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CONTROL OF A CONDITIONALLY CORRECT SYSTEM DESCRIBED BY THE NEUMANN PROBLEM FOR AN ELLIPTIC-TYPE EQUATION UNDER CONJUGATION CONDITIONS

2.1 DISTRIBUTED CONTROL WITH OBSERVATION THROUGHOUT A WHOLE DOMAIN

Assume that the elliptic equation

$$Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(k_{ij}(x) \frac{\partial y}{\partial x_j} \right) = f(x) \tag{1.1}$$

is specified in a domain Ω that consists of two bounded convex domains, namely, Ω_1 and $\Omega_2 \in R^n$, where R^n is an n-dimensional real linear space. The second-type boundary Neumann condition

$$\sum_{i,j=1}^{n} k_{ij}(x) \frac{\partial y}{\partial x_j} \cos(\nu, x_i) = g(x) \tag{1.2}$$

is specified, in its turn, on a boundary $\Gamma = (\partial\Omega_1 \bigcup \partial\Omega_2)\backslash\gamma$ $\quad (\gamma = \partial\Omega_1 \bigcap \bigcap \partial\Omega_2 \neq \varnothing)$; in this case, ν is an outer normal to Γ, $k_{ij}(x)\big|_{\bar{\Omega}_l} = k_{ji}\big|_{\bar{\Omega}_l} = k_{ij}\big|_{\bar{\Omega}_l} \in C(\bar{\Omega}_l) \bigcap C^1(\Omega_l)$, $\quad f(x)\big|_{\Omega_l} = f\big|_{\Omega_l} \in C(\Omega_l)$, $\left|D^1 k_{ij}\right| < \infty$, $\; g(x)\big|_{\Gamma\bigcap\partial\Omega_l} = g\big|_{\Gamma\bigcap\partial\Omega_l} \in C(\Gamma \bigcap \partial\Omega_l)$, $\quad i,j = \overline{1,n}$; $\quad l = 1,2$, $|f| \le c_1 < \infty$,

$$\sum_{i,j=1}^{n} k_{ij}(x)\xi_i\xi_j \geq \alpha_0 \sum_{i=1}^{n} \xi_i^2 \quad \forall x \in \Omega,$$

$$\forall \xi_i, \xi_j \in R^1,\ i,j = \overline{1,n},\ \ \alpha_0 = \text{const} > 0\,; \tag{1.2'}$$

and the conjugation conditions

$$[y] = 0 \tag{1.3}$$

and

$$\left[\sum_{i,j=1}^{n} k_{ij} \frac{\partial y}{\partial x_j} \cos(\nu, x_i)\right] = \omega \tag{1.4}$$

are specified, also in their turn, on a section γ of the domain $\bar{\Omega}$; in this case, $\omega \in C(\gamma)$, $[\varphi] = \varphi^+ - \varphi^-$, $\varphi^\pm = \{\varphi\}^\pm = \varphi(x)$ under $x \in \gamma^\pm$, $\gamma^+ = \gamma \cap \partial\Omega_2$, $\gamma^- = \gamma \cap \partial\Omega_1$, ν is a normal to γ and such normal is directed into the domain Ω_2.

Let $y(x) \in \bar{M} = \left\{v(x):\ v|_{\bar{\Omega}_l} \in C^1(\bar{\Omega}_l) \cap C^2(\Omega_l),\ l = 1, 2,\ \left|D^2 v\right| < \infty\right\}$ be a classical solution to boundary-value problem (1.1)–(1.4). It is easy to see that a solution $y + c$ is also classical to it for an arbitrary constant c.

The necessary condition for the existence of the classical solution y to problem (1.1)–(1.4) is the one under which the equality

$$\int_\Omega f\,dx + \int_\Gamma g\,d\Gamma = \int_\gamma \omega\,d\gamma \tag{1.5}$$

is met. Find this solution under the constraint

$$\int_\Omega y\,dx = Q\,, \tag{1.6}$$

where Q is some known real number. Assume the following: $\bar{H} = \left\{ v(x):\ v|_{\Omega_i} \in W_2^1(\Omega_i),\ i=1,2 \right\}$, $V_Q = \left\{ v \in \bar{H}:\ [v]=0,\ (v,1)=Q \right\}$, $(\varphi,\psi) = \int\limits_{\Omega} \varphi\psi dx$.

Let there be a control Hilbert space $\mathcal{U}$ and mapping $B \in \mathcal{L}(\mathcal{U}; V')$, where V' is a space dual with respect to a state Hilbert space V. Assume the following: $\mathcal{U} = L_2(\Omega)$.

For every control $u \in \mathcal{U}$, determine system state $y = y(u)$ as a generalized solution to the boundary-value problem specified by the equation

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(k_{ij}(x) \frac{\partial y}{\partial x_j} \right) = f(x) + Bu,\ y \in V_Q, \tag{1.7}$$

and by conditions (1.2)–(1.4) and (1.6).

Specify the observation

$$Z(u) = C\,y(u), \tag{1.8}$$

where $C \in \mathcal{L}(V; \mathcal{H})$ and $\mathcal{H}$ is some Hilbert space. Assume the following:

$$C\,y(u) \equiv y(u),\ \mathcal{H} = V \subset L_2(\Omega). \tag{1.9}$$

Bring a value of the cost functional

$$J(u) = \left\| Cy(u) - z_g \right\|_{\mathcal{H}}^2 + (\mathcal{N}u,u)_{\mathcal{U}} \tag{1.10}$$

in correspondence with every control $u \in \mathcal{U}$; in this case, z_g is some known element of $\mathcal{H}$, $\mathcal{N} \in \mathcal{L}(\mathcal{U};\mathcal{U})$, $(\mathcal{N}u,u)_{\mathcal{U}} \geq \nu_0 \|u\|_{\mathcal{U}}^2$, $\nu_0 = \text{const} > 0$ $\forall u \in \mathcal{U}$.

Assume the following: $f \in L_2(\Omega)$, $Bu \equiv u \in L_2(\Omega)$, $\mathcal{N}u = \bar{a}(x)u$, $0 < a_0 \leq \bar{a}(x) \leq a_1 < \infty$, $\bar{a}(x)|_{\Omega_l} \in C(\Omega_l)$, $l = 1, 2$; $a_0, a_1 = \text{const}$, $(\varphi,\psi)_{\mathcal{U}} = \int\limits_{\Omega} \varphi\psi\, dx$. A unique state, namely, a function $y(u) \in V_Q$

corresponds to every control $u \in \mathcal{U}$, delivers the minimum to the energy functional [21]

$$\Phi_1(v) = a_1(v,v) - 2l_1(v) \tag{1.11}$$

on V_Q, and it is the unique solution in V_Q to the weakly stated problem: Find an element $y \in V_Q$ that meets the equation

$$a_1(y,v) = l_1(v) \ \ \forall v \in V_0, \tag{1.12}$$

where $V_0 = \{v \in \bar{H} : [v] = 0, (v,1) = 0\}$, $a_1(u,v) = \int\limits_{\Omega} \sum\limits_{i,j=1}^{n} k_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx,$

$$l_1(v) = \int\limits_{\Omega} (f+u) v\, dx + \int\limits_{\Gamma} g\, v\, d\Gamma - \int\limits_{\gamma} \omega\, v\, d\gamma .$$

The following statement is valid [21].

Lemma 1.1. *Problems (1.11) and (1.12) are equivalent* $\forall f \in L_2(\Omega)$, $\forall \omega \in L_2(\gamma)$, $\forall u \in \mathcal{U}$ *and have a unique solution* $y = y(u) \in V_Q$.

Remark 1.1. If a solution $y \in V_Q$ to problems (1.11) and (1.12) belongs to a set $\bar{M}$, then y is classical to boundary-value problem (1.7), (1.2)–(1.4), (1.6) under the constraint

$$\int\limits_{\Omega} (f+u)\, dx + \int\limits_{\Gamma} g\, d\Gamma = \int\limits_{\gamma} \omega\, d\gamma . \tag{1.13}$$

Remark 1.2. If a solution y to problems (1.11) and (1.12) exists, it is not necessary to meet constraint (1.13).

Remark 1.3. If equality (1.5) takes place, then, to meet constraint (1.13), it is necessary for a control u to satisfy the condition

$$\int\limits_{\Omega} u\, dx = 0 . \tag{1.14}$$

Rewrite cost functional (1.10) as

$$J(u) = \pi(u,u) - 2L(u) + \left\| z_g - y(0) \right\|^2 ; \tag{1.15}$$

in this case, $\|\varphi\| = \|\varphi\|_{L_2(\Omega)} = (\varphi, \varphi)^{1/2}$ and the bilinear form $\pi(\cdot,\cdot)$ and linear functional $L(\cdot)$ are expressed as

$$\pi(u,v) = \left(y(u) - y(0),\ y(v) - y(0)\right) + (\bar{a}\,u, v)$$

$$L(v) = \left(z_g - y(0),\ y(v) - y(0)\right). \tag{1.16}$$

Let $\tilde{y}' = \tilde{y}(u')$ and $\tilde{y}'' = \tilde{y}(u'')$ be solutions from V_Q to problem (1.12) under $f = 0$, $g = 0$ and $\omega = 0$ and under a function $u = u(x)$ that is equal, respectively, to u' and u''. Then, take the ellipticity condition and generalized Poincare inequality into account, and the inequality

$$\bar{\alpha}_0 \|\tilde{y}' - \tilde{y}''\|^2 \le \bar{\alpha}_0 \|\tilde{y}' - \tilde{y}''\|_V^2 \le a_1\left(\tilde{y}' - \tilde{y}'', \tilde{y}' - \tilde{y}''\right) \le$$

$$\le \|u' - u''\| \cdot \|\tilde{y}' - \tilde{y}''\|_V, \quad \bar{\alpha}_0 = \text{const} > 0,$$

is derived, where $\|v\|_V = \left\{ \sum_{i=1}^{2} \|v\|^2_{W_2^1(\Omega_i)} \right\}^{1/2}$ and $\|\cdot\|_{W_2^1(\Omega_i)}$ is the norm of the Sobolev space $W_2^1(\Omega_i)$.

