# Chapter 2

# SYSTEMS OF EQUATIONS

In the previous chapter the reader was acquainted with equations having one unknown function. However, almost all partial differential equations can be reduced to a system of quasilinear equations.

The first section of this chapter is devoted to explaining classical knowledge related to systems of quasilinear partial differential equations. The main definitions of quasilinear systems, as well as the notions of characteristics and relations along characteristics are considered in this section.

Systems written in Riemann invariants play an important role in continuum mechanics. In particular, homogeneous systems have solutions, called Riemann waves, where only one Riemann invariant is changed. The well-known problem of the decay of arbitrary discontinuity is solved in terms of Riemann waves. Chapter 2 also contains an application of Riemann waves for describing one-dimensional motion of an elastic-plastic material.

Another method playing a very important role in gas dynamics is the hodograph method. For some problems this method allows linearization. If a hodograph is degenerate, such solutions form a class of solutions called solutions with degenerate hodograph. This class of solutions is considered in the next chapter. Here it is worth mentioning that solutions with degenerate hodograph have a group-invariant nature: they are partially invariant solutions. Another class of solutions which also has group-invariant nature is the class of self-similar solutions. This class of invariant solutions, very well-known in continuum mechanics, is based on the analysis of dimensions of studied quantities. The approach used in the book for introducing self-similar solutions is related to admitted scale groups. This way of studying self-similar solutions can also be considered as an introduction to group analysis method. The theory of self-similar solutions is followed by applications to one-dimensional gas dynamics, in particular to the problem of an intense gas explosion.

Among the approaches using a simple representation of the dependent variables through the independent variables are travelling waves and solutions with linear dependence of velocity with respect to spatial (all or part) independent variables.

The chapter ends with a general study of completely integrable systems.

## 1. Basic definitions

Most mathematical problems in science are described by systems of partial differential equations. The vast majority of these problems are reduced to problems involving the study of systems of quasilinear partial differential equations. In this section one can find sufficient notations for further reading[1] related with these types of systems.

Let $x = (x_1, x_2, \dots, x_n) \in R^n$ be the independent variables and $u = (u_1, u_2, \dots, u_n), \in R^m$ be the dependent functions.

**Definition 2.1.** *A system of first order partial differential equations of the form*

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{m} a_{i\beta}^{\alpha}(x,u)\frac{\partial u_\beta}{\partial x_\alpha} = f_i(x,u), \ \ (i = 1, 2, \dots, N) \tag{2.1}$$

*is called a system of quasilinear partial differential equations.*

Any system of quasilinear partial differential equations can be written in the matrix form

$$\sum_{\alpha=1}^{n} A_\alpha \frac{\partial u}{\partial x_\alpha} = f, \tag{2.2}$$

where $A_\alpha$ are rectangle $N \times m$ matrices with the entries $a_{i\beta}^{\alpha}$ ($i$ is the row number, $\beta$ is the column number), and the vector-columns

$$f = (f_1, f_2, \dots, f_m)^t, \ \ \partial u/\partial x_\alpha = (\partial u_1/\partial x_\alpha, \partial u_2/\partial x_\alpha, \dots, \partial u_m/\partial x_\alpha)^t.$$

System (2.1) or (2.2) is called a determined system if $N = m$, and overdetermined if $N > m$. If the vector $f$ and the matrices $A_\alpha$, $(\alpha = 1, 2, ..., n)$ do not depend on the independent variables $x = (x_1, x_2, \dots, x_n)$, then the system is called an autonomous system. If the vector $f = 0$, then it is called a homogeneous system.

The matrix

$$A(\xi) = \sum_{\alpha=1}^{n} A_\alpha \xi_\alpha,$$

[1]For more detail study of properties of systems of quasilinear partial differential equations one can read, for example, [147].

associated with system (2.2) is called the characteristic matrix. Here $\xi = (\xi_1, \xi_2, ..., \xi_n) \in R^n$.

**Definition 2.2.** *A vector $\xi$ is called a normal characteristic vector of system (2.2) at the point $(x, u)$, if*

$$\det( A(\xi) ) = 0. \tag{2.3}$$

*Equation (2.3) is called the characteristic equation.*

For any normal characteristic vector $\xi$ there exists a left eigenvector $l(\xi) = (l_1(\xi), l_2(\xi), \ldots, l_m(\xi))$ of the matrix $A(\xi)$

$$l(\xi)A(\xi) = 0. \tag{2.4}$$

Let $\eta \in R^n$ be a unit fixed vector, then any vector $\xi$ can be decomposed

$$\xi = z\eta + \sigma,$$

where $\sigma$ is a vector orthogonal to $\eta$.

**Definition 2.3.** *System (2.2) is called a hyperbolic system at a point $(x, u)$, if there exists a vector $\eta$, such that for any vector $\sigma$, which is orthogonal to $\eta$, the characteristic equation*

$$\det (A(z\eta + \sigma)) = 0$$

*has $m$ real roots with respect to $z$ and the set of eigenvectors $l^k$, $(k = 1, 2, \ldots, m)$ corresponding to these roots, composes a basis in $R^m$. A system is called a hyperbolic system in a domain $D$, if it is hyperbolic at any point $(x, u) \in D$ (the vector $\eta$ can depend on $(x, u)$). A direction defined by the vector $\eta$ is called a hyperbolic direction.*

Let a solution $u = \phi(x)$ of system (2.2) be given. The point $(x, u) = (x, \phi(x))$ is defined by the point $x$.

**Definition 2.4.** *A surface $\Gamma \subset R^n(x)$ such that a tangent hyperplane at any $x \in \Gamma$ has a characteristic direction is called a characteristic surface of system (2.2) on this solution, or simply a characteristic.*

The problem of finding a characteristic given by the equation $h(x) = 0$ is reduced to the problem of solving the equation

$$\det ( A(\nabla h)) = 0.$$

This equation is a first order partial differential equation. The characteristics of this equation are called bicharacteristics of system (2.2).

**Remark 2.1.** *If one of the matrices $A_i$, $(i = 1, 2, ..., n)$ is nonsingular, then without loss of generality one can assume that $A_1$ is the unit matrix.*

**Remark 2.2.** *The concept of characteristic surfaces plays an important role in the process of constructing solutions. Their allocations give information about the qualitative behavior of the solution, and about the correctness of initial and boundary value problems. Moreover, they play a crucial role in defining the so called "simple" solutions. Sewing together simple solutions one can find more "complicated" solutions. The classical example of this construction is the solution of the decay of an arbitrary discontinuity problem in gas dynamics.*

## 2. Riemann invariants

In this section the Riemann invariants for a hyperbolic system of quasilinear partial differential equations with two independent variables[2]

$$\frac{\partial u}{\partial x_1} + A\frac{\partial u}{\partial x_2} = f. \tag{2.5}$$

are considered. The direction of hyperbolicity of system (2.5) is assumed $\eta = (1, 0)$. Hence, the matrix $A$ has $m$ real eigenvalues $\lambda_k(x, u)$ with the left eigenvectors $l^k(x, u)$, $(k = 1, \ldots, m)$ of the matrix $A$. The eigenvectors $l^k(x, u)$, $(k = 1, \ldots, m)$ form a basis for $R^m$. Hence, the matrix $L = (l_j^k)$ comprising the coordinates $l_j^k(x, u)$ of the vectors $l^k(x, u)$ is nonsingular. If the characteristic, corresponding to the eigenvalue $\lambda_k(x, u)$ of system (2.5) is defined by the equation $h_k = x_2 - \phi_k(x_1) = 0$, then the function $\phi_k(x_1)$ satisfies the equation $d\phi_k/dx_1 = \lambda_k$.

