
CONTROL OF A CONDITIONALLY CORRECT SYSTEM DESCRIBED BY THE NEUMANN PROBLEM FOR AN ELLIPTIC-TYPE EQUATION UNDER CONJUGATION CONDITIONS

2.1 DISTRIBUTED CONTROL WITH OBSERVATION THROUGHOUT A WHOLE DOMAIN

Assume that the elliptic equation

$$Lu \equiv - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij}(x) \frac{\partial y}{\partial x_j} \right) = f(x) \quad (1.1)$$

is specified in a domain Ω that consists of two bounded convex domains, namely, Ω_1 and $\Omega_2 \in R^n$, where R^n is an n -dimensional real linear space. The second-type boundary Neumann condition

$$\sum_{i,j=1}^n k_{ij}(x) \frac{\partial y}{\partial x_j} \cos(v, x_i) = g(x) \quad (1.2)$$

is specified, in its turn, on a boundary $\Gamma = (\partial\Omega_1 \cup \partial\Omega_2) \setminus \gamma$ ($\gamma = \partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$); in this case, v is an outer normal to Γ , $k_{ij}(x)|_{\bar{\Omega}_l} = k_{ji}|_{\bar{\Omega}_l} = k_{ij}|_{\bar{\Omega}_l} \in C(\bar{\Omega}_l) \cap C^1(\Omega_l)$, $f(x)|_{\Omega_l} = f|_{\Omega_l} \in C(\Omega_l)$, $|D^1 k_{ij}| < \infty$, $g(x)|_{\Gamma \cap \partial\Omega_l} = g|_{\Gamma \cap \partial\Omega_l} \in C(\Gamma \cap \partial\Omega_l)$, $i, j = \overline{1, n}$; $l = 1, 2$, $|f| \leq c_1 < \infty$,

$$\sum_{i,j=1}^n k_{ij}(x) \xi_i \xi_j \geq \alpha_0 \sum_{i=1}^n \xi_i^2 \quad \forall x \in \Omega,$$

$$\forall \xi_i, \xi_j \in R^1, \quad i, j = \overline{1, n}, \quad \alpha_0 = \text{const} > 0; \quad (1.2')$$

and the conjugation conditions

$$[y] = 0 \quad (1.3)$$

and

$$\left[\sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(v, x_i) \right] = \omega \quad (1.4)$$

are specified, also in their turn, on a section γ of the domain $\bar{\Omega}$; in this case, $\omega \in C(\gamma)$, $[\varphi] = \varphi^+ - \varphi^-$, $\varphi^\pm = \{\varphi\}^\pm = \varphi(x)$ under $x \in \gamma^\pm$, $\gamma^+ = \gamma \cap \partial\Omega_2$, $\gamma^- = \gamma \cap \partial\Omega_1$, v is a normal to γ and such normal is directed into the domain Ω_2 .

Let $y(x) \in \bar{M} = \left\{ v(x): v|_{\bar{\Omega}_l} \in C^1(\bar{\Omega}_l) \cap C^2(\Omega_l), \quad l = 1, 2, \quad |D^2 v| < \infty \right\}$

be a classical solution to boundary-value problem (1.1)–(1.4). It is easy to see that a solution $y + c$ is also classical to it for an arbitrary constant c .

The necessary condition for the existence of the classical solution y to problem (1.1)–(1.4) is the one under which the equality

$$\int_{\Omega} f \, dx + \int_{\Gamma} g \, d\Gamma = \int_{\gamma} \omega \, d\gamma \quad (1.5)$$

is met. Find this solution under the constraint

$$\int_{\Omega} y \, dx = Q, \quad (1.6)$$

where Q is some known real number. Assume the following: $\bar{H} = \left\{ v(x) : v|_{\Omega_i} \in W_2^1(\Omega_i), i=1,2 \right\}$, $V_Q = \left\{ v \in \bar{H} : [v] = 0, (v,1) = Q \right\}$, $(\varphi, \psi) = \int_{\Omega} \varphi \psi dx$.

Let there be a control Hilbert space \mathcal{U} and mapping $B \in \mathcal{L}(\mathcal{U}; V')$, where V' is a space dual with respect to a state Hilbert space V . Assume the following: $\mathcal{U} = L_2(\Omega)$.

For every control $u \in \mathcal{U}$, determine system state $y = y(u)$ as a generalized solution to the boundary-value problem specified by the equation

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij}(x) \frac{\partial y}{\partial x_j} \right) = f(x) + Bu, y \in V_Q, \quad (1.7)$$

and by conditions (1.2)–(1.4) and (1.6).

Specify the observation

$$Z(u) = C y(u), \quad (1.8)$$

where $C \in \mathcal{L}(V; \mathcal{H})$ and \mathcal{H} is some Hilbert space. Assume the following:

$$C y(u) \equiv y(u), \mathcal{H} = V \subset L_2(\Omega). \quad (1.9)$$

Bring a value of the cost functional

$$J(u) = \|Cy(u) - z_g\|_{\mathcal{H}}^2 + (\mathcal{N}u, u)_{\mathcal{U}} \quad (1.10)$$

in correspondence with every control $u \in \mathcal{U}$; in this case, z_g is some known element of \mathcal{H} , $\mathcal{N} \in \mathcal{L}(\mathcal{U}; \mathcal{U})$, $(\mathcal{N}u, u)_{\mathcal{U}} \geq \nu_0 \|u\|_{\mathcal{U}}^2$, $\nu_0 = \text{const} > 0$ $\forall u \in \mathcal{U}$.

Assume the following: $f \in L_2(\Omega)$, $Bu \equiv u \in L_2(\Omega)$, $\mathcal{N}u = \bar{a}(x)u$, $0 < a_0 \leq \bar{a}(x) \leq a_1 < \infty$, $\bar{a}(x)|_{\Omega_l} \in C(\Omega_l)$, $l=1, 2$; $a_0, a_1 = \text{const}$,

$(\varphi, \psi)_{\mathcal{U}} = \int_{\Omega} \varphi \psi dx$. A unique state, namely, a function $y(u) \in V_Q$

corresponds to every control $u \in \mathcal{U}$, delivers the minimum to the energy functional [21]

$$\Phi_1(v) = a_1(v, v) - 2l_1(v) \quad (1.11)$$

on V_Q , and it is the unique solution in V_Q to the weakly stated problem:

Find an element $y \in V_Q$ that meets the equation

$$a_1(y, v) = l_1(v) \quad \forall v \in V_0, \quad (1.12)$$

where $V_0 = \{v \in \bar{H} : [v] = 0, (v, 1) = 0\}$, $a_1(u, v) = \int_{\Omega} \sum_{i,j=1}^n k_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx$,

$$l_1(v) = \int_{\Omega} (f + u)v dx + \int_{\Gamma} g v d\Gamma - \int_{\gamma} \omega v d\gamma.$$

The following statement is valid [21].

Lemma 1.1. *Problems (1.11) and (1.12) are equivalent $\forall f \in L_2(\Omega)$, $\forall \omega \in L_2(\gamma)$, $\forall u \in \mathcal{U}$ and have a unique solution $y = y(u) \in V_Q$.*

Remark 1.1. If a solution $y \in V_Q$ to problems (1.11) and (1.12) belongs to a set \bar{M} , then y is classical to boundary-value problem (1.7), (1.2)–(1.4), (1.6) under the constraint

$$\int_{\Omega} (f + u) dx + \int_{\Gamma} g d\Gamma = \int_{\gamma} \omega d\gamma. \quad (1.13)$$

Remark 1.2. If a solution y to problems (1.11) and (1.12) exists, it is not necessary to meet constraint (1.13).

Remark 1.3. If equality (1.5) takes place, then, to meet constraint (1.13), it is necessary for a control u to satisfy the condition

$$\int_{\Omega} u dx = 0. \quad (1.14)$$

Rewrite cost functional (1.10) as

$$J(u) = \pi(u, u) - 2L(u) + \|z_g - y(0)\|^2; \quad (1.15)$$

in this case, $\|\varphi\| = \|\varphi\|_{L_2(\Omega)} = (\varphi, \varphi)^{1/2}$ and the bilinear form $\pi(\cdot, \cdot)$ and linear functional $L(\cdot)$ are expressed as

$$\begin{aligned}\pi(u, v) &= (y(u) - y(0), y(v) - y(0)) + (\bar{a} u, v) \\ L(v) &= (z_g - y(0), y(v) - y(0)).\end{aligned}\quad (1.16)$$

Let $\tilde{y}' = \tilde{y}(u')$ and $\tilde{y}'' = \tilde{y}(u'')$ be solutions from V_Q to problem (1.12) under $f = 0$, $g = 0$ and $\omega = 0$ and under a function $u = u(x)$ that is equal, respectively, to u' and u'' . Then, take the ellipticity condition and generalized Poincare inequality into account, and the inequality

$$\begin{aligned}\bar{\alpha}_0 \|\tilde{y}' - \tilde{y}''\|^2 &\leq \bar{\alpha}_0 \|\tilde{y}' - \tilde{y}''\|_V^2 \leq a_1 (\tilde{y}' - \tilde{y}'', \tilde{y}' - \tilde{y}'') \leq \\ &\leq \|u' - u''\| \cdot \|\tilde{y}' - \tilde{y}''\|_V, \quad \bar{\alpha}_0 = \text{const} > 0,\end{aligned}$$

is derived, where $\|v\|_V = \left\{ \sum_{i=1}^2 \|v\|_{W_2^1(\Omega_i)}^2 \right\}^{1/2}$ and $\|\cdot\|_{W_2^1(\Omega_i)}$ is the norm of the Sobolev space $W_2^1(\Omega_i)$.