On the basis of [58, Theorem 1.1, Chapter 1], the validity of the following statement is proved.

Theorem 1.1. *Let a system state be determined as a solution to equivalent problems (1.11) and (1.12). Then, there exists a unique element u of a convex set $\mathcal{U}_\partial$ that is closed in $\mathcal{U}$, and*

$$J(u) = \inf_{v \in \mathcal{U}_\partial} J(v) \tag{1.17}$$

takes place for u.

Definition 1.1. If an element $u \in \mathcal{U}_\partial$ meets condition (1.17), it is called an optimal control.

Let the equation

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(k_{ij} \frac{\partial y}{\partial x_j}\right) + \int_\Omega y\,dx = f(x) + Q + u \tag{1.18}$$

be specified on the domain Ω instead of equation (1.7). Neumann condition (1.2) and conjugation conditions (1.3)–(1.4) are specified, in their turn, respectively, on the boundary Γ and section γ.

If y is a classical solution to boundary-value problem (1.7), (1.2)–(1.4), (1.6) (Problem 1), then it is easy to see that y is classical to problem (1.18), (1.2)–(1.4) (Problem 1′). It can be shown [21] that a classical solution to Problem 1′ is also classical to Problem 1 if constraint (1.13) is satisfied.

Let observation (1.8) be specified, where the operator C is given by expression (1.9). Cost functional (1.10) is specified, in its turn, for every control $u \in \mathcal{U}$. Then, a unique state, namely, a function $y(u) \in$ $\in V = \{v \in \bar{H}: \ [v]=0\}$, corresponds to every $u \in \mathcal{U}$, minimizes the energy functional

$$\Phi(v) = a_1'(v,v) - 2\,l_1'(v) \tag{1.19}$$

on V, and it is the unique solution in V to the weakly stated problem: Find an element $y \in V$ that meets the equation

$$a_1'(y,v) = l_1'(v), \quad \forall v \in V, \tag{1.20}$$

where $a_1'(y,v) = a_1(y,v) + (y,1)(v,1)$ and $l_1'(v) = l_1(v) + Q(v,1)$.

Lemma 1.2. *Problems (1.19) and (1.20) are equivalent* $\forall f \in L_2(\Omega)$, $\forall u \in \mathcal{U}$ *and have a unique solution* $y(u) \in V$.

Remark 1.4. If a solution $y \in V$ to problems (1.19) and (1.20) belongs to a set $\bar{M}$, then y is classical to boundary-value Problem 1′, and it is also classical to Problem 1 if constraint (1.13) is met.

Therefore, there exists such an operator A generated by problems (1.19), (1.20) and acting from V into $L_2(\Omega)$, that

$$y(u) = A^{-1}(f + Q + Bu), \quad \forall u \in L_2 = L_2(\Omega).$$

Let $\tilde{y}' = \tilde{y}(u')$ and $\tilde{y}'' = \tilde{y}(u'')$ be solutions from V to Problem 1′ under $f = 0$, $g = 0$ and $\omega = 0$ and under a function $u = u(x)$ that is equal, respectively, to u' and u''.

Then, on the basis of the generalized Poincare inequality, the following one, i.e.

$$\bar{\alpha}_1 \|\tilde{y}' - \tilde{y}''\|^2 \le \bar{\alpha}_1 \|\tilde{y}' - \tilde{y}''\|_V^2 \le a\,(\tilde{y}' - \tilde{y}'', \tilde{y}' - \tilde{y}'') \le$$

$$\le \|u' - u''\| \cdot \|\tilde{y}' - \tilde{y}''\|, \;\; \bar{\alpha}_1 = \text{const} > 0,$$

is derived that provides the continuity of the linear functional $L(\cdot)$ and bilinear form $\pi(\cdot,\cdot)$ of expressions (1.16) on $\mathcal{U}$.

On the basis of [58, Theorem 1.1, Chapter 1], the validity of the following statement is proved.

Theorem 1.2. *Let a system state be determined as a solution to equivalent problems (1.19) and (1.20). Then, there exists a unique element* u *of a convex set* $\mathcal{U}_\partial$ *that is closed in* $\mathcal{U}$, *and relation like (1.17) takes place for* u.

Remark 1.5. If equality (1.13) is satisfied, then problems (1.11) and (1.19) are equivalent. Therefore, optimal controls coincide when states are described by boundary-value Problems 1 and 1′.

Here is the problem of finding the control $u \in \mathcal{U}_\partial$ that satisfies relation (1.17). It is optimization Problem 1 if a system state is a generalized solution to boundary-value Problem 1, and it is optimization Problem 1′ if a system state is a generalized solution to boundary-value Problem 1′.

Remark 1.6. If constraint (1.5) is met and $\mathcal{U}_\partial = \left\{ u \in L_2(\Omega) : \int_\Omega u\,dx = 0 \right\}$, then optimization Problems 1 and 1′ are equivalent.

If $u \in \mathcal{U}_\partial$ is the optimal control, then the following inequality is true $\forall v \in \mathcal{U}_\partial$:

$$\big(y(u) - z_g,\, y(v) - y(u)\big) + (\bar{a}\,u, v - u) \ge 0\;. \tag{1.21}$$

As for the control $v \in \mathcal{U}$, the conjugate state $p(v) \in V^* = V$ is specified by the relations

$$A^* p(v) = y(v) - z_g, \;\; x \in \Omega_1 \cup \Omega_2,$$

$$\frac{\partial p}{\partial \nu_{A^*}} = 0, \;\; x \in \Gamma,$$

$$[p] = 0, \;\; \left[\frac{\partial p}{\partial \nu_{A^*}} \right] = 0, \;\; x \in \gamma, \tag{1.22}$$

where

$$A^* p = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial p}{\partial x_j} \right) + \int_{\Omega} p \, dx \,,$$

$$\frac{\partial p}{\partial \nu_{A^*}} = \sum_{i,j=1}^{n} k_{ij} \frac{\partial p}{\partial x_j} \cos(\nu, x_i) \,. \tag{1.23}$$

The equality

$$\left(A^* p(u), y(v) - y(u) \right) = \left(y(u) - z_g, y(v) - y(u) \right) = a_1' \left(p, y(v) - y(u) \right) =$$

$$= \sum_{l=1}^{2} \int_{\partial \Omega_l} \sum_{i,j=1}^{n} k_{ij} \frac{\partial (y(v) - y(u))}{\partial x_j} \cos(\nu, x_i) \, p(u) \, d\partial\Omega_l +$$

$$+ \left(p, \, v - u \right) = \left(p, \, v - u \right),$$

i.e. $\left(y(u) - z_g, y(v) - y(u) \right) = \left(p, \, v - u \right)$ is obtained. Take it into account, and the inequality

$$\left(p + \bar{a} u, v - u \right) \geq 0, \;\; \forall v \in \mathcal{U}_\partial, \tag{1.24}$$

is derived from inequality (1.21).

To make the element $u \in \mathcal{U}_\partial$ the optimal control of a state described by boundary-value Problem $1'$, it is necessary and sufficient to meet inequality (1.24) and the relations

$$a_1'(y, v) = l_1(u, v), \;\; y \in V, \;\; \forall v \in V, \tag{1.25}$$

and

$$a_1'(p, v) = l_2(y, v), \;\; p \in V, \;\; \forall v \in V, \tag{1.26}$$

where

$$l_1(u,v) = (f+Q,\ v)+(u,v)+\int_{\Gamma} gv\,d\Gamma - \int_{\gamma} \omega v\,d\gamma$$

and

$$l_2(y,v) = (y,v)-(z_g,v).$$

If the constraints are absent, i.e. when $\mathcal{U}_\partial=\mathcal{U}$, then the equality

$$p+\bar{a}\,u=0 \tag{1.27}$$

follows from condition (1.24). Therefore, when the constraints are absent, the control u can be excluded from equality (1.25) by means of equality (1.27). On the basis of equalities (1.25) and (1.26), the problem

$$Ay+p/\bar{a}=f,\ \ y\in V, \tag{1.28}$$

$$A^* p-y=-z_g,\ \ p\in V^*, \tag{1.29}$$

is derived, and the vector solution $(y,p)^{\mathrm{T}}$ is found from this problem along with the optimal control $u=-p/\bar{a}$ of the system specified by boundary-value Problem 1′.

