

## Chapter II.

### STEINHAUS TYPE THEOREMS

The generalizations of Steinhaus's theorem and its Baire category analogy, Piccard's theorem, play a key role in several later sections as well as certain applications. This chapter deals with these generalizations.

#### 3. GENERALIZATIONS OF A THEOREM OF STEINHAUS

A famous theorem of Steinhaus [189] from 1920 asserts that, for any measurable set  $A \subset \mathbb{R}$  with positive Lebesgue measure the set  $A - A$  contains an interval. More generally, if  $A, B \subset \mathbb{R}^n$  are measurable sets with positive Lebesgue measure,  $A + B$  has an interior point; see for example the paper of Kemperman [123].

The proof can be based on Weil's idea [206] that the convolution of the characteristic functions  $\chi_A$  and  $\chi_B$  (in case  $A$  and  $B$  has finite measure) is a continuous function, hence the function

$$t \mapsto \lambda(A \cap (t - B)) = \int \chi_B(t - y) \chi_A(y) d\lambda(y)$$

is continuous and as follows from Fubini's theorem, not everywhere zero. This means that  $A + B$  contains a nonvoid open set. This proof works directly when  $\lambda$  is a Haar measure on a locally compact group.

This theorem allows various generalizations and modifications. In the generalizations the following problem is treated: if we replace the addition by a binary operation  $F(x, y)$ , under what conditions on  $F$  can we prove that  $F(A, B)$  contains a nonvoid open set? The first step was done by Erdős and Oxtoby [52] proving in the case  $x, y \in \mathbb{R}$  that, if  $F$  is a continuously differentiable function with nonvanishing partial derivatives, then  $F(A, B)$  contains a nonvoid open set.

Further generalizations detail the case when  $x$  and  $y$  are from different topological measure spaces and  $F$  satisfies certain solvability conditions in  $x$  and  $y$ . See in this direction Kuczma [130] and Sander [178]. Sander has pointed out that one of the sets  $A, B$  may be nonmeasurable. These results apply to the case when  $x, y \in \mathbb{R}^n$  and  $F$  is a continuously differentiable function of which the partial derivatives are nonsingular.

In this § we will treat a generalization for function  $F$  with more than two variables. Of course, if  $F$  maps  $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$  into  $\mathbb{R}$  we obtain a problem already solved by the theorem of Erdős and Oxtoby. To obtain a really interesting new problem, we have to consider a function with values in  $\mathbb{R}^2$ . The condition about the nonvanishing partial derivatives will be substituted with the condition that the null space of the derivative (as linear mapping) is in general position. Theorem 3.11 and Remark 3.13 are such generalizations of Steinhaus' theorem. As a corollary we obtain the well-known special case  $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

Our proof depends on a very general version of the theorem about the continuity of the convolution formulated in Theorem 3.4. As corollary we obtain a result about the continuity of a mapping

$$(1) \quad t \mapsto \nu \left( \bigcap_{i=1}^n g_{i,t}^{-1}(A_i) \right)$$

where the functions  $g_{i,t}$  do not map sets with positive measure into zero sets and depend smoothly on the parameter  $t$ . The investigation of this mapping occurs in an implicit way in Járαι [80] and in an explicit way in Krausz [125].

As we explained in the introduction, the lower semicontinuity of mapping (1) and several other variants of Steinhaus' theorem have applications in the theory of functional equations. The results of this § are used in §§ 5, 6, 8, 10, and when discussing applications in § 23.

More detailed references about earlier results may be found in Kuczma [130], Kuczma and Kuczma [131], Sander [178], and Grosse-Erdmann [61].

Another kind of refinement of Steinhaus' theorem using one-sided lower densities in  $\mathbb{R}$  is given by Raikoff [168] and by Matkowski and Świątkowski [156]. The results of this § are published in the paper J  rai [99].

We start with the investigation of an important condition.

**3.1. Condition.** In this § and in §§ 8, 10, and 19 a measure theoretical condition will play an important r  le. In general form this may be formulated as follows.

*Let  $X$  and  $Y$  be sets with measures  $\mu$  and  $\nu$ , respectively, let  $T$  be a set,  $D \subset T \times Y$ , and  $g : D \rightarrow X$  a function. Our condition is the following:*

- (1) *For each  $\varepsilon > 0$  there exists a  $\delta > 0$  such that if  $B \subset D_t$ ,  $\nu(B) \geq \varepsilon$ ,  $t \in T$ , then  $\mu(g_t(B)) \geq \delta$ .*

Concerning this condition we summarize some simple properties. We treat this condition in 3.10 in the most important case when  $X$  and  $Y$  are open subsets of Euclidean spaces. Further results can be found in J  rai [80].

**3.2. Remarks.** *With the notation of the previous point,*

- (1) *if  $D_1 \subset D_2 \subset T \times Y$ ,  $g_2 : D_2 \rightarrow X$ ,  $g_1 = g_2|_{D_1}$ , and  $g_2$  satisfies condition 3.1.(1), then  $g_1$  too;*
- (2) *if  $D = D_1 \cup D_2$ ,  $g_1 = g|_{D_1}$ ,  $g_2 = g|_{D_2}$ , moreover  $g_1$  and  $g_2$  satisfies condition 3.1.(1), then  $g$  too;*
- (3) *if for all  $\varepsilon > 0$  there exists a decomposition  $D = D_1 \cup D_2$  such that  $g|_{D_2}$  satisfies condition 3.1.(1) and  $\mu(g_t(D_{1,t})) < \varepsilon$  for all  $t \in T$ , then  $g$  also satisfies condition 3.1.(1);*
- (4) *condition 3.1.(1) is equivalent with the following one: for all  $\varepsilon > 0$  there exists a  $\delta > 0$ , such that if  $A \subset X$ ,  $\mu(A) < \delta$ , then  $\nu(g_t^{-1}(A)) < \varepsilon$  for all  $t \in T$ ;*
- (5) *if  $\mu$  is a regular measure, then it is enough if (4) is satisfied for all measurable  $A$ ;*
- (6) *if  $X$  and  $Y$  are Hausdorff spaces,  $\mu$  and  $\nu$  are Radon measures, all  $g_t$  is continuous on the Borel set  $D_t$  and the condition 3.1.(1) is satisfied, then  $g_t^{-1}(A)$  is measurable for all  $\sigma$ -finite measurable set  $A \subset X$  whenever  $t \in T$ ;*
- (7) *under conditions of (6) if  $B \subset A$  and  $A$  is a  $\mu$  hull of  $B$ , then  $g_t^{-1}(A)$  is a  $\nu$  hull of  $g_t^{-1}(B)$  whenever  $t \in T$ ;*

**Proof.** (1)–(5) are trivial. To prove (6) let us represent  $A$  as the union of a Borel set  $B$  and a set  $C$  having measure zero. Then  $g_t^{-1}(A) = g_t^{-1}(B) \cup$

$g_t^{-1}(C)$  where  $g_t^{-1}(B)$  is a Borel set and  $g_t^{-1}(C)$  has measure zero. To prove (7) suppose indirectly that for some measurable set  $E \subset Y$  we have

$$\nu(E \cap g_t^{-1}(A)) > \nu(E \cap g_t^{-1}(B)).$$

Then by the measurability of  $g_t^{-1}(A)$  and since  $\nu$  is a Radon measure there exists a compact set  $C \subset E \cap g_t^{-1}(A \setminus B)$  such that  $\nu(C) > 0$ ; but by  $g_t(C) \subset A \setminus B$  we should have  $\mu(g_t(C)) = 0$ .