On the basis of [58, Theorem 1.1, Chapter 1], the validity of the following statement is proved.

Theorem 1.1. *Let a system state be determined as a solution to equivalent problems (1.11) and (1.12). Then, there exists a unique element u of a convex set \mathcal{U}_Q that is closed in \mathcal{U} , and*

$$J(u) = \inf_{v \in \mathcal{U}_Q} J(v) \quad (1.17)$$

takes place for u .

Definition 1.1. If an element $u \in \mathcal{U}_Q$ meets condition (1.17), it is called an optimal control.

Let the equation

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial y}{\partial x_j} \right) + \int_{\Omega} y dx = f(x) + Q + u \quad (1.18)$$

be specified on the domain Ω instead of equation (1.7). Neumann condition (1.2) and conjugation conditions (1.3)–(1.4) are specified, in their turn, respectively, on the boundary Γ and section γ .

If y is a classical solution to boundary-value problem (1.7), (1.2)–(1.4), (1.6) (Problem 1), then it is easy to see that y is classical to problem (1.18), (1.2)–(1.4) (Problem 1'). It can be shown [21] that a classical solution to Problem 1' is also classical to Problem 1 if constraint (1.13) is satisfied.

Let observation (1.8) be specified, where the operator C is given by expression (1.9). Cost functional (1.10) is specified, in its turn, for every control $u \in \mathcal{U}$. Then, a unique state, namely, a function $y(u) \in V = \{v \in \bar{H} : [v] = 0\}$, corresponds to every $u \in \mathcal{U}$, minimizes the energy functional

$$\Phi(v) = a'_1(v, v) - 2l'_1(v) \quad (1.19)$$

on V , and it is the unique solution in V to the weakly stated problem: Find an element $y \in V$ that meets the equation

$$a'_1(y, v) = l'_1(v), \quad \forall v \in V, \quad (1.20)$$

where $a'_1(y, v) = a_1(y, v) + (y, 1)(v, 1)$ and $l'_1(v) = l_1(v) + Q(v, 1)$.

Lemma 1.2. *Problems (1.19) and (1.20) are equivalent $\forall f \in L_2(\Omega)$, $\forall u \in \mathcal{U}$ and have a unique solution $y(u) \in V$.*

Remark 1.4. If a solution $y \in V$ to problems (1.19) and (1.20) belongs to a set \bar{M} , then y is classical to boundary-value Problem 1', and it is also classical to Problem 1 if constraint (1.13) is met.

Therefore, there exists such an operator A generated by problems (1.19), (1.20) and acting from V into $L_2(\Omega)$, that

$$y(u) = A^{-1}(f + Q + Bu), \quad \forall u \in L_2 = L_2(\Omega).$$

Let $\tilde{y}' = \tilde{y}(u')$ and $\tilde{y}'' = \tilde{y}(u'')$ be solutions from V to Problem 1' under $f = 0$, $g = 0$ and $\omega = 0$ and under a function $u = u(x)$ that is equal, respectively, to u' and u'' .

Then, on the basis of the generalized Poincare inequality, the following one, i.e.

$$\begin{aligned}\bar{\alpha}_1 \|\tilde{y}' - \tilde{y}''\|^2 &\leq \bar{\alpha}_1 \|\tilde{y}' - \tilde{y}''\|_V^2 \leq a (\tilde{y}' - \tilde{y}'', \tilde{y}' - \tilde{y}'') \leq \\ &\leq \|u' - u''\| \cdot \|\tilde{y}' - \tilde{y}''\|, \quad \bar{\alpha}_1 = \text{const} > 0,\end{aligned}$$

is derived that provides the continuity of the linear functional $L(\cdot)$ and bilinear form $\pi(\cdot, \cdot)$ of expressions (1.16) on \mathcal{U} .

On the basis of [58, Theorem 1.1, Chapter 1], the validity of the following statement is proved.

Theorem 1.2. *Let a system state be determined as a solution to equivalent problems (1.19) and (1.20). Then, there exists a unique element u of a convex set \mathcal{U}_∂ that is closed in \mathcal{U} , and relation like (1.17) takes place for u .*

Remark 1.5. If equality (1.13) is satisfied, then problems (1.11) and (1.19) are equivalent. Therefore, optimal controls coincide when states are described by boundary-value Problems 1 and 1'.

Here is the problem of finding the control $u \in \mathcal{U}_\partial$ that satisfies relation (1.17). It is optimization Problem 1 if a system state is a generalized solution to boundary-value Problem 1, and it is optimization Problem 1' if a system state is a generalized solution to boundary-value Problem 1'.

Remark 1.6. If constraint (1.5) is met and $\mathcal{U}_\partial = \left\{ u \in L_2(\Omega) : \int_{\Omega} u dx = 0 \right\}$, then optimization Problems 1 and 1' are equivalent.

If $u \in \mathcal{U}_\partial$ is the optimal control, then the following inequality is true $\forall v \in \mathcal{U}_\partial$:

$$(y(u) - z_g, y(v) - y(u)) + (\bar{a}u, v - u) \geq 0. \quad (1.21)$$

As for the control $v \in \mathcal{U}$, the conjugate state $p(v) \in V^* = V$ is specified by the relations

$$A^* p(v) = y(v) - z_g, \quad x \in \Omega_1 \cup \Omega_2,$$

$$\begin{aligned} \frac{\partial p}{\partial \nu_{A^*}} &= 0, \quad x \in \Gamma, \\ [p] &= 0, \quad \left[\frac{\partial p}{\partial \nu_{A^*}} \right] = 0, \quad x \in \gamma, \end{aligned} \quad (1.22)$$

where

$$\begin{aligned} A^* p &= - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial p}{\partial x_j} \right) + \int_{\Omega} p \, dx, \\ \frac{\partial p}{\partial \nu_{A^*}} &= \sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \cos(\nu, x_i). \end{aligned} \quad (1.23)$$

The equality

$$\begin{aligned} \left(A^* p(u), y(v) - y(u) \right) &= \left(y(u) - z_g, y(v) - y(u) \right) = a'_1(p, y(v) - y(u)) = \\ &= \sum_{l=1}^2 \int_{\partial \Omega_l} \sum_{i,j=1}^n k_{ij} \frac{\partial (y(v) - y(u))}{\partial x_j} \cos(\nu, x_i) p(u) \, d\partial \Omega_l + \\ &\quad + (p, v - u) = (p, v - u), \end{aligned}$$

i.e. $(y(u) - z_g, y(v) - y(u)) = (p, v - u)$ is obtained. Take it into account, and the inequality

$$(p + \bar{a}u, v - u) \geq 0, \quad \forall v \in \mathcal{U}_{\partial}, \quad (1.24)$$

is derived from inequality (1.21).

To make the element $u \in \mathcal{U}_{\partial}$ the optimal control of a state described by boundary-value Problem 1', it is necessary and sufficient to meet inequality (1.24) and the relations

$$a'_1(y, v) = l_1(u, v), \quad y \in V, \quad \forall v \in V, \quad (1.25)$$

and

$$a'_1(p, v) = l_2(y, v), \quad p \in V, \quad \forall v \in V, \quad (1.26)$$

where

$$l_1(u, v) = (f + Q, v) + (u, v) + \int_{\Gamma} g v d\Gamma - \int_{\gamma} \omega v d\gamma$$

and

$$l_2(y, v) = (y, v) - (z_g, v).$$

If the constraints are absent, i.e. when $\mathcal{U}_{\bar{\partial}} = \mathcal{U}$, then the equality

$$p + \bar{a} u = 0 \quad (1.27)$$

follows from condition (1.24). Therefore, when the constraints are absent, the control u can be excluded from equality (1.25) by means of equality (1.27). On the basis of equalities (1.25) and (1.26), the problem

$$A y + p/\bar{a} = f, \quad y \in V, \quad (1.28)$$

$$A^* p - y = -z_g, \quad p \in V^*, \quad (1.29)$$

is derived, and the vector solution $(y, p)^T$ is found from this problem along with the optimal control $u = -p/\bar{a}$ of the system specified by boundary-value Problem 1'.