Multiplying system (2.5) by the left eigenvectors $l^k(x, u)$, one obtains:

$$l^k(x, u)\left(\frac{\partial u}{\partial x_1} + \lambda_k(x, u)\frac{\partial u}{\partial x_2}\right) = l^k f, \quad (k = 1, 2, \ldots, m). \tag{2.6}$$

Let each of the differential forms

$$\omega_k(x, u, du) = l_\alpha^k\, du_\alpha, \quad k = 1, 2, \ldots, m$$

with fixed $x_1, x_2$ have the integrating factor[3] $\mu_k(x, u)$ , i. e.,

$$\mu_k(x, u)\omega_k(x, u, du) = \frac{\partial r_k(x, u)}{\partial u_\alpha} du_\alpha, \quad (k = 1, 2, \ldots, m).$$

Hence, after multiplying equations (2.6) by $\mu_k$, they take on the form

$$D_{x_1} r_k + \lambda_k D_{x_2} r_k = g_k(x, u), \quad (k = 1, 2, \ldots, m),$$

[2] A notation of Riemann invariants in the case of more than two independent variables is given in the next Chapter.

[3] In the case $m = 2$ the integrating factor always exists. This is not so for $m > 2$.

where $g_k = \mu_k f_k + r'_{kx_1} + \lambda_k r'_{kx_2}$, $D_{x_i}$ is the total derivative operator with respect to $x_i$, and the value $r'_{kx_i}$ is the partial derivative of the function $r_k$ with respect to $x_i$ with fixed values of the dependent variables $u_1, u_2, \ldots, u_m$. Since the Jacobian

$$\frac{\partial(r_1, \ldots, r_m)}{\partial(u_1, \ldots, u_m)} = \det(L) \neq 0,$$

changing the dependent variables $(u_1, \ldots, u_m)$ into the new dependent functions $(r_1, \ldots, r_m)$, one reduces system (2.5) to the system of quasilinear partial differential equations

$$\frac{\partial r_k}{\partial x_1} + \lambda_k(x, u)\frac{\partial r_k}{\partial x_2} = g_k, \quad (k = 1, 2, \ldots, m). \tag{2.7}$$

The values $r_k$ are called Riemann invariants and system (2.7) is a system written in invariants. If system (2.5) is homogeneous and autonomous ($A = A(u)$, $f = 0$), then the system written in invariants (2.7) is also homogeneous. If $g_k = 0$ in (2.7), then the Riemann invariant $r_k$ is constant along the characteristic $\frac{dx}{dt} = \lambda_k$.

Let a system written in Riemann invariants (2.7) be homogeneous $g_i = 0$, $(i = 1, 2, \ldots, m)$. Assume that all Riemann invariants except one, for example $r_k$, are constant, i. e.,

$$r_i = c_i, \ (1 \leq i \leq m, \ i \neq k).$$

The Riemann invariant $r_k$ satisfies the equation

$$\frac{\partial r_k}{\partial x_1} + \lambda_k(\hat{x}, r_k)\frac{\partial r_k}{\partial x_2} = 0,$$

where $\lambda_k(\hat{x}, r_k) = \lambda_k(x, c_1, \ldots, c_{k-1}, r_k, c_{k+1}, \ldots, c_m)$. Such a solution is called a simple wave or a Riemann wave.

## 2.1 The problem of stretching an elastic–plastic bar

The equations, governing a one–dimensional time dependent motion of an elastic–plastic material with the state equation $\sigma = f(\varepsilon)$, $f' = \phi^2(\varepsilon) > 0$, are

$$v_t = \sigma_x, \quad \varepsilon_t = v_x. \tag{2.8}$$

Taking linear combinations of the first equation and second multiplied by $\phi(\varepsilon)$, one obtains

$$(v_t + \phi(\varepsilon)\varepsilon_t) - \phi(\varepsilon)(v_x + \phi(\varepsilon)\varepsilon_x) = 0,$$
$$(v_t - \phi(\varepsilon)\varepsilon_t) + \phi(\varepsilon)(v_x - \phi(\varepsilon)\varepsilon_x) = 0.$$

Hence, the Riemann invariants and characteristic eigenvalues are

$$r_1 = v + F(\varepsilon), \quad \lambda_1 = -\phi(\varepsilon),$$
$$r_2 = v - F(\varepsilon), \quad \lambda_2 = \phi(\varepsilon),$$

where $F(\varepsilon) = \int_{\varepsilon_0}^{\varepsilon} \phi(\eta)d\eta$. The original system of partial differential equations (2.8) becomes

$$\frac{\partial r_1}{\partial t} + \lambda_1(r_1, r_2)\frac{\partial r_1}{\partial x} = 0,$$
$$\frac{\partial r_2}{\partial t} + \lambda_2(r_1, r_2)\frac{\partial r_2}{\partial x} = 0,$$

The dependent variables are recovered through the Riemann invariants

$$v = \frac{r_1 + r_2}{2}, \quad F(\varepsilon) = \frac{r_1 - r_2}{2}.$$

Let us pose the problem of stretching a semi-infinite bar. Assume that at the initial time $t = 0$ an elastic–plastic half–infinitely long bar is in the unperturbed state:

$$v(x, 0) = 0, \varepsilon(x, 0) = \varepsilon_0, \quad x \geq 0.$$

The end of the bar $x = 0$ at the time $t = 0$ starts stretching with the velocity

$$v(0, t) = -u(t). \tag{2.9}$$

It is assumed that $u(0) = 0$ and $u'(t) \geq 0$. The problem is to find the loading wave propagating in the bar.

The solution of this problem can be constructed with the help of Riemann waves.

Since the Riemann invariants are constant along their own characteristics, the solution in the domain, joining to the initial data $V_1 = \{(x, t) \mid 0 \leq t \leq \infty, \ 0 \leq x \leq t\phi(\varepsilon_0)\}$, is constant: $v = 0, \varepsilon = \varepsilon_0$. Because the Riemann invariant $r_1$ is constant along the characteristics $dx/dt = -\phi$ (which cross the characteristic curve $x = t\phi(\varepsilon_0)$) and have a constant value on the characteristic $x = t\phi(\varepsilon_0)$, this invariant is also constant in the domain $V_2$, joining to $V_1$. This means, that in this domain one obtains the Riemann wave: $r_1 = v + F(\varepsilon) = F(\varepsilon_0)$. In this Riemann wave the other Riemann invariant $r_2$ constant along the characteristics $dx/dt = \phi$, which are straight lines. Hence, the solution in the domain $V_2$ is defined by the values $v = -u(t)$ and $F(\varepsilon) = F(\varepsilon_0) + u(t)$ at the point $(x_0(t), t)$, where $x_0(t) = -\int_0^t u(\tau)\, d\tau$. The relation $u'(t) \geq 0$ provides the condition that characteristics $\frac{dx}{dt} = \phi$ intersect in the domain $V_2$. If the condition $u'(t) \geq 0$ is broken, this leads to the formation of a gradient catastrophe. The relation $u(0) = 0$ gives a nonsingular Riemann wave. If $u(0) > 0$, then the part of the domain $V_2$ is occupied by the rarefaction Riemann wave. This part is bounded by the characteristic $x = t\phi(\varepsilon_1)$, where $F(\varepsilon_1) = F(\varepsilon_0) + u(0)$. The deformation $\varepsilon$ in this domain is defined by the equation $\phi(\varepsilon) = \frac{x}{t}$. The characteristics $\frac{dx}{dt} = \phi$ issue from the origin $(x, t) = (0, 0)$.

