Chapter 2

A DIRECT METHOD FOR OPEN-LOOP DYNAMIC GAMES FOR AFFINE CONTROL SYSTEMS

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Abstract Recently in Carlson and Leitmann (2004) some improvements on Leitmann's direct method, first presented for problems in the calculus of variations in Leitmann (1967), for open-loop dynamic games in Dockner and Leitmann (2001) were given. In these papers each player has its own state which it controls with its own control inputs. That is, there is a state equation for each player. However, many applications involve the players competing for a single resource (e.g., two countries competing for a single species of fish). In this note we investigate the utility of the direct method for a class of games whose dynamics are described by a single equation for which the state dynamics are affine in the players strategies. An illustrative example is also presented

1. The direct method

In Carlson and Leitmann (2004) a direct method for finding open-loop Nash equilibria for a class of differential N-player games is presented. A particular case included in this study concerns the situation in which the j-th player's dynamics at any time $t \in [t_0, t_f]$ is a vector-valued function $t \to x_j(t) \in \mathbb{R}^{n_j}$ that is described by an ordinary control system of the form

$$\dot{x}_j(t) = f_j(t, \mathbf{x}(t)) \quad + \quad g_j(t, \mathbf{x}(t))u_j(t) \text{ a.e. } t_0 \leq t \leq t_f \tag{2.1}$$

$$x_j(t_0) = x_{jt_0} \quad \text{and} \quad x_j(t_f) = x_{jt_f} \tag{2.2}$$

with control constraints

$$u_j(t) \in U_j(t) \subset \mathbb{R}^{m_j} \quad \text{a.e.} \quad t \in [t_0, t_f], \tag{2.3}$$

and state constraints

$$\mathbf{x}(t) \in \mathbf{X}(t) \subset \mathbb{R}^{\mathbf{n}} \quad \text{for} \quad t \in [t_0, t_f], \tag{2.4}$$

in which for each $j = 1, 2, \ldots N$ the function $f_j(\cdot, \cdot) : [t_0, t_f] \times \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{n_j}$ is continuous, $g_j(\cdot, \cdot) : [t_0, t_f] \times \mathbb{R}^{m_j \times n_j}$ is a continuous $m_j \times n_j$ matrix-valued function having a left inverse, and $U_j(\cdot)$ is set-valued mapping, and $\mathbf{X}(t)$ is a given set in $\mathbb{R}^{\mathbf{n}}$ for $t \in [t_0, t_f]$. Here we use the notation $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_N} = \mathbb{R}^{\mathbf{n}}$, where $\mathbf{n} = n_1 + n_2 + \ldots + n_N$; similarly $\mathbf{u} = (u_1, u_2 \ldots, u_N) \in \mathbb{R}^{\mathbf{m}}$, $\mathbf{m} = m_1 + m_2 \ldots + m_N$. Additionally we assume that the sets, $M_j = \{(t, \mathbf{x}, u_j) \in [t_0, t_f] \times \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{m_j} : \quad u_j \in U_j(t)\}$ are closed and nonempty. The objective of each player is to minimize an objective function of the form,

$$J_j(\mathbf{x}(\cdot), u_j(\cdot)) = \int_{t_0}^{t_f} f_j^0(t, \mathbf{x}(t), u_j(t))\, dt, \tag{2.5}$$

where we assume that for each $j = 1, 2, \ldots, N$ the function $f_j^0(\cdot, \cdot, \cdot) : M_j \times \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{m_j}$ is continuous.

With the above model description we now define the feasible set of admissible trajectory-strategy pairs.

DEFINITION 2.1 *We say a pair of functions* $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\} : [t_0, t_f] \to \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{m}}$ *is an admissible trajectory-strategy pair iff* $t \to \mathbf{x}(t)$ *is absolutely continuous on* $[t_0, t_f]$, $t \to \mathbf{u}(t)$ *is Lebesgue measurable on* $[t_0, t_f]$, *for each* $j = 1, 2, \ldots, N$, *the relations (2.1)–(2.3) are satisfied, and for each* $j = 1, 2, \ldots, N$, *the functionals (2.5) are finite Lebesgue integrals.*

REMARK 2.1 For brevity we will refer to an admissible trajectory-strategy pair as an admissible pair. Also, for a given admissible pair, $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\}$, we will follow the traditional convention and refer to $\mathbf{x}(\cdot)$ as an admissible trajectory and $\mathbf{u}(\cdot)$ as an admissible strategy.

For a fixed $j = 1, 2, \ldots, N$, $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$, and $y_j \in \mathbb{R}^{n_j}$ we use the notation $[\mathbf{x}^j, y_j]$ to denote a new vector in $\mathbb{R}^{\mathbf{n}}$ in which $x_j \in \mathbb{R}^{n_j}$ is replaced by $y_j \in \mathbb{R}^{n_j}$. That is,

$$[\mathbf{x}^j, y_j] \doteq (x_1, x_2, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_N).$$

Analogously $[\mathbf{u}^j, v_j] \doteq (u_1, u_2, \ldots, u_{j-1}, v_j, u_{j+1}, \ldots, u_N)$ for all $\mathbf{u} \in \mathbb{R}^{\mathbf{m}}$, $v_j \in \mathbb{R}^{m_j}$, and $j = 1, 2, \ldots, N$. With this notation we now have the following two definitions.

DEFINITION 2.2 *Let* $j = 1, 2, \ldots, N$ *be fixed and let* $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\}$ *be an admissible pair. We say that the pair of functions* $\{y_j(\cdot), v_j(\cdot)\} : [t_0, t_f]$

$\to \mathbb{R}^{n_j} \times \mathbb{R}^{m_j}$ is an admissible trajectory-strategy pair for player j relative to $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\}$ iff the pair

$$\{[\mathbf{x}(\cdot)^j, y_j(\cdot)], [\mathbf{u}(\cdot)^j, v_j(\cdot)]\}$$

is an admissible pair.

DEFINITION 2.3 *An admissible pair $\{\mathbf{x}^*(\cdot), \mathbf{u}^*(\cdot)\}$ is a Nash equilibrium iff for each $j = 1, 2, \ldots, N$ and each pair $\{y_j(\cdot), v_j(\cdot)\}$ that is admissible for player j relative to $\{\mathbf{x}^*(\cdot), \mathbf{u}^*(\cdot)\}$, it is the case that*

$$\begin{aligned} J_j(\mathbf{x}^*(\cdot), u_j^*(\cdot)) &= \int_{t_0}^{t_f} f_j^0(t, \mathbf{x}^*(t), u_j^*(t))\, dt \\ &\leq \int_{t_0}^{t_f} f_j^0(t, [\mathbf{x}^*(t)^j, y_j(t)], v_j(t))\, dt \\ &= J_j([\mathbf{x}^*(\cdot)^j, y_j(\cdot)], v_j(\cdot)). \end{aligned}$$

Our goal in this paper is to provide a "direct method" which in some cases will enable us to determine a Nash equilibrium. We point out that relative to a fixed Nash equilibrium $\{\mathbf{x}^*(\cdot), \mathbf{u}^*(\cdot)\}$ each of the players in the above game solves an optimization problem taking the form of a standard problem of optimal control. Thus, under suitable additional assumptions, it is relatively easy to derive a set of necessary conditions (in the form of a Pontryagin-type maximum principle) that must be satisfied by all Nash equilibria. Unfortunately these conditions are only necessary and not sufficient. Further, it is well known that non-uniqueness is always a source of difficulty in dynamic games so that in general the necessary conditions are not uniquely solvable (as is often the case in optimal control theory, when sufficient convexity is imposed). Therefore it is important to be able to find usable sufficient conditions for Nash equilibria.