If the vector solution $(y, p)^T$ to problem (1.28), (1.29) is smooth enough on $\bar{\Omega}_l$, viz., $y|_{\bar{\Omega}_l}, p|_{\bar{\Omega}_l} \in C^1(\bar{\Omega}_l) \cap C^2(\Omega_l)$, $l = 1, 2$, then the differential problem of finding the vector-function $(y, p)^T$, that satisfies the relations

$$\begin{aligned} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial y}{\partial x_j} \right) + p/\bar{a} + \int_{\Omega} y dx &= f + Q, \quad x \in \Omega_1 \cup \Omega_2, \\ - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial p}{\partial x_j} \right) + \int_{\Omega} p dx - y &= -z_g, \quad x \in \Omega_1 \cup \Omega_2, \end{aligned}$$

$$\begin{aligned}\sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(v, x_i) &= g, \quad x \in \Gamma, \\ \sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \cos(v, x_i) &= 0, \quad x \in \Gamma, \\ [y] &= 0, \quad [p] = 0, \quad x \in \gamma,\end{aligned}$$

$$\left[\sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(v, x_i) \right] = \omega, \quad \left[\sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \cos(v, x_i) \right] = 0, \quad x \in \gamma, \quad (1.30)$$

corresponds to problem (1.28), (1.29).

Definition 1.2. A generalized (weak) solution to boundary-value problem (1.30) is called a vector-function $(y, p)^T \in H = \left\{ v = (v_1, v_2)^T : v_i|_{\Omega_j} \in W_2^1(\Omega_j), i, j = 1, 2; [v] = 0 \right\}$ that satisfies the following integral equation $\forall z \in H$:

$$\begin{aligned}& \int_{\Omega} \left\{ \sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \frac{\partial z_1}{\partial x_i} + p z_1 / \bar{a} + \sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \frac{\partial z_2}{\partial x_i} - y z_2 \right\} dx + \\& \quad + \int_{\Omega} y dx \int_{\Omega} z_1 dx + \int_{\Omega} p dx \int_{\Omega} z_2 dx = \\& \quad = \int_{\Omega} ((f + Q) z_1 - z_g z_2) dx + \int_{\Gamma} g z_1 d\Gamma - \int_{\gamma} \omega z_1 d\gamma. \quad (1.31)\end{aligned}$$

Let $u = (u_1, u_2)^T$ and $v = (v_1, v_2)^T$ be arbitrary elements of the complete Hilbert space H with the norm $\|\cdot\|_H = \left\{ \sum_{i=1}^2 \|\cdot\|_{W_2^1(\Omega_i)}^2 \right\}^{1/2}$. Specify the bilinear form

$$a(u, v) = \int_{\Omega} \left\{ \sum_{l=1}^2 \sum_{i,j=1}^n k_{ij} \frac{\partial u_l}{\partial x_j} \frac{\partial v_l}{\partial x_i} + u_2 v_1 / \bar{a} - u_1 v_2 \right\} dx + \sum_{l=1}^2 \int_{\Omega} u_l dx \int_{\Omega} v_l dx$$

and linear functional

$$l(v) = \int_{\Omega} ((f + Q)v_1 - z_g v_2) dx + \int_{\Gamma} g v_1 d\Gamma - \int_{\gamma} \omega v_1 d\gamma$$

on H .

Assume that the constraint $\alpha_1 = \min \left\{ \frac{\alpha_0}{2}, 1 \right\} \mu - \frac{1}{2} \left\{ \frac{1}{a_0} + 1 \right\} > 0$ is met,

where μ is the constant in the generalized Poincare inequality. Take the generalized Poincare inequality [21] and Cauchy-Bunyakovsky one into account, and the relations

$$a(v, v) \geq \bar{\alpha}_1 \|v\|_H^2 \quad \forall v \in H, \quad \bar{\alpha}_1 = \text{const} > 0,$$

and

$$|a(u, v)| \leq c_1 \|u\|_H \|v\|_H \quad \forall u, v \in H, \quad c_1 = \text{const} > 0,$$

are true for the bilinear form $a(\cdot, \cdot)$, i.e. this form is H -elliptic and continuous [49] on H .

Consider the Cauchy-Bunyakovsky inequality and embedding theorems [55], and the following inequality is obtained $\forall v \in H$:

$$|l(v)| \leq c_2 \|v\|_H, \quad c_2 = \text{const}.$$

Use the Lax-Milgram lemma [16], and it is concluded that the unique solution (v, p) to problem (1.31) exists in H .

Problem (1.31) can be solved approximately by means of the finite-element method. For this purpose, divide the domains $\bar{\Omega}_i$ into N_i finite elements \bar{e}_i^j ($j = \overline{1, N_i}$, $i = 1, 2$) of the regular family [16]. Specify the subspace $H_k^N \subset H$ ($N = N_1 + N_2$) of the vector-functions $V_k^N(x)$. The

components $v_{1k}^N|_{\bar{\Omega}_i}, v_{2k}^N|_{\bar{\Omega}_i} \in C(\bar{\Omega}_i)$ ($i=1,2$) of $V_k^N(x)$ are the complete polynomials of the power k that contain the variables x_1, x_2, \dots, x_n at every \bar{e}_i^j , and $[V_k^N] = 0$. Then, the linear algebraic equation system

$$A\bar{U} = B \quad (1.32)$$

follows from equation (1.31), and the solution \bar{U} to system (1.32) exists and such solution is unique. The vector \bar{U} specifies the unique approximate solution $U_k^N \in H_k^N$ to problem (1.31) as the unique one to the equation

$$a(U_k^N, V_k^N) = l(V_k^N), \quad \forall V_k^N \in H_k^N. \quad (1.33)$$

Let $U = U(x) \in H$ be the solution to problem (1.31). Then:

$$a(U - U_k^N, V_k^N) = 0, \quad \forall V_k^N \in H_k^N.$$

Therefore,

$$\begin{aligned} \bar{\alpha}_1 \|U - U_k^N\|_H^2 &\leq a(U - U_k^N, U - U_k^N) = \\ &= a(U - U_k^N, U - \tilde{U}), \quad \forall \tilde{U} \in H_k^N, \end{aligned}$$

and the inequality

$$\|U - U_k^N\|_H \leq c_0 \|U - \tilde{U}\|_H, \quad c_0 = \text{const}, \quad (1.34)$$

is thus derived since the bilinear form $a(\cdot, \cdot)$ is continuous on H .

Suppose that $\tilde{U} \in H_k^N$ is a complete interpolation polynomial for the solution U at every \bar{e}_i^j . Take the interpolation estimates [16] into account, assume that every component U_1 and U_2 of the solution U on Ω_l belongs to the Sobolev space $W_2^{k+1}(\Omega_l)$ ($l=1,2$), and the estimate

$$\|U - U_k^N\|_H \leq ch^k, \quad (1.35)$$

where h is a maximum diameter of all the finite elements \bar{e}_i^j , $c = \text{const}$, follows from inequality (1.34).

Take estimate (1.35) into consideration, and the estimate

$$\|u - u_k^N\|_{W_2^1} \leq c_2 \|p - p_k^N\|_{W_2^1} \leq c_3 h^k, \quad (1.35')$$

where $\|\cdot\|_{W_2^1} = \left\{ \sum_{i=1}^2 \|\cdot\|_{W_2^1(\Omega_i)}^2 \right\}^{1/2}$, takes place for the approximation $u_k^N(x) = -p_k^N / \bar{a}(x)$ of the control $u = u(x)$ of a state described by Problem 1' and $p_k^N = u_{2k}^N$ is the second component of the vector U_k^N .

Remark 1.7. If constraint (1.13) is met, then the first component of a classical solution to problem (1.30) is a classical solution to boundary-value Problem 1.

2.2 DISTRIBUTED CONTROL WITH OBSERVATION ON A THIN INCLUSION

Assume that equation (1.1), where the coefficients and right-hand side meet conditions (1.2'), is specified in the bounded, continuous and strictly Lipschitz domains Ω_1 and Ω_2 . Condition (1.2) is specified, in its turn, on the boundary Γ and the conjugation conditions have the form of expressions (1.3) and (1.4).

For every control $u \in \mathcal{U} = L_2(\Omega)$, determine a system state as a generalized solution to the boundary-value problem specified by equation (1.7) and by conditions (1.2)–(1.4), where $Bu \equiv u$ and $u \in L_2(\Omega)$.

Equality (1.13) is the necessary condition under which there exists a classical solution $y = y(u)$ to boundary-value problem (1.7), (1.2)–(1.4) (Problem 2). Find this solution under constraint (1.6).

Bring a value of the cost functional

$$J(u) = \int_{\gamma} (y(u) - z_g)^2 d\gamma + (\mathcal{N}u, u)_{\mathcal{U}} \quad (2.1)$$

in correspondence with every control $u \in \mathcal{U} = L_2(\Omega)$; in this case, z_g is a known element from the space $L_2(\gamma)$; $\mathcal{N}u = \bar{a}u$, $0 < a_0 \leq \bar{a}(x) \leq a_1^0 < \infty$, $\bar{a}(x)|_{\Omega_i} \in C(\Omega_i)$, $i = 1, 2$; $a_0, a_1^0 = \text{const}$.