## 3. Hodograph method

The basic idea of the hodograph method consists of interchanging the role of the dependent and independent variables. For some classes of equations this method reduces them to a system of linear partial differential equations. The essence of the hodograph method is described by the system of the equations, governing two–dimensional irrotational isentropic flows of a gas ($\nu = 0, 1$):

$$u_y - v_x = 0, \ \ (u^2 - c^2)u_x + 2uvu_y + (v^2 - c^2)v_y = \frac{\nu}{y}c^2 v, \tag{2.10}$$

where $(u, v)$ is the velocity, $c$ is the sound speed, which is expressed through the value $q^2 = u^2 + v^2$. The plane $R^2(u, v)$ is called a hodograph plane, and $(u, v)$ are the hodograph variables.

Assume that the Jacobian[4] $\Delta = \frac{\partial(u,v)}{\partial(x,y)} \neq 0$. Choosing $(u, v)$ as the new independent variables, one can find

$$x = x(u, v), \quad y = y(u, v).$$

Differentiating these relations with respect to $x$ and $y$, one obtains

$$\begin{aligned} 1 &= x_u u_x + x_v v_x, \quad 0 = y_u u_x + y_v v_x, \\ 0 &= x_u u_y + x_v v_y, \quad 1 = y_u u_y + y_v v_y. \end{aligned}$$

Since $\Delta \neq 0$, one finds

$$u_x = \Delta y_v, \quad v_x = -\Delta y_u, \quad u_y = -\Delta v_v, \quad v_y = \Delta x_u. \tag{2.11}$$

The two-dimensional gas dynamics equations (2.10) may then be written

$$x_v - y_u = 0, \ (u^2 - c^2)y_v + 2uvx_v + (v^2 - c^2)x_u = \frac{\nu}{y}c^2 v(x_u y_v - x_v y_u). \tag{2.12}$$

The first equation of (2.12) leads to the existence of a potential: a function $\phi = \phi(u, v)$ such that

$$x = \phi_u, \quad y = \phi_v.$$

The second equation of (2.12) becomes the equation for the function $\phi$

$$(u^2 - c^2)\phi_{vv} + 2uv\phi_{uv} + (v^2 - c^2)\phi_{uu} = \frac{\nu c^2 v}{\phi_v}(\phi_{uu}\phi_{vv} - \phi_{uv}^2). \tag{2.13}$$

Equation (2.13) assumes a specially simple form in the case $\nu = 0$ in the polar coordinates ($u = q\cos\theta$, $v = q\sin\theta$):

$$(M^2 - 1)\phi_{\theta\theta} - q^2\phi_{qq} + (M^2 - 1)q\phi_q = 0, \tag{2.14}$$

where $M = q/c$ is the Mach number. The most important property of equation (2.14) consists of its linearity. Thus the hodograph transformation can simplify

[4] Vanishing of the Jacobian $\Delta$ defines a class of solutions which are called solutions with degenerate hodograph. A detailed study of these solutions is given in the next Chapter.

the original system of equations. It should be also noted that equation (2.14) allows finding solutions with separated variables $q$ and $\theta$. In fact, substituting the representation of the solution

$$\phi = h(\theta)g(q)$$

into (2.14), one obtains

$$\frac{h''}{h} - \frac{q}{g(M^2-1)}\big(g'' - g'(M^2-1)\big) = 0.$$

Alongside the complete interchange of the dependent and independent variables, sometimes it is convenient to make only a partial change of them. For example, for the equations of a stationary boundary layer

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U\frac{dU}{dx} + \nu\frac{\partial^3 \psi}{\partial y^3}, \quad U = U(x),$$

applying the transformation from the independent variables $(x, y)$ to the Prandtl–Mises variables: $\xi = x$, $\eta = \psi(x, y)$, one obtains the equations

$$\frac{\partial u^2}{\partial \xi} - \frac{dU^2}{d\xi} = 2\nu u\frac{\partial}{\partial \eta}(u\frac{\partial u}{\partial \eta}),$$

where $u = 1/(\partial y/\partial \eta)$. Introducing the variable $2z = U^2 - u^2$, one finds the equation of a boundary layer in the Mises form

$$\frac{\partial z}{\partial \xi} = \nu\sqrt{U^2 - 2z}\frac{\partial^2 z}{\partial \eta^2}.$$

# 4. Self–similar solutions

## 4.1 Definitions and basic properties

One of the modelling stages of a problem in continuum mechanics is the dimensional analysis of the quantities of the variables involved. This analysis also allows forming representations of solutions, which are called self-similar solutions. One example of a self-similar solution was presented in the first chapter. Here the main definitions and properties of self–similar solutions are given. Since the basis for dimensional analysis is a scale group, the given approach is based on the concept of an admitted scale group.

Let $(x_1, \ldots, x_n)$ and $(u_1, \ldots, u_m)$ be the independent and dependent variables.

**Definition 2.5.** *A transformation $h_a : R^{n+m} \to R^{n+m}$ of the form*

$$x_i' = x_i \textstyle\prod_{\alpha=1}^{r}(a_\alpha)^{\lambda_\alpha^i}, \quad u_k' = u_k \textstyle\prod_{\alpha=1}^{r}(a_\alpha)^{\mu_\alpha^k}, \quad (i = 1, \ldots, n; \quad k = 1, \ldots, m) \tag{2.15}$$

*is called a scale group $H^r$ of transformations of the space $R^{n+m}(x,u)$. The variables $a_\alpha$ $(\alpha = 1,\dots,r)$ are called its parameters.*

It is natural to require

$$\equiv rank \begin{pmatrix} \lambda_1^1 & \dots & \lambda_1^n & \mu_1^1 & \dots & \mu_1^m \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda_r^1 & \dots & \lambda_r^n & \mu_r^1 & \dots & \mu_r^m \end{pmatrix} = r. \tag{2.16}$$

Otherwise, introducing new scale parameters, it is possible to reduce the number $r$.

Under action of the transformation (2.15) the first order derivatives are transformed according to the formulae

$$\frac{\partial u_k'}{\partial x_i'} = \frac{\partial u_k}{\partial x_i} \prod_{\alpha=1}^{r} (a_\alpha)^{\mu_\alpha^k - \lambda_\alpha^i}. \tag{2.17}$$

Similar formulae are valid for higher order derivatives.

**Definition 2.6.** *The group of transformations, consisting of the transformations of the independent, dependent variables (2.15) and the derivatives (2.17), is called a prolonged group of $H^r$.*

For any function $F : R^{n+m} \to R$ the total derivative with respect to the parameter $a_j$ is

$$\frac{\partial F(x',u')}{\partial a_j} = \frac{1}{a_j}\Big(\sum_{\alpha=1}^{n} \lambda_j^\alpha x_\alpha' \frac{\partial F(x',u')}{\partial x_\alpha'} + \sum_{\gamma=1}^{m} \mu_j^\gamma u_\gamma' \frac{\partial F(x',u')}{\partial u_\gamma'}\Big). \tag{2.18}$$

**Definition 2.7.** *The linear differential operator*

$$\zeta_j \partial = \sum_{\alpha=1}^{n} \lambda_j^\alpha x_\alpha \frac{\partial}{\partial x_\alpha} + \sum_{\gamma=1}^{m} \mu_j^\gamma u_\gamma \frac{\partial}{\partial u_\gamma}$$