If the vector solution $(y,p)^{\mathrm{T}}$ to problem (1.28), (1.29) is smooth enough on $\bar{\Omega}_l$, viz., $y|_{\bar{\Omega}_l}$, $p|_{\bar{\Omega}_l}\in C^1(\bar{\Omega}_l)\cap C^2(\Omega_l)$, $l=1,2$, then the differential problem of finding the vector-function $(y,p)^{\mathrm{T}}$, that satisfies the relations

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial y}{\partial x_j}\right)+p/\bar{a}+\int_{\Omega} y\,dx = f+Q,\ \ x\in\Omega_1\cup\Omega_2,$$

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial p}{\partial x_j}\right)+\int_{\Omega} p\,dx - y = -z_g,\ \ x\in\Omega_1\cup\Omega_2,$$

$$\sum_{i,j=1}^{n} k_{ij} \frac{\partial y}{\partial x_j} \cos(\nu, x_i) = g, \quad x \in \Gamma,$$

$$\sum_{i,j=1}^{n} k_{ij} \frac{\partial p}{\partial x_j} \cos(\nu, x_i) = 0, \quad x \in \Gamma,$$

$$[y] = 0, \; [p] = 0, \; x \in \gamma,$$

$$\left[\sum_{i,j=1}^{n} k_{ij} \frac{\partial y}{\partial x_j} \cos(\nu, x_i)\right] = \omega, \; \left[\sum_{i,j=1}^{n} k_{ij} \frac{\partial p}{\partial x_j} \cos(\nu, x_i)\right] = 0, \; x \in \gamma, \quad (1.30)$$

corresponds to problem (1.28), (1.29).

Definition 1.2. A generalized (weak) solution to boundary-value problem (1.30) is called a vector-function $(y,p)^{\mathrm{T}} \in H = \left\{v = (v_1, v_2)^{\mathrm{T}} : v_i\big|_{\Omega_j} \in W_2^1(\Omega_j),\; i,j = 1,2;\; [v] = 0\right\}$ that satisfies the following integral equation $\forall z \in H$:

$$\int_{\Omega} \left\{ \sum_{i,j=1}^{n} k_{ij} \frac{\partial y}{\partial x_j} \frac{\partial z_1}{\partial x_i} + p z_1 / \bar{a} + \sum_{i,j=1}^{n} k_{ij} \frac{\partial p}{\partial x_j} \frac{\partial z_2}{\partial x_i} - y z_2 \right\} dx +$$

$$+ \int_{\Omega} y\, dx \int_{\Omega} z_1 dx + \int_{\Omega} p\, dx \int_{\Omega} z_2 dx =$$

$$= \int_{\Omega} \big((f + Q) z_1 - z_g z_2 \big) dx + \int_{\Gamma} g z_1 d\Gamma - \int_{\gamma} \omega z_1 d\gamma . \quad (1.31)$$

Let $u = (u_1, u_2)^{\mathrm{T}}$ and $v = (v_1, v_2)^{\mathrm{T}}$ be arbitrary elements of the complete Hilbert space H with the norm $\|\cdot\|_H = \left\{ \sum_{i=1}^{2} \|\cdot\|_{W_2^1(\Omega_i)} \right\}^{1/2}$. Specify the bilinear form

$$a(u,v)=\int_{\Omega}\left\{\sum_{l=1}^{2}\sum_{i,j=1}^{n}k_{ij}\frac{\partial u_l}{\partial x_j}\frac{\partial v_l}{\partial x_i}+u_2 v_1/\bar{a}-u_1 v_2\right\}dx+\sum_{l=1}^{2}\int_{\Omega}u_l dx\int_{\Omega}v_l dx$$

and linear functional

$$l(v)=\int_{\Omega}\left((f+Q)v_1-z_g v_2\right)dx+\int_{\Gamma}g\,v_1 d\Gamma-\int_{\gamma}\omega\,v_1 d\gamma$$

on H.

Assume that the constraint $\alpha_1=\min\left\{\frac{\alpha_0}{2},1\right\}\mu-\frac{1}{2}\left\{\frac{1}{a_0}+1\right\}>0$ is met, where μ is the constant in the generalized Poincare inequality. Take the generalized Poincare inequality [21] and Cauchy-Bunyakovsky one into account, and the relations

$$a(v,v)\geq\bar{\alpha}_1\|v\|_H^2\quad\forall v\in H,\ \bar{\alpha}_1=\text{const}>0,$$

and

$$|a(u,v)|\leq c_1\|u\|_H\|v\|_H\quad\forall u,v\in H,\ c_1=\text{const}>0,$$

are true for the bilinear form $a(\cdot,\cdot)$, i.e. this form is H-elliptic and continuous [49] on H.

Consider the Cauchy-Bunyakovsky inequality and embedding theorems [55], and the following inequality is obtained $\forall v\in H$:

$$|\,l(v)\,|\leq c_2\|v\|_H,\ c_2=\text{const}.$$

Use the Lax-Milgramm lemma [16], and it is concluded that the unique solution (y,p) to problem (1.31) exists in H.

Problem (1.31) can be solved approximately by means of the finite-element method. For this purpose, divide the domains $\bar{\Omega}_i$ into N_i finite elements $\bar{e}_i^j$ $(j=\overline{1,N_i},\ i=1,2)$ of the regular family [16]. Specify the subspace $H_k^N\subset H$ $(N=N_1+N_2)$ of the vector-functions $V_k^N(x)$. The components $v_{1k}^N\big|_{\bar{\Omega}_i}$, $v_{2k}^N\big|_{\bar{\Omega}_i} \in C(\bar{\Omega}_i)$ $(i=1,2)$ of $V_k^N(x)$ are the complete polynomials of the power k that contain the variables $x_1, x_2, \ldots, x_n$ at every $\bar{e}_i^j$, and $\left[V_k^N\right]=0$. Then, the linear algebraic equation system

$$A\bar{U}=B \tag{1.32}$$

follows from equation (1.31), and the solution $\bar{U}$ to system (1.32) exists and such solution is unique. The vector $\bar{U}$ specifies the unique approximate solution $U_k^N \in H_k^N$ to problem (1.31) as the unique one to the equation

$$a\left(U_k^N, V_k^N\right)=l\left(V_k^N\right), \ \forall V_k^N \in H_k^N. \tag{1.33}$$

Let $U=U(x)\in H$ be the solution to problem (1.31). Then:

$$a\left(U-U_k^N, V_k^N\right)=0, \ \forall V_k^N \in H_k^N.$$

Therefore,

$$\bar{\alpha}_1\left\|U-U_k^N\right\|_H^2 \le a\left(U-U_k^N, U-U_k^N\right)=$$

$$=a\left(U-U_k^N, U-\tilde{U}\right), \ \forall \tilde{U}\in H_k^N,$$

and the inequality

$$\left\|U-U_k^N\right\|_H \le c_0\left\|U-\tilde{U}\right\|_H, \ c_0=\text{const}, \tag{1.34}$$

is thus derived since the bilinear form $a(\cdot,\cdot)$ is continuous on *H*.

Suppose that $\tilde{U}\in H_k^N$ is a complete interpolation polynomial for the solution *U* at every $\bar{e}_i^j$. Take the interpolation estimates [16] into account, assume that every component U_1 and U_2 of the solution *U* on Ω_l belongs to the Sobolev space $W_2^{k+1}(\Omega_l)$ $(l=1,2)$, and the estimate

$$\left\|U - U_k^N\right\|_H \le ch^k, \tag{1.35}$$

where h is a maximum diameter of all the finite elements $\overline{e}_i^j$, $c = \text{const}$, follows from inequality (1.34).

Take estimate (1.35) into consideration, and the estimate

$$\left\|u - u_k^N\right\|_{W_2^1} \le c_2 \left\|p - p_k^N\right\|_{W_2^1} \le c_3 h^k, \tag{1.35'}$$

where $\|\cdot\|_{W_2^1} = \left\{ \sum_{i=1}^{2} \|\cdot\|^2_{W_2^1(\Omega_i)} \right\}^{1/2}$, takes place for the approximation $u_k^N(x) = -p_k^N / \overline{a}(x)$ of the control $u = u(x)$ of a state described by Problem 1′ and $p_k^N = u_{2k}^N$ is the second component of the vector U_k^N.

Remark 1.7. If constraint (1.13) is met, then the first component of a classical solution to problem (1.30) is a classical solution to boundary-value Problem 1.

2.2 DISTRIBUTED CONTROL WITH OBSERVATION ON A THIN INCLUSION

Assume that equation (1.1), where the coefficients and right-hand side meet conditions (1.2′), is specified in the bounded, continuous and strictly Lipschitz domains Ω_1 and Ω_2. Condition (1.2) is specified, in its turn, on the boundary Γ and the conjugation conditions have the form of expressions (1.3) and (1.4).

For every control $u \in \mathcal{U} = L_2(\Omega)$, determine a system state as a generalized solution to the boundary-value problem specified by equation (1.7) and by conditions (1.2)–(1.4), where $Bu \equiv u$ and $u \in L_2(\Omega)$.

Equality (1.13) is the necessary condition under which there exists a classical solution $y = y(u)$ to boundary-value problem (1.7), (1.2)–(1.4) (Problem 2). Find this solution under constraint (1.6).

Bring a value of the cost functional

$$J(u) = \int_\gamma (y(u) - z_g)^2 d\gamma + (\mathcal{N}u, u)_{\mathcal{U}} \tag{2.1}$$

in correspondence with every control $u \in \mathcal{U} = L_2(\Omega)$; in this case, z_g is a known element from the space $L_2(\gamma)$; $\mathcal{N}u = \bar{a}\,u$, $0 < a_0 \le \bar{a}(x) \le \le a_1^0 < \infty$, $\bar{a}(x)\big|_{\Omega_i} \in C(\Omega_i)$, $i = 1,2$; a_0, $a_1^0 = \text{const}$.

A unique state, namely, a function $y(u) \in V_Q$ corresponds to every control $u \in \mathcal{U}$, minimizes energy functional (1.11) on V_Q, and it is the unique solution in V_Q to weakly stated problem (1.12). Lemma 1.1 and Remarks 1.1 and 1.2 hold here.