A unique state, namely, a function $y(u) \in V_Q$ corresponds to every control $u \in \mathcal{U}$, minimizes energy functional (1.11) on V_Q , and it is the unique solution in V_Q to weakly stated problem (1.12). Lemma 1.1 and Remarks 1.1 and 1.2 hold here.

Rewrite cost functional (2.1) as

$$J(u) = \pi(u, u) - 2L(u) + \|z_g - y(0)\|_{L_2(\gamma)}^2, \quad (2.2)$$

where

$$\begin{aligned} \|v\|_{L_2(\gamma)} &= (v, v)_{L_2(\gamma)}^{1/2}, \quad (\varphi, \psi)_{L_2(\gamma)} = \int_{\gamma} \varphi \psi d\gamma, \\ \pi(u, v) &= (y(u) - y(0), y(v) - y(0))_{L_2(\gamma)} + (\bar{a}u, v) \end{aligned}$$

and

$$L(v) = (z_g - y(0), y(v) - y(0))_{L_2(\gamma)}.$$

Take the embedding theorems, ellipticity condition and generalized Poincare inequality into account, and the inequality

$$\begin{aligned} \bar{\alpha}_2 \|\tilde{y}' - \tilde{y}''\|_{L_2(\gamma)}^2 &\leq \bar{\alpha}_1 \|\tilde{y}' - \tilde{y}''\|_V^2 \leq \\ &\leq a_1 (\tilde{y}' - \tilde{y}'', \tilde{y}' - \tilde{y}'') \leq \|u' - u''\| \|\tilde{y}' - \tilde{y}''\|_V, \end{aligned}$$

i.e.

$$\|\tilde{y}' - \tilde{y}''\|_{L_2(\gamma)} \leq \frac{1}{\sqrt{\bar{\alpha}_1 \bar{\alpha}_2}} \|u' - u''\|, \quad \alpha_1, \alpha_2 = \text{const} > 0,$$

is derived, where $\tilde{y}' = \tilde{y}(u')$ and $\tilde{y}'' = \tilde{y}(u'')$ are the generalized solutions from V_Q to boundary-value Problem 2 under $f = 0$, $g = 0$ and $\omega = 0$ and under a function $u = u(x)$ that is equal, respectively, to u' and u'' .

On the basis of the derived inequality and [58, Chapter 1, Theorem 1.1], the validity of the following statement is proved.

Theorem 2.1. *Let a system state be determined as a solution to equivalent problems (1.11) and (1.12). Then, there exists a unique element u of a convex set \mathcal{U}_∂ that is closed in $\mathcal{U} = L_2(\Omega)$, and relation like (1.17) takes place for u , where the cost functional $J(u)$ is specified by expression (2.1).*

Let equation (1.18) be specified on Ω instead of equation (1.7). Neumann condition (1.2) and conjugation conditions (1.3) and (1.4) are specified, in their turn, respectively, on Γ and γ . I.e., boundary-value problem (1.18), (1.2)–(1.4) (Problem 2') is obtained.

Remark 2.1. Boundary-value Problems 1, 2 and 1', 2' coincide pairwise. Optimization Problems do not coincide because their cost functionals $J(u)$ are different.

Consider optimization Problem 2': Find a control $u \in \mathcal{U}_\partial \subset \mathcal{U} = L_2(\Omega)$, for which relation like (1.17) is satisfied, where the cost functional $J(u)$ is specified by expression (2.1), and a state $y(u)$ is a generalized solution to boundary-value Problem 2'.

If $u \in \mathcal{U}_\partial$ is the optimal control for optimization Problem 2', then the following inequality is true:

$$\left(y(u) - z_g, y(v) - y(u) \right)_{L_2(\gamma)} + (\bar{a}u, v - u) \geq 0, \quad \forall v \in \mathcal{U}_\partial. \quad (2.3)$$

As for the control $v \in \mathcal{U}$, the conjugate state $p(v) \in V^* = V$ is specified by the relations

$$A^* p(v) = 0, \quad x \in \Omega_1 \cup \Omega_2,$$

$$\begin{aligned} \frac{\partial p}{\partial v_{A^*}} &= 0, \quad x \in \Gamma, \\ [p] &= 0, \quad \left[\frac{\partial p}{\partial v_{A^*}} \right] = -y(v) + z_g, \quad x \in \gamma, \end{aligned} \quad (2.4)$$

where the operators A^* and $\frac{\partial}{\partial v_{A^*}}$ are specified, in their turn, by expressions (1.23).

The equality

$$\begin{aligned} 0 &= \left(A^* p(u), y(v) - y(u) \right) = a'_1(p, y(v) - y(u)) - \\ &\quad - \left(y(u) - z_g, y(v) - y(u) \right)_{L_2(\gamma)} = \\ &= - \left(y(u) - z_g, y(v) - y(u) \right)_{L_2(\gamma)} + (p, v - u), \end{aligned}$$

i.e. $\left(y(u) - z_g, y(v) - y(u) \right)_{L_2(\gamma)} = (p, v - u) \quad \forall v \in \mathcal{U}_\partial$ is obtained. Take it into account, and the inequality

$$(\bar{a}u + p, v - u) \geq 0, \quad \forall v \in \mathcal{U}_\partial, \quad (2.5)$$

is derived from inequality (2.3).

An element $u \in \mathcal{U}_\partial$ is an optimal control for optimization Problem 2' if and only if inequality (2.5) and the equalities

$$a'_1(y, v) = l_1(u, v), \quad y \in V, \quad \forall v \in V, \quad (2.6)$$

and

$$a'_1(p, v) = l_2(y, v), \quad p \in V, \quad \forall v \in V, \quad (2.7)$$

are met; the bilinear form $a'_1(\cdot, \cdot)$ and functional $l_1(u, v)$ are specified in point 2.1, and

$$l_2(y, v) = - \int_{\gamma} (z_g - y) v d\gamma.$$

If the constraints are absent, i.e. when $\mathcal{U}_{\partial} = \mathcal{U}$, then the equality

$$p + \bar{a} u = 0 \quad (2.8)$$

follows from condition (2.5). Therefore, when the constraints are absent, the control u can be excluded from equality (2.6) by means of equality (2.8). Let the solution $(y, p)^T$ to problem (2.6), (2.7), where $l_1(u, y) = l_1(u(p), y)$, be sufficiently smooth on $\bar{\Omega}_1$ and $\bar{\Omega}_2$. Then, such solution satisfies the relations

$$- \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial y}{\partial x_j} \right) + p/\bar{a} + \int_{\Omega} y dx = f + Q, \quad x \in \Omega_1 \cup \Omega_2,$$

$$- \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial p}{\partial x_j} \right) + \int_{\Omega} p dx = 0, \quad x \in \Omega_1 \cup \Omega_2,$$

$$\sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(v, x_i) = g, \quad x \in \Gamma,$$

$$\sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \cos(v, x_i) = 0, \quad x \in \Gamma,$$

$$[y] = 0, [p] = 0, \quad x \in \gamma,$$

$$\left[\sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(v, x_i) \right] = \omega, \quad x \in \gamma,$$

$$\left[\sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \cos(v, x_i) \right] = -y + z_g, \quad x \in \gamma. \quad (2.9)$$

Definition 2.1. A generalized (weak) solution to boundary-value problem (2.9) is called a vector-function $(y, p)^T \in H$ that satisfies the following integral equation $\forall z \in H$:

$$\begin{aligned} & \int_{\Omega} \left\{ \sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \frac{\partial z_1}{\partial x_i} + p z_1 / \bar{a} + \sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \frac{\partial z_2}{\partial x_i} \right\} dx + \\ & + \int_{\Omega} y dx \int_{\Omega} z_1 dx + \int_{\Omega} p dx \int_{\Omega} z_2 dx = \\ & = \int_{\Omega} (f + Q) z_1 dx + \int_{\Gamma} g z_1 d\Gamma - \int_{\gamma} \omega z_1 d\gamma + \int_{\gamma} (y - z_g) z_2 d\gamma. \quad (2.10) \end{aligned}$$

Let $u = (u_1, u_2)^T$ and $v = (v_1, v_2)^T$ be arbitrary elements of the complete Hilbert space H . Specify the bilinear form

$$\begin{aligned} a(u, v) = & \int_{\Omega} \left\{ \sum_{l=1}^2 \sum_{i,j=1}^n k_{ij} \frac{\partial u_l}{\partial x_j} \frac{\partial v_l}{\partial x_i} + u_2 v_1 / \bar{a} \right\} dx + \\ & + \sum_{l=1}^2 \int_{\Omega} u_l dx \int_{\Omega} v_l dx - \int_{\gamma} u_1 v_2 d\gamma \end{aligned}$$

and linear functional

$$l(v) = \int_{\Omega} (f + Q) v_1 dx + \int_{\Gamma} g v_1 d\Gamma - \int_{\gamma} \omega v_1 d\gamma - \int_{\gamma} z_g v_2 d\gamma$$

on H . If the constraint

$$\min \left\{ \frac{\alpha_0}{2}, \min \left\{ \frac{\alpha_0}{2}, 1 \right\} \mu \right\} - \frac{1}{2a_0} - \frac{c_0^2}{2} > 0, \quad (2.11)$$

where μ and c_0 are the positive constants, respectively, in the generalized Poincare inequality and embedding theorem, is met, then the unique solution $(y, p)^T$ to problem (2.10) exists in H . Problem (2.10) can be solved by means of the finite-element method. Estimates like (1.35) and

(1.35') are true, respectively, for its approximate solution $U_k^N \in H_k^N \subset H$ and for the approximation $u_k^N(x)$ of the control u .