*is called an infinitesimal operator of the group $H^r$.*

**Definition 2.8.** *A function $F : R^{n+m} \to R$ is called an invariant of the group $H^r$, if for any transformation $h_a \in H^r$:*

$$F(x',u') = F(x,u). \tag{2.19}$$

**Theorem 2.1.** *A function $F : R^{n+m} \to R$ is an invariant of the group $H^r$ if and only if*

$$\zeta_j \partial F = 0, \quad (j = 1,\dots,r). \tag{2.20}$$

*Proof.*
If the function $F(x, u)$ is invariant, then $F(x', u') = F(x, u)$. Differentiating it with respect to the parameter $a_j$, $(j = 1, \ldots, r)$, and setting the parameters $a_i = 1$, $(i = 1, 2, ..., r)$, one obtains (2.20). Conversely, if (2.20) is valid, then by virtue of (2.18) one has $\partial F(x', u')/\partial a_j = 0$, $(j = 1, \ldots, r)$. This means that $F(x', u')$ does not depend on the parameters $a_j$, $(j = 1, \ldots, r)$. Since for $a_i = 1$, $(i = 1, 2, ..., r)$ its value is equal to $F(x, u)$, hence, $F(x', u') = F(x, u)$. Therefore $F$ is an invariant of the group $H^r$.•

**Theorem 2.2.** *For a scale group $H^r$ of the space $R^{n+m}(x, u)$ with the condition $r < n + m$ there exist $n + m - r$ independent invariants. They are the monomials*

$$J^k(x, u) = \prod_{\alpha=1}^{n} (x_\alpha)^{\theta_\alpha^k} \cdot \prod_{\beta=1}^{m} (u_\beta)^{\sigma_\beta^k}, \quad (k = 1, \ldots, m + n - r). \tag{2.21}$$

*Proof.*
By virtue of the criterion, the monomial

$$J(x, u) = \prod_{\alpha=1}^{n} (x_\alpha)^{\theta_\alpha} \cdot \prod_{\beta=1}^{m} (u_\beta)^{\sigma_\beta}$$

is an invariant of the group $H^r$, if and only if

$$(\zeta_j \partial) J = J(\sum_{\alpha=1}^{n} \lambda_j^\alpha \theta_\alpha + \sum_{\gamma=1}^{m} \mu_j^\gamma \sigma_\gamma) = 0, \quad (j = 1, \ldots, r). \tag{2.22}$$

Because of (2.16), the system of linear algebraic equations (2.22) with $n + m$ unknown $\theta_\alpha$, $\sigma_\gamma$ has $n+m-r$ linearly independent solutions. Let the solutions be

$$(\theta_1^k, \ldots, \theta_n^k, \sigma_1^k, \ldots, \sigma_m^k), \quad (k = 1, \ldots, m + n - r). \tag{2.23}$$

Then the monomials (2.21) with exponents (2.23) are also independent. By virtue of the exponent representation of the monomials (2.21), it is enough to prove the independence of their exponent.

Assume that there exist constants $c_k$, $(k = 1, \ldots, m + n - r)$, at least one of them is not equal to zero $(c_1^2 + c_2^2 + \ldots + c_{m+n-r}^2 \neq 0)$, and for which one has the equality

$$\prod_{k=1}^{n+m-r} (J^k)^{c_k} = 1.$$

This is only possible if

$$\sum_{k=1}^{n+m-r} \theta_j^k c_k = 0, \quad \sum_{k=1}^{n+m-r} \sigma_\gamma^k c_k = 0, \quad (j = 1, \ldots, n;\ \gamma = 1, \ldots, m),$$

But by virtue of the linear independence of (2.23), one obtains $c_k = 0, \ (k = 1, \dots, m+n-r)$. This contradicts to the assumption $c_1^2+c_2^2+\dots+c_{m+n-r}^2 \neq 0$.●

**Definition 2.9.** *A manifold assigned by the equations* $\phi_k(x,u) = 0, \ (k = 1, 2, \dots, l)$*, is called an invariant manifold of the group* $H^r$*, if*

$$\phi_k(x', u') = 0, (k = 1, 2, \dots, l).$$

The concept of a scale group is closely related with the dimensional analysis theory of physical quantities [130].

Each physical quantity $\phi$ is characterized by a unit of its measurement $E$ and a numerical value $|\phi|$, hence, $\phi = |\phi|E$. Let $E_\alpha, \ (\alpha = 1, \dots, r)$ be some independent units of measurements that $E$ is expressed through them $E = \prod_{\alpha=1}^{r} E_\alpha^{\lambda_\alpha}$. The value $E$ is called the dimension of the physical quantity $\phi$ in the terms of the units $E_\alpha$ and it is denoted $[\phi]$.

If one changes the scales of the units $E_i$ into the new units $E_i'$

$$E_i = a_i E_i', \quad (i = 1, \dots, r),$$

the numerical value of the physical quantity $\phi$ in the new units is

$$\phi = |\phi|[\phi] = |\phi|(\prod_{\alpha=1}^{r}(a_\alpha)^{\lambda_\alpha})(\prod_{\alpha=1}^{r}(E_\alpha')^{\lambda_\alpha}) = |\phi|'[\phi]'.$$

This means that $|\phi|' = |\phi|(\prod_{\alpha=1}^{r}(a_\alpha)^{\lambda_\alpha})$, i.e., the change of the numerical values of the physical quantity $\phi$ is similar with the scale group $H^r$. When constructing exact solutions by the dimensional analysis theory one can use the theory of invariant solutions with respect to the scale group $H^r$. This theory is explained next.

**Definition 2.10.** *A scale group* $H^r$ *is said to be an admitted by a system of partial differential equations if the manifold assigned by this system is invariant with respect to the prolonged group* $H^r$*.*

Nonsingular invariant solutions are constructed as follows. First, one finds all of the independent invariants $J^k, \ \ (k = 1, \dots, m+n-r)$. They should be such that it is possible to solve $m$ of them (for example, $J^k, \ (k = 1, \dots, m)$) with respect to all dependent variables. A sufficient condition for this is the inequality

$$\frac{\partial(J^1, \dots, J^m)}{\partial(u^1, \dots, u^m)} = (\prod_{k=1}^{m} J^k / \prod_{\alpha=1}^{m} u_\alpha) \det \begin{pmatrix} \sigma_1^1 & \dots & \sigma_m^1 \\ \dots & \dots & \dots \\ \sigma_1^m & \dots & \sigma_m^m \end{pmatrix} \neq 0.$$

Without loss of generality one can let $\sigma^i_j = \delta_{ij}$ $(i, j = 1, \ldots, m)$. The remaining invariants $J^{m+k}$, $(k = 1, \ldots, n - r)$ can be chosen depending only of the independent variables (the case where $r = n$ is also possible). Hence,

$$rank \begin{pmatrix} \lambda^1_1 & \ldots & \lambda^n_1 \\ \ldots & \ldots & \ldots \\ \lambda^1_r & \ldots & \lambda^n_r \end{pmatrix} = r \leq n.$$

After obtaining the independent invariants $J^k$, $(k = 1, \ldots, m + n - r)$, one supposes the dependence of the first invariants $J^k$, $(k = 1, \ldots, m)$ of the remaining, i.e.,

$$J^k = \varphi_k(J^{m+1}, \ldots, J^{m+n-r}), \ (k = 1, \ldots, m).$$

Since the invariants $J^k$, $(k = 1, \ldots, m)$ can be solved with respect to all dependent variables $u^i$, $(i = 1, ..., m)$, defining the dependent variables from the last equations, one obtains a representation of the invariant solution. Substituting the representation of the functions $u^i$, $(i = 1, ..., m)$ into the initial system of partial differential equations, one obtains the system of equations for the unknown functions $\varphi_k$, $(k = 1, \ldots, m)$. This system involves a smaller number of independent variables.