Rewrite cost functional (2.1) as

$$J(u) = \pi(u,u) - 2L(u) + \left\| z_g - y(0) \right\|^2_{L_2(\gamma)}, \tag{2.2}$$

where

$$\|v\|_{L_2(\gamma)} = (v,v)^{1/2}_{L_2(\gamma)}, \quad (\varphi,\psi)_{L_2(\gamma)} = \int_\gamma \varphi\psi\, d\gamma,$$

$$\pi(u,v) = \big(y(u) - y(0),\ y(v) - y(0)\big)_{L_2(\gamma)} + (\bar{a}\,u, v)$$

and

$$L(v) = \big(z_g - y(0),\ y(v) - y(0)\big)_{L_2(\gamma)}.$$

Take the embedding theorems, ellipticity condition and generalized Poincare inequality into account, and the inequality

$$\bar{\alpha}_2 \|\tilde{y}' - \tilde{y}''\|^2_{L_2(\gamma)} \le \bar{\alpha}_1 \|\tilde{y}' - \tilde{y}''\|^2_V \le$$

$$\le a_1(\tilde{y}' - \tilde{y}'', \tilde{y}' - \tilde{y}'') \le \|u' - u''\|\, \|\tilde{y}' - \tilde{y}''\|_V,$$

i.e.

$$\|\tilde{y}' - \tilde{y}''\|_{L_2(\gamma)} \leq \frac{1}{\sqrt{\bar{\alpha}_1 \bar{\alpha}_2}} \|u' - u''\|, \quad \alpha_1, \alpha_2 = \text{const} > 0,$$

is derived, where $\tilde{y}' = \tilde{y}(u')$ and $\tilde{y}'' = \tilde{y}(u'')$ are the generalized solutions from V_Q to boundary-value Problem 2 under $f = 0$, $g = 0$ and $\omega = 0$ and under a function $u = u(x)$ that is equal, respectively, to u' and u''.

On the basis of the derived inequality and [58, Chapter 1, Theorem 1.1], the validity of the following statement is proved.

Theorem 2.1. *Let a system state be determined as a solution to equivalent problems (1.11) and (1.12). Then, there exists a unique element u of a convex set* $\mathcal{U}_\partial$ *that is closed in* $\mathcal{U} = L_2(\Omega)$, *and relation like (1.17) takes place for u, where the cost functional J(u) is specified by expression (2.1).*

Let equation (1.18) be specified on Ω instead of equation (1.7). Neumann condition (1.2) and conjugation conditions (1.3) and (1.4) are specified, in their turn, respectively, on Γ and γ. I.e., boundary-value problem (1.18), (1.2)–(1.4) (Problem 2′) is obtained.

Remark 2.1. Boundary-value Problems 1, 2 and 1′, 2′ coincide pairwise. Optimization Problems do not coincide because their cost functionals $J(u)$ are different.

Consider optimization Problem 2′: Find a control $u \in \mathcal{U}_\partial \subset \mathcal{U} = L_2(\Omega)$, for which relation like (1.17) is satisfied, where the cost functional *J(u)* is specified by expression (2.1), and a state *y*(*u*) is a generalized solution to boundary-value Problem 2′.

If $u \in \mathcal{U}_\partial$ is the optimal control for optimization Problem 2′, then the following inequality is true:

$$\left(y(u) - z_g,\, y(v) - y(u)\right)_{L_2(\gamma)} + (\bar{a}\, u, v - u) \geq 0, \ \forall v \in \mathcal{U}_\partial. \qquad (2.3)$$

As for the control $v \in \mathcal{U}$, the conjugate state $p(v) \in V^* = V$ is specified by the relations

$$A^* p(v) = 0, \ \ x \in \Omega_1 \cup \Omega_2,$$

$$\frac{\partial p}{\partial \nu_{A^*}} = 0, \;\; x \in \Gamma,$$

$$[p] = 0, \;\; \left[\frac{\partial p}{\partial \nu_{A^*}} \right] = -y(v) + z_g, \;\; x \in \gamma, \tag{2.4}$$

where the operators A^* and $\dfrac{\partial}{\partial \nu_{A^*}}$ are specified, in their turn, by expressions (1.23).

The equality

$$0 = \left(A^* p(u),\, y(v) - y(u) \right) = a_1'\left(p,\, y(v) - y(u)\right) -$$

$$-\left(y(u) - z_g,\, y(v) - y(u) \right)_{L_2(\gamma)} =$$

$$= -\left(y(u) - z_g, y(v) - y(u) \right)_{L_2(\gamma)} + \left(p,\, v - u\right),$$

i.e. $\left(y(u) - z_g, y(v) - y(u) \right)_{L_2(\gamma)} = \left(p,\, v - u\right) \;\; \forall v \in \mathcal{U}_\partial$ is obtained. Take it into account, and the inequality

$$\left(\bar{a}u + p, v - u \right) \geq 0, \;\; \forall v \in \mathcal{U}_\partial, \tag{2.5}$$

is derived from inequality (2.3).

An element $u \in \mathcal{U}_\partial$ is an optimal control for optimization Problem 2′ if and only if inequality (2.5) and the equalities

$$a_1'(y, v) = l_1(u, v), \;\; y \in V, \;\; \forall v \in V, \tag{2.6}$$

and

$$a_1'(p, v) = l_2(y, v), \;\; p \in V, \;\; \forall v \in V, \tag{2.7}$$

are met; the bilinear form $a_1'(\cdot, \cdot)$ and functional $l_1(u, v)$ are specified in point 2.1, and

$$l_2(y,v) = -\int\limits_{\gamma}(z_g - y)v\,d\gamma .$$

If the constraints are absent, i.e. when $\mathcal{U}_\partial = \mathcal{U}$, then the equality

$$p + \bar{a}\,u = 0 \tag{2.8}$$

follows from condition (2.5). Therefore, when the constraints are absent, the control u can be excluded from equality (2.6) by means of equality (2.8). Let the solution $(y,p)^{\mathrm{T}}$ to problem (2.6), (2.7), where $l_1(u,y) = l_1(u(p),y)$, be sufficiently smooth on $\bar{\Omega}_1$ and $\bar{\Omega}_2$. Then, such solution satisfies the relations

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial y}{\partial x_j}\right) + p/\bar{a} + \int\limits_{\Omega} y\,dx = f + Q, \quad x \in \Omega_1 \cup \Omega_2,$$

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial p}{\partial x_j}\right) + \int\limits_{\Omega} p\,dx = 0, \quad x \in \Omega_1 \cup \Omega_2,$$

$$\sum_{i,j=1}^{n} k_{ij}\frac{\partial y}{\partial x_j}\cos(\nu, x_i) = g, \quad x \in \Gamma,$$

$$\sum_{i,j=1}^{n} k_{ij}\frac{\partial p}{\partial x_j}\cos(\nu, x_i) = 0, \quad x \in \Gamma,$$

$$[y] = 0, \ [p] = 0, \ x \in \gamma,$$

$$\left[\sum_{i,j=1}^{n} k_{ij}\frac{\partial y}{\partial x_j}\cos(\nu, x_i)\right] = \omega, \ x \in \gamma,$$

$$\left[\sum_{i,j=1}^{n} k_{ij}\frac{\partial p}{\partial x_j}\cos(\nu, x_i)\right] = -y + z_g, \ x \in \gamma. \tag{2.9}$$

Definition 2.1. A generalized (weak) solution to boundary-value problem (2.9) is called a vector-function $(y,p)^{\mathrm{T}} \in H$ that satisfies the following integral equation $\forall z \in H$:

$$\int_{\Omega} \left\{ \sum_{i,j=1}^{n} k_{ij} \frac{\partial y}{\partial x_j} \frac{\partial z_1}{\partial x_i} + p\, z_1 / \bar{a} + \sum_{i,j=1}^{n} k_{ij} \frac{\partial p}{\partial x_j} \frac{\partial z_2}{\partial x_i} \right\} dx +$$
$$+ \int_{\Omega} y\, dx \int_{\Omega} z_1 dx + \int_{\Omega} p\, dx \int_{\Omega} z_2 dx =$$
$$= \int_{\Omega} (f+Q) z_1\, dx + \int_{\Gamma} g\, z_1\, d\Gamma - \int_{\gamma} \omega\, z_1\, d\gamma + \int_{\gamma} \left(y - z_g \right) z_2\, d\gamma\,. \qquad (2.10)$$

Let $u = (u_1, u_2)^{\mathrm{T}}$ and $v = (v_1, v_2)^{\mathrm{T}}$ be arbitrary elements of the complete Hilbert space H. Specify the bilinear form

$$a(u,v) = \int_{\Omega} \left\{ \sum_{l=1}^{2} \sum_{i,j=1}^{n} k_{ij} \frac{\partial u_l}{\partial x_j} \frac{\partial v_l}{\partial x_i} + u_2\, v_1 / \bar{a} \right\} dx +$$
$$+ \sum_{l=1}^{2} \int_{\Omega} u_l\, dx \int_{\Omega} v_l\, dx - \int_{\gamma} u_1\, v_2\, d\gamma$$

and linear functional

$$l(v) = \int_{\Omega} (f+Q) v_1\, dx + \int_{\Gamma} g\, v_1\, d\Gamma - \int_{\gamma} \omega\, v_1\, d\gamma - \int_{\gamma} z_g v_2\, d\gamma$$

on H. If the constraint

$$\min\left\{ \frac{\alpha_0}{2},\ \min\left\{ \frac{\alpha_0}{2}, 1 \right\} \mu \right\} - \frac{1}{2a_0} - \frac{c_0^2}{2} > 0\,, \qquad (2.11)$$

where μ and c_0 are the positive constants, respectively, in the generalized Poincare inequality and embedding theorem, is met, then the unique solution $(y,p)^{\mathrm{T}}$ to problem (2.10) exists in H. Problem (2.10) can be solved by means of the finite-element method. Estimates like (1.35) and

(1.35′) are true, respectively, for its approximate solution $U_k^N \in H_k^N \subset H$ and for the approximation $u_k^N(x)$ of the control u.