The associated variational game

We observe that, under our assumptions, the algebraic equations,

$$z_j = f_j(t, \mathbf{x}) + g_j(t, \mathbf{x}) u_j \quad j = 1, 2, \ldots N, \tag{2.6}$$

can be solved for u_j in terms of t, z_j, and $\mathbf{x}$ to obtain

$$u_j = g_j(t, \mathbf{x})^{-1} \left(z_j - f_j(t, \mathbf{x})\right), \quad j = 1, 2, \ldots N, \tag{2.7}$$

where $g_j(t, \mathbf{x})^{-1}$ denotes the inverse of the matrix $g_j(t, \mathbf{x})$. As a consequence we can define the extended real-valued functions $L_j(\cdot, \cdot, \cdot) : [t_0, t_f] \times \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{n_j} \to \mathbb{R} \cup +\infty$ as

$$L_j(t, \mathbf{x}, z_j) = f_j^0(t, \mathbf{x}, g_j(t, \mathbf{x})^{-1}(z_j - f_j(t, \mathbf{x}))) \tag{2.8}$$

if $g_j(t, \mathbf{x})^{-1}(z_j - f_j(t, \mathbf{x})) \in U_j(t)$ with $L_j(t, \mathbf{x}, z_j) = +\infty$ otherwise.

With these functions we can consider the N-player variational game in which the objective functional for the jth player is defined by,

$$I_j(\mathbf{x}(\cdot)) = \int_{t_0}^{t_f} L_j(t, \mathbf{x}(t), \dot{x}_j(t))\, dt. \tag{2.9}$$

With this notation we have the following additional definitions.

DEFINITION 2.4 *An absolutely continuous function* $\mathbf{x}(\cdot) : [t_0, t_f] \to \mathbb{R}^{\mathbf{n}}$ *is said to be admissible for the variational game iff it satisfies the boundary conditions given in equation (2.2) and such that the map* $t \to L_j(t, \mathbf{x}(t), \dot{x}_j(t))$ *is finitely Lebesgue integrable on* $[t_0, t_f]$ *for each* $j = 1, 2, \ldots, N$.

DEFINITION 2.5 *Let* $\mathbf{x}(\cdot) : [t_0, t_f] \to \mathbb{R}^{\mathbf{n}}$ *be admissible for the variational game and let* $j \in \{1, 2, \ldots, N\}$ *be fixed. We say that* $y_j(\cdot) : [t_0, t_f] \to \mathbb{R}^{n_j}$ *is admissible for player* j *relative to* $\mathbf{x}(\cdot)$ *iff* $[\mathbf{x}^j(\cdot), y_j(\cdot)]$ *is admissible for the variational game.*

DEFINITION 2.6 *We say that* $\mathbf{x}^*(\cdot) : [t_0, t_f] \to \mathbb{R}^{\mathbf{n}}$ *is a Nash equilibrium for the variational game iff for each* $j = 1, 2, \ldots, N$,

$$I_j(\mathbf{x}^*(\cdot)) \leq I_j([\mathbf{x}^{*j}(\cdot), y_j(\cdot)])$$

for all functions $y_j(\cdot) : [t_0, t_f] \to \mathbb{R}^{n_j}$ *that are admissible for player* j *relative to* $\mathbf{x}^*(\cdot)$.

Clearly the variational game and our original game are related. In particular we have the following theorem given in Carlson and Leitmann (2004).

THEOREM 2.1 *Let* $\mathbf{x}^*(\cdot)$ *be a Nash equilibrium for the variational game defined above. Then there exists a measurable function* $\mathbf{u}^*(\cdot) : [t_0, t_f] \to \mathbb{R}^{\mathbf{m}}$ *such that the pair* $\{\mathbf{x}^*(\cdot), \mathbf{u}^*(\cdot)\}$ *is an admissible trajectory-strategy pair for the original dynamic game. Moreover, it is a Nash equilibrium for the original game as well.*

Proof. See Carlson and Leitmann (2004), Theorem 7.1. □

REMARK 2.2 The above result holds in a much more general setting than indicated above. We chose the restricted setting since it is sufficient for our needs in the analysis of the model we will consider in the next section.

With the above result we now focus our attention on the variational game. In 1967, for the case of one player variational games (i.e., the calculus of variations), Leitmann (1967) presented a technique (the "direct method") for determining solutions of these games by comparing their solutions to that of an equivalent problem whose solution is more easily determined than that of the original game. This equivalence was obtained through a coordinate transformation. Since then this method has been used successfully to solve a variety of problems. Recently, Carlson (2002) presented an extension of this method that expands the utility of the approach as well as made some useful comparisons with a technique originally presented by Carathéodory in the early twentieth century (see Carathéodory (1982)). Also, Dockner and Leitmann (2001) extended the original direct method to include the case of open-loop dynamic games. Finally, the extension of Carlson to the method was also modified in Leitmann (2004) to the include the case of open-loop differential games in Carlson and Leitmann (2004).

We begin by stating the following lemma found in Carlson and Leitmann (2004).

LEMMA 2.1 *Let* $x_j = z_j(t, \tilde{x}_j)$ *be a transformation of class* C^1 *having a unique inverse* $\tilde{x}_j = \tilde{z}_j(t, x_j)$ *for all* $t \in [t_0, t_f]$ *such that there is a one-to-one correspondence* $\mathbf{x}(t) \Leftrightarrow \tilde{\mathbf{x}}(t)$*, for all admissible trajectories* $\mathbf{x}(\cdot)$ *satisfying the boundary conditions (2.2) and for all* $\tilde{\mathbf{x}}(\cdot)$ *satisfying*

$$\tilde{x}_j(t_0) = \tilde{z}_j(t_0, x_{0j}) \quad \textit{and} \quad \tilde{x}_j(t_f) = \tilde{z}_j(t_f, x_{t_f j})$$

for all $j = 1, 2, \ldots, N$*. Furthermore, for each* $j = 1, 2, \ldots, N$ *let* $\tilde{L}_j(\cdot, \cdot, \cdot) : [t_0, t_f] \times \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{n_j} \to \mathbb{R}$ *be a given integrand. For a given admissible* $\mathbf{x}^*(\cdot) : [t_0, t_f] \to \mathbb{R}^{\mathbf{n}}$ *suppose the transformations* $x_j = z_j(t, \tilde{x}_j)$ *are such that there exists a* C^1 *function* $H_j(\cdot, \cdot) : [t_0, t_f] \times \mathbb{R}^{n_j} \to \mathbb{R}$ *so that the functional identity*