**Definition 2.11.** *An invariant solution of an admitted scale group $H^r$ is called a self–similar solution*[5].

Let us apply the theory to the nonlinear diffusion equation[6]

$$r^{1-q}\frac{\partial(r^{q-1}c^n c_r)}{\partial r} = \frac{\partial c}{\partial t}, \quad (n > 0) \tag{2.24}$$

The dependent and independent variables are scaled as follows

$$c = a^{\alpha_1}\widehat{c}, \ \ r = a^{\alpha_2}\widehat{r}, \ \ t = a^{\alpha_3}\widehat{t}.$$

Equation (2.24) in the new variables becomes

$$a^{n\alpha_1 - 2\alpha_2 + \alpha_3}\widehat{r}^{1-q}\frac{\partial}{\partial\widehat{r}}(\widehat{r}^{q-1}\widehat{c}^n\frac{\partial\widehat{c}}{\partial\widehat{r}}) - \frac{\partial\widehat{c}}{\partial\widehat{t}} = 0.$$

If equation (2.24) is invariant with respect to scaling, then it is necessary that

$$n\alpha_1 - 2\alpha_2 + \alpha_3 = 0.$$

For $\alpha_3 \neq 0$ the combinations $J^1 = ct^{-\alpha_1/\alpha_3}$, $J^2 = rt^{-\alpha_2/\alpha_3}$ are invariant with respect to this scaling. Assuming

$$ct^{-\alpha_1/\alpha_3} = V(rt^{-\alpha_2/\alpha_3}),$$

one obtains the representation of an invariant (self–similar) solution of (1.56).

[5]With a group point of view such solutions are called self–similar solutions in narrow sense [130].
[6]Self-similar solutions of this equation were considered in the previous chapter.

## 4.2 Self–similar solutions in an inviscid gas

The one-dimensional gas dynamics equations are considered to illustrate the method of constructing self-similar solutions. Notice that the solution of the problem of a strong explosion in a gas was found with the help of self–similar solutions.

The system of equations describing a one-dimensional motion of a gas is

$$\begin{gathered}\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0,\\ \frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\nu\frac{\rho u}{x}=0,\\ \frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right)+v\frac{\partial}{\partial x}\left(\frac{p}{\rho^\gamma}\right)=0.\end{gathered} \tag{2.25}$$

Here $\gamma$ is the exponent of the adiabatic curve, $\nu$ characterizes a geometrical structure of the problem: $\nu=0$ for plane flows, $\nu=1$ for cylindrical flows, and $\nu=2$ for spherical flows. Because the number of the independent variables $n=2$, by virtue of the condition $n\geq r$ only two cases are possible: either $r=1$ or $r=2$.

Let $r=1$, and define the one-parameter scale group $H^1$ admitted by system (2.25) with the equations ($a\equiv a_1$):

$$u'=ua^{\mu^1},\ \ \rho'=\rho a^{\mu^2},\ \ p'=pa^{\mu^3},\ \ t'=ta^{\lambda^1},\ \ x'=xa^{\lambda^2}.$$

For the transformations that remain invariant the manifold, assigned by equations (2.25), one obtains

$$\mu^1=\lambda^2-\lambda^1,\ \ \mu^3=\mu^2+2(\lambda^2-\lambda^1).$$

Further it is assumed that $(\lambda^1)^2+(\lambda^2)^2\neq 0$. Since $m+n-r=4$, when forming the independent invariants $J^k=t^{\theta_1^k}x^{\theta_2^k}u^{\sigma_1^k}\rho^{\sigma_2^k}p^{\sigma_3^k}$, $(k=1,2,3,4)$ it is enough to find independent solutions of the linear equation

$$\lambda^1\theta_1^k+\lambda^2\theta_2^k+(\lambda^2-\lambda^1)\sigma_1^k+\mu^2\sigma_2^k+(\mu^2+2(\lambda^2-\lambda^1))\sigma_3^k=0.$$

If $\lambda^1=0$, one can choose the following independent invariants

$$J^1=ux^{-1},\ \ J^2=\rho x^{-\alpha},\ \ J^3=px^{-(\alpha+2)},\ J^4=t,\ \ (\alpha=\mu^2/\lambda^2).$$

The self–similar solution has the representation[7]

$$u=x\phi_1(t),\ \ \rho=x^\alpha\phi_2(t),\ \ p=x^{\alpha+2}\phi_3(t).$$

The system of equations (2.25) becomes

$$\begin{gathered}\phi_1'+\phi_1^2+(\alpha+2)\phi\phi_2^{\gamma-1}=0,\\ \phi_2'+(\alpha+1+\nu)\phi_1\phi_2=0,\\ \phi'+(\alpha+2-\gamma\alpha)\phi_1\phi=0,\end{gathered}$$

[7] Such solutions are called solutions with a linear profile of velocity.

where $\phi \equiv \phi_3/\phi_2^\gamma$. The solution of this system of equations can be found in terms of quadratures.

If $\lambda^1 \neq 0$, the invariants can be chosen as follows

$$J^1 = ut^{\alpha+1}, \quad J^2 = \rho t^\beta, \quad J^3 = pt^{2(\alpha+1)+\beta}, \, J^4 = xt^\alpha,$$

where $\alpha = -\lambda^2/\lambda^1$, $\beta = -\mu^2/\lambda^1$. The self–similar solution in this case has the representation

$$u = t^{-(1+\alpha)}\phi_1(\lambda), \quad \rho = t^{-\beta}\phi_2(\lambda), \quad p = t^{-(\beta+2(\alpha+1))}\phi_3(\lambda),$$

where $\lambda = xt^\alpha$. There is another equivalent representation

$$u = \frac{x}{t}U(\lambda), \quad \rho = x^{-(k+3)}t^{-s}R(\lambda), \quad p = x^{-(k+1)}t^{-(s+2)}P(\lambda). \qquad (2.26)$$

Substituting these expressions in equations (2.25), one obtains the system of ordinary differential equations for the functions $U,\ R,\ P$

$$\lambda[(\alpha+U)U' + \tfrac{P'}{R}] = -U^2 + U + (k+1)\tfrac{P}{R},$$
$$\lambda[U' + (\alpha+U)\tfrac{R'}{R}] = s + (k-\nu+2)U,$$
$$\lambda(\alpha+U)[\tfrac{P'}{P} - \gamma\tfrac{R'}{R}] = s(1-\gamma) + 2 + [k(1-\gamma)+1-3\gamma]U.$$

This system of equations is split into two equations of the form

$$\lambda\frac{dU}{d\lambda} = f_1(U,Z), \; \lambda\frac{dR}{d\lambda} = Rf_2(U,Z), \qquad (2.27)$$

and one differential equation

$$\frac{dZ}{dU} = \frac{Z}{U+\alpha}\frac{M(Z,U)}{N(Z,U)},$$

where $Z = P/R$. Notice that the last equation in the case where $k=-3,\ s=-(2+\alpha(\nu+3))$ has the integral

$$ZU + (U+\alpha)(\frac{1}{2}U^2 + \frac{1}{\gamma-1}Z) = 0. \qquad (2.28)$$

The values of the constant $\alpha$ and $\beta$ in each particular self–similar solution are chosen by analysis of the initial parameters of the problem.

## 4.3 An intense explosion in a gas

The problem of a strong explosion in a gas is formulated as follows[8].

[8]Detailed analysis of this problem can be found in [149] and references therein.