**3.3. Lemma.** *Let  $T$  be a topological space,  $X$  a uniform space,  $C$  a compact uniform space,  $D \subset T \times C$ ,  $t_0 \in T$ ,  $\{t_0\} \times C \subset D$ , and let  $g : D \rightarrow X$  be a continuous function. Then for any relation  $\alpha$  from the uniformity of  $X$  there exists a relation  $\beta$  from the uniformity of  $C$  and a neighborhood  $V$  of  $t_0$  such that if  $t, t' \in V$ , the points  $y, y'$  are  $\beta$ -near in  $C$  and  $(t, y), (t', y') \in D$ , then the points  $g(t, y)$  and  $g(t', y')$  are  $\alpha$ -near in  $X$ .*

**Proof.** Let  $\delta$  be a symmetric relation from the uniformity of  $X$  for which  $\delta \circ \delta \subset \alpha$ . For all  $y \in C$  there exists a neighborhood  $V_y$  of  $t_0$  and a symmetric relation  $\gamma_y$  such that if  $t \in V_y$  and the point  $y'$  is  $\gamma_y$ -near to  $y$ , then  $g(t, y')$  and  $g(t_0, y)$  are  $\delta$ -near in  $X$ . Let us choose for each  $\gamma_y$  a symmetric relation  $\beta_y$  for which  $\beta_y \circ \beta_y \subset \gamma_y$ . The open kernels of the sets  $\beta_y(y)$  give an open cover of  $C$ . Let us choose a finite subcover and let  $\beta = \bigcap_{i=1}^n \beta_{y_i}$ ,  $V = \bigcap_{i=1}^n V_{y_i}$ . If the points  $y$  and  $y'$  are  $\beta$ -near in  $C$ , then there exists a  $y_i$  such that  $y$  and  $y_i$ , moreover  $y'$  and  $y_i$  are  $\gamma_{y_i}$ -near, hence  $g(t, y)$  and  $g(t_0, y_i)$  moreover  $g(t', y')$  and  $g(t_0, y_i)$  are  $\delta$ -near. This means that  $g(t, y)$  and  $g(t', y')$  are  $\alpha$ -near in  $X$ .

**3.4. Theorem.** *Let  $T$  be a topological space,  $Y$  a Hausdorff space,  $X_i$  ( $i = 1, 2, \dots, n$ ) be completely regular spaces, and let  $Z, Z_i$  ( $i = 1, 2, \dots, n$ ) be Banach spaces. Let  $\nu$  and  $\mu_i$  be finite Radon measures over  $Y$  and  $X_i$ , respectively. Consider the functions  $f_i : X_i \rightarrow Z_i$ ,  $g_i : T \times Y \rightarrow X_i$ ,  $h : Z_1 \times \dots \times Z_n \rightarrow Z$ . Suppose that the following conditions hold:*

- (1)  $h$  maps bounded subsets into bounded subsets and is continuous;
- (2)  $f_i$  is  $\mu_i$  measurable and is in  $\mathcal{L}^\infty(\mu_i)$  ( $i = 1, 2, \dots, n$ );
- (3)  $g_i$  is continuous and for each  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $\mu_i(g_{i,t}(B)) \geq \delta$  whenever  $B \subset Y$ ,  $\nu(B) \geq \varepsilon$ ,  $t \in T$ , and  $1 \leq i \leq n$ .

Then the function

$$f(t) = \int_Y h(f_1(g_1(t, y)), \dots, f_n(g_n(t, y))) d\nu(y)$$

is continuous on  $T$ .

**Proof.** First we prove that the integral exists. We may replace each  $f_i$  by a bounded Borel function defined on all of  $X_i$ , which is almost equal to  $f_i$ , and is separable valued. This switch does not change the integral, because, by (3), the set of points  $y$  for which the value of  $f_i(g_i(t, y))$  are changed, has measure 0. Hence we may assume that the functions  $f_i$  are separable valued bounded Borel functions. By (1) the function

$$(4) \quad y \mapsto h(f_1(g_1(t, y)), \dots, f_n(g_n(t, y)))$$

is a separable valued Borel function whenever  $t \in T$  is fixed, and its image is a bounded subset of  $Z$ . Hence the integral exists whenever  $t \in T$ .

Now let  $\varepsilon > 0$  and  $t_0 \in T$ . Let us choose a real number  $M > 0$ , for which the image of (4) is contained in the closed ball with center 0 and radius  $M$ . By (3), there exists a  $\delta > 0$  such that  $B \subset Y$ ,  $\nu(B) \geq \varepsilon' = \varepsilon/(16Mn)$ ,  $t \in T$ , and  $1 \leq i \leq n$  implies  $\mu_i(g_{i,t}(B)) \geq \delta$ . Let us choose a compact set  $C \subset Y$  for which  $\nu(Y \setminus C) < \varepsilon/(8M)$ . There exists a compact set  $C_i$  in  $X_i$  for which  $\mu_i(X_i \setminus C_i) < \delta$  and  $f_i|_{C_i}$  is continuous. Let us choose uniformities on the spaces  $C, X_1, \dots, X_n$  compatible with their topology. By (1) there exists an  $\alpha > 0$  such that

$$|h(z_1, \dots, z_n) - h(z'_1, \dots, z'_n)| < \frac{\varepsilon}{2\nu(Y)}$$

whenever  $z_i, z'_i \in f_i(C_i)$  and  $|z_i - z'_i| < \alpha$ . Because of the uniform continuity of  $f_i|_{C_i}$  there exists a reflexive symmetric relation  $\beta_i$  in the uniformity of  $X_i$  such that

$$|f_i(x_i) - f_i(x'_i)| < \alpha$$

whenever  $x_i, x'_i \in X_i$  and  $x_i$  and  $x'_i$  are  $\beta_i$ -near, that is,  $(x_i, x'_i) \in \beta_i$ . By Lemma 3.3 there exists an open neighborhood  $V$  of  $t_0$  in  $T$  and a reflexive symmetric relation  $\gamma$  in the uniformity of  $C$  such that  $g_i(t_0, y)$  and  $g_i(t, y')$  are  $\beta_i$ -near in  $X_i$  whenever  $t \in V$  and  $y$  and  $y'$  are  $\gamma$ -near in  $C$ . Now let  $t$  be an element of  $V$  and let

$$K = \bigcap_{i=1}^n g_{i,t_0}^{-1}(C_i) \cap \bigcap_{i=1}^n g_{i,t}^{-1}(C_i) \cap C.$$

