)j}$$
$$- \frac{q^{k(k+1)}(-1)^k}{(q^{2\nu+2}; q^2)_k (q^2; q^2)_k} C = 0, \quad k = 0, 1, 2, \ldots.$$

Dividing out common factors and letting $k \to \infty$ gives $C = 0$. We can now conclude that $G(w) \equiv 0$, or that is,

$$\int_0^1 g(x) J_\mu(wqx) d_q x \equiv 0.$$

We complete the proof with a simple argument that gives $g(q^m) = 0$, $m = 0, 1, \ldots$. If $G(w) \equiv 0$ then

$$\sum_{j=0}^{\infty} g\left(q^j\right) q^{(2k+1)j} = 0.$$

Letting $k \to 0$ gives $g(1) = 0$. Then dividing by q^{2k} and again letting $k \to \infty$ gives $g(q) = 0$. Continuing this process we have $g(q^m) \equiv 0$ and the proof of the theorem is complete. □

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