

## Chapter 2

# ADVERTISING AND ADVERTISING CLAIMS OVER TIME

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**Abstract** Advertising budget allocation with carryover effects over time is a problem that was treated extensively by economists. Additional developments were carried out by Sethi who has also provided some outstanding review papers. The model treated by Sethi were essentially defined in terms of optimal control problems using deterministic advertising models while my own were essentially sales response stochastic models with advertising budget determined by stochastic control problems. These problems continue to be of academic and practical interest. Issues relating to the “advertising message” such as truthful claims advertising directed to first time buyers has not attracted much attention however.

The purpose of this paper is to address issues relating to advertising and their messages by suggesting a stochastic advertising-repeat purchase model. In this model, advertising directed to first time buyers is essentially defined by two factors: the advertising budget and the advertising message (such as statement regarding the characteristics of a product, its lifetime etc.). Consumers experience in case they buy the product will define the advertising message “reliability”, namely that the probability that advertised message are confirmed or not. Repeat purchasers, however, are influenced by two factors, on the one hand the advertising messages that are directed to experienced consumers and of course the effects of their own experience (where past advertising claims whether truthful, or not, interact with customers’ personal experience). Advertising claims that underestimate products characteristics might be “reliable” but then they might not entice first time purchasers, while overly optimistic advertising messages might entice first time purchasers but be perceived as unreliable by repeat purchasers who might switch to other competing brands. In this sense, the decision to advertise is necessarily appended by the decision to “what to advertise”, which may turn out to be far more important for a firm. This paper provides a theoretical approach to deal with this issue.

## 1. Introduction

Advertising budget allocation with carryover effects over time is a problem treated extensively by economists (Dorfman and Steiner (1954); Nerlove and Arrow (1962); Gould (1970); Nelson (1970); Nelson (1974); Schmalensee (1972); Katowitz and Mathewson (1979)), marketers (Vidale and Wolfe (1957); Ehrenberg (1972); Feichtinger (1982); Feichtinger et al. (1988); Schmittlein et al. (1985)). Additional developments were carried by both Sethi (Sethi (1973); Sethi (1974); Sethi (1975); Sethi (1977a); Sethi (1981); Sethi (1983a); Sethi (1983b)), who has also provided some outstanding review papers, Sethi (1977b), Feichtinger, Hartl and Sethi (1994) and myself (Tapiero (1975a); Tapiero (1975b); Tapiero (1977); Tapiero (1978); Tapiero (1979); Tapiero (1981); Tapiero (1982a); Tapiero (1982b); Tapiero (1982c); Tapiero (1982d)) as well as Farley and Tapiero (1981); Farley and Tapiero (1982) and Tapiero, Elyashberg and Wind (1987). As the Sethi, Feichtinger and Hartl papers attest, the number of references related to these problems is indeed extremely large. The models treated by Sethi are essentially defined in terms of deterministic optimal control problems while my own were essentially sales response stochastic models defining the optimal advertising policy in terms of stochastic control problems. A number of such studies value advertising expenses in terms of their contribution to the firm profit objectives or their effects on competitive posture and market structure such as the market response, the effects of memory, competition and other important topics that differentiate advertising models by the hypotheses they make about the sales response to advertising (through goodwill-capital accumulation, word of mouth and their like).

These problems continue to be of academic and practical interest both by raising new hypotheses regarding the effects of carry-over (memory) effects of advertising and the market competition structure (leading thereby to differential games for example, Tapiero (1978)). Issues relating to advertising claims (such as truthfulness in advertising) and the effects of experience and advertising efficiency on repeat purchasers has attracted relatively little attention however. This is in contradiction to strong empirical evidence that advertising weights (quantities) do not always matter while advertising copy may have a greater effect on sales response (Lodish et al. (1995)). Explicitly, Lodish et al. (1995), using extensive and shared data on advertising on TV claim that increasing advertising budgets in relation to competitors does not increase sales in general. However, changing brand, copy and media strategies in categories can in many cases lead to a sales response to advertising. Furthermore, they conclude that “New brands or line extensions tend to be

more responsive to alternative T.V. advertising plans than established products". Such a claim supports the hypothesis that advertising acts essentially on "first time purchasers" and less on purchasers of well established brands that derive their essential sales from repeat purchasers.

The purpose of this paper is to address issues relating to the relationships between advertising budgets and advertising claims based on a stochastic model of advertising-quality and quantity (see also Tapiero (2000)). We clearly distinguish between first time purchasers and repeat purchasers, the former determined in terms of a stochastic "Capital Goodwill Model" (Nerlove and Arrow (1962); Tapiero (1978)) while the latter is based on a probability response to ex-post product consumption and satisfaction. Further, unlike previous optimal and stochastic control models of advertising, we recognize that in many products a firm derives benefits not only from product sales to customers but also from the services they sell to these customers. Such an assumption is increasingly realistic and emphasizes as well the importance of repeat