Then

$$Y \setminus K = Y \setminus C \cup \left( \bigcup_{i=1}^n g_{i,t_0}^{-1}(X_i \setminus C_i) \right) \cup \left( \bigcup_{i=1}^n g_{i,t}^{-1}(X_i \setminus C_i) \right),$$

and hence (using (3) and that  $\mu_i(X_i \setminus C_i) < \delta$ ),

$$(5) \quad \nu(Y \setminus K) < \frac{\varepsilon}{8M} + n \frac{\varepsilon}{16Mn} + n \frac{\varepsilon}{16Mn} = \frac{\varepsilon}{4M}.$$

Using this, we have with the notation

$$H(t, y) = h\left(f_1(g_1(t, y)), \dots, f_n(g_n(t, y))\right)$$

that

$$\begin{aligned} |f(t) - f(t_0)| &\leq \int_Y |H(t, y) - H(t_0, y)| d\nu(y) \\ &= \int_{Y \setminus K} |H(t, y) - H(t_0, y)| d\nu(y) + \int_K |H(t, y) - H(t_0, y)| d\nu(y). \end{aligned}$$

By (5) the first term on the right side is not greater than  $2M\varepsilon/(4M) = \varepsilon/2$ . By the choice of  $K$ ,  $\alpha$ ,  $\beta_1, \dots, \beta_n$ ,  $\gamma$ , and  $V$ , the second term on the right side is not greater than  $\nu(Y)\varepsilon/(2\nu(Y)) = \varepsilon/2$ .

**3.5. Corollary.** *Let  $T$ ,  $Y$ ,  $X_i$ ,  $\nu$ ,  $\mu_i$ , and  $g_i$  be the same as in the previous theorem. Suppose that condition (3) of the previous theorem is satisfied, and let  $A_i$  be a subset of  $X_i$ . Suppose that  $A_i$  is  $\mu_i$  measurable if  $2 \leq i \leq n$ . Then the function*

$$f(t) = \nu\left(\bigcap_{i=1}^n g_{i,t}^{-1}(A_i)\right) \quad \text{if } t \in T$$

*is continuous on  $T$ .*

**Proof.** Condition (3) of the previous theorem by 3.2.(7) implies that the set  $g_{1,t}^{-1}(B_1)$  is a  $\nu$  hull of  $g_{1,t}^{-1}(A_1)$  whenever  $B_1$  is a  $\mu_1$  hull of  $A_1$ . Hence

$$f(t) = \int_Y \chi_{B_1}(g_1(t, y)) \chi_{A_2}(g_2(t, y)) \cdots \chi_{A_n}(g_n(t, y)) d\nu(y),$$

where  $\chi_{A_i}$  is the characteristic function of  $A_i$  and  $\chi_{B_1}$  is the characteristic function of  $B_1$ .

**3.6. Theorem.** Let  $T$  be a topological space,  $X$  and  $Y$  be Hausdorff spaces with Radon measures  $\mu$  and  $\nu$ , respectively. Let  $D$  be an open subset of  $T \times Y$ ,  $g : D \rightarrow X$ ,  $t_0 \in T$ , and let  $K \subset X$  be a compact set. Suppose, that

- (1) the mappings  $g$  and  $(t, x) \mapsto g_t^{-1}(x)$  are continuous and  $g_t$  is a homeomorphism of  $D_t$  onto  $X$  if  $t \in T$ ;
- (2) for each  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $\mu(g_t(B)) > \delta$  whenever  $B \subset Y$ ,  $\nu(B) \geq \varepsilon$ , and  $t \in T$ .

Then

$$\nu(g_t^{-1}(K) \triangle g_{t_0}^{-1}(K)) \rightarrow 0 \quad \text{if } t \rightarrow t_0.$$

**Proof.** Let  $\varepsilon > 0$  and let us choose a  $\delta > 0$  for  $\varepsilon/2$ . Let  $C = g_{t_0}^{-1}(K)$ , and let  $W$  be an open subset of  $D_{t_0}$  containing  $C$  for which  $\nu(W \setminus C) < \varepsilon/2$ . Similarly, let  $U$  be an open subset of  $X$  containing  $K$  for which  $\mu(U \setminus K) < \delta$ . Let us choose an open neighborhood  $V$  of  $t_0$  such that if  $t \in V$ ,  $y \in C$ , then  $g_t(y)$  is defined and  $g_t(y) \in U$ ; moreover if  $t \in V$ ,  $x \in K$ , then  $g_t^{-1}(x) \in W$ . Then by  $g_t^{-1}(K) \setminus g_{t_0}^{-1}(K) \subset V \setminus C$  we have  $\nu(g_t^{-1}(K) \setminus g_{t_0}^{-1}(K)) < \varepsilon/2$ . Moreover  $\nu(g_{t_0}^{-1}(K) \setminus g_t^{-1}(K)) < \varepsilon/2$  because the mapping  $g_t$  maps the set  $g_{t_0}^{-1}(K) \setminus g_t^{-1}(K)$  into  $U \setminus K$ , for which  $\mu(U \setminus K) < \delta$ . Summarizing,  $\nu(g_t^{-1}(K) \triangle g_{t_0}^{-1}(K)) < \varepsilon$  if  $t \in V$ .

**3.7. Corollary.** Let  $G$  be a locally compact group and let  $\lambda$  be a left Haar measure on  $G$ . Let  $A_i$ ,  $i = 1, 2, \dots, n$  be subsets of  $G$  with finite measure. Suppose that  $A_i$  is  $\lambda$  measurable if  $2 \leq i \leq n$ . Then the mapping

$$(t_1, \dots, t_n) \mapsto \lambda(t_1 A_1 \cap \dots \cap t_n A_n)$$

of  $G^n$  into  $\mathbb{R}$  is continuous.

**Proof.** Since the replacement of  $A_1$  by a  $\lambda$  hull does not change this function, we may suppose that  $A_1$  is measurable too. Let  $\varepsilon > 0$  and  $T = G^n$ . Let us choose the compact sets  $K_i$  such that  $K_i \subset A_i$  and  $\lambda(A_i \setminus K_i) < \varepsilon$  are satisfied. Let  $Y = X_1 = X_2 = \dots = X_n = G$  and

$$g_i(t, y) = t_i^{-1}y \quad \text{if } (t_1, \dots, t_n) \in T \text{ and } y \in Y.$$

Since for any measurable subsets  $B, C$  with finite measure

$$|\lambda(B) - \lambda(C)| \leq \lambda(B \triangle C)$$

and

$$\left( \bigcap_{i=1}^n B_i \right) \triangle \left( \bigcap_{i=1}^n C_i \right) \subset \bigcup_{i=1}^n (B_i \triangle C_i),$$

by the previous theorem we obtain

$$\begin{aligned} & \left| \lambda(t_1 K_1 \cap t_2 K_2 \cap \cdots \cap t_n K_n) - \lambda(t_1^0 K_1 \cap t_2^0 K_2 \cap \cdots \cap t_n^0 K_n) \right| \\ & \leq \sum_{i=1}^n \lambda(t_i K_i \triangle t_i^0 K_i) \rightarrow 0 \quad \text{whenever } t_i \rightarrow t_i^0, \end{aligned}$$

that is

$$t \mapsto \lambda(t_1 K_1 \cap t_2 K_2 \cap \cdots \cap t_n K_n)$$

is continuous on  $T$ . But

$$\begin{aligned} 0 & \leq \lambda(t_1 A_1 \cap t_2 A_2 \cap \cdots \cap t_n A_n) - \lambda(t_1 K_1 \cap t_2 K_2 \cap \cdots \cap t_n K_n) \\ & \leq \sum_{i=1}^n \lambda(t_i A_i \setminus t_i K_i) \leq n\varepsilon. \end{aligned}$$

Hence

$$(t_1, \dots, t_n) \mapsto \lambda(t_1 A_1 \cap \cdots \cap t_n A_n)$$

is the uniform limit of continuous functions, and so itself is continuous.