$$\begin{aligned} L_j(t, [\mathbf{x}^{*j}(t), x_j(t)], \dot{x}_j(t)) \quad &- \quad \tilde{L}_j(t, [\mathbf{x}^{*j}(t), \tilde{x}_j(t)], \dot{\tilde{x}}_j(t)) \\ &= \quad \frac{d}{dt} H_j(t, \tilde{x}_j(t)) \end{aligned} \tag{2.10}$$

holds on $[t_0, t_f]$*. If* $\tilde{x}_j^*(\cdot)$ *yields an extremum of* $\tilde{I}_j([\mathbf{x}^{*j}(\cdot), \cdot])$ *with* $\tilde{x}_j^*(\cdot)$ *satisfying the transformed boundary conditions, then* $x_j^*(\cdot)$ *with* $x_j^*(t) = z_j(t, \tilde{x}^*(t))$ *yields an extremum for* $I_j([\mathbf{x}^{*j}(\cdot), \cdot])$ *with the boundary conditions (2.2).*

Moreover, the function $\mathbf{x}^*(\cdot)$ *is an open-loop Nash equilibrium for the variational game.*

Proof. See Carlson and Leitmann (2004), Lemma 5.1. □

This lemma has three useful corollaries which we state below.

COROLLARY 2.1 *The existence of* $H_j(\cdot,\cdot)$ *in (2.9) implies that the following identities hold for* $(t,\tilde{x}_j) \in (t_0,t_f) \times \mathbb{R}^{n_j}$ *and for* $j = 1,2,\ldots,N$*:*

$$\begin{aligned} L_j(t,[\mathbf{x}^{*j}(t), z_j(t,\tilde{x}_j)], \frac{\partial z_j(t,\tilde{x}_j))}{\partial t} \quad &+ \quad \langle \nabla_{\tilde{x}_j} z_j(t,\tilde{x}_j), \tilde{p}_j \rangle) \\ -\tilde{L}_j(t,[\mathbf{x}^{*j}(t),\tilde{x}_j],\tilde{p}_j) \quad &\equiv \quad \frac{\partial H_j(t,\tilde{x}_j)}{\partial t} + \langle \nabla_{\tilde{x}_j} H_j(t,\tilde{x}_j), \tilde{p}_j \rangle, \end{aligned} \tag{2.11}$$

in which $\nabla_{\tilde{x}_j} H_j(\cdot,\cdot)$ *denotes the gradient of* $H_j(\cdot,\cdot)$ *with respect to the variables* $\tilde{x}_j$ *and* $\langle\cdot,\cdot\rangle$ *denotes the usual scalar or inner product in* $\mathbb{R}^{n_j}$*.*

COROLLARY 2.2 *For each* $j = 1,2,\ldots,N$ *the left-hand side of the identity, (2.11) is linear in* $\tilde{p}_j$*, that is, it is of the form,*

$$\theta_j(t,\tilde{x}_j) + \langle \psi_j(t,\tilde{x}_j), \tilde{p}_j \rangle$$

and,

$$\frac{\partial H_j(t,\tilde{x}_j)}{\partial t} = \theta_j(t,\tilde{x}_j) \quad \text{and} \quad \nabla_{\tilde{x}_j} H_j(t,\tilde{x}_j) = \psi(t,\tilde{x}_j)$$

on $[t_0,t_f] \times \mathbb{R}^{n_j}$*.*

COROLLARY 2.3 *For integrands* $L_j(\cdot,\cdot,\cdot)$ *of the form,*

$$\begin{aligned} L_j(t,[\mathbf{x}^{*j}(t),x_j(t)],\dot{x}_j(t)) \quad &= \quad \dot{x}_j'(t) a_j(t,[\mathbf{x}^{*j}(t),x_j(t)])\dot{x}_j(t) \\ &\quad +b_j(t,[\mathbf{x}^{*j}(t),x_j(t)])'\dot{x}_j(t) \\ &\quad +c_j(t,[\mathbf{x}^{*j}(t),x_j(t)]), \end{aligned}$$

and

$$\begin{aligned} \tilde{L}_j(t,[\mathbf{x}^{*j}(t),x_j(t)],\dot{x}_j(t)) \quad &= \quad \dot{x}_j'(t) \alpha_j(t,[\mathbf{x}^{*j}(t),x_j(t)])\dot{x}_j(t) \\ &\quad +\beta_j(t,[\mathbf{x}^{*j}(t),x_j(t)])'\dot{x}_j(t) \\ &\quad +\gamma_j(t,[\mathbf{x}^{*j}(t),x_j(t)]), \end{aligned}$$

with $a_j(t,[\mathbf{x}^{*j}(t),x_j(t)]) \neq 0$ *and* $\alpha_j(t,[\mathbf{x}^{*j}(t),x_j(t)]) \neq 0$*, the class of transformations that permit us to obtain (2.11) must satisfy,*

$$\left[\frac{\partial z_j(t,\tilde{x}_j)}{\partial \tilde{x}_j}\right]' a_j(t,[\mathbf{x}^*(t)^j, z_j(t,\tilde{x}_j)]) \left[\frac{\partial z_j(t,\tilde{x}_j)}{\partial \tilde{x}_j}\right] = \alpha_j(t,[\mathbf{x}^*(t)^j,\tilde{x}_j])$$

for $(t,x_j) \in [t_0,t_1] \times \mathbb{R}^{n_j}$*.*

A class of dynamic games to which the above method has not been applied is that in which there is a single state equation which is controlled by all of the players. A simple example of such a problem is the competitive harvesting of a renewable resource (e.g., a single species fishery model). In the next section we show how the direct method described above can be applied to a class of these types of models.

2. The model

Consider an N-player game where a single state $x(t) \in \mathbb{R}^n$ satisfies an ordinary control system of the form

$$\dot{x}(t) = F(t, x(t)) + \sum_{i=1}^{N} G_i(t, x(t)) u_i(t) \quad \text{a.e.} \quad t_0 \leq t \leq t_f, \tag{2.12}$$

with initial and terminal conditions

$$x(t_0) = x_{t_0} \quad \text{and} \quad x(t_f) = x_{t_f}, \tag{2.13}$$

a fixed state constraint,

$$x(t) \in X(t) \subset \mathbb{R}^n \quad \text{for} \quad t_0 \leq t \leq t_f, \tag{2.14}$$

with $X(t)$ a convex set for each $t_0 \leq t \leq t_f$, and control constraints,

$$u_i(t) \in U_i(t) \subset \mathbb{R}^{m_i} \quad \text{a.e.} \quad t_0 \leq t \leq t_f \quad i = 1, 2, \ldots N. \tag{2.15}$$

In this system each player has a strategy, $u_i(\cdot)$, which influences the state variable $x(\cdot)$ over time.