2.3 DISTRIBUTED CONTROL WITH BOUNDARY OBSERVATION

Assume that equation (1.1), where the coefficients and right-hand side meet conditions (1.2'), is specified in the bounded, continuous and strictly Lipschitz domains Ω_1 and Ω_2 . The conjugation conditions have the form of expressions (1.3) and (1.4) and the boundary condition has the form of expression (1.2).

For every control $u \in \mathcal{U} = L_2(\Omega)$, determine a system state as a generalized solution to the boundary-value problem specified by equation (1.7) and by conditions (1.2)–(1.4). Equality (1.13) is the necessary condition under which there exists a classical solution y to boundary-value problem (1.7), (1.2)–(1.4) (Problem 3): Find a solution y that meets constraint (1.6).

Bring a value of the cost functional

$$J(u) = \int_{\Gamma} (y(u) - z_g)^2 d\Gamma + (\mathcal{N}u, u)_{\mathcal{U}} \quad (3.1)$$

in correspondence with every control $u \in \mathcal{U} = L_2(\Omega)$; in this case, z_g is a known element from the space $L_2(\Gamma)$, $\mathcal{N}u = \bar{a}u$, $0 < a_0 \leq \bar{a}(x) \leq a_1^0 < \infty$, $\bar{a}(x)|_{\Omega_i} \in C(\Omega_i)$, $i = 1, 2$; $a_0, a_1^0 = \text{const}$.

A unique state, namely, a function $y(u) \in V_Q$ corresponds to every control $u \in \mathcal{U}$, delivers the minimum to energy functional (1.11) on V_Q , and it is the unique solution in V_Q to weakly stated problem (1.12). Lemma 1.1 and Remarks 1.1 and 1.2 take place here.

Rewrite cost functional (3.1) as

$$J(u) = \pi(u, u) - 2L(u) + \|z_g - y(0)\|_{L_2(\Gamma)}^2, \quad (3.2)$$

where

$$\begin{aligned} \|v\|_{L_2(\Gamma)} &= (v, v)_{L_2(\Gamma)}^{1/2}, \quad (\varphi, \psi)_{L_2(\Gamma)} = \int_{\Gamma} \varphi \psi d\Gamma, \\ \pi(u, v) &= (y(u) - y(0), y(v) - y(0))_{L_2(\Gamma)} + (\bar{a} u, v) \end{aligned}$$

and

$$L(v) = (z_g - y(0), y(v) - y(0))_{L_2(\Gamma)}.$$

Take the embedding theorems, ellipticity condition and generalized Poincare inequality into account, and the inequality

$$\begin{aligned} \bar{\alpha}_2 \|\tilde{y}' - \tilde{y}''\|_{L_2(\Gamma)}^2 &\leq \bar{\alpha}_1 \|\tilde{y}' - \tilde{y}''\|_{\nu}^2 \leq \\ &\leq a_1 (\tilde{y}' - \tilde{y}'', \tilde{y}' - \tilde{y}'') \leq \|u' - u''\| \|\tilde{y}' - \tilde{y}''\|_{\nu}, \end{aligned}$$

i.e.

$$\|\tilde{y}' - \tilde{y}''\|_{L_2(\Gamma)} \leq \frac{1}{\sqrt{\bar{\alpha}_1 \bar{\alpha}_2}} \|u' - u''\|, \quad \bar{\alpha}_1, \bar{\alpha}_2 = \text{const} > 0,$$

is derived, where $\tilde{y}' = \tilde{y}(u')$ and $\tilde{y}'' = \tilde{y}(u'')$ are the generalized solutions from V_Q to boundary-value Problem 3 under $f = 0$, $g = 0$ and $\omega = 0$ and under a function $u = u(x)$ that is equal, respectively, to u' and u'' .

On the basis of the derived inequality and [58, Chapter 1, Theorem 1.1], the validity of the following statement is proved.

Theorem 3.1. *Let a system state be determined as a solution to equivalent problems (1.11) and (1.12). Then, there exists a unique element u of a convex set \mathcal{U}_∂ that is closed in \mathcal{U} , and relation like (1.17) takes place for u , where the cost functional $J(u)$ is specified by expression (3.1).*

Let equation (1.18) be specified on the domain Ω instead of equation (1.7). Neumann condition (1.2) and conjugation conditions (1.3) and (1.4) are specified, in their turn, respectively, on Γ and γ . I.e., boundary-value problem (1.18), (1.2)–(1.4) (Problem 3') is obtained.

Remark 3.1. Boundary-value Problem 3 coincides with Problems 1 and 2. Problem 3' coincides with Problems 1' and 2'. Optimization Problems do not coincide because their cost functionals $J(u)$ are different.

Consider optimization Problem 3': Find a control $u \in \mathcal{U}_\partial \subset \mathcal{U} = L_2(\Omega)$, for which relation like (1.17) is satisfied and where the cost functional $J(u)$ is specified by expression (3.1). A state $y = y(u)$ is a generalized solution to boundary-value Problem 3' and such solution is unique for one of equivalent problems (1.19) and (1.20).

If $u \in \mathcal{U}_\partial$ is the optimal control for optimization Problem 3', then the following inequality is true:

$$\left(y(u) - z_g, y(v) - y(u) \right)_{L_2(\Gamma)} + (\bar{a}u, v - u) \geq 0, \quad \forall v \in \mathcal{U}_\partial. \quad (3.3)$$

As for the control $v \in \mathcal{U}$, the conjugate state $p(v) \in V^* = V$ is specified by the relations

$$\begin{aligned} A^* p(v) &= 0, \quad x \in \Omega_1 \cup \Omega_2, \\ \frac{\partial p}{\partial v_{A^*}} &= -z_g + y, \quad x \in \Gamma, \\ [p] &= 0, \quad \left[\frac{\partial p}{\partial v_{A^*}} \right] = 0, \quad x \in \gamma, \end{aligned} \quad (3.4)$$

where the operators A^* and $\frac{\partial}{\partial v_{A^*}}$ are specified, in their turn, by expressions (1.23).

The equality

$$\begin{aligned} 0 &= \left(A^* p(u), y(v) - y(u) \right) = a'_1(p, y(v) - y(u)) - \\ &\quad - \left(y(u) - z_g, y(v) - y(u) \right)_{L_2(\Gamma)} = \\ &= - \left(y(u) - z_g, y(v) - y(u) \right)_{L_2(\Gamma)} + (p, v - u), \end{aligned}$$

i.e. $(y(u) - z_g, y(v) - y(u))_{L_2(\Gamma)} = (p, v - u) \quad \forall v \in \mathcal{U}_\partial$ is obtained. Take it into account, and the inequality

$$(p + \bar{a}u, v - u) \geq 0, \quad \forall v \in \mathcal{U}_\partial, \quad (3.5)$$

is derived from inequality (3.3).

To make the element $u \in \mathcal{U}_\partial$ the optimal control for optimization Problem 3', it is necessary and sufficient for inequality (3.5) and the equalities

$$a'_1(y, v) = l_1(u, v), \quad y \in V, \quad \forall v \in V, \quad (3.6)$$

and

$$a'_1(p, v) = l_2(y, v), \quad p \in V, \quad \forall v \in V, \quad (3.7)$$

to be met; the bilinear form $a'_1(\cdot, \cdot)$ and linear functional $l_1(\cdot, \cdot)$ are specified in point 2.1, and

$$l_2(y, v) = - \int_{\Gamma} (z_g - y)v \, d\Gamma.$$

If the constraints are absent, i.e. when $\mathcal{U}_\partial = \mathcal{U}$, then the equality

$$p + \bar{a}u = 0 \quad (3.8)$$

follows from condition (3.5).