In the following lemma which will be needed for the proof of our main result, we give sufficient conditions for the validity of condition (3) in Theorem 3.4.

**3.8. Lemma.** *Let  $Y$  be an open subset of  $\mathbb{R}^k$ , let  $T$  be a topological space,  $y_0 \in Y$ , and  $t_0 \in T$ . Let  $g : T \times Y \rightarrow \mathbb{R}^r$  be a continuous function and suppose that  $\frac{\partial g}{\partial y}$  is continuous and*

$$\text{rank} \left( \frac{\partial g}{\partial y}(t_0, y_0) \right) = r.$$

*Then there exist open neighborhoods  $Y^*$  and  $T^*$  of  $y_0$  and  $t_0$ , respectively, and there exists a constant  $0 < C < \infty$  such that  $Y^* \subset Y$ ,  $T^* \subset T$ , and*

$$(1) \quad \lambda^k(B) \leq \lambda^r(g_t(B)) C (\text{diam } B)^{k-r}$$

*whenever  $B \subset Y^*$  and  $t \in T^*$ . (Here  $\text{diam } B$  denotes the diameter of the set  $B$ .)*

**Proof.** Let  $q = k - r$  and let us divide the coordinates of  $y = (y_1, \dots, y_k)$  into two groups  $y' = (y'_1, \dots, y'_q)$  and  $y'' = (y''_1, \dots, y''_r)$  so that the condition

$$\det \left( \frac{\partial g}{\partial y''}(t_0, y_0) \right) = \det \left( \frac{\partial g}{\partial y''}(t_0, y'_0, y''_0) \right) \neq 0$$



is satisfied. Let us introduce the notation

$$L(t, y') = \frac{\partial g}{\partial y''}(t, y', y_0'').$$

Using the proof of the inverse function theorem (see Rudin [172], theorem 9.24) we obtain that, if  $Y''$  is an open ball with center  $y_0''$  in  $\mathbb{R}^r$ ,  $t \in T$ ,  $(y', y'') \in Y$ , and

$$(2) \quad \left\| \frac{\partial g}{\partial y''}(t, y', y'') - L(t, y') \right\| < \frac{1}{2\|L(t, y')^{-1}\|}$$

whenever  $y'' \in Y''$ , then  $g_{t, y'}$  is a homeomorphic mapping of  $Y''$  onto an open subset  $U(t, y')$  of  $\mathbb{R}^r$ . Now let

$$0 < \beta < \frac{1}{2\|L(t_0, y_0')^{-1}\|}$$

and

$$(3) \quad 0 < \gamma < \left| \det \frac{\partial g}{\partial y''}(t_0, y_0', y_0'') \right|.$$

Using the continuity of the expressions in (2) and (3) we can choose an open ball  $Y''$  with center  $y_0''$  and open sets  $Y'$  and  $T^*$  such that  $t_0 \in T^*$ ,  $y_0' \in Y'$ ,  $Y^* = Y' \times Y'' \subset Y$ , moreover  $t \in T^*$ ,  $y' \in Y'$ ,  $y'' \in Y''$  implies that

$$\begin{aligned} \left\| \frac{\partial g}{\partial y''}(t, y', y'') - L(t, y') \right\| &< \beta; \\ \beta &< \frac{1}{2\|L(t, y')^{-1}\|}; \\ \gamma &< \left| \det \frac{\partial g}{\partial y''}(t, y', y'') \right|. \end{aligned}$$

Let  $\alpha(q)$  denote the  $\lambda^q$  measure of the  $q$  dimensional unit ball ( $\alpha(0) = 1$ ). We are going to prove that

$$\lambda^k(B) \leq \lambda^r(g_t(B)) \frac{\alpha(q)}{\gamma} (\text{diam } B)^{k-r}$$

whenever  $B \subset Y^*$  and  $t \in T^*$ . Let  $R = \text{diam } B$ . Then there exists a closed ball  $V$  with radius  $R$  in  $\mathbb{R}^q$  such that  $B \subset (V \cap Y') \times Y''$ . Suppose to the contrary that there exists a  $t \in T^*$  for which

$$\lambda^k(B) > \lambda^r(g_t(B))CR^q,$$

where  $C = \alpha(q)/\gamma$ . Then we can choose an open set  $U$  for which  $g_t(B) \subset U$  and

$$\lambda^k(B) > \lambda^r(U)CR^q.$$

Let

$$B^* = g_t^{-1}(U) \cap ((V \cap Y') \times Y'').$$

Then  $B \subset B^*$ ,  $B^*$  is a Borel set and  $g_t(B^*) \subset U$ , that is

$$\lambda^r(g_t(B^*))CR^q < \lambda^k(B) \leq \lambda^k(B^*).$$

We are going to prove that this is impossible. Let

$$B_{y'}^* = \{y'' : (y', y'') \in B^*\} \quad \text{if } y' \in V \cap Y'.$$

Using the theorem concerning transformation of integrals we have that

$$\lambda^r(g_t(B^*)) \geq \lambda^r(g_{t,y'}(B_{y'}^*)) = \int_{B_{y'}^*} \left| \det \frac{\partial g}{\partial y''}(t, y', y'') \right| d\lambda^r(y'') \geq \gamma \lambda^r(B_{y'}^*)$$

whenever  $y' \in V \cap Y'$ . By Fubini's theorem

$$\begin{aligned} \lambda^k(B^*) &= \int_{V \cap Y'} \lambda^r(B_{y'}^*) d\lambda^q(y') \\ &\leq \frac{\lambda^r(g_t(B^*))}{\gamma} \lambda^q(V) = \lambda^r(g_t(B^*))CR^q \end{aligned}$$

which is a contradiction. Hence the proof is complete.