2.3 DISTRIBUTED CONTROL WITH BOUNDARY OBSERVATION

Assume that equation (1.1), where the coefficients and right-hand side meet conditions (1.2′), is specified in the bounded, continuous and strictly Lipschitz domains Ω_1 and Ω_2. The conjugation conditions have the form of expressions (1.3) and (1.4) and the boundary condition has the form of expression (1.2).

For every control $u \in \mathcal{U} = L_2(\Omega)$, determine a system state as a generalized solution to the boundary-value problem specified by equation (1.7) and by conditions (1.2)–(1.4). Equality (1.13) is the necessary condition under which there exists a classical solution y to boundary-value problem (1.7), (1.2)–(1.4) (Problem 3): Find a solution y that meets constraint (1.6).

Bring a value of the cost functional

$$J(u) = \int_{\Gamma} (y(u) - z_g)^2 d\Gamma + (\mathcal{N}u, u)_{\mathcal{U}} \tag{3.1}$$

in correspondence with every control $u \in \mathcal{U} = L_2(\Omega)$; in this case, z_g is a known element from the space $L_2(\Gamma)$, $\mathcal{N}u = \bar{a}\,u$, $0 < a_0 \le \bar{a}(x) \le$ $\le a_1^0 < \infty$, $\bar{a}(x)\big|_{\Omega_i} \in C(\Omega_i)$, $i = 1,2$; a_0, $a_1^0 = \text{const}$.

A unique state, namely, a function $y(u) \in V_Q$ corresponds to every control $u \in \mathcal{U}$, delivers the minimum to energy functional (1.11) on V_Q, and it is the unique solution in V_Q to weakly stated problem (1.12). Lemma 1.1 and Remarks 1.1 and 1.2 take place here.

Rewrite cost functional (3.1) as

$$J(u)=\pi(u,u)-2L(u)+\left\|z_g-y(0)\right\|^2_{L_2(\Gamma)}, \tag{3.2}$$

where

$$\|v\|_{L_2(\Gamma)}=(v,v)^{1/2}_{L_2(\Gamma)},\ (\varphi,\psi)_{L_2(\Gamma)}=\int_\Gamma \varphi\psi\, d\Gamma,$$

$$\pi(u,v)=\big(y(u)-y(0),\ y(v)-y(0)\big)_{L_2(\Gamma)}+(\bar{a}\,u,v)$$

and

$$L(v)=\big(z_g-y(0),\ y(v)-y(0)\big)_{L_2(\Gamma)}.$$

Take the embedding theorems, ellipticity condition and generalized Poincare inequality into account, and the inequality

$$\bar{\alpha}_2\|\tilde{y}'-\tilde{y}''\|^2_{L_2(\Gamma)}\le\bar{\alpha}_1\|\tilde{y}'-\tilde{y}''\|^2_V\le$$
$$\le a_1(\tilde{y}'-\tilde{y}'',\tilde{y}'-\tilde{y}'')\le\|u'-u''\|\,\|\tilde{y}'-\tilde{y}''\|_V,$$

i.e.

$$\|\tilde{y}'-\tilde{y}''\|_{L_2(\Gamma)}\le\frac{1}{\sqrt{\bar{\alpha}_1\bar{\alpha}_2}}\|u'-u''\|,\ \bar{\alpha}_1,\bar{\alpha}_2=\text{const}>0,$$

is derived, where $\tilde{y}'=\tilde{y}(u')$ and $\tilde{y}''=\tilde{y}(u'')$ are the generalized solutions from V_Q to boundary-value Problem 3 under $f=0$, $g=0$ and $\omega=0$ and under a function $u=u(x)$ that is equal, respectively, to u' and u''.

On the basis of the derived inequality and [58, Chapter 1, Theorem 1.1], the validity of the following statement is proved.

Theorem 3.1. *Let a system state be determined as a solution to equivalent problems (1.11) and (1.12). Then, there exists a unique element u of a convex set $\mathcal{U}_\partial$ that is closed in $\mathcal{U}$, and relation like (1.17) takes place for u, where the cost functional J(u) is specified by expression (3.1).*

Let equation (1.18) be specified on the domain Ω instead of equation (1.7). Neumann condition (1.2) and conjugation conditions (1.3) and (1.4) are specified, in their turn, respectively, on Γ and γ. I.e., boundary-value problem (1.18), (1.2)–(1.4) (Problem 3′) is obtained.

Remark 3.1. Boundary-value Problem 3 coincides with Problems 1 and 2. Problem 3′ coincides with Problems 1′ and 2′. Optimization Problems do not coincide because their cost functionals $J(u)$ are different.

Consider optimization Problem 3′: Find a control $u \in \mathcal{U}_\partial \subset \mathcal{U} = L_2(\Omega)$, for which relation like (1.17) is satisfied and where the cost functional $J(u)$ is specified by expression (3.1). A state $y = y(u)$ is a generalized solution to boundary-value Problem 3′ and such solution is unique for one of equivalent problems (1.19) and (1.20).

If $u \in \mathcal{U}_\partial$ is the optimal control for optimization Problem 3′, then the following inequality is true:

$$\left(y(u) - z_g,\, y(v) - y(u)\right)_{L_2(\Gamma)} + (\bar{a}\, u, v - u) \geq 0, \;\; \forall v \in \mathcal{U}_\partial. \tag{3.3}$$

As for the control $v \in \mathcal{U}$, the conjugate state $p(v) \in V^* = V$ is specified by the relations

$$\begin{gathered} A^* p(v) = 0, \;\; x \in \Omega_1 \cup \Omega_2, \\ \frac{\partial p}{\partial \nu_{A^*}} = -z_g + y, \;\; x \in \Gamma, \\ [p] = 0, \;\; \left[\frac{\partial p}{\partial \nu_{A^*}}\right] = 0, \;\; x \in \gamma, \end{gathered} \tag{3.4}$$

where the operators A^* and $\dfrac{\partial}{\partial \nu_{A^*}}$ are specified, in their turn, by expressions (1.23).

The equality

$$\begin{aligned} 0 = \left(A^* p(u),\, y(v) - y(u)\right) = a_1'\left(p,\, y(v) - y(u)\right) - \\ -\left(y(u) - z_g,\, y(v) - y(u)\right)_{L_2(\Gamma)} = \\ = -\left(y(u) - z_g, y(v) - y(u)\right)_{L_2(\Gamma)} + \left(p,\, v - u\right), \end{aligned}$$

i.e. $\left(y(u)-z_g, y(v)-y(u)\right)_{L_2(\Gamma)}=\left(p, v-u\right)\ \forall v\in\mathcal{U}_\partial$ is obtained. Take it into account, and the inequality

$$\left(p+\bar{a}u, v-u\right)\geq 0,\ \ \forall v\in\mathcal{U}_\partial, \tag{3.5}$$

is derived from inequality (3.3).

To make the element $u\in\mathcal{U}_\partial$ the optimal control for optimization Problem 3′, it is necessary and sufficient for inequality (3.5) and the equalities

$$a_1'(y,v)=l_1(u,v),\ \ y\in V,\ \ \forall v\in V, \tag{3.6}$$

and

$$a_1'(p,v)=l_2(y,v),\ \ p\in V,\ \ \forall v\in V, \tag{3.7}$$

to be met; the bilinear form $a_1'(\cdot,\cdot)$ and linear functional $l_1(\cdot,\cdot)$ are specified in point 2.1, and

$$l_2(y,v)=-\int_\Gamma (z_g-y)v\,d\Gamma\,.$$

If the constraints are absent, i.e. when $\mathcal{U}_\partial=\mathcal{U}$, then the equality

$$p+\bar{a}\,u=0 \tag{3.8}$$

follows from condition (3.5).