Therefore, when the constraints are absent, the control u can be excluded from equality (3.6) by means of equality (3.8), and the following may be written: $l_1(u, y) = l_1(u(p), y)$. If the solution $(y, p)^T$ to problem (3.6), (3.7), where $l_1(u, y) = l_1(u(p), y)$, is smooth enough on $\bar{\Omega}_1$ and $\bar{\Omega}_2$, then such solution satisfies the relations

$$- \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial y}{\partial x_j} \right) + p/\bar{a} + \int_{\Omega} y \, dx = f + Q, \quad x \in \Omega_1 \cup \Omega_2,$$

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial p}{\partial x_j} \right) + \int_{\Omega} p \, dx = 0, \quad x \in \Omega_1 \cup \Omega_2,$$

$$\sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(\nu, x_i) = g, \quad x \in \Gamma,$$

$$\sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \cos(\nu, x_i) = y - z_g, \quad x \in \Gamma,$$

$$[y] = 0, \quad [p] = 0, \quad x \in \gamma,$$

$$\left[\sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(\nu, x_i) \right] = \omega, \quad \left[\sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \cos(\nu, x_i) \right] = 0, \quad x \in \gamma. \quad (3.9)$$

Definition 3.1. A generalized (weak) solution to boundary-value problem (3.9) is called a vector-function $(y, p)^T \in H$ that satisfies the following integral equation $\forall z \in H$:

$$\begin{aligned} & \int_{\Omega} \left\{ \sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \frac{\partial z_1}{\partial x_i} + p z_1 / \bar{a} + \sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \frac{\partial z_2}{\partial x_i} \right\} dx + \\ & + \int_{\Omega} y \, dx \int_{\Omega} z_1 \, dx + \int_{\Omega} p \, dx \int_{\Omega} z_2 \, dx = \\ & = \int_{\Omega} (f + Q) z_1 \, dx + \int_{\Gamma} g z_1 \, d\Gamma + \\ & + \int_{\Gamma} (y - z_g) z_2 \, d\Gamma - \int_{\gamma} \omega z_1 \, d\gamma. \end{aligned} \quad (3.10)$$

Let $u = (u_1, u_2)^T$ and $v = (v_1, v_2)^T$ be arbitrary elements of the complete Hilbert space H . Specify the bilinear form

$$a(u, v) = \int_{\Omega} \left\{ \sum_{l=1}^2 \sum_{i,j=1}^n k_{ij} \frac{\partial u_l}{\partial x_j} \frac{\partial v_l}{\partial x_i} + u_2 v_1 / \bar{a} \right\} dx + \\ + \sum_{l=1}^2 \int_{\Omega} u_l dx \int_{\Omega} v_l dx - \int_{\Gamma} u_1 v_2 d\Gamma$$

and linear functional

$$l(v) = \int_{\Omega} (f + Q)v_1 dx + \int_{\Gamma} g v_1 d\Gamma - \int_{\gamma} \omega v_1 d\gamma - \int_{\Gamma} z_g v_2 d\Gamma$$

on H .

Let constraint like (2.11) be met. Then, the unique solution $(y, p)^T$ to problem (3.10) exists in H . Problem (3.10) can be solved by means of the finite-element method. Estimates like (1.35) and (1.35') take place, respectively, for the approximate solution $U_k^N \in H_k^N$ to problem like (3.10) and for the approximation $u_k^N(x)$ of the control u .

2.4 CONTROL UNDER CONJUGATION CONDITION WITH BOUNDARY OBSERVATION

Assume that equation (1.1), where the coefficients and right-hand side meet conditions (1.2'), is specified in the domains Ω_1 and Ω_2 . Neumann condition (1.2), constraint (1.3) and the condition

$$\left[\sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(v, x_i) \right] = \omega + u, \quad x \in \gamma, \quad (4.1)$$

where ω is a fixed function from $L_2(\gamma)$ and the control is $u \in \mathcal{U} = L_2(\gamma)$, are specified, in their turn, on the boundary Γ .

For every control $u \in \mathcal{U}$, determine a system state as a generalized solution to the boundary-value problem specified by equation (1.1) and by conditions (1.2), (1.3) and (4.1) (Problem 4). The equality

$$\int_{\Omega} f dx + \int_{\Gamma} g d\Gamma = \int_{\gamma} (\omega + u) d\gamma \quad (4.2)$$

is the necessary condition under which there exists a classical solution to the latter problem. Find this solution under constraint (1.6).

Bring a value of the cost functional

$$J(u) = \int_{\Gamma} (y(u) - z_g)^2 d\Gamma + (\mathcal{N}u, u)_{L_2(\gamma)} \quad (4.3)$$

in correspondence with every control $u \in \mathcal{U} = L_2(\gamma)$; in this case, z_g is a known element from the space $L_2(\Gamma)$, $\mathcal{N}u = \bar{a}u$, $0 < a_0 \leq \bar{a}(x) \leq a_1 < \infty$, $\bar{a} \in L_2(\gamma)$, $a_0, a_1 = \text{const}$.

A unique state, namely, a function $y(u) \in V_Q$ corresponds to every control $u \in \mathcal{U}$, minimizes the functional

$$\Phi(v) = a_4(v, v) - 2l_4(v) \quad (4.4)$$

on V_Q , and it is the unique solution in V_Q to the weakly stated problem:

Find a function $y \in V_Q$ that meets the equation

$$a_4(y, v) = l_4(v) \quad \forall v \in V_0, \quad (4.5)$$

where

$$a_4(y, v) = \int_{\Omega} \sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \frac{\partial v}{\partial x_i} dx, \quad (4.6)$$

$$l_4(v) = \int_{\Omega} f v dx + \int_{\Gamma} g v d\Gamma - \int_{\gamma} (\omega + u) v d\gamma.$$

Lemma 4.1. Problems (4.4) and (4.5) are equivalent $\forall f \in L_2(\Omega)$, $\forall \omega \in L_2(\gamma)$, $\forall u \in \mathcal{U}$ and have a unique solution $y \in V_Q$.

Rewrite cost functional (4.3) in the form of expression (3.2), where

$$\pi(u, v) = (y(u) - y(0), y(v) - y(0))_{L_2(\Gamma)} + (\bar{a} u, v)_{L_2(\gamma)}$$

and

$$L(v) = (z_g - y(0), y(v) - y(0))_{L_2(\Gamma)}.$$

Take the embedding theorems, ellipticity condition and generalized Poincare inequality into account, and the inequality

$$\begin{aligned} \bar{\alpha}_2 \|\tilde{y}' - \tilde{y}''\|_{L_2(\Gamma)}^2 &\leq \bar{\alpha}_1 \|\tilde{y}' - \tilde{y}''\|_V^2 \leq a_4 (\tilde{y}' - \tilde{y}'', \tilde{y}' - \tilde{y}'') \leq \\ &\leq c_1 \|u' - u''\|_{L_2(\gamma)} \|\tilde{y}' - \tilde{y}''\|_V, \quad \bar{\alpha}_1, \bar{\alpha}_2, c_1 = \text{const} > 0, \end{aligned}$$

i.e.

$$\|\tilde{y}' - \tilde{y}''\|_{L_2(\Gamma)} \leq \frac{c_1}{\sqrt{\bar{\alpha}_1 \bar{\alpha}_2}} \|u' - u''\|_{L_2(\gamma)},$$

is derived, where $\tilde{y}' = \tilde{y}(u')$ and $\tilde{y}'' = \tilde{y}(u'')$ are the generalized solutions to boundary-value Problem 4 under $f = 0$, $g = 0$ and $\omega = 0$ and under a function $u = u(x)$ that is equal, respectively, to u' and u'' .

On the basis of the derived inequality and [58, Chapter 1, Theorem 1.1], the validity of the following statement is proved.

Theorem 4.1. Let a system state y be determined as a solution to equivalent problems (4.4) and (4.5). Then, there exists a unique element $u = u(x)$ of a convex set \mathcal{U}_δ that is closed in \mathcal{U} , and relation like (1.17) takes place for $u = u(x)$, where the cost functional $J(u)$ is specified by expression (4.3).

Let the equation

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial y}{\partial x_j} \right) + \int_{\Omega} y dx = f(x) + Q \quad (4.7)$$

be specified on Ω instead of equation (1.1). Neumann condition (1.2) and conjugation conditions (1.3) and (4.1) are specified, in their turn, respectively, on Γ and γ . I.e., boundary-value problem (4.7), (1.2), (1.3), (4.1) (Problem 4') is obtained.