**3.9. Lemma.** *Under the conditions of the previous lemma, if a subset  $D$  of  $\mathbb{R}$  has density 1 in the point  $g(t_0, y_0)$ , then  $g_{t_0}^{-1}(D) \cap Y^*$  has density 1 in the point  $y_0$ .*

**Proof.** By the continuity of  $\frac{\partial g}{\partial y}(t_0, y)$  the function  $g_{t_0}$  satisfies the Lipschitz condition on a neighborhood of  $y_0$ . Hence there exist a  $\gamma > 0$  and an  $0 < M < \infty$  such that  $y \in Y^*$  and  $|g(t_0, y) - g(t_0, y_0)| \leq M|y - y_0|$  whenever

$|y - y_0| < \gamma$ . Let  $\alpha(k)$  and  $\alpha(r)$  denote the  $\lambda^k$  and  $\lambda^r$  measures of the  $k$  and  $r$  dimensional unit balls, respectively. Let  $\varepsilon > 0$ , and let

$$0 < \delta < \frac{\varepsilon \alpha(k)}{CM^r 2^{k-r} \alpha(r)},$$

where  $C$  is the constant from the previous lemma.

Let us choose a  $\beta > 0$  such that, whenever  $V$  is a closed ball with center  $g(t_0, y_0)$  and radius less than  $\beta$ , then  $\lambda^r(V \cap D) \geq (1 - \delta)\lambda^r(V)$ . We prove that if  $W$  is a closed ball in  $Y^*$  with center  $y_0$  and radius less than  $\gamma$  and  $\beta/M$ , then

$$\lambda^k(W \cap g_{t_0}^{-1}(D)) \geq (1 - \varepsilon)\lambda^k(W).$$

Suppose to the contrary that for such a  $W$  with radius  $R$ ,

$$\lambda^k(W \cap g_{t_0}^{-1}(D)) < (1 - \varepsilon)\lambda^k(W).$$

Then there exists a compact subset  $B \subset W \setminus g_{t_0}^{-1}(D)$  for which  $\lambda^k(B) > \varepsilon\lambda^k(W)$ . Hence by the previous lemma,

$$\varepsilon R^k \alpha(k) < \lambda^k(B) \leq C 2^{k-r} R^{k-r} \lambda^r(g_{t_0}(B)).$$

But  $g_{t_0}(B)$  is a compact subset of  $V \setminus D$ , where  $V$  is the closed ball in  $\mathbb{R}^r$  with center  $g(t_0, y_0)$  and radius  $MR < \beta$ . Since  $\lambda^r(V \setminus D) < \delta\lambda^r(V)$  we get

$$\varepsilon R^k \alpha(k) < C 2^{k-r} \delta \lambda^r(V) = C 2^{k-r} R^{k-r} M^r R^r \alpha(r) \delta,$$

which contradicts the choice of  $\delta$ .

**3.10. Lemma.** *Let  $Y$  be an open subset of  $\mathbb{R}^k$ ,  $T$  a topological space,  $D$  an open subset of  $T \times Y$ , and  $(t_0, y_0) \in D$ . Suppose that the function  $g : D \rightarrow \mathbb{R}^r$  is continuous and continuously differentiable with respect to  $y$ . If the rank of the matrix  $\frac{\partial g}{\partial y}(t_0, y_0)$  is  $r$ , then there exist such neighborhoods  $T^*$  and  $Y^*$  of  $t_0$  and  $y_0$ , respectively for which*

- (1) *for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\lambda^r(g_t(B)) \geq \delta$  whenever  $t \in T^*$ ,  $B \subset Y^*$   $\lambda^k(B) \geq \varepsilon$ ;*
- (2) *if  $A$  is a  $\lambda^r$  measurable subset of  $\mathbb{R}^r$ , then  $g_t^{-1}(A) \cap Y^*$  is a  $\lambda^k$  measurable subset of  $Y$  for each  $t \in T^*$ .*

**Proof.** Let us use Lemma 3.8. Shrinking the so obtained set  $Y^*$  we may suppose that it is bounded. Now from the statement of Lemma 3.8 and from 3.2.(4) we obtain (1) and from 3.2.(7) we obtain (2).

**3.11. Theorem.** *Let  $X$  be an  $r$ -dimensional Euclidean space, and let  $X_1, \dots, X_n$  be orthogonal subspaces of  $X$  with dimensions  $r_1, \dots, r_n$ . Suppose that  $r_i \geq 1$  ( $1 \leq i \leq n$ ) and  $\sum_{i=1}^n r_i = r$ . Let  $U$  be an open subset of  $X$  and  $F : U \rightarrow \mathbb{R}^m$  be a continuously differentiable function. For each  $x \in U$  let  $N_x$  denote the null space of  $F'(x)$ . Let  $A_i$  be a subset of  $X_i$  ( $i = 1, \dots, n$ ) and suppose that  $A_i$  is  $\lambda^{r_i}$  measurable for  $2 \leq i \leq n$ . Let  $a \in U$ ,  $\dim N_a = r - m$ . Let  $p_i$  denote the orthogonal projection of  $X$  onto  $X_i$ . Suppose that  $p_i(N_a) = X_i$  and  $A_i$  has density 1 in the point  $p_i(a)$  whenever  $1 \leq i \leq n$ . Then  $F(A_1 \times \dots \times A_n)$  is a neighborhood of  $F(a)$ .*

**Proof.** Let  $k = r - m$ . Since  $x \mapsto \text{rank } F'(x)$  is lower semicontinuous, and  $\text{rank}(F'(a)) = m$ , we may suppose that  $\text{rank}(F'(x)) = m$  whenever  $x \in U$ . Similarly, choosing a smaller  $U$  if necessary, we may suppose that  $p_i(N_x) = X_i$  whenever  $x \in U$  and  $1 \leq i \leq n$ ; to prove this, suppose to the contrary that there exists an  $i$  and for each natural number  $j$  there exists an  $x_j \in U$  and there exist orthonormal vectors  $e_1^{(j)}, \dots, e_{k-r_i+1}^{(j)}$  in  $N_{x_j}$  such that  $x_j \rightarrow a$  and

$$p_i(e_s^{(j)}) = 0 \quad \text{whenever } j = 1, 2, \dots \text{ and } 1 \leq s \leq k - r_i + 1.$$

Using the compactness of the unit sphere we can pass to a subsequence and suppose that

$$e_s^{(j)} \rightarrow e_s \quad \text{if } j \rightarrow \infty.$$

But this proves that the vectors  $e_s$  are orthonormal in  $N_a$  and  $p_i(e_s) = 0$  whenever  $1 \leq s \leq k - r_i + 1$ , which is a contradiction.

Now, choosing a smaller  $U$  if necessary and using the rank theorem (see Dieudonné [49], 10.3.1), we have that there exist mappings  $u$ ,  $p$ , and  $v$  and an open neighborhood  $V$  of  $b = F(a)$  in  $\mathbb{R}^m$  with the following properties:  $u$  maps  $U$  onto the open cube  $I^r$ , where  $I = ]-1, 1[$ ,  $u$  is invertible and moreover  $u$  and  $u^{-1}$  are continuously differentiable;  $v$  maps  $I^m$  onto  $V$ ,  $v$  is invertible and moreover  $v$  and  $v^{-1}$  are continuously differentiable;  $p$  is the projection

$$p : (x_1, \dots, x_r) \mapsto (x_1, \dots, x_m)$$

of  $I^r$  onto  $I^m$ ; and finally  $F = v \circ p \circ u$ . We may write  $I^r$  as  $I^r = T \times Y$  where  $T = I^m$  and  $Y = I^k$ . Let  $u(a) = (t_0, y_0) \in T \times Y$ . Now let us use some facts from differential geometry (see Dieudonné [49], mainly 16.8.8).