At the moment $t = 0$ in an undisturbed gas ($u_1 = 0$) with the initial density $\rho_1$ and zero pressure $p_1 = 0$ at the center of symmetry ($x = 0$) an explosion occurs, i.e., a final energy $E_0$ is instantly released. The areas of the disturbed and undisturbed parts of a gas are separated by a shock wave. The gas dynamics values in front of the wave $\rho_1,\ p_1,\ u_1$ and behind it $\rho_2,\ p_2,\ u_2$ are related by the conditions across the shock wave (the Hugoniot relations):

$$\rho_2(u_2 - D) = \rho_1(u_1 - D),$$
$$p_2 + \rho_2(u_2 - D)^2 = p_1 + \rho_1(u_1 - D)^2,$$
$$\frac{\gamma}{(\gamma-1)}\frac{p_2}{\rho_2} + \frac{(u_2-D)^2}{2} = \frac{\gamma}{(\gamma-1)}\frac{p_1}{\rho_1} + \frac{(u_1-D)^2}{2}.$$

These relations express the conservation of mass, impulse and energy laws on the shock wave $x = x_b(t)$. Here $D = dx_b/dt$ is the velocity of the shock wave propagation. From the Hugoniot relations one can obtain the values of the density, pressure and velocity behind the front of the wave

$$\rho_2 = \frac{(\gamma+1)}{(\gamma-1)}\rho_1, \quad p_2 = \frac{2}{(\gamma+1)}\rho_1 D^2, \quad u_2 = \frac{2}{(\gamma+1)}D.$$

The disturbed part of the gas is located in the interval $(0, x_b)$. Because the energy of the volume of the gas is equal to $\rho v^2/2 + p/(\gamma - 1)$, according to the conservation of energy

$$\frac{2\nu\pi}{(\nu+1)}\int_0^{x_b}[\rho u^2/2 + p/(\gamma-1)]x^\nu dx = E_0. \tag{2.29}$$

The motion of the gas after the explosion ($t > 0$) is defined by the dimension of the parameters $E_0,\ \rho_1,\ x$ and $t$. They have the dimensions

$$[E_0] = ML^\nu T^{-2}, \quad [\rho_1] = ML^{-3}, \quad [x] = L, \quad [t] = T.$$

Here $M$ is the dimension of the mass, $L$ is the dimension of the length and $T$ is the dimension of the time. In this problem there is only one dimensionless variable parameter

$$\lambda = (E/\rho_1)^{-\frac{1}{(\nu+3)}} x t^{-\frac{2}{(\nu+3)}},$$

where $E = hE_0$ with some constant $h$. This constant is chosen to scale the variable $\lambda$. Assume that the disturbed part of the gas is governed by the self-similar solution of the type (2.26). The front of the shock wave is $x_b = \lambda_* t^{-\alpha}(E/\rho_1)^{\frac{1}{(\nu+3)}}$. Without loss of generality one can let $\lambda_* = 1$. According to the dependence of the variable $\lambda$ of $t$ and $x$, in (2.26) one has to choose $\alpha = -2/(\nu+3)$. Analyzing the dimension of the density $[\rho]$ it also follows that $\beta = 0$. Therefore when seeking the solution of the problem of an intense explosion, one can try to find it in the class of self–similar solutions (2.26) with $s = 0, k = -3$:

$$\rho = \rho_1 R(\lambda), \quad u = \frac{x}{t}U(\lambda), \quad p = \rho_1\frac{x^2}{t^2}P(\lambda).$$

Notice that for these parameters there is the integral (2.28). Because the coordinate of the shock wave front is $x_b = t^{-\alpha}(E/\rho_1)^{\frac{1}{(\nu+3)}}$, then $D = -\alpha x_b t^{-1}$. This gives the initial data at $\lambda = 1$ (on the front of the shock wave)

$$U(1) = -\frac{2\alpha}{(\gamma+1)},\ R(1) = \frac{(\gamma+1)}{(\gamma-1)},\ P(1) = \frac{2\alpha^2}{(\gamma+1)},\ Z(1) = \frac{2\alpha^2(\gamma-1)}{(\gamma+1)^2}.$$

Equation (2.29) in the self-similar variables is reduced to the equation

$$\frac{2\nu\pi E}{(\nu+1)}\int_0^1 \lambda^{2+\nu}\left(RU^2 + \frac{P}{(\gamma-1)}\right)d\lambda = E_0.$$

This equation serves for finding the constant

$$h = \frac{2\nu\pi}{(\nu+1)}\int_0^1 \lambda^{2+\nu}\left(RU^2 + \frac{P}{(\gamma-1)}\right)d\lambda.$$

Therefore, the solution of intense explosion can be found in the form of quadratures of (2.27).

## 5. Solutions with a linear profile of velocity

Among the approaches for obtaining classes of exact solutions in continuum mechanics there is the method where the velocity vector is required to be linear with respect to the spatial independent variables, with respect to all or their part of them. Such solutions were studied a long time ago by Dirichlet, Dedekind and Riemann. One example of such a solution was obtained in the previous section[9]. Assuming linearity of the velocity with respect to some independent variables, one usually obtains polynomial equations with respect to them. Splitting these equations leads to an overdetermined system of partial differential equations. The main problem in these studies is the compatibility problem for the overdetermined system of equations.

Let us consider the equations describing the isentropic motion of a polytropic gas. For the sake of simplicity[10] we consider the two-dimensional case

$$\frac{d\theta}{dt} + \kappa\theta\, div(u) = 0,\quad \frac{du_i}{dt} + \frac{\partial\theta}{\partial x_i} = 0,\quad (i = 1, 2). \tag{2.30}$$

We will assume that the velocity vector has the representation

$$u_i = m_{i\alpha}(t)x_\alpha,\quad (i = 1, 2). \tag{2.31}$$

The last two equations of (2.30) define the derivatives

$$\frac{\partial\theta}{\partial x_i} = -a_{i\alpha}x_\alpha,\quad (i = 1, 2), \tag{2.32}$$

[9]Applications of this method to the Navier-Stokes equations can be found in [159]

[10]Similar, but more cumbersome, one can obtain solutions in the three-dimensional case.

where

$$a_{ii} = m'_{ii} + m^2_{ii} + m_{12}m_{21}, \ (i = 1, 2),$$

$$a_{12} = m'_{12} + m_{12}(m_{11} + m_{22}), \ a_{21} = m'_{21} + m_{21}(m_{11} + m_{22}).$$

The condition

$$\partial^2\theta/\partial x_1\partial x_2 = \partial^2\theta/\partial x_2\partial x_1$$

gives $a_{12} = a_{21}$. Integrating (2.32) with respect to $x_1$ and $x_2$, one finds

$$\theta = -\frac{1}{2}a_{\alpha\beta}x_\alpha x_\beta + \phi(t), \tag{2.33}$$

where $\phi(t)$ is an arbitrary function of integration. Substituting the expression for $\theta$ and the velocity in the first equation of (2.30), one obtains the squared form with respect to $x_1$ and $x_2$

$$b_{\alpha\beta}x_\alpha x_\beta - b = 0 \tag{2.34}$$

with the coefficients

$$b_{11} = a'_{11} + 2m_{11}a_{11} + 2m_{21}a_{12} + \kappa(m_{11} + m_{22})a_{11},$$

$$b_{22} = a'_{22} + 2m_{22}a_{22} + 2m_{12}a_{12} + \kappa(m_{11} + m_{22})a_{22},$$

$$b_{12} = b_{21} = a'_{12} + m_{12}a_{11} + m_{21}a_{22} + (\kappa + 1)(m_{11} + m_{22})a_{12},$$

$$b = -2[\phi' + \kappa(m_{11} + m_{22})\phi].$$

Factors of the squared form (2.34) are independent of the independent variables $x_1$ and $x_2$. Hence, splitting equation (2.34), one has a system of five ordinary differential equations for five functions: $\phi(t)$ and $m_{ij}(t), \ (i, j = 1, 2)$.