Consider optimization Problem 4': Find a control $u \in \mathcal{U}_\partial \subset \mathcal{U} = L_2(\gamma)$, for which relation like (1.17) is satisfied and where the cost functional $J(u)$ is specified by expression (4.3). A state $y = y(u)$ is a generalized solution to boundary-value Problem 4', where the energy functional is

$$\Phi(v) = a'_4(v, v) - 2 l'_4(v), \quad \forall v \in V, \quad (4.8)$$

and the weakly stated problem is to find a function $y \in V$ that meets the following equation $\forall z \in V$:

$$a'_4(y, z) = l'_4(z); \quad (4.8')$$

in this case:

$$\begin{aligned} a'_4(y, v) &= \int_{\Omega} \sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \frac{\partial v}{\partial x_i} dx + \int_{\Omega} y dx \int_{\Omega} v dx, \\ l'_4(v) &= \int_{\Omega} (f + Q)v dx + \int_{\Gamma} g v d\Gamma - \int_{\gamma} (\omega + u)v d\gamma. \end{aligned} \quad (4.9)$$

If $u \in \mathcal{U}_\partial$ is the optimal control for optimization Problem 4', then the following inequality is true $\forall v \in \mathcal{U}_\partial$:

$$\left(y(u) - z_g, y(v) - y(u) \right)_{L_2(\Gamma)} + (\bar{a} u, v - u)_{L_2(\gamma)} \geq 0. \quad (4.10)$$

As for the control $v \in \mathcal{U}$, the conjugate state $p(v) \in V^* = V$ is specified by relations like (3.4). The equality

$$\begin{aligned} 0 &= \left(A^* p(u), y(v) - y(u) \right) = a'_4(p, y(v) - y(u)) + \\ &\quad + \left(z_g - y, y(v) - y(u) \right)_{L_2(\Gamma)} = \end{aligned}$$

$$= -(p, v - u)_{L_2(\gamma)} + (z_g - y, y(v) - y(u))_{L_2(\Gamma)},$$

i.e.

$$(y(u) - z_g, y(v) - y(u))_{L_2(\Gamma)} = -(p, v - u)_{L_2(\gamma)}$$

is obtained. Take it into account, and the inequality

$$(-p + \bar{a}u, v - u)_{L_2(\gamma)} \geq 0, \quad \forall v \in \mathcal{U}_\partial, \quad (4.11)$$

is derived from inequality (4.10).

An element $u \in \mathcal{U}_\partial$ is an optimal control for optimization Problem 4' if and only if inequality (4.11) and the equalities

$$a'_4(y, v) = l_1(u, v), \quad y \in V, \quad \forall v \in V, \quad (4.12)$$

and

$$a'_4(p, v) = l_2(y, v), \quad p \in V, \quad \forall v \in V, \quad (4.13)$$

are met; in this case, the bilinear form $a'_4(\cdot, \cdot)$ is specified by the first formula of expressions (4.9) and the functionals $l_1(\cdot, \cdot)$ and $l_2(\cdot, \cdot)$ are

$$l_1(u, v) = \int_{\Omega} (f + Q)v \, dx + \int_{\Gamma} gv \, d\Gamma - \int_{\gamma} (\omega + u)v \, d\gamma$$

and

$$l_2(y, v) = - \int_{\Gamma} (z_g - y)v \, d\Gamma.$$

If the constraints are absent, i.e. when $\mathcal{U}_\partial = \mathcal{U}$, then the equality

$$-p + \bar{a}u = 0, \quad x \in \gamma, \quad (4.14)$$

follows from condition (4.11).

Therefore, when the constraints are absent and if the solution $(y, p)^\top$ to problem (4.12)–(4.14) is smooth enough on $\bar{\Omega}_i$ ($i=1,2$), then the boundary-value problem is obtained:

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial y}{\partial x_j} \right) + \int_{\Omega} y \, dx = f + Q, \quad x \in \Omega_1 \cup \Omega_2,$$

$$\begin{aligned}
& - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial p}{\partial x_j} \right) + \int_{\Omega} p \, dx = 0, \quad x \in \Omega_1 \cup \Omega_2, \\
& \sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(\nu, x_i) = g, \quad x \in \Gamma, \\
& \sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \cos(\nu, x_i) = y - z_g, \quad x \in \Gamma, \\
& [y] = 0, \quad [p] = 0, \quad x \in \gamma, \\
& \left[\sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(\nu, x_i) \right] = \omega + p/\bar{a}, \quad x \in \gamma, \\
& \left[\sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \cos(\nu, x_i) \right] = 0, \quad x \in \gamma. \tag{4.15}
\end{aligned}$$

Definition 4.1. A generalized (weak) solution to boundary-value problem (4.15) is called a vector-function $(y, p)^T \in H$ that satisfies the following integral equation $\forall z \in H$:

$$\begin{aligned}
& \int_{\Omega} \left\{ \sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \frac{\partial z_1}{\partial x_i} + \sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \frac{\partial z_2}{\partial x_i} \right\} dx + \\
& + \int_{\Omega} y \, dx \int_{\Omega} z_1 \, dx + \int_{\Omega} p \, dx \int_{\Omega} z_2 \, dx = \int_{\Omega} (f + Q) z_1 \, dx + \\
& + \int_{\Gamma} g z_1 \, d\Gamma + \int_{\Gamma} (y - z_g) z_2 \, d\Gamma - \int_{\gamma} (\omega + p/\bar{a}) z_1 \, d\gamma. \tag{4.16}
\end{aligned}$$

Let $u = (u_1, u_2)^T$ and $v = (v_1, v_2)^T$ be arbitrary elements of the complete Hilbert space H . Specify the bilinear form

$$a(u, v) = \int_{\Omega} \left\{ \sum_{l=1}^2 \sum_{i,j=1}^n k_{ij} \frac{\partial u_l}{\partial x_j} \frac{\partial v_l}{\partial x_i} \right\} dx + \\ + \sum_{l=1}^2 \int_{\Omega} u_l dx \int_{\Omega} v_l dx - \int_{\Gamma} u_1 v_2 d\Gamma + \int_{\gamma} u_2 v_1 / \bar{a} d\gamma$$

and linear functional

$$l(v) = \int_{\Omega} (f + Q) v_1 dx + \int_{\Gamma} g v_1 d\Gamma - \int_{\Gamma} z_g v_2 d\Gamma - \int_{\gamma} \omega v_1 d\gamma$$

on H .

If the constraint

$$\min \left\{ \frac{\alpha_0}{2}, \min \left\{ \frac{\alpha_0}{2}, 1 \right\} \mu \right\} - \frac{c_0'^2}{2a_0} - \frac{c_0^2}{2} > 0, \quad (4.16')$$

where μ is the constant in the Poincare inequality and c_0' and c_0 are the positive constants derived on the basis of the inequalities proved within the framework of the embedding theorems, is met, then the unique solution $(y, p)^T$ to problem (4.16) exists in H . Estimate like (1.35) is true for its approximate solution $U_k^N \in H_k^N$ and the estimate

$$\|u - u_k^N\|_{L_2(\gamma)} \leq c h^k \quad (4.17)$$

takes place for the approximation $u_k^N(x)$ of the control u .

2.5 BOUNDARY CONTROL WITH OBSERVATION ON A THIN INCLUSION

Assume that equation (1.1), where the coefficients and right-hand side meet conditions (1.2'), is specified in the domains Ω_1 and Ω_2 . The condition

$$\sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(v, x_i) = g + u \quad (5.1)$$

is specified, in its turn, on the boundary Γ and the conjugation conditions have the form of expressions (1.3) and (1.4) on γ , where g is a fixed function from $L_2(\Gamma)$ and the control is $u \in \mathcal{U} = L_2(\Gamma)$.

For every control $u \in \mathcal{U}$, determine a system state as a generalized solution to the boundary-value problem specified by equation (1.1) and by constraints (1.3), (1.4) and (5.1) (Problem 5). The equality

$$\int_{\Omega} f dx + \int_{\Gamma} g d\Gamma + \int_{\Gamma} u d\Gamma = \int_{\gamma} \omega d\gamma \quad (5.2)$$

is the necessary condition under which there exists a classical solution y to Problem 5: Find this solution under constraint (1.6).

Bring a value of the cost functional

$$J(u) = \int_{\gamma} (y(u) - z_g)^2 d\gamma + (\mathcal{N}u, u)_{L_2(\Gamma)} \quad (5.3)$$

in correspondence with every control $u \in \mathcal{U}$; in this case, z_g is a known element from the space $L_2(\gamma)$, $\mathcal{N}u = \bar{a}u$, $0 < a_0 \leq \bar{a}(x) \leq a_1 < \infty$, $\bar{a} \in L_2(\Gamma)$; $a_0, a_1 = \text{const}$.

A unique state, namely, a function $y(u) \in V_Q$ corresponds to every control $u \in \mathcal{U}$, delivers the minimum to functional (4.4) on V_Q , and it is the unique solution in V_Q to the weakly stated problem specified by equation like (4.5), where

$$l_4(v) = \int_{\Omega} f v dx + \int_{\Gamma} (g + u)v d\Gamma - \int_{\gamma} \omega v d\gamma. \quad (5.4)$$

Lemma 5.1. Problems like (4.4) and (4.5), where the bilinear form $a_4(\cdot, \cdot)$ is specified by the first formula of expressions (4.6) and the linear

functional $l_4(\cdot)$ is specified by formula (5.4), are equivalent $\forall f \in L_2(\Omega)$, $\forall \omega \in L_2(\gamma)$, $\forall u \in \mathcal{U}$ and have a unique solution $y(u) \in V_Q$.