$U \cap F^{-1}(v(t))$  is a closed submanifold of  $U$  whenever  $t \in T$ . The tangent space of this submanifold in a point  $x \in U \cap F^{-1}(v(t))$  is equal to the subspace  $N_x$  of  $X$ . Clearly  $u^{-1}$  is a diffeomorphism of the closed submanifold  $\{t\} \times Y$

of  $T \times Y$  onto  $U \cap F^{-1}(v(t))$ . Let  $g_i = p_i \circ u^{-1}$  if  $1 \leq i \leq n$ . By the choice of  $U$ ,  $p_i$  is a submersion of  $U \cap F^{-1}(v(t))$  into  $X_i$ . Hence the mapping  $g_{i,t} : Y \rightarrow X_i$  is a submersion too, that is, its derivative has rank  $r_i$  whenever  $y \in Y$  and  $t \in T$ .

Now, by Lemma 3.8, there exist open sets  $T^*$  and  $Y^*$  and there exists a  $0 < K < \infty$  such that  $t_0 \in T^* \subset T$ ,  $y_0 \in Y^* \subset Y$ , and

$$\lambda^k(B) \leq K\lambda^{r_i}(g_{i,t}(B))$$

whenever  $B \subset Y^*$ ,  $t \in T^*$ . Let  $X_i^* = X_i$ ,  $A_i^* = A_i$ , and  $g_i^*$  the restriction of  $g_i$  onto  $T^* \times Y^*$ . Applying Corollary 3.5 to the sets and functions marked by stars we have that the function

$$f(t) = \lambda^k \left( \bigcap_{i=1}^n g_{i,t}^{*-1}(A_i) \right) \quad \text{if } t \in T^*$$

is continuous on  $T^*$ . By Lemma 3.9,  $g_{i,t_0}^{*-1}(A_i)$  has density 1 in the point  $y_0$ . Since  $g_{i,t_0}^{*-1}(A_i) \cap Y^*$  is measurable by Lemma 3.10 if  $2 \leq i \leq n$ , we have that

$$\bigcap_{i=1}^n g_{i,t_0}^{*-1}(A_i)$$

has density 1 in the point  $y_0$ . Hence  $f(t_0) > 0$  and we have that there exists a neighborhood  $V$  of  $t_0$  for which  $f(t) > 0$  if  $t \in V$ . Clearly  $v(V)$  is a neighborhood of  $b$  in  $\mathbb{R}^m$ . If  $z \in v(V)$ , then  $t := v^{-1}(z) \in V$  and hence the set

$$\bigcap_{i=1}^n g_{i,t}^{*-1}(A_i)$$

is nonvoid. If  $y$  is an element of this set, then  $F(u^{-1}(t, y)) = v(p(t, y)) = v(t) = z$  and  $x_i = p_i(u^{-1}(t, y)) = g_{i,t}^*(y) \in A_i$  if  $1 \leq i \leq n$ . This means that  $F(x_1, \dots, x_n) = z$ .

**3.12. Corollary.** *Let  $U$  be an open subset of  $\mathbb{R}^r \times \mathbb{R}^r$  and  $F : (x, y) \mapsto F(x, y)$  a continuously differentiable mapping of  $U$  into  $\mathbb{R}^r$ . Let  $A, B \subset \mathbb{R}^r$  and suppose that  $B$  is  $\lambda^r$  measurable. If  $(a, b) \in U$ ,*

$$\det \frac{\partial F}{\partial x}(a, b) \neq 0, \quad \det \frac{\partial F}{\partial y}(a, b) \neq 0,$$

*$A$  has density 1 in the point  $a$  and  $B$  has density 1 in the point  $b$ , then  $F(A, B)$  is a neighborhood of  $F(a, b)$ .*

**Proof.** By Theorem 3.11 we have to prove only that  $p_1(N_{a,b}) = \mathbb{R}^r$  and  $p_2(N_{a,b}) = \mathbb{R}^r$  where  $N_{a,b}$  is the null space of  $F'(a, b)$ . Let  $(x, y) \in N_{a,b}$ . If  $p_1(x, y) = 0$ , then  $x = 0$ . Hence

$$0 = F'(a, b)(x, y) = \frac{\partial F}{\partial y}(a, b)(y).$$

But  $\det \frac{\partial F}{\partial y}(a, b) \neq 0$ , hence  $y = 0$ . This proves that  $p_1 : N_{a,b} \rightarrow \mathbb{R}^r$  is a one-to-one mapping, that is,  $p_1(N_{a,b}) = \mathbb{R}^r$ . Similarly  $p_2(N_{a,b}) = \mathbb{R}^r$ .

**3.13. Remark.** *The previous theorem may be stated in the following global form: If  $N_x$  is in general position, i.e.,  $\dim N_x = r - m$  and  $p_i(N_x) = X_i$  for all  $x \in U$  ( $i = 1, 2, \dots, n$ ), moreover  $A_1 \times A_2 \times \dots \times A_n \subset U$ ,  $\lambda^{r_i}(A_i) > 0$  ( $i = 1, 2, \dots, n$ ), and the set  $A_i$  is  $\lambda^{r_i}$  measurable for  $2 \leq i \leq n$ , then  $F(A_1 \times \dots \times A_n)$  contains a nonvoid open set.*

**Proof.** Let us choose a point  $a \in U$  for which  $A_i$  has density 1 at  $p_i(a)$  whenever  $1 \leq i \leq n$ , and let us apply the previous theorem.

## 4. GENERALIZATIONS OF A THEOREM OF PICCARD

Piccard's result [167] analogous to the theorem of Steinhaus states that the sum of two Baire sets having second Baire category has an inner point. Very strong generalizations exist; in this case also addition can be replaced by a two variable function with weak solvability conditions. These results are useful in the proof of "Baire property implies continuity" and "Baire property implies boundedness" type regularity theorems for functional equations. We refer the reader to the papers Sander [179], [181], [183], Kominek [124], J  rai [86], Grosse-Erdmann [61], Lindberg and Szymanski [139], and the references cited in them.

The purpose of this section is to give a generalization of Piccard's theorem analogous to the results of the previous section. These results were published in J  rai [103].

The following theorem is an abstract version of our generalization of the theorem of Piccard.