Therefore, when the constraints are absent, the control u can be excluded from equality (3.6) by means of equality (3.8), and the following may be written: $l_1(u,y)=l_1(u(p),y)$. If the solution $(y,p)^{\mathrm{T}}$ to problem (3.6), (3.7), where $l_1(u,y)=l_1(u(p),y)$, is smooth enough on $\bar{\Omega}_1$ and $\bar{\Omega}_2$, then such solution satisfies the relations

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial y}{\partial x_j}\right)+p/\bar{a}+\int_\Omega y\,dx=f+Q,\ \ x\in\Omega_1\cup\Omega_2,$$

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial p}{\partial x_j}\right)+\int_{\Omega} p\,dx=0,\quad x\in\Omega_1\cup\Omega_2,$$

$$\sum_{i,j=1}^{n} k_{ij}\frac{\partial y}{\partial x_j}\cos(\nu,x_i)=g,\quad x\in\Gamma,$$

$$\sum_{i,j=1}^{n} k_{ij}\frac{\partial p}{\partial x_j}\cos(\nu,x_i)=y-z_g,\quad x\in\Gamma,$$

$$[y]=0,\ [p]=0,\quad x\in\gamma,$$

$$\left[\sum_{i,j=1}^{n} k_{ij}\frac{\partial y}{\partial x_j}\cos(\nu,x_i)\right]=\omega,\ \left[\sum_{i,j=1}^{n} k_{ij}\frac{\partial p}{\partial x_j}\cos(\nu,x_i)\right]=0,\quad x\in\gamma. \quad (3.9)$$

Definition 3.1. A generalized (weak) solution to boundary-value problem (3.9) is called a vector-function $(y,p)^{\mathrm{T}}\in H$ that satisfies the following integral equation $\forall z\in H$:

$$\int_{\Omega}\left\{\sum_{i,j=1}^{n} k_{ij}\frac{\partial y}{\partial x_j}\frac{\partial z_1}{\partial x_i}+p\,z_1/\bar{a}+\sum_{i,j=1}^{n} k_{ij}\frac{\partial p}{\partial x_j}\frac{\partial z_2}{\partial x_i}\right\}dx+$$

$$+\int_{\Omega} y\,dx\int_{\Omega} z_1\,dx+\int_{\Omega} p\,dx\int_{\Omega} z_2\,dx=$$

$$=\int_{\Omega}(f+Q)\,z_1\,dx+\int_{\Gamma} g\,z_1\,d\Gamma+$$

$$+\int_{\Gamma}(y-z_g)\,z_2\,d\Gamma-\int_{\gamma}\omega\,z_1\,d\gamma. \quad (3.10)$$

Let $u=(u_1,u_2)^{\mathrm{T}}$ and $v=(v_1,v_2)^{\mathrm{T}}$ be arbitrary elements of the complete Hilbert space H. Specify the bilinear form

$$a(u,v)=\int\limits_{\Omega}\left\{\sum_{l=1}^{2}\sum_{i,j=1}^{n}k_{ij}\frac{\partial u_l}{\partial x_j}\frac{\partial v_l}{\partial x_i}+u_2\,v_1/\bar{a}\right\}dx+$$

$$+\sum_{l=1}^{2}\int\limits_{\Omega}u_l\,dx\int\limits_{\Omega}v_l\,dx-\int\limits_{\Gamma}u_1\,v_2\,d\Gamma$$

and linear functional

$$l(v)=\int\limits_{\Omega}(f+Q)v_1\,dx+\int\limits_{\Gamma}g\,v_1\,d\Gamma-\int\limits_{\gamma}\omega\,v_1\,d\gamma-\int\limits_{\Gamma}z_g v_2\,d\Gamma$$

on H.

Let constraint like (2.11) be met. Then, the unique solution $(y,p)^{\mathrm{T}}$ to problem (3.10) exists in H. Problem (3.10) can be solved by means of the finite-element method. Estimates like (1.35) and (1.35′) take place, respectively, for the approximate solution $U_k^N\in H_k^N$ to problem like (3.10) and for the approximation $u_k^N(x)$ of the control u.

2.4 CONTROL UNDER CONJUGATION CONDITION WITH BOUNDARY OBSERVATION

Assume that equation (1.1), where the coefficients and right-hand side meet conditions (1.2′), is specified in the domains Ω_1 and Ω_2. Neumann condition (1.2), constraint (1.3) and the condition

$$\left[\sum_{i,j=1}^{n}k_{ij}\frac{\partial y}{\partial x_j}\cos(\nu,x_i)\right]=\omega+u,\ \ x\in\gamma, \tag{4.1}$$

where ω is a fixed function from $L_2(\gamma)$ and the control is $u \in \mathcal{U} = L_2(\gamma)$, are specified, in their turn, on the boundary Γ.

For every control $u \in \mathcal{U}$, determine a system state as a generalized solution to the boundary-value problem specified by equation (1.1) and by conditions (1.2), (1.3) and (4.1) (Problem 4). The equality

$$\int_\Omega f\,dx + \int_\Gamma g\,d\Gamma = \int_\gamma (\omega + u)\,d\gamma \tag{4.2}$$

is the necessary condition under which there exists a classical solution to the latter problem. Find this solution under constraint (1.6).

Bring a value of the cost functional

$$J(u) = \int_\Gamma \left(y(u) - z_g\right)^2 d\Gamma + (\mathcal{N}u, u)_{L_2(\gamma)} \tag{4.3}$$

in correspondence with every control $u \in \mathcal{U} = L_2(\gamma)$; in this case, z_g is a known element from the space $L_2(\Gamma)$, $\mathcal{N}u = \bar{a}\,u$, $0 < a_0 \le \bar{a}(x) \le$ $\le a_1 < \infty$, $\bar{a} \in L_2(\gamma)$, a_0, $a_1 = \text{const}$.

A unique state, namely, a function $y(u) \in V_Q$ corresponds to every control $u \in \mathcal{U}$, minimizes the functional

$$\Phi(v) = a_4(v,v) - 2\,l_4(v) \tag{4.4}$$

on V_Q, and it is the unique solution in V_Q to the weakly stated problem: Find a function $y \in V_Q$ that meets the equation

$$a_4(y,v) = l_4(v) \quad \forall v \in V_0, \tag{4.5}$$

where

$$a_4(y,v) = \int_\Omega \sum_{i,j=1}^{n} k_{ij} \frac{\partial y}{\partial x_j} \frac{\partial v}{\partial x_i}\,dx,$$

$$l_4(v) = \int_\Omega fv\,dx + \int_\Gamma gv\,d\Gamma - \int_\gamma (\omega + u)v\,d\gamma. \tag{4.6}$$

Lemma 4.1. Problems *(4.4) and (4.5) are equivalent* $\forall f \in L_2(\Omega)$, $\forall \omega \in L_2(\gamma)$, $\forall u \in \mathcal{U}$ *and have a unique solution* $y \in V_Q$.

Rewrite cost functional (4.3) in the form of expression (3.2), where

$$\pi(u,v) = \big(y(u)-y(0),\ y(v)-y(0)\big)_{L_2(\Gamma)} + \big(\bar{a}\,u, v\big)_{L_2(\gamma)}$$

and

$$L(v) = \big(z_g - y(0),\ y(v)-y(0)\big)_{L_2(\Gamma)}.$$

Take the embedding theorems, ellipticity condition and generalized Poincare inequality into account, and the inequality

$$\bar{\alpha}_2 \|\tilde{y}' - \tilde{y}''\|^2_{L_2(\Gamma)} \le \bar{\alpha}_1 \|\tilde{y}' - \tilde{y}''\|^2_V \le a_4(\tilde{y}' - \tilde{y}'', \tilde{y}' - \tilde{y}'') \le$$

$$\le c_1 \|u' - u''\|_{L_2(\gamma)} \|\tilde{y}' - \tilde{y}''\|_V,\ \bar{\alpha}_1, \bar{\alpha}_2, c_1 = \text{const} > 0,$$

i.e.

$$\|\tilde{y}' - \tilde{y}''\|_{L_2(\Gamma)} \le \frac{c_1}{\sqrt{\bar{\alpha}_1 \bar{\alpha}_2}}, \|u' - u''\|_{L_2(\gamma)},$$

is derived, where $\tilde{y}' = \tilde{y}(u')$ and $\tilde{y}'' = \tilde{y}(u'')$ are the generalized solutions to boundary-value Problem 4 under $f = 0$, $g = 0$ and $\omega = 0$ and under a function $u = u(x)$ that is equal, respectively, to u' and u''.

On the basis of the derived inequality and [58, Chapter 1, Theorem 1.1], the validity of the following statement is proved.

Theorem 4.1. *Let a system state y be determined as a solution to equivalent problems (4.4) and (4.5). Then, there exists a unique element* $u = u(x)$ *of a convex set* $\mathcal{U}_\partial$ *that is closed in* $\mathcal{U}$, *and relation like (1.17) takes place for* $u = u(x)$, *where the cost functional J(u) is specified by expression (4.3).*

Let the equation

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial y}{\partial x_j}\right) + \int_\Omega y\,dx = f(x) + Q \qquad (4.7)$$

be specified on Ω instead of equation (1.1). Neumann condition (1.2) and conjugation conditions (1.3) and (4.1) are specified, in their turn, respectively, on Γ and γ. I.e., boundary-value problem (4.7), (1.2), (1.3), (4.1) (Problem 4′) is obtained.