DEFINITION 2.7 *A set of functions*

$$\{x(\cdot), \mathbf{u}(\cdot)\} \doteq \{x(\cdot), u_1(\cdot), u_2(\cdot), \ldots, u_N(\cdot)\}$$

defined for $t_0 \leq t \leq t_f$ is called an admissible trajectory-strategy pair iff $x(\cdot)$ is absolutely continuous on its domain, $\mathbf{u}(\cdot)$ is Lebesgue measurable on its domain, and the equations (2.12)– (2.15) are satisfied.

We assume that $F(\cdot,\cdot) : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $G_i(\cdot,\cdot,\cdot) : [t_0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^{m_j} \to \mathbb{R}^n$ is sufficiently smooth so that for each selection of strategies $\mathbf{u}(\cdot)$ (i.e., measurable functions) the initial value problem given by (2.12)–(2.13) has a unique solution $x_{\mathbf{u}}(\cdot)$. These conditions can be made more explicit for particular models and are not unduly restrictive. For brevity we do not to indicate these explicitly.

Each of the players in the dynamic game wishes to minimize a performance criterion given of the form,

$$J_j(x(\cdot), u_j(\cdot)) = \int_{t_0}^{t_f} f_j(t, x(t), u_j(t))\, dt, \quad j = 1, 2, \ldots, N, \tag{2.16}$$

in which we assume that $f_j(\cdot,\cdot,\cdot) : [t_0, t_f] \times \mathbb{R}^n \times \mathbb{R}^{m_j} \to \mathbb{R}$ is continuous.

To place the above dynamic game into a form amenable to the direct method consider a set of strictly positive weights, say $\alpha_i > 0$,

$i = 1, 2, \ldots N$, which satisfy $\sum_{i=1}^{N} \alpha_i = 1$ and consider the related ordinary control system

$$\dot{x}_i(t) = F\left(t, \sum_{i=1}^{N} \alpha_i x_i(t)\right) + \frac{1}{\alpha_i} G\left(t, \sum_{i=1}^{N} \alpha_i x_i(t)\right) u_i(t) \quad \text{a.e.} \quad t \geq t_0, \tag{2.17}$$

$i = 1, 2, \ldots, N$, with boundary conditions,

$$x_i(t_0) = x_{t_0} \quad \text{and} \quad x_i(t_f) = x_{t_f}, \quad i = 1, 2, \ldots N, \tag{2.18}$$

and control constraints and state constraints,

$$u_i(t) \in U_i(t) \subset \mathbb{R}^{m_i} \quad \text{a.e.} \quad t_0 \leq t \leq t_f \quad i = 1, 2, \ldots N, \tag{2.19}$$

$$x_i(t) \in X_i(t) \doteq X(t) \subset \mathbb{R}^n \quad \text{for} \quad t_0 \leq t \leq t_f \quad i = 1, 2, \ldots N. \tag{2.20}$$

DEFINITION 2.8 *A set of functions*

$$\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\} \doteq \{x_1(\cdot), x_2(\cdot), \ldots x_N(\cdot), u_1(\cdot), u_2(\cdot), \ldots, u_N(\cdot)\}$$

defined for $t_0 \leq t \leq t_f$ is called an admissible trajectory-strategy pair for the related system iff $\mathbf{x}(\cdot) : [t_0, +\infty) \to \mathbb{R}^{\mathbf{n}}$, where $\mathbf{n} = nN$, is absolutely continuous on its domain, $\mathbf{u}(\cdot) : [t_0, +\infty) \to \mathbb{R}^{\mathbf{m}}$, where $\mathbf{m} = m_1 + m_2 + \ldots + m_N$, is Lebesgue measurable on its domain, and the equations (2.17)–(2.19) are satisfied.

For this related system it is easy to see that the conditions guaranteeing uniqueness for the original system would also insure the existence of the solution $\mathbf{x}(\cdot)$ for a fixed set of strategies $u_i(\cdot)$.

PROPOSITION 2.1 *Let $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\}$ be an admissible trajectory-strategy pair for the related control system. Then the pair, $\{x(\cdot), \mathbf{u}(\cdot)\}$, with $x(t) \doteq \sum_{i=1}^{N} \alpha_i x_i(t)$ is an admissible trajectory-strategy pair for the original control system. Conversely, if $\{x(\cdot), \mathbf{u}(\cdot)\}$ is an admissible trajectory-strategy pair for the original control system, then there exists a function $\mathbf{x}(\cdot) = (x_1(\cdot), \ldots, x_N(\cdot))$ so that $x(t) \doteq \sum_{i=1}^{N} \alpha_i x_i(t)$ for $i = 1, 2, \ldots N$ and $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\}$ is an admissible trajectory-strategy pair for the related control system.*

Proof. We begin by first letting $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\}$ be an admissible trajectory-strategy pair for the related control system. Then defining $x(t) = \sum_{i=1}^{N} \alpha_i x_i(t)$ for $t_0 \leq t \leq t_f$ we observe that

$$\dot{x}(t) \;=\; \sum_{i=1}^{N} \alpha_i \dot{x}_i(t)$$

$$\begin{aligned}
&= \sum_{i=1}^{N} \alpha_i \left(F(t,x(t)) + \frac{1}{\alpha_i} G_i(t,x(t))u_i(t) \right) \\
&= \sum_{i=1}^{N} \alpha_i F(t,x(t)) + \sum_{i=1}^{N} G_i(t,x(t))u_i(t) \\
&= F(t,x(t)) + \sum_{i=1}^{N} G_i(t,x(t))u_i(t),
\end{aligned}$$

since $\sum_{i=1}^{N} \alpha_i = 1$. Further we also have that

$$\begin{aligned}
x(t_0) &= \sum_{i=1}^{N} \alpha_i x_{t_0} = x_{t_0}, \\
u_j(t) &\in U_j(t) \quad \text{for almost all} \quad t_0 \le t \le t_f \quad \text{and} \quad j = 1,2,\dots N, \\
x_j(t) &\in X(t) \quad \text{for} \quad t_0 \le t \le t_f \quad \text{and} \quad j = 1,2,\dots N,
\end{aligned}$$

implying that $\{x(\cdot), \mathbf{u}(\cdot)\}$ is an admissible trajectory-strategy pair.

Now assume that $\{x(\cdot), \mathbf{u}(\cdot)\}$ is an admissible trajectory-strategy pair for the original dynamical system (2.12–2.15) and consider the system of differential equations given by (2.17) with the initial conditions (2.18). By our hypotheses this system has a unique solution $\mathbf{x}(\cdot) : [t_0, +\infty) \to \mathbb{R}^N$. Furthermore, from the above computation we know that the function, $y(\cdot) \doteq \sum_{i=1}^{N} \alpha_i x_i(\cdot)$, along with the strategies, $\mathbf{u}(\cdot)$ satisfy the differential equation (2.12) as well as the initial condition (2.13). However, this initial value problem has a unique solution, namely $x(\cdot)$, so that we must have $y(t) \equiv x(t)$ for all $t_0 \le t \le t_f$. Further, we also have the constraints, (2.19) and (2.20), holding as well. Hence we have, $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\}$ is an admissible trajectory-strategy pair for the related system as desired. □

In light of the above theorem it is clear that to use the direct method to solve the dynamic game described by (2.12)–(2.16) we consider the game described by the dynamic equations (2.17)–(2.19) where now the objective for player j, $j = 1,2,\dots N$, is given as

$$\mathcal{J}_j(\mathbf{x}(\cdot), u_j(\cdot)) = \int_{t_0}^{t_f} f_j^0 \left(t, \sum_{i=1}^{N} \alpha_i x(t), u_j(t) \right) dt. \tag{2.21}$$

In the next section we demonstrate this process with an example from mathematical economics.