**4.1. Theorem.** *Let  $T$ ,  $Y$ , and  $X_i$  be topological spaces,  $g_i : T \times Y \rightarrow X_i$  continuous functions, and suppose that  $g_{i,t}(B)$  has second Baire category whenever  $B \subset Y$  is a subset of  $Y$  with second Baire category. Suppose that  $A_i \subset X_i$  and  $A_i$  is a Baire set whenever  $1 \leq i \leq n$ . Then the set  $V$  of points  $t \in T$  for which*

$$\bigcap_{i=1}^n g_{i,t}^{-1}(A_i)$$

*is of second category is an open subset of  $T$ .*

**Proof.** The sets  $A_i$  can be written in the form  $A_i = E_i \triangle M_i$ , where  $E_i$  is open and  $M_i$  is of first category. Suppose that  $t_0 \in V$ , and let  $K = \bigcap_{i=1}^n g_{i,t_0}^{-1}(A_i)$ ,  $K' = K \cap (\bigcap_{i=1}^n g_{i,t_0}^{-1}(E_i))$ . Since  $g_{i,t_0}^{-1}(M_i)$  is of first category in  $Y$  whenever  $i = 1, 2, \dots, n$ , we have that

$$K \setminus K' \subset \bigcup_{i=1}^n g_{i,t_0}^{-1}(M_i)$$

is of first category, hence  $K'$  is of second category in  $Y$ . Let  $y_0$  be a point of  $K'$  for which  $W \cap K'$  is of second category in  $Y$  for each open neighborhood  $W$  of  $y_0$ . Clearly  $g_i(t_0, y_0) \in E_i$  if  $i = 1, 2, \dots, n$ . Since the sets  $E_i$  are open and the functions  $g_i$  are continuous, the sets  $g_i^{-1}(E_i)$  are open and contain the point  $(t_0, y_0)$ , hence there exist open sets  $V'$  and  $W'$ , such that  $t_0 \in V'$ ,  $y_0 \in W'$ , and  $V' \times W' \subset \bigcap_{i=1}^n g_i^{-1}(E_i)$ . We will prove that

$$W' \cap \left( \bigcap_{i=1}^n g_{i,t}^{-1}(A_i) \right)$$

is of second category for each  $t \in V'$ . Were this not true, the sets

$$W' \setminus g_{i,t}^{-1}(A_i), \quad i = 1, 2, \dots, n$$

would cover — except for a set of first category — the set  $W'$ . If we prove that these sets are of first category, then we have a contradiction. But this follows from the inclusion

$$W' \setminus g_{i,t}^{-1}(A_i) \subset g_{i,t}^{-1}(M_i \cap E_i),$$

which is a consequence of  $W' \subset g_{i,t}^{-1}(E_i)$ .

**4.2. Remark.** *If we suppose that  $Y$  is a complete separable metric space and  $X_1$  is metrizable, then we may omit the condition that the set  $A_1$  has the Baire property.*

To prove this, let  $C_1$  denote the set of all points  $x_1 \in X_1$  such that for each neighborhood  $U_1$  of  $x_1$  the set  $U_1 \cap A_1$  is of second Baire category. It is known that  $C_1$  is a closed set and  $A_1 \setminus C_1$  is of first category. Let  $B_1$  denote the set of inner points of  $C_1$ . Then  $B_1$  is open and  $A_1 \setminus B_1$  is also of first category. As in the previous proof we obtain that  $W' \setminus g_{1,t}^{-1}(B_1)$  and  $W' \setminus g_{i,t}^{-1}(A_i)$ ,  $2 \leq i \leq n$  are of first category. It is enough to prove that  $W' \cap g_{1,t}^{-1}(A_1)$  is of second category, because then it follows that

$$W' \cap \left( \bigcap_{i=1}^n g_{i,t}^{-1}(A_i) \right)$$

cannot be of first category.

Suppose that  $W' \setminus g_{1,t}^{-1}(A_1)$  is of first category. Then, using that  $W' \cap g_{1,t}^{-1}(B_1)$  is an open set of second category, we obtain that

$$(W' \cap g_{1,t}^{-1}(B_1)) \setminus g_{1,t}^{-1}(A_1) = (W' \cap g_{1,t}^{-1}(B_1)) \setminus (W' \cap g_{1,t}^{-1}(A_1))$$

is a Baire set of second category. Let  $G$  be a  $\mathcal{G}_\delta$  subset of second category of the set above. Then  $g_{1,t}(G)$  is of second category as a subset of  $X_1$ . By Bourbaki [38], IX, §6, Exercise 10,  $g_{1,t}(G)$  is a Baire set in  $X_1$ . Clearly  $g_{1,t}(G) \subset B_1 \setminus A_1$ . Writing  $g_{1,t}(G) = U \triangle F$  where  $U$  is open and  $F$  is of first category, we see that  $U \cap B_1$  is a nonvoid open set for which the intersection with  $A_1$  is of first category. This contradicts the definition of  $B_1$ .

In Laczkovich [135], II.9.9 it is proved that the continuous image of a Polish space in a Hausdorff space is a Baire set. This shows that it is enough to suppose that  $X_1$  is Hausdorff.

The following lemma allows us to use derivatives to verify that the conditions on the functions  $g_i$  in the previous theorem are satisfied.

**4.3. Lemma.** *Let  $Y$  be an open subset of  $\mathbb{R}^k$ ,  $T$  a topological space,  $D$  an open subset of  $T \times Y$ , and  $(t_0, y_0) \in D$ . Suppose that the function  $g : D \rightarrow \mathbb{R}^r$  is continuous and has continuous partial derivative with respect to  $y$ . If the rank of  $\frac{\partial g}{\partial y}(t_0, y_0)$  is  $r$ , then there exist open neighborhoods  $T^*$  and  $Y^*$  of  $t_0$  and  $y_0$ , respectively, such that*



- (1) if  $B$  has second category in  $Y^*$ , then  $g_t(B)$  has second category in  $\mathbb{R}^r$  for each  $t \in T^*$ ;
- (2) if  $A$  is a Baire set in  $\mathbb{R}^r$ , then  $g_t^{-1}(A) \cap Y^*$  is a Baire set in  $Y$  for each  $t \in T^*$ .

**Proof.** We have proved in the proof of Lemma 3.8, that there exist open sets  $T^*$  and  $Y'$  and an open ball  $Y''$  centered at  $y_0''$  such that  $t_0 \in T^*$ ,  $y_0' \in Y'$ ,  $T^* \times Y' \times Y'' \subset D$ , and  $g_{t,y'}$  is a homeomorphism of  $Y''$  onto an open subset  $U(t, y')$  of  $\mathbb{R}^r$  whenever  $t \in T^*$  and  $y' \in Y'$ .