Consider optimization Problem 4′: Find a control $u \in \mathcal{U}_\partial \subset \mathcal{U} = L_2(\gamma)$, for which relation like (1.17) is satisfied and where the cost functional $J(u)$ is specified by expression (4.3). A state $y = y(u)$ is a generalized solution to boundary-value Problem 4′, where the energy functional is

$$\Phi(v) = a_4'(v,v) - 2\, l_4'(v),\ \forall v \in V, \tag{4.8}$$

and the weakly stated problem is to find a function $y \in V$ that meets the following equation $\forall z \in V$:

$$a_4'(y,\, z) = l_4'(z); \tag{4.8′}$$

in this case:

$$a_4'(y,v) = \int\limits_\Omega \sum_{i,j=1}^n k_{ij} \frac{\partial y}{\partial x_j} \frac{\partial v}{\partial x_i} dx + \int\limits_\Omega y\, dx \int\limits_\Omega v\, dx\,,$$

$$l_4'(v) = \int\limits_\Omega (f + Q) v\, dx + \int\limits_\Gamma g v\, d\Gamma - \int\limits_\gamma (\omega + u) v\, d\gamma\,. \tag{4.9}$$

If $u \in \mathcal{U}_\partial$ is the optimal control for optimization Problem 4′, then the following inequality is true $\forall v \in \mathcal{U}_\partial$:

$$\big(y(u) - z_g,\ y(v) - y(u)\big)_{L_2(\Gamma)} + \big(\bar{a}\, u, v - u\big)_{L_2(\gamma)} \ge 0. \tag{4.10}$$

As for the control $v \in \mathcal{U}$, the conjugate state $p(v) \in V^* = V$ is specified by relations like (3.4). The equality

$$0 = \big(A^* p(u),\ y(v) - y(u)\big) = a_4'\big(p,\ y(v) - y(u)\big) +$$
$$+ \big(z_g - y,\ y(v) - y(u)\big)_{L_2(\Gamma)} =$$

$$= -\left(p,\, v-u\right)_{L_2(\gamma)} + \left(z_g - y,\, y(v) - y(u)\right)_{L_2(\Gamma)},$$

i.e.

$$\left(y(u) - z_g, y(v) - y(u)\right)_{L_2(\Gamma)} = -\left(p,\, v-u\right)_{L_2(\gamma)}$$

is obtained. Take it into account, and the inequality

$$\left(-p + \bar{a}u, v-u\right)_{L_2(\gamma)} \geq 0, \quad \forall v \in \mathcal{U}_\partial, \tag{4.11}$$

is derived from inequality (4.10).

An element $u \in \mathcal{U}_\partial$ is an optimal control for optimization Problem 4′ if and only if inequality (4.11) and the equalities

$$a_4'(y,v) = l_1(u,v), \quad y \in V, \quad \forall v \in V, \tag{4.12}$$

and

$$a_4'(p,v) = l_2(y,v), \quad p \in V, \quad \forall v \in V, \tag{4.13}$$

are met; in this case, the bilinear form $a_4'(\cdot,\cdot)$ is specified by the first formula of expressions (4.9) and the functionals $l_1(\cdot,\cdot)$ and $l_2(\cdot,\cdot)$ are

$$l_1(u,v) = \int_\Omega (f+Q)v\,dx + \int_\Gamma gv\,d\Gamma - \int_\gamma (\omega + u)v\,d\gamma$$

and

$$l_2(y,v) = -\int_\Gamma (z_g - y)v\,d\Gamma .$$

If the constraints are absent, i.e. when $\mathcal{U}_\partial = \mathcal{U}$, then the equality

$$-p + \bar{a}\,u = 0, \quad x \in \gamma, \tag{4.14}$$

follows from condition (4.11).

Therefore, when the constraints are absent and if the solution $(y,p)^{\mathrm{T}}$ to problem (4.12)–(4.14) is smooth enough on $\bar{\Omega}_i$ $(i=1,2)$, then the boundary-value problem is obtained:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial y}{\partial x_j}\right) + \int_\Omega y\,dx = f + Q, \quad x \in \Omega_1 \cup \Omega_2,$$

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial p}{\partial x_j}\right)+\int_{\Omega}p\,dx=0,\quad x\in\Omega_1\cup\Omega_2,$$

$$\sum_{i,j=1}^{n}k_{ij}\frac{\partial y}{\partial x_j}\cos(\nu,x_i)=g,\quad x\in\Gamma,$$

$$\sum_{i,j=1}^{n}k_{ij}\frac{\partial p}{\partial x_j}\cos(\nu,x_i)=y-z_g,\quad x\in\Gamma,$$

$$[y]=0,\ \ [p]=0,\quad x\in\gamma,$$

$$\left[\sum_{i,j=1}^{n}k_{ij}\frac{\partial y}{\partial x_j}\cos(\nu,x_i)\right]=\omega+p/\bar{a},\quad x\in\gamma,$$

$$\left[\sum_{i,j=1}^{n}k_{ij}\frac{\partial p}{\partial x_j}\cos(\nu,x_i)\right]=0,\quad x\in\gamma. \tag{4.15}$$

Definition 4.1. A generalized (weak) solution to boundary-value problem (4.15) is called a vector-function $(y,p)^{\mathrm{T}}\in H$ that satisfies the following integral equation $\forall z\in H$:

$$\int_{\Omega}\left\{\sum_{i,j=1}^{n}k_{ij}\frac{\partial y}{\partial x_j}\frac{\partial z_1}{\partial x_i}+\sum_{i,j=1}^{n}k_{ij}\frac{\partial p}{\partial x_j}\frac{\partial z_2}{\partial x_i}\right\}dx+$$

$$+\int_{\Omega}y\,dx\int_{\Omega}z_1\,dx+\int_{\Omega}p\,dx\int_{\Omega}z_2\,dx=\int_{\Omega}(f+Q)z_1\,dx+$$

$$+\int_{\Gamma}g\,z_1\,d\Gamma+\int_{\Gamma}\left(y-z_g\right)z_2\,d\Gamma-\int_{\gamma}(\omega+p/\bar{a})z_1\,d\gamma. \tag{4.16}$$

Let $u=(u_1,u_2)^{\mathrm{T}}$ and $v=(v_1,v_2)^{\mathrm{T}}$ be arbitrary elements of the complete Hilbert space *H*. Specify the bilinear form

$$a(u,v) = \int\limits_{\Omega} \left\{ \sum_{l=1}^{2} \sum_{i,j=1}^{n} k_{ij} \frac{\partial u_l}{\partial x_j} \frac{\partial v_l}{\partial x_i} \right\} dx +$$

$$+ \sum_{l=1}^{2} \int\limits_{\Omega} u_l \, dx \int\limits_{\Omega} v_l \, dx - \int\limits_{\Gamma} u_1 \, v_2 \, d\Gamma + \int\limits_{\gamma} u_2 \, v_1 / \bar{a} \, d\gamma$$

and linear functional

$$l(v) = \int\limits_{\Omega} (f + Q) v_1 \, dx + \int\limits_{\Gamma} g \, v_1 \, d\Gamma - \int\limits_{\Gamma} z_g v_2 \, d\Gamma - \int\limits_{\gamma} \omega \, v_1 \, d\gamma$$

on H.

If the constraint

$$\min\left\{ \frac{\alpha_0}{2}, \min\left\{ \frac{\alpha_0}{2}, 1 \right\} \mu \right\} - \frac{c_0'^2}{2a_0} - \frac{c_0^2}{2} > 0, \qquad (4.16')$$

where μ is the constant in the Poincare inequality and c_0' and c_0 are the positive constants derived on the basis of the inequalities proved within the framework of the embedding theorems, is met, then the unique solution $(y, p)^{\mathrm{T}}$ to problem (4.16) exists in H. Estimate like (1.35) is true for its approximate solution $U_k^N \in H_k^N$ and the estimate

$$\left\| u - u_k^N \right\|_{L_2(\gamma)} \le c \, h^k \qquad (4.17)$$

takes place for the approximation $u_k^N(x)$ of the control u.

2.5 BOUNDARY CONTROL WITH OBSERVATION ON A THIN INCLUSION

Assume that equation (1.1), where the coefficients and right-hand side meet conditions (1.2′), is specified in the domains Ω_1 and Ω_2. The condition

$$\sum_{i,j=1}^{n} k_{ij} \frac{\partial y}{\partial x_j} \cos(\nu, x_i) = g + u \tag{5.1}$$

is specified, in its turn, on the boundary Γ and the conjugation conditions have the form of expressions (1.3) and (1.4) on γ, where g is a fixed function from $L_2(\Gamma)$ and the control is $u \in \mathcal{U} = L_2(\Gamma)$.

For every control $u \in \mathcal{U}$, determine a system state as a generalized solution to the boundary-value problem specified by equation (1.1) and by constraints (1.3), (1.4) and (5.1) (Problem 5). The equality

$$\int_{\Omega} f\, dx + \int_{\Gamma} g\, d\Gamma + \int_{\Gamma} u\, d\Gamma = \int_{\gamma} \omega\, d\gamma \tag{5.2}$$

is the necessary condition under which there exists a classical solution y to Problem 5: Find this solution under constraint (1.6).

Bring a value of the cost functional

$$J(u) = \int_{\gamma} \left(y(u) - z_g \right)^2 d\gamma + (\mathcal{N} u, u)_{L_2(\Gamma)} \tag{5.3}$$

in correspondence with every control $u \in \mathcal{U}$; in this case, z_g is a known element from the space $L_2(\gamma)$, $\mathcal{N} u = \bar{a}\, u$, $0 < a_0 \le \bar{a}(x) \le a_1 < \infty$, $\bar{a} \in L_2(\Gamma)$; a_0, $a_1 = \text{const}$.

A unique state, namely, a function $y(u) \in V_Q$ corresponds to every control $u \in \mathcal{U}$, delivers the minimum to functional (4.4) on V_Q, and it is the unique solution in V_Q to the weakly stated problem specified by equation like (4.5), where

$$l_4(v) = \int_{\Omega} f\, v\, dx + \int_{\Gamma} (g + u) v\, d\Gamma - \int_{\gamma} \omega\, v\, d\gamma\,. \tag{5.4}$$

Lemma 5.1. *Problems like (4.4) and (4.5), where the bilinear form* $a_4(\cdot,\cdot)$ *is specified by the first formula of expressions (4.6) and the linear* *functional* $l_4(\cdot)$ *is specified by formula (5.4), are equivalent* $\forall f \in L_2(\Omega)$, $\forall \omega \in L_2(\gamma)$, $\forall u \in \mathcal{U}$ *and have a unique solution* $y(u) \in V_Q$.