Remark 2.3 In solving constrained optimization or dynamic games problems one of the biggest difficulties is finding reasonable candidates for the solution that meet the constraints. Perhaps the most often used method is to solve the unconstrained problem and hope that it satisfies the constraints. To understand why this technique works we observe that either in a game or in an optimization problem the set of admissible trajectory-strategy pairs that satisfy the constraints is a subset of the set of all admissible for pairs for the problem without constraints. Consequently, if you can find an admissible trajectory-strategy pair which is an optimal (or Nash equilibrium) solution for the problem without constraints (say via the direct method for the unconstrained problem) and if additionally it actually satisfies the constraints you indeed have a solution for the original problem with constraints. It is this technique that is used in the next section to obtain the Nash equilibrium.

3. Example

We consider two firms which produce an identical product. The production cost for each firm is given by the total cost function,

$$C(u_j) = \frac{1}{2}u_j^2, \quad j = 1, 2,$$

in which u_j refers to a jth firm's production level. Each firm supplies all that it produces to the market at all times. The amount supplied at each time effects the price, $P(t)$, and the total inventory of the market determines the price according to the ordinary control system,

$$\dot{P}(t) = s[a - u_1(t) - u_2(t) - P(t)] \quad \text{a.e.} \quad t \in [t_0, t_f]. \tag{2.22}$$

Here $s > 0$ refers to the speed at which the price adjusts to the price corresponding to the total quantity (i.e., $u_1(t) + u_2(t)$). The model assumes a linear demand rate given by $\Pi = a - X$ where X denotes total supply related to a price P. Thus the dynamics above says that the rate of change of price at time t is proportional to the difference between the actual price $P(t)$ and the idealized price $\Pi(t) = a - u_1(t) - u_2(t)$. We assume that (through negotiation perhaps) the firms have agreed to move from the price P_0 at time t_0 to a price P_f at time t_f. This leads to the boundary conditions,

$$P(t_0) = P_0 \quad \text{and} \quad P(t_f) = P_f. \tag{2.23}$$

Additionally we also impose the constraints

$$u_j(t) \geq 0 \quad \text{for almost all} \quad t \in [t_0, t_f]. \tag{2.24}$$

and

$$P(t) \geq 0 \quad \text{for} \quad t \in [t_0, t_f]. \tag{2.25}$$

The goal of each firm is to maximize its accumulated profit, assuming that it sells all that it produces, over the interval, $[t_0, t_f]$ given by the integral functional,

$$J_j(P(\cdot), u_j(\cdot)) = \int_{t_0}^{t_f} \left[P(t)u_j(t) - \frac{1}{2}u_j^2(t) \right] dt. \tag{2.26}$$

To put the above dynamic game into the framework to use the direct method let $\alpha, \beta > 0$ satisfy $\alpha + \beta = 1$ and consider the ordinary 2-dimensional control system,

$$\dot{x}(t) = -s(\alpha x(t) + \beta y(t) - a) - \frac{s}{\alpha}u_1(t), \text{ a.e. } t_0 \leq t \leq t_f \tag{2.27}$$

$$\dot{y}(t) = -s(\alpha x(t) + \beta y(t) - a) - \frac{s}{\beta}u_2(t), \text{ a.e. } t_0 \leq t \leq t_f \tag{2.28}$$

with the boundary conditions,

$$x(t_0) = y(t_0) = P_0 \tag{2.29}$$

$$x(t_f) = y(t_f) = P_f, \tag{2.30}$$

and of course the control constraints given by (2.24) and state constraints (2.25). The payoffs for each of the player now become,

$$J_j(x(\cdot), y(\cdot), u_j(\cdot)) = \int_{t_0}^{t_f} \left[(\alpha x(t) + \beta y(t))u_j(t) - \frac{1}{2}u_j(t)^2 \right] dt \tag{2.31}$$

for $j = 1, 2$. This gives a dynamic game for which the direct method can be applied.

We now put the above game in the equivalent variational form by solving the dynamic equations (2.27) and (2.28) for the individual strategies. That is we have,

$$u_1 = \alpha(a - (\alpha x + \beta y) - \frac{1}{s}p) \tag{2.32}$$

$$u_2 = \beta(a - (\alpha x + \beta y) - \frac{1}{s}q) \tag{2.33}$$

which gives (after a number of elementary steps of algebra) the new objectives (with negative sign to pose the variational problems as minimization problems) to get

$$\begin{aligned} \mathcal{J}_1(x(\cdot), y(\cdot), \dot{x}(\cdot)) = \int_{t_0}^{t_f} \Bigg\{ & \frac{\alpha^2}{2s^2}\dot{x}(t)^2 + \frac{\alpha^2 a^2}{2} \\ & + \left(\frac{\alpha^2}{2} + \alpha \right) (\alpha x(t) + \beta y(t))^2 \end{aligned}$$

$$
\begin{aligned}
&+ \left[\frac{\alpha}{s}(\alpha x(t)+\beta y(t)) - \frac{\alpha^2}{s}(a-(\alpha x(t)+\beta y(t)))\right]\dot{x}(t) \\
&- a(\alpha^2+\alpha)(\alpha x(t)+\beta y(t))\Big\}\, dt \qquad (2.34)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{J}_2(x(\cdot), y(\cdot), \dot{y}(t)) = & \int_{t_0}^{t_f} \Big\{ \frac{\beta^2}{2s^2}\dot{y}(t)^2 + \frac{\beta^2 a^2}{2} \\
&+ \left(\frac{\beta^2}{2}+\beta\right)(\alpha x(t)+\beta y(t))^2 \\
&+ \left[\frac{\beta}{s}(\alpha x(t)+\beta y(t)) - \frac{\beta^2}{s}(a-(\alpha x(t)+\beta y(t)))\right]\dot{y}(t) \\
&- a(\beta^2+\alpha)(\alpha x(t)+\beta y(t))\Big\}\, dt. \qquad (2.35)
\end{aligned}
$$

For the remainder of our discussion we focus on the first player as the computation of the second player is the same. We begin by observing that the integrand for player 1 is

$$
\begin{aligned}
L_1(x, y, p) = & \Big\{ \frac{\alpha^2}{2s^2}p^2 + \frac{\alpha^2 a^2}{2} + \left(\frac{\alpha^2}{2}+\alpha\right)(\alpha x+\beta y)^2 \\
&+ \left[\frac{\alpha}{s}(\alpha x+\beta y) - \frac{\alpha^2}{s}(a-(\alpha x+\beta y))\right] p \\
&- a(\alpha^2+\alpha)(\alpha x+\beta y)\Big\}. \qquad (2.36)
\end{aligned}
$$