More general class of solutions is obtained if one assumes linearity only with respect to one independent variable.

## 6. Travelling waves

The idea of a travelling wave was presented in the previous chapter. The concept of travelling waves can be generalized for many independent variables $x \in R^n$ and many dependent variables $u \in R^m$. In this section the generalization for the multidimensional case is given.

Let $Lx$ be a linear form of the independent variables

$$Lx = L_\alpha x_\alpha.$$

The representation of a solution is assumed in the form

$$u(x) = v(Lx),$$

where the function $v(\xi)$ depends on one independent variable $\xi$. The value $\xi$ is called a phase of the wave. Fixing the phase $\xi = Lx$, one obtains the front of the wave, where the values of the dependent variables are constant. Hence, the front of the wave is a plane propagating in the space $R^n$. Usually one of the independent variables plays the role of time $t$ (for example, $t = x_n$), the phase of the wave is represented as $\xi = \eta y - Dt$, where $\eta \in R^{n-1}$ is a unit vector, $y = (x_1, x_2, ..., x_{n-1})$, and $\eta y$ is a scalar product.

**Definition 2.12.** *A solution $u(x_1, x_2, \ldots, x_n)$ is called an $r$–multiple travelling wave, if it has the representation*

$$u(x) = v(Lx),$$

*where $Lx$ be a vector, which coordinates are linear forms of the independent variables*

$$(Lx)_i = L_{i\alpha}x_\alpha, \ \ (i = 1, 2, ..., r, \ r < n).$$

*The variables $\xi = Lx \in R^r$ are called parameters of the wave.*

Here the vector function $v(\xi)$ depends on a smaller number of the independent variables $\xi$. Fixing all but one of the components of the vector $\xi$ defines a wave propagating in the subspace of $R^n$ of the dimension[11] $n - (r - 1)$. Equations for the function $v(\xi)$ are obtained by substituting the representation of the solution into the initial system of equations.

Notice that for an $r$-multiple travelling wave the rank of the Jacobi matrix of the dependent variables with respect to the independent variables is less or equal than $r$. This provides an idea for further generalization of the travelling wave concept. The generalization is achieved by rejecting the requirement that the parameters of the wave are linear forms. These solutions are called solutions with a degenerate hodograph, and they are studied in the next chapter.

Let us apply the method to two-dimensional flows of a fluid, described by the Navier-Stokes equations

$$\begin{gathered} u_t + uu_x + vu_y + p_x = u_{xx} + u_{yy}, \\ v_t + uv_x + vv_y + p_y = v_{xx} + v_{yy}, \\ u_x + u_y = 0. \end{gathered}$$

Without loss of generality one can assume that the travelling wave type solution has the representation

$$u = u(\xi), \ \ v = v(\xi), \ \ p = p(\xi), \ \ \xi = x + ay + Dt.$$

Substituting the representation of the solution into the Navier-Stokes equations, one obtains

$$\begin{gathered} p' + (u + av + D)u' = (a^2 + 1)u'', \\ ap' + (u + av + D)v' = (a^2 + 1)v'', \\ u' + av' = 0. \end{gathered} \tag{2.35}$$

[11] Without loss of generality it is assumed that the rank of the matrix $L = (L_{\alpha\beta})$ is equal to $k$.

Taking the linear combination of the first equation and the second equation multiplied by the constant $a$, and using the third equation, one finds

$$p' = 0.$$

Integrating the third equation of (2.35), one gets

$$u = av + c,$$

where $c$ is constant. The second equation of (2.35) becomes

$$(c + D)v' = (a^2 + 1)v''.$$

Notice that by virtue of a Galilean transformation and a rotation one can assume that $c = 0$ and $a = 0$.

## 7. Completely integrable systems

One class of overdetermined systems, for which the problem of compatibility is solved, is the class of completely integrable systems. One particular case of such a system was considered in the first chapter. Here the theory of completely integrable systems is developed in the general case.

**Definition 2.13.** *A system*

$$\frac{\partial z^i}{\partial a^j} = f^i_j(a, z), \ \ (i = 1, 2, \dots, N; \ \ j := 1, 2, \dots, r) \tag{2.36}$$

*is called a completely integrable if it has a solution for any initial values $a_o$, $z_o$ in some open domain $D$.*

**Lemma 2.1.** *Any system of the type (2.36) is completely integrable if and only if the equalities*

$$\begin{gathered}\frac{\partial f^i_j}{\partial a^\beta} + f^\gamma_\beta \frac{\partial f^i_j}{\partial z^\gamma} = \frac{\partial f^i_\beta}{\partial a^j} + f^\gamma_j \frac{\partial f^i_\beta}{\partial z^\gamma}, \\ (i = 1, 2, , \dots, N; \ \beta, j = 1, 2, \dots, r)\end{gathered} \tag{2.37}$$

*are identically satisfied with respect to the variables $(a, z) \in D$.*

*Proof.*

Let $z = z(a)$ be a solution of the initial value problem

$$z(a_o) = z_o.$$

Calculating the derivatives

$$\frac{\partial}{\partial a^j}\left(\frac{\partial z^i}{\partial a^\beta}\right) = \frac{\partial}{\partial a^\beta}\left(\frac{\partial z^i}{\partial a^j}\right)$$

at the point $a_o$, one obtains

$$\frac{\partial f^i_j}{\partial a^\beta}(a_o, z_o) + f^\gamma_\beta(a_o, z_o)\frac{\partial f^i_j}{\partial z^\gamma}(a_o, z_o) = \frac{\partial f^i_\beta}{\partial a^j}(a_o, z_o) + f^\gamma_j(a_o, z_o)\frac{\partial f^i_\beta}{\partial z^\gamma}(a_o, z_o).$$

Since the initial values $a_o, z_o$ are arbitrary, the first part of the lemma is proven.

For proving the second part of the lemma, the theorems of existence, uniqueness and continuity with respect to parameters of a solution of an initial value problem are applied. Let $e$ be an arbitrary vector in the closed unit ball $\overline{B_1(0)}$, and pose the problem

$$\frac{\partial v^i}{\partial t}(t, e) = e^\alpha f^i_\alpha(a_o + te, v(t, e)), \ \ v^i(0) = z^i_o, \ \ (i = 1, 2, , \ldots, N). \quad (2.38)$$

This problem has a unique solution $v(t, e)$, which is defined in the maximal interval $t \in (0, \tau_e)$. Direct calculations show that the function $u(t, e) = v(\lambda t, e)$ is a solution of the problem

$$\frac{\partial u^i}{\partial t}(t, e) = \lambda e^\alpha f^i_\alpha(a_o + \lambda te, u(t, e)), \ \ u^i(0) = z^i_o, \ \ (i = 1, 2, , \ldots, N).$$

Because of the uniqueness of the solution for $\lambda e \in \overline{B_1(0)}$, the vector function $u(t, e) = v(t, \lambda e)$. Hence, $v(t, \lambda e)$ is defined in the interval $t \in (0, \tau_{\lambda e})$, and $\lambda \tau_{\lambda e} = \tau_e$.