Rewrite cost functional (5.3) as

$$J(u) = \pi(u, u) - 2L(u) + \|z_g - y(0)\|_{L_2(\gamma)}^2,$$

where

$$\pi(u, v) = (y(u) - y(0), y(v) - y(0))_{L_2(\gamma)} + (\bar{a}u, v)_{L_2(\Gamma)}$$

and

$$L(v) = (z_g - y(0), y(v) - y(0))_{L_2(\gamma)}.$$

Take the embedding theorems, ellipticity condition and generalized Poincare inequality into account, and the inequality

$$\begin{aligned} \bar{\alpha}_2 \|\tilde{y}' - \tilde{y}''\|_{L_2(\gamma)}^2 &\leq \bar{\alpha}_1 \|\tilde{y}' - \tilde{y}''\|_V^2 \leq a_4 (\tilde{y}' - \tilde{y}'', \tilde{y}' - \tilde{y}'') \leq \\ &\leq c_0 \|u' - u''\|_{L_2(\Gamma)} \|\tilde{y}' - \tilde{y}''\|_V, \quad \bar{\alpha}_1, \bar{\alpha}_2, c_0 = \text{const} > 0, \end{aligned}$$

i.e. $\|\tilde{y}' - \tilde{y}''\|_{L_2(\gamma)} \leq c_1 \|u' - u''\|_{L_2(\Gamma)}$ is derived, where $\tilde{y}' = \tilde{y}(u')$ and $\tilde{y}'' = \tilde{y}(u'')$.

On the basis of the derived inequality and [58, Chapter 1, Theorem 1.1], the validity of the following statement is proved.

Theorem 5.1. *If a system state y is determined as a solution to equivalent problems (4.4) and (4.5) that correspond to boundary-value Problem 5, then there exists a unique element $u = u(x)$ of a convex set \mathcal{U}_∂ that is closed in \mathcal{U} , and relation like (1.17) takes place for $u = u(x)$, where the cost functional $J(u)$ is specified by expression (5.3).*

Let equation (4.7) be specified on Ω instead of equation (1.1). Condition (5.1) and constraints (1.3) and (1.4) are specified, in their turn, respectively, on Γ and γ . I.e., boundary-value problem (4.7), (1.3), (1.4), (5.1) (Problem 5') is obtained.

Consider the following problem: Find a control $u \in \mathcal{U}_\partial \subset \mathcal{U} = L_2(\Gamma)$ for which relation like (1.17) is satisfied and where the cost functional $J(u)$ is

specified by expression (5.3); a state $y = y(u)$ is a generalized solution to boundary-value Problem 5', where the energy functional and weakly stated problem are given, respectively, by expression (4.8) and equality (4.8'). The form $a'_4(\cdot, \cdot)$ is specified by expression (4.9) and the linear functional is

$$l'_4(v) = \int_{\Omega} (f + Q)v dx + \int_{\Gamma} (g + u)v d\Gamma - \int_{\gamma} \omega v d\gamma.$$

If $u \in \mathcal{U}_{\partial}$ is the optimal control for optimization Problem 5', then the following inequality is true $\forall v \in \mathcal{U}_{\partial}$:

$$(y(u) - z_g, y(v) - y(u))_{L_2(\gamma)} + (\bar{a}u, v - u)_{L_2(\Gamma)} \geq 0. \quad (5.5)$$

As for the control $v \in \mathcal{U}$, the conjugate state $p(v) \in V^* = V$ is specified by the relations

$$\begin{aligned} A^* p(v) &= 0, \quad x \in \Omega_1 \cup \Omega_2, \\ \frac{\partial p}{\partial v_{A^*}} &= 0, \quad x \in \Gamma, \\ [p] &= 0, \quad \left[\frac{\partial p}{\partial v_{A^*}} \right] = z_g - y, \quad x \in \gamma, \end{aligned} \quad (5.6)$$

where the operators A^* and $\frac{\partial}{\partial v_{A^*}}$ are specified, in their turn, by expressions (1.23).

The equality

$$\begin{aligned} 0 &= (A^* p(u), y(v) - y(u)) = a'_4(p, y(v) - y(u)) + \\ &\quad + (z_g - y(u), y(v) - y(u))_{L_2(\gamma)} = \\ &= (p, v - u)_{L_2(\Gamma)} + (z_g - y(u), y(v) - y(u))_{L_2(\gamma)}, \end{aligned}$$

i.e. $(y(u) - z_g, y(v) - y(u))_{L_2(\Gamma)} = (p, v - u)_{L_2(\Gamma)}$ is obtained. Take it into account, and the inequality

$$(p + \bar{a}u, v - u)_{L_2(\Gamma)} \geq 0 \quad (5.7)$$

is derived from inequality (5.5).

To make the element $u \in \mathcal{U}_\partial$ an optimal control for optimization Problem 5', it is necessary and sufficient to meet inequality (5.7) and equalities like (4.12) and (4.13), where the bilinear form $a'_4(\cdot, \cdot)$ is specified by expression (4.9) and, besides this, the linear functionals are

$$l_1(u, v) = \int_{\Omega} (f + Q)v \, dx + \int_{\Gamma} (g + u)v \, d\Gamma - \int_{\gamma} \omega v \, d\gamma$$

and

$$l_2(y, v) = - \int_{\gamma} (z_g - y)v \, d\gamma.$$

If the constraints are absent, i.e. when $\mathcal{U}_\partial = \mathcal{U}$, then the equality

$$p + \bar{a}u = 0, \quad x \in \Gamma, \quad (5.8)$$

follows from condition (5.7).

Therefore, when the constraints are absent and if the solution $(y, p)^T$ to problem (4.12), (4.13), (5.8) is smooth enough on $\bar{\Omega}_i$ ($i = 1, 2$), then, take equality (5.8) into account, and the boundary-value problem is obtained:

$$\begin{aligned} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial y}{\partial x_j} \right) + \int_{\Omega} y \, dx &= f + Q, \quad x \in \Omega_1 \cup \Omega_2, \\ - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial p}{\partial x_j} \right) + \int_{\Omega} p \, dx &= 0, \quad x \in \Omega_1 \cup \Omega_2, \\ \sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(v, x_i) &= g - p/\bar{a}, \quad x \in \Gamma, \end{aligned}$$

$$\begin{aligned}
\sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \cos(v, x_i) &= 0, \quad x \in \Gamma, \\
[y] &= 0, \quad [p] = 0, \quad x \in \gamma, \\
\left[\sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \cos(v, x_i) \right] &= \omega, \quad x \in \gamma, \\
\left[\sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \cos(v, x_i) \right] &= z_g - y, \quad x \in \gamma.
\end{aligned} \tag{5.9}$$

Definition 5.1. A generalized (weak) solution to boundary-value problem (5.9) is called a vector function $(y, p)^T \in H$ that satisfies the following integral equation $\forall z \in H$:

$$\begin{aligned}
&\int_{\Omega} \left\{ \sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \frac{\partial z_1}{\partial x_i} + \sum_{i,j=1}^n k_{ij} \frac{\partial p}{\partial x_j} \frac{\partial z_2}{\partial x_i} \right\} dx + \\
&+ \int_{\Omega} y dx \int_{\Omega} z_1 dx + \int_{\Omega} p dx \int_{\Omega} z_2 dx = \int_{\Omega} (f + Q) z_1 dx + \\
&+ \int_{\Gamma} (g - p/\bar{a}) z_1 d\Gamma - \int_{\gamma} \omega z_1 d\gamma - \int_{\gamma} (z_g - y) z_2 d\gamma.
\end{aligned} \tag{5.10}$$

Let $u = (u_1, u_2)^T$ and $v = (v_1, v_2)^T$ be arbitrary elements of the complete Hilbert space H . Specify the bilinear form

$$\begin{aligned}
a(u, v) &= \int_{\Omega} \left\{ \sum_{l=1}^2 \sum_{i,j=1}^n k_{ij} \frac{\partial u_l}{\partial x_j} \frac{\partial v_l}{\partial x_i} \right\} dx + \\
&+ \sum_{l=1}^2 \int_{\Omega} u_l dx \int_{\Omega} v_l dx + \int_{\Gamma} u_2 v_1 / \bar{a} d\Gamma - \int_{\gamma} u_1 v_2 d\gamma
\end{aligned}$$

and linear functional

$$l(v) = \int_{\Omega} (f + Q)v_1 dx + \int_{\Gamma} g v_1 d\Gamma - \int_{\gamma} \omega v_1 d\gamma - \int_{\gamma} z_g v_2 d\gamma$$

on H .

If constraint like (4.16') is met, then the unique solution $(y, p)^T$ to problem (5.10) exists in H . Estimate like (1.35) is true for its approximate solution $U_k^N \in H_k^N$ and the estimate

$$\|u - u_k^N\|_{L_2(\Gamma)} \leq c h^k$$

takes place for the approximation $u_k^N(x)$ of the control u .

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