Inspecting this integrand we choose $\tilde{L}(\cdot,\cdot,\cdot)$ to be,

$$
\tilde{L}(\tilde{x}, \tilde{y}, \tilde{p}) = \frac{\alpha^2}{2s^2}\tilde{p}^2 + \frac{\alpha^2 a^2}{2}
$$

from which we immediately deduce, applying Corollary 2.3, that the appropriate transformation, $z_1(\cdot,\cdot)$, must satisfy the partial differential equation,

$$
\left(\frac{\partial z_1}{\partial \tilde{x}}\right)^2 = 1
$$

giving us that $z_1(t, \tilde{x}) = f(t) \pm \tilde{x}$ and that

$$
\frac{\partial z_1}{\partial t} + \frac{\partial z_1}{\partial \tilde{x}}\tilde{p} = \dot{f}(t) \pm \tilde{p}.
$$

From this we now compute,

$$
\begin{aligned}
\Delta L_1 &= L_1(f(t) \pm \tilde{x}, y^*(t), \dot{f}(t) \pm \tilde{p}) - \tilde{L}_1(\tilde{x}, y^*(t), \tilde{p}) \\
&= \left\{ \frac{\alpha^2}{2s^2}(\dot{f}(t) \pm \tilde{p})^2 + \frac{\alpha^2 a^2}{2} + \left(\frac{\alpha^2}{2} + \alpha\right)(\alpha(f(t) \pm \tilde{x}) + \beta y^*(t))^2 \right. \\
&\quad + \left[\frac{\alpha}{s}(\alpha(f(t) \pm \tilde{x}) + \beta y^*(t)) \right. \\
&\quad \left. - \frac{\alpha^2}{s}(a - (\alpha(f(t) \pm \tilde{x}) + \beta y^*(t))) \right] (\dot{f}(t) \pm \tilde{p}) - a(\alpha^2 + \alpha) \times \\
&\quad \left. (\alpha(f(t) \pm \tilde{x}) + \beta y^*(t)) \right\} \\
&\quad - \left\{ \frac{\alpha^2}{2s^2}\tilde{p}^2 + \frac{\alpha^2 a^2}{2} \right\} \\
&= \left\{ \frac{\alpha^2}{2s^2}\dot{f}(t)^2 + \left(\frac{\alpha^2}{2} + \alpha\right) [\alpha(f(t) \pm \tilde{x}) + \beta y^*(t)]^2 - \left(\alpha^2 + \alpha\right) \times \right. \\
&\quad [\alpha(f(t) \pm \tilde{x}) + \beta y^*(t)] \\
&\quad \left. + \left[\left(\frac{\alpha^2}{s} + \frac{\alpha}{s}\right) [\alpha(f(t) \pm \tilde{x}) \quad + \beta y^*(t)] - \frac{\alpha^2 a}{s} \right] \dot{f}(t) \right\} \\
&\quad \pm \left\{ \frac{\alpha^2}{s^2}\dot{f}(t) + \left(\frac{\alpha^2}{s} + \frac{\alpha}{s}\right) \times \ [\alpha(f(t) \pm \tilde{x}) + \beta y^*(t)] - \frac{\alpha^2 a}{s} \right\} \tilde{p} \\
\doteq \quad & \frac{\partial H_1(t, \tilde{x})}{\partial t} + \frac{\partial H_1(t, \tilde{x})}{\partial \tilde{x}}\tilde{p}.
\end{aligned}
$$

From this we compute the mixed partial derivatives to obtain,

$$
\begin{aligned}
\frac{\partial^2 H_1}{\partial \tilde{x} \partial t}(t, \tilde{x}) &= \pm 2 \left(\frac{\alpha^2}{2} + \alpha\right) [\alpha(f(t) \pm \tilde{x}) + \beta y^*(t)] \, \alpha \\
&\quad \mp a\alpha(\alpha^2 + \alpha) \pm \alpha \left(\frac{\alpha^2}{s} + \frac{\alpha}{s}\right) \dot{f}(t) \\
&= \pm \left\{ \alpha^3(\alpha + 2)(f(t) \pm \tilde{x}) + \alpha^2 \beta(\alpha + 2) y^*(t) \right. \\
&\quad \left. - \alpha^2(\alpha + 1)a + \frac{\alpha^2}{s}(\alpha + 1)\dot{f}(t) \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 H_1}{\partial t \partial \tilde{x}}(t, \tilde{x}) &= \pm \left\{ \frac{\alpha^2}{s^2}\ddot{f}(t) + \left(\frac{\alpha^2}{s} + \frac{\alpha}{s}\right) \left[\alpha \dot{f}(t) + \beta \dot{y}^*(t)\right] \right\} \\
&= \pm \left\{ \frac{\alpha^2}{s^2}\ddot{f}(t) + \frac{\alpha^2}{s}(\alpha + 1)\dot{f}(t) + \frac{\alpha\beta}{s}(\alpha + 1)\dot{y}^*(t) \right\}.
\end{aligned}
$$

Assuming sufficient smoothness and equating the mixed partial derivatives we obtain the following equation:

$$\begin{aligned}\ddot{f}(t) - \alpha s^2(\alpha+2)f(t) &= \beta s^2(\alpha+2)y^*(t) - \frac{\beta s}{\alpha}(\alpha+1)\dot{y}^*(t) \\ &\quad \pm \alpha s^2(\alpha+2)\tilde{x} - as^2(\alpha+1).\end{aligned}$$

A similar analysis for player 2 yields:

$$\begin{aligned}L_2(x,y,q) &= \left\{ \frac{\beta^2}{2s^2}q^2 + \frac{\beta^2 a^2}{2} + \left(\frac{\beta^2}{2}+\beta\right)(\alpha x+\beta y)^2 \right. \\ &\qquad \left[\frac{\beta}{s}(\alpha x+\beta y) - \frac{\beta^2}{s}(a-(\alpha x+\beta y)\right] q \qquad (2.37) \\ &\quad \left. - a(\beta^2+\beta)(\alpha x+\beta y) \right\},\end{aligned}$$

and so choosing

$$\tilde{L}_2(\tilde{x},\tilde{y},\tilde{q}) = \left\{ \frac{\beta^2}{2s^2}q^2 + \frac{\beta^2 a^2}{2} \right\}$$

gives us that the transformation $z_2(\cdot,\cdot)$ is obtained by solving the partial differential equation

$$\left(\frac{\partial z_2}{\partial \tilde{y}}\right)^2 = 1,$$

which of course gives us, $z_2(t,\tilde{y}) = g(t) \pm \tilde{y}$. Proceeding as above we arrive at the following differential equation for $g(\cdot)$,

$$\begin{aligned}\ddot{g}(t) - \beta s^2(\beta+2)g(t) &= \alpha s^2(\beta+2)x^*(t) - \frac{\alpha s}{\beta}(1+\beta)\dot{x}^*(t) \\ &\quad \pm \beta s^2(\beta+2)\tilde{y} - as^2(\beta+1).\end{aligned}$$