Rewrite cost functional (5.3) as

$$J(u) = \pi(u,u) - 2L(u) + \left\| z_g - y(0) \right\|^2_{L_2(\gamma)},$$

where

$$\pi(u,v) = \big(y(u) - y(0),\ y(v) - y(0)\big)_{L_2(\gamma)} + \big(\bar{a}\,u, v\big)_{L_2(\Gamma)}$$

and

$$L(v) = \big(z_g - y(0),\ y(v) - y(0)\big)_{L_2(\gamma)}.$$

Take the embedding theorems, ellipticity condition and generalized Poincare inequality into account, and the inequality

$$\bar{\alpha}_2 \left\| \tilde{y}' - \tilde{y}'' \right\|^2_{L_2(\gamma)} \le \bar{\alpha}_1 \left\| \tilde{y}' - \tilde{y}'' \right\|^2_V \le a_4\big(\tilde{y}' - \tilde{y}'', \tilde{y}' - \tilde{y}''\big) \le$$
$$\le c_0 \left\| u' - u'' \right\|_{L_2(\Gamma)} \left\| \tilde{y}' - \tilde{y}'' \right\|_V, \quad \bar{\alpha}_1, \bar{\alpha}_2, c_0 = \text{const} > 0,$$

i.e. $\left\| \tilde{y}' - \tilde{y}'' \right\|_{L_2(\gamma)} \le c_1 \left\| u' - u'' \right\|_{L_2(\Gamma)}$ is derived, where $\tilde{y}' = \tilde{y}(u')$ and $\tilde{y}'' = \tilde{y}(u'')$.

On the basis of the derived inequality and [58, Chapter 1, Theorem 1.1], the validity of the following statement is proved.

Theorem 5.1. *If a system state y is determined as a solution to equivalent problems (4.4) and (4.5) that correspond to boundary-value Problem 5, then there exists a unique element* $u = u(x)$ *of a convex set* $\mathcal{U}_\partial$ *that is closed in* $\mathcal{U}$, *and relation like (1.17) takes place for* $u = u(x)$, *where the cost functional J(u) is specified by expression (5.3).*

Let equation (4.7) be specified on Ω instead of equation (1.1). Condition (5.1) and constraints (1.3) and (1.4) are specified, in their turn, respectively, on Γ and γ. I.e., boundary-value problem (4.7), (1.3), (1.4), (5.1) (Problem 5$'$) is obtained.

Consider the following problem: Find a control $u \in \mathcal{U}_\partial \subset \mathcal{U} = L_2(\Gamma)$ for which relation like (1.17) is satisfied and where the cost functional $J(u)$ is specified by expression (5.3); a state $y = y(u)$ is a generalized solution to boundary-value Problem 5′, where the energy functional and weakly stated problem are given, respectively, by expression (4.8) and equality (4.8′). The form $a_4'(\cdot\,,\,\cdot)$ is specified by expression (4.9) and the linear functional is

$$l_4'(v) = \int_\Omega (f+Q)v\,dx + \int_\Gamma (g+u)v\,d\Gamma - \int_\gamma \omega v\,d\gamma .$$

If $u \in \mathcal{U}_\partial$ is the optimal control for optimization Problem 5′, then the following inequality is true $\forall v \in \mathcal{U}_\partial$:

$$\big(y(u) - z_g,\, y(v) - y(u)\big)_{L_2(\gamma)} + \left(\bar{a}\,u, v - u\right)_{L_2(\Gamma)} \geq 0. \tag{5.5}$$

As for the control $v \in \mathcal{U}$, the conjugate state $p(v) \in V^* = V$ is specified by the relations

$$A^* p(v) = 0,\;\; x \in \Omega_1 \cup \Omega_2,$$

$$\frac{\partial p}{\partial \nu_{A^*}} = 0,\;\; x \in \Gamma,$$

$$[p] = 0,\;\; \left[\frac{\partial p}{\partial \nu_{A^*}}\right] = z_g - y,\;\; x \in \gamma, \tag{5.6}$$

where the operators A^* and $\dfrac{\partial}{\partial \nu_{A^*}}$ are specified, in their turn, by expressions (1.23).

The equality

$$0 = \left(A^* p(u),\, y(v) - y(u)\right) = a_4'\left(p,\, y(v) - y(u)\right) +$$

$$+ \left(z_g - y(u),\, y(v) - y(u)\right)_{L_2(\gamma)} =$$

$$= \left(p,\, v - u\right)_{L_2(\Gamma)} + \left(z_g - y(u), y(v) - y(u)\right)_{L_2(\gamma)},$$

i.e. $\big(y(u)-z_g, y(v)-y(u)\big)_{L_2(\gamma)} = \big(p,\, v-u\big)_{L_2(\Gamma)}$ is obtained. Take it into account, and the inequality

$$\big(p+\bar{a}u, v-u\big)_{L_2(\Gamma)} \geq 0 \tag{5.7}$$

is derived from inequality (5.5).

To make the element $u \in \mathcal{U}_\partial$ an optimal control for optimization Problem 5′, it is necessary and sufficient to meet inequality (5.7) and equalities like (4.12) and (4.13), where the bilinear form $a_4'(\cdot,\cdot)$ is specified by expression (4.9) and, besides this, the linear functionals are

$$l_1(u,v) = \int_\Omega (f+Q)v\,dx + \int_\Gamma (g+u)\,v\,d\Gamma - \int_\gamma \omega v\,d\gamma$$

and

$$l_2(y,v) = -\int_\gamma \big(z_g - y\big)v\,d\gamma.$$

If the constraints are absent, i.e. when $\mathcal{U}_\partial = \mathcal{U}$, then the equality

$$p + \bar{a}\,u = 0, \;\; x \in \Gamma, \tag{5.8}$$

follows from condition (5.7).

Therefore, when the constraints are absent and if the solution $(y,p)^{\mathrm{T}}$ to problem (4.12), (4.13), (5.8) is smooth enough on $\bar{\Omega}_i$ $(i=1,2)$, then, take equality (5.8) into account, and the boundary-value problem is obtained:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial y}{\partial x_j}\right) + \int_\Omega y\,dx = f+Q, \;\; x \in \Omega_1 \cup \Omega_2,$$

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial p}{\partial x_j}\right) + \int_\Omega p\,dx = 0, \;\; x \in \Omega_1 \cup \Omega_2,$$

$$\sum_{i,j=1}^{n} k_{ij}\frac{\partial y}{\partial x_j}\cos(\nu, x_i) = g - p/\bar{a}, \;\; x \in \Gamma,$$

$$\sum_{i,j=1}^{n} k_{ij} \frac{\partial p}{\partial x_j} \cos(\nu, x_i) = 0, \quad x \in \Gamma, \tag{5.9}$$

$$[y] = 0, \quad [p] = 0, \quad x \in \gamma,$$

$$\left[\sum_{i,j=1}^{n} k_{ij} \frac{\partial y}{\partial x_j} \cos(\nu, x_i) \right] = \omega, \quad x \in \gamma,$$

$$\left[\sum_{i,j=1}^{n} k_{ij} \frac{\partial p}{\partial x_j} \cos(\nu, x_i) \right] = z_g - y, \quad x \in \gamma.$$

Definition 5.1. A generalized (weak) solution to boundary-value problem (5.9) is called a vector function $(y, p)^{\mathrm{T}} \in H$ that satisfies the following integral equation $\forall z \in H$:

$$\int_{\Omega} \left\{ \sum_{i,j=1}^{n} k_{ij} \frac{\partial y}{\partial x_j} \frac{\partial z_1}{\partial x_i} + \sum_{i,j=1}^{n} k_{ij} \frac{\partial p}{\partial x_j} \frac{\partial z_2}{\partial x_i} \right\} dx +$$
$$+ \int_{\Omega} y\, dx \int_{\Omega} z_1\, dx + \int_{\Omega} p\, dx \int_{\Omega} z_2\, dx = \int_{\Omega} (f + Q) z_1\, dx +$$
$$+ \int_{\Gamma} (g - p/\bar{a}) z_1\, d\Gamma - \int_{\gamma} \omega z_1\, d\gamma - \int_{\gamma} (z_g - y) z_2\, d\gamma. \tag{5.10}$$

Let $u = (u_1, u_2)^{\mathrm{T}}$ and $v = (v_1, v_2)^{\mathrm{T}}$ be arbitrary elements of the complete Hilbert space H. Specify the bilinear form

$$a(u, v) = \int_{\Omega} \left\{ \sum_{l=1}^{2} \sum_{i,j=1}^{n} k_{ij} \frac{\partial u_l}{\partial x_j} \frac{\partial v_l}{\partial x_i} \right\} dx +$$
$$+ \sum_{l=1}^{2} \int_{\Omega} u_l\, dx \int_{\Omega} v_l\, dx + \int_{\Gamma} u_2 v_1 / \bar{a}\, d\Gamma - \int_{\gamma} u_1 v_2\, d\gamma$$

and linear functional

$$l(v) = \int_{\Omega}(f+Q)v_1\,dx + \int_{\Gamma} g\,v_1\,d\Gamma - \int_{\gamma}\omega v_1\,d\gamma - \int_{\gamma} z_g v_2\,d\gamma$$

on H.

If constraint like (4.16′) is met, then the unique solution $(y,p)^{\mathrm{T}}$ to problem (5.10) exists in H. Estimate like (1.35) is true for its approximate solution $U_k^N \in H_k^N$ and the estimate

$$\left\|u - u_k^N\right\|_{L_2(\Gamma)} \le c\,h^k$$

takes place for the approximation $u_k^N(x)$ of the control u.

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