Set $\tau_* = \inf_e \tau_e$, and assume that $\tau_* = 0$. This means that for any $\varepsilon > 0$ there exists a vector $e \in \overline{B_1(0)}$ such that $\tau_* = 0 \leq \tau_e < \varepsilon$. Choosing $\varepsilon = 1/k$ one constructs a sequence of the vectors $\{e_k\}$ such that $\tau_{e_k} \to 0$. Because the unit ball $\overline{B_1(0)}$ is a compact, there exists a convergent subsequence $\{e_{k_l}\} \to e_* \in \overline{B_1(0)}$. For the vector $e_*$ the solution of the problem (2.38) is defined in the interval $(0, \tau_{e_*})$, where $\tau_{e_*} > 0$. Because of the continuity with respect to the parameter $e$, there exists an interval $(0, \tau)$ and a neighborhood $U_{e_*}$ of the vector $e_*$ such that $0 < \tau < \tau_{e_*}$ and for any $e \in U_{e_*}$ the solution $v(t, e)$ is defined in the interval $(0, \tau)$. This contradicts to the condition that $\tau_{e_{k_l}} \to 0$. Hence, $\tau_* > 0$.

Set $\tau = \min(1, \tau_*)$. The functions $v^i(t, a - a_o)$ are defined for any $a \in B_\tau(a_o)$ and $t \in (0, 1]$. Let us prove this statement.

If $\tau_* \geq 1$, this follows from the definition of $\tau_*$. In fact, in this case $\tau = 1$, and for any $a \in \overline{B_\tau(a_o)}$ the functions $v^i(t, e)$ with $e = a - a_o \in \overline{B_1(0)}$ are defined in the interval $(0, \tau)$, where $\tau \geq \tau_* \geq 1$. If $\tau_* < 1$, then $\tau = \tau_*$. For any vector $a \in B_{\tau_*}(a_o)$ such that $a - a_o \neq 0$ the vector $e = \lambda^{-1}(a - a_o)$, with $\lambda = |a - a_o|^{-1}$, belongs to the ball $\overline{B_1(0)}$. Notice that $\lambda e \in \overline{B_\tau(0)} \subset \overline{B_1(0)}$. Since $\tau_{\lambda e} = \lambda^{-1}\tau_e \geq \lambda^{-1}\tau_* = \lambda^{-1}\tau \geq 1$, then for any $a \in B_\tau(a_o)$ the values $v^i(1, a - a_o)$ are defined. Thus, the functions $v^i(t, a - a_o)$ are defined at the point $t = 1$ for any $a \in B_\tau(a_o)$.

Let us show that the functions $z^i(a) = v^i(1, a - a_o)$, $\forall a \in B_\tau(a_o)$ satisfy the equations

$$R^i_j(a) = \frac{\partial z^i}{\partial a^j}(a) - f^i_j(a, z(a)) = 0, \ \forall a \in B_\tau(a_o).$$

For this purpose the functions $S^i_j(t) = tR^i_j(a_o + te)$, where $e = a - a_o \in B_\tau(0)$, are studied.

First one can obtain some useful identities. Notice that

$$z^i(a_o + te) = v^i(1, te) = v^i(t, e).$$

Differentiating these equalities with respect to $t$, and using equations (2.38), one has

$$e^\beta \frac{\partial z^i}{\partial a^\beta}(a_o + te) = e^\beta f^i_\beta(a_o + te, z(a_o + te)),$$

which at the point $t = 1$ become

$$(a^\beta - a^\beta_o)\frac{\partial z^i}{\partial a^\beta}(a) = (a^\beta - a^\beta_o) f^i_\beta(a, z(a)).$$

Differentiating these relations with respect to the coordinate $a^j$, one obtains

$$\frac{\partial z^i}{\partial a^j}(a) + (a^\beta - a^\beta_o)\frac{\partial^2 z^i}{\partial a^\beta a^j}(a) =$$
$$f^i_j(a, z(a)) + (a^\beta - a^\beta_o)\left(\frac{\partial f^i_\beta}{\partial a^j}(a, z(a)) + \frac{\partial f^i_\beta}{\partial z^\gamma}(a, z(a))\frac{\partial z^\gamma}{\partial a^j}(a, z(a))\right).$$

Substituting the expressions $\frac{\partial z^i}{\partial a^j}(a) = f^i_j(a, z(a)) + R^i_j(a)$ into these equations, they become

$$(a^\beta - a^\beta_o)\frac{\partial^2 z^i}{\partial a^\beta a^j}(a) =$$
$$-R^i_j(a) + (a^\beta - a^\beta_o)\left(\frac{\partial f^i_\beta}{\partial a^j}(a, z(a)) + \frac{\partial f^i_\beta}{\partial z^\gamma}(a, z(a))\frac{\partial z^\gamma}{\partial a^j}(a, z(a))\right).$$

Because of the definition $S^i_j(t) = tR^i_j(a_o + te)$, at the point $a = a_o + te$ the last equations are

$$\begin{aligned} te^\beta\left(\frac{\partial^2 z^i}{\partial a^j \partial a^\beta}(a_o + te)\right) &= -R^i_j(a_o + te) + te^\beta \frac{\partial f^i_\beta}{\partial a^j}(a_o + te) + \\ & e^\beta\left(S^\gamma_j(t) + tf^\gamma_j(a_o + te)\right)\frac{\partial f^i_\beta}{\partial z^\gamma}(a_o + te). \end{aligned} \tag{2.39}$$

From another point of view, differentiating the relations $tR^i_j(a_o + te) = S^i_j(t)$ with respect to $t$, one obtains

$$\frac{dS^i_j}{dt} = R^i_j + te^\beta\left(\frac{\partial^2 z^i}{\partial a^j \partial a^\beta} - \frac{\partial f^i_j}{\partial a^\beta} - (R^\gamma_\beta + f^\gamma_\beta)\frac{\partial f^i_j}{\partial z^\gamma}\right).$$

Using (2.39), these equations are reduced to the equations

$$\frac{dS_j^i(t)}{dt} = -te^\beta\left(\frac{\partial f_j^i}{\partial a^\beta} + (R_\beta^\gamma + f_\beta^\gamma)\frac{\partial f_j^i}{\partial z^\gamma}\right) + e^\beta\left(t\frac{\partial f_\beta^i}{\partial a^j} + \left(S_j^\gamma + tf_j^\gamma\right)\frac{\partial f_\beta^i}{\partial z^\gamma}\right).$$

Changing $tR_j^i$ with $S_j^i(t)$, one finds

$$\frac{dS_j^i(t)}{dt} = -e^\beta S_\beta^\gamma\frac{\partial f_j^i}{\partial z^\gamma} + e^\beta S_j^\gamma\frac{\partial f_\beta^i}{\partial z^\gamma} + te^\beta\left(\frac{\partial f_\beta^i}{\partial a^j} + f_j^\gamma\frac{\partial f_\beta^i}{\partial z^\gamma} - \frac{\partial f_j^i}{\partial a^\beta} - f_\beta^\gamma\frac{\partial f_j^i}{\partial z^\gamma}\right)$$

Because of the relations (2.37), one finds that the functions $S_j^i(t)$ satisfy the problem

$$\frac{dS_j^i}{dt} = -e^\beta\left(S_\beta^\gamma\frac{\partial f_j^i}{\partial z^\gamma} - S_j^\gamma\frac{\partial f_\beta^i}{\partial z^\gamma}\right), \quad S_j^i(0) = 0.$$

By virtue of the uniqueness of the solution of this problem, one finds that $S_j^i(t) = 0$ or $R_j^i(a) = 0$.

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