Now the auxiliary variational problem we must solve consists of minimizing the two functionals,

$$\int_{t_0}^{t_f} \left(\frac{\alpha^2}{2s^2}\dot{\tilde{x}}^2(t) + \frac{\alpha a^2}{2}\right) dt \quad \text{and} \quad \int_{t_0}^{t_f} \left(\frac{\beta^2}{2s^2}\dot{\tilde{y}}^2(t) + \frac{\beta a^2}{2}\right) dt$$

over some appropriately chosen boundary conditions. We observe that these two minimization problems are easily solved if these conditions take the form,

$$\tilde{x}(t_0) = \tilde{x}(t_f) = c_1 \quad \text{and} \quad \tilde{y}(t_0) = \tilde{y}(t_f) = c_2$$

for arbitrary but fixed constants c_1 and c_2. The solutions are in fact,

$$\tilde{x}^*(t) \equiv c_1 \quad \text{and} \quad \tilde{y}^*(t) \equiv c_2$$

According to our theory we then have that the solution to our variational game is,

$$x^*(t) = f(t) \pm c_1 \quad \text{and} \quad y^*(t) = g(t) \pm c_2.$$

In particular, using this information in the equations for $f(\cdot)$ and $g(\cdot)$ with $\tilde{x} = c_1$ and with $\tilde{y} = c_2$ we obtain the following equations for $x^*(\cdot)$ and $y^*(\cdot)$,

$$\begin{aligned}
\ddot{x}^*(t) - \alpha s^2(\alpha+2)x^*(t) &= \beta s^2(\alpha+2)y^*(t) \\
&\quad -\frac{\beta s}{\alpha}(\alpha+1)\dot{y}^*(t) - as^2(\alpha+1) \\
\ddot{y}^*(t) - \beta s^2(\beta+2)y^*(t) &= \alpha s^2(\beta+2)x^*(t) \\
&\quad -\frac{\alpha s}{\beta}(1+\beta)\dot{x}^*(t) - as^2(\beta+1),
\end{aligned}$$

with the end conditions,

$$x^*(t_0) = y^*(t_0) = P_0 \quad \text{and} \quad x^*(t_f) = y^*(t_f) = P_f.$$

These equations coincide exactly with the Euler-Lagrange equations, as derived by the Maximum Principle for the open-loop variational game without constraints. Additionally we note that as these equations are derived here via the direct method we see that they become sufficient conditions for a Nash equilibrium of the unconstrained system, and hence for the constrained system for solutions which satisfy the constraints (see the comments in Remark 2.3). Moreover, we also observe that we can recover the functions $H_j(\cdot,\cdot)$, for $j = 1, 2$, since we can recover both $f(\cdot)$ and $g(\cdot)$ by the formulas

$$f(t) = x^*(t) \mp c_1 \quad \text{and} \quad g(t) = y^*(t) \mp c_2.$$

The required functions are now recovered by integrating the partial derivatives of $H_1(\cdot,\cdot)$ and $H_2(\cdot,\cdot)$ which can be computed. Consequently, we see that in this instance the solution to our variational game is given by the solutions of the above Euler-Lagrange system, provided the resulting strategies and the price satisfy the requisite constraints. Finally, we can obtain the solution to the original problem by taking,

$$P^*(t) = \alpha x^*(t) + \beta y^*(t),$$

$$u_1^*(t) = \alpha\left(a - P^*(t) - \frac{1}{s}\dot{x}^*(t)\right),$$

and

$$u_2^*(t) = \beta\left(a - P^*(t) - \frac{1}{s}\dot{y}^*(t)\right).$$

Of course, we still must check that these functions meet whatever constraints are required (i.e., $u_i(t) \geq 0$ and $P(t) \geq 0$).

There is one special case of the above analysis in which the solution can be obtained easily. This is the case when $\alpha = \beta = \frac{1}{2}$. In this case the above Euler-Lagrange system becomes,

$$\begin{aligned}
\ddot{x}^*(t) - \frac{5}{4}s^2 x^*(t) &= \frac{5}{4}s^2 y^*(t) - \frac{3}{2}s\dot{y}^*(t) - \frac{3}{2}as^2 \\
\ddot{y}^*(t) - \frac{5}{4}s^2 y^*(t) &= \frac{5}{4}s^2 x^*(t) - \frac{3}{2}s\dot{x}^*(t) - \frac{3}{2}as^2.
\end{aligned}$$

Using the fact that $P^*(t) = \frac{1}{2}(x^*(t) + y^*(t))$ for all $t \in [t_0, t_f]$ we can multiply each of these equations by $\frac{1}{2}$ and add them together to obtain the following equation for $P^*(\cdot)$,

$$\ddot{P}^*(t) + \frac{3}{2}s\dot{P}^*(t) - \frac{5}{2}s^2 P^*(t) = -\frac{3}{2}as^2,$$

for $t_0 \leq t \leq t_f$. This equation is an elementary non-homogeneous second order linear equation with constant coefficients whose general solution is given by

$$P^*(t) = Ae^{r_+(t-t_0)} + Be^{r_-(t-t_0)} + \frac{3}{5}a$$

in which $r_\pm$ are the characteristics roots of the equation and A and B are arbitrary constants. More specifically, the characteristic roots are roots of the polynomial

$$r^2 + \frac{3}{2}sr - \frac{5}{2}s^2 = 0$$

and are given by

$$r_+ = s \quad \text{and} \quad r_- = -\frac{5}{2}s.$$

Thus, to solve the dynamic game in this case we select A and B so that $P^*(\cdot)$ satisfies the fixed boundary conditions. Further we note that we can also take

$$x^*(t) = y^*(t) = \frac{1}{2}P^*(t)$$

and so obtain the optimal strategies as

$$u_1^*(t) = u_2^*(t) = \frac{1}{2}\left(a - P^*(t) - \frac{1}{s}\dot{P}^*(t)\right).$$

It remains to verify that there exists some choice of parameters for which the optimal price, $P^*(\cdot)$, and the optimal strategies, $u_1^*(\cdot), u_2^*(\cdot)$

remain nonnegative. To this end we observe that we impose the fixed boundary conditions to obtain the following linear system of equations for the unknowns, A and B:

$$\begin{pmatrix} 1 & 1 \\ e^{s(t_f-t_0)} & e^{-\frac{5}{2}s(t_f-t_0)} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} P_0 - \frac{3}{5}a \\ P_f - \frac{3}{5}a \end{pmatrix}.$$