Suppose that there exists a subset  $B$  of  $Y^* = Y' \times Y''$  of second category, and a  $t \in T^*$ , such that  $g_t(B)$  is of first category in  $\mathbb{R}^r$ . Let us choose a Borel set  $U$  of first category in  $\mathbb{R}^r$ , for which  $g_t(B) \subset U \subset g_t(Y^*)$  and let  $B^* = g_t^{-1}(U) \cap Y^*$ . This set  $B^*$  is a Baire set and is of second category in  $Y^*$ , but  $g_t(B^*)$  is of first category in  $\mathbb{R}^r$ . By the Kuratowski-Ulam theorem the set of all points  $y' \in Y'$  for which  $B_{y'}^*$  is of second category is a set of second category. On the other hand, by the same theorem, the set of all points  $y' \in Y'$  for which  $B_{y'}^*$  is not a Baire set, is of first category. From this it follows that there exists  $y' \in Y'$  for which  $B_{y'}^*$  is a Baire set of second category in  $Y''$ . Since  $g_{t,y'}$  is a homeomorphism of  $Y''$  onto  $U(t, y')$ , the set  $g_{t,y'}(B_{y'}^*)$  is of second category in  $\mathbb{R}^r$ . This is a contradiction, because  $g_{t,y'}(B_{y'}^*) \subset g_t(B^*)$ . Hence (1) is proved.

To prove (2) suppose that  $A$  is a Baire set in  $\mathbb{R}^r$ , and let us choose a Borel set  $B$  for which  $A \subset B$  and  $B \setminus A$  is of first category. Then

$$g_t^{-1}(A) \cap Y^* = (g_t^{-1}(B) \cap Y^*) \setminus (g_t^{-1}(B \setminus A) \cap Y^*).$$

Using that  $g_t^{-1}(B)$  is a Borel set and  $g_t^{-1}(B \setminus A) \cap Y^*$  has first category by (1), we have that  $g_t^{-1}(A) \cap Y^*$  is a Baire set.

Now we are prepared to prove the local version of our generalization of the theorem of Piccard for a function from an open subset of  $\mathbb{R}^r$  into  $\mathbb{R}^m$ . The condition of the following theorem means, roughly speaking, that the null space of the derivative is large enough and is in general position.

**4.4. Theorem.** *Let  $X$  be the  $r$ -dimensional Euclidean space, and let  $X_1, \dots, X_n$  be orthogonal subspaces of  $X$  with dimensions  $r_1, \dots, r_n$ , respectively. Suppose that  $r_i \geq 1$  whenever  $1 \leq i \leq n$  and  $\sum_{i=1}^n r_i = r$ . Let  $U$  be an open subset of  $X$  and  $F : U \rightarrow \mathbb{R}^m$  a continuously differentiable function. For each  $x \in U$  let  $N_x$  denote the null space of  $F'(x)$ . Let  $A_i$  be a Baire subset of  $X_i$  ( $i = 1, 2, \dots, n$ ), and suppose that  $a \in U$  and*

$\dim N_a = r - m$ . Let  $p_i$  denote the orthogonal projection of  $X$  onto  $X_i$ . Suppose, that  $p_i(N_a) = X_i$  and  $p_i(a)$  has a neighborhood  $U_i$  such that  $U_i \setminus A_i$  is of first category if  $1 \leq i \leq n$ . Then  $F(A_1 \times \cdots \times A_n)$  is a neighborhood of  $F(a)$ .

**Proof.** Let us define  $t_0, y_0, T, Y$ , and  $g_i$  as in the proof of 3.11. By the above lemma there exist sets  $T^*$  and  $Y^*$  such that  $t_0 \in T^* \subset T, y_0 \in Y^* \subset Y$ , and  $g_{i,t}(B)$  is of second category whenever  $B \subset Y^*$  has second category and  $t \in T^*$ . Let  $X_i^* = X_i, A_i^* = A_i$ , and  $g_i^*$  be the restriction of  $g_i$  to  $T^* \times Y^*$ . Applying the theorem above to the sets marked by a star we have that the set  $V^*$  of points  $t$  for which

$$\bigcap_{i=1}^n g_{i,t}^{-1}(A_i)$$

is of second category is open in  $T^*$ . Since  $g_{i,t_0}^*$  maps  $Y^*$  onto an open neighborhood of  $p_i(a)$  and  $g^*(t_0, y_0) = p_i(a)$ , there exists an open neighborhood  $W$  of  $y_0$  in  $Y^*$  such that  $W \setminus g_{i,t_0}^{*-1}(A_i)$  is of first category if  $1 \leq i \leq n$ . This proves that  $t_0 \in V^*$ . Clearly  $v(V^*)$  is an open neighborhood of  $b$  in  $\mathbb{R}^m$ . If  $z \in v(V^*)$ , then  $v^{-1}(z) \in V^*$ , and hence the set

$$\bigcap_{i=1}^n g_{i,t}^{*-1}(A_i)$$

is nonvoid. If  $y$  is an element of this set, then  $u^{-1}(t, y) \in F^{-1}(z)$  and  $x_i = p_i(u^{-1}(t, y)) \in A_i$  whenever  $1 \leq i \leq n$ . This implies  $F(x_1, \dots, x_n) = z$ , which is enough since  $v(V^*)$  is an open neighborhood of  $b = F(a)$ .

**4.5. Corollary.** Let  $W$  be an open subset of  $\mathbb{R}^r \times \mathbb{R}^r$ , and let  $F : (x, y) \mapsto F(x, y)$  be a continuously differentiable mapping of  $W$  into  $\mathbb{R}^r$ . Suppose, that  $A, B \subset \mathbb{R}^r$  and  $A, B$  are Baire sets. If  $(a, b) \in W$ ,

$$\det \frac{\partial F}{\partial x}(a, b) \neq 0, \quad \det \frac{\partial F}{\partial y}(a, b) \neq 0,$$

and there exist neighborhoods  $U$  and  $V$  of  $a$  and  $b$  respectively such that  $U \setminus A$  and  $V \setminus B$  is of first category, then  $F(A, B)$  contains a neighborhood of  $F(a, b)$ .

**Proof.** Similar to Corollary 3.12 of the previous section.

The above theorem can be formulated in the following global form:

**4.6. Theorem.** *Let  $X$  be the  $r$ -dimensional Euclidean space, and let  $X_1, \dots, X_n$  be orthogonal subspaces of  $X$  with dimensions  $r_1, \dots, r_n$ , respectively. Let  $p_i$  denote the orthogonal projection of  $X$  onto  $X_i$ . Suppose that  $r_i \geq 1$  whenever  $1 \leq i \leq n$  and  $\sum_{i=1}^n r_i = r$ . Let  $U$  be an open subset of  $X$  and  $F : U \rightarrow \mathbb{R}^m$  a continuously differentiable function. For each  $x \in U$  let  $N_x$  denote the null space of  $F'(x)$ . If  $\dim N_x = r - m$  and  $p_i(N_x) = X_i$  whenever  $x \in U$  and  $i = 1, 2, \dots, n$ , moreover  $A_1 \times \dots \times A_n \subset U$  and  $A_i$  is a Baire set having second category if  $1 \leq i \leq n$ , then  $F(A_1 \times \dots \times A_n)$  contains a nonvoid open set.*

**Proof.** A Baire set  $A_i$  can be written in the form  $U_i \triangle F_i$  where  $U_i$  is a nonvoid open set and  $F_i$  is of first category. For any  $a \in U_1 \times U_2 \times \dots \times U_n$  we may apply the previous theorem.

**4.7. Remark.** *Using the previous Remark, we may omit the condition that  $A_1$  is a Baire set.*

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