Using Cramer's rule we obtain the following formulas for A and B,

$$\begin{aligned} A &= \frac{1}{D}\left[\left(P_0 - \frac{3}{5}a\right)e^{-\frac{5}{2}s(t_f-t_0)} - \left(P_f - \frac{3}{5}a\right)\right] \\ B &= \frac{1}{D}\left[\left(P_f - \frac{3}{5}a\right) - \left(P_0 - \frac{3}{5}a\right)e^{s(t_f-t_0)}\right] \end{aligned}$$

in which D is the determinant of the coefficient matrix and is given by

$$D = e^{-\frac{5}{2}s(t_f-t_0)} - e^{s(t_f-t_0)} = e^{s(t_f-t_0)}\left(e^{-\frac{7}{2}s(t_f-t_0)} - 1\right).$$

We observe that D is clearly negative since $t_f > t_0$. Also, to insure that $P^*(t)$ is nonnegative for $t \in [t_0, t_f]$ it is sufficient to insure that A and B are both positive. This means we must have,

$$\begin{aligned} 0 &> \left[\left(P_0 - \frac{3}{5}a\right)e^{-\frac{5}{2}s(t_f-t_0)} - \left(P_f - \frac{3}{5}a\right)\right] \\ 0 &> \left[\left(P_f - \frac{3}{5}a\right) - \left(P_0 - \frac{3}{5}a\right)e^{s(t_f-t_0)}\right] \end{aligned}$$

which can be equivalently expressed as,

$$\left(P_0 - \frac{3}{5}a\right)e^{-\frac{5}{2}s(t_f-t_0)} < P_f - \frac{3}{5}a < \left(P_0 - \frac{3}{5}a\right)e^{s(t_f-t_0)}. \tag{2.38}$$

Observe that as long as P_0 and P_f are chosen to be larger than $\frac{3}{5}a$ this last inequality can be satisfied if we choose $t_f - t_0$ sufficiently large. In this case we have explicitly given the optimal price, $P^*(\cdot)$ in terms of the model parameters P_0, P_f, t_0, t_f, a, and s (all strictly positive). It remains to check that the strategies are nonnegative. To this end we notice that,

$$\dot{P}^*(t) = Ase^{s(t-t_0)} - \frac{5}{2}Bse^{-\frac{5}{2}s(t-t_0)}$$

so that we have, the admissible strategies given by, for $j = 1, 2$,

$$u_j^*(t) = \frac{1}{2}\left[a - P^*(t) - \frac{1}{2s}\dot{P}^*(t)\right]$$

$$\begin{aligned}
&= \frac{1}{2}\left[a - \left(Ae^{s(t-t_0)} + Be^{-\frac{5}{2}s(t-t_0)}\right)\right. \\
&\quad - \left.\frac{1}{2s}\left(Ase^{s(t-t_0)} - \frac{5}{2}Bse^{-\frac{5}{2}s(t-t_0)}\right)\right] \\
&= \frac{1}{2}\left[a - \frac{3}{2}Ae^{s(t-t_0)} + \frac{1}{4}Be^{-\frac{5}{2}(t-t_0)}\right].
\end{aligned}$$

Taking the time derivative of $u_j^*(\cdot)$ we obtain

$$\dot{u}_j^*(t) = \frac{1}{2}\left[-\frac{3}{2}Ase^{s(t-t_0)} - \frac{5}{8}Bse^{-\frac{5}{2}(t-t_0)}\right] < 0,$$

since A and B are positive. This implies that $u_j^*(t) \geq u_j^*(t_f)$ for all $t \in [t_0, t_f]$. Thus to insure that $u_j^*(\cdot)$ is nonnegative it is sufficient to insure $u_j^*(t_f) \geq 0$ which holds if we have

$$a - \frac{3}{2}Ae^{s(t_f-t_0)} + \frac{1}{4}Be^{-\frac{5}{2}(t_f-t_0)} \geq 0.$$

To investigate this inequality we first observe that we have, from the solution $P^*(\cdot)$, that

$$P_f = Ae^{s(t_f-t_0)} + Be^{\frac{-5}{2}s(t_f-t_0)} + \frac{3}{5}a.$$

This allows us to rewrite the last inequality in the form,

$$a - \frac{7}{4}Ae^{-s(t_f-t_0)} + \frac{1}{4}\left(P_f - \frac{3}{5}a\right) \geq 0$$

or equivalently (using the explicit expression for A),

$$P_f - \frac{3}{5}a \geq 7\frac{1}{e^{-\frac{7}{2}s(t_f-t_0)} - 1}\left[\left(P_0 - \frac{3}{5}a\right)e^{-\frac{5}{2}s(t_f-t_0)} - \left(P_f - \frac{3}{5}a\right)\right] - 4a.$$

Solving this inequality for $P_f - \frac{3}{5}a$ we obtain the inequality,

$$P_f - \frac{3}{5}a \leq \frac{7e^{-\frac{5}{2}s(t_f-t_0)}}{6 + e^{-\frac{7}{2}s(t_f-t_0)}}\left(P_0 - \frac{3}{5}a\right) + 4a\frac{1 - e^{-\frac{7}{2}s(t_f-t_0)}}{6 + e^{-\frac{7}{2}s(t_f-t_0)}}. \quad (2.39)$$

Thus to insure that the state and control constraints, $P^*(t) \geq 0$ and $u_i(t) \geq 0$ for $t \in [t_0, t_f]$, hold, we must check that the parameters of the system satisfy inequalities (2.38) and (2.39). We have already observed that for $P_0, P_f \geq \frac{3}{5}a$ we can choose $t_f - t_0$ sufficiently large to insure that (2.38) holds. Further, we observe that as $t_f - t_0 \to +\infty$ the right side of (2.39) tends $\frac{2}{3}a$ so that we can always find $t_f - t_0$ sufficiently large so that we have

$$\frac{7e^{-\frac{5}{2}s(t_f-t_0)}}{6+e^{-\frac{7}{2}s(t_f-t_0)}}\left(P_0-\frac{3}{5}a\right)+4a\frac{1-e^{-\frac{7}{2}s(t_f-t_0)}}{6+e^{-\frac{7}{2}s(t_f-t_0)}} \leq \left(P_0-\frac{3}{5}a\right)e^{s(t_f-t_0)}$$

Moreover, it is easy to see that

$$\frac{7e^{-\frac{5}{2}s(t_f-t_0)}}{6+e^{-\frac{7}{2}s(t_f-t_0)}}\left(P_0-\frac{3}{5}a\right)+4a\frac{1-e^{-\frac{7}{2}s(t_f-t_0)}}{6+e^{-\frac{7}{2}s(t_f-t_0)}} \geq \left(P_0-\frac{3}{5}a\right)e^{-\frac{5}{2}s(t_f-t_0)}$$

holds whenever $t_f - t_0$ is sufficiently large. Combining these observations allows us to conclude that for $t_f - t_0$ sufficiently large $\{P^*(\cdot), u_1^*(\cdot), u_2^*(\cdot)\}$ is a Nash equilibrium for the original dynamic game.

4. Conclusion

In this paper we have presented means to utilize the direct method to obtain open-loop Nash equilibria for differential games for which there is a single state whose time evolution is determined by the competitive strategies of several players appearing linearly in the equation. That is a so called affine control system with "many inputs and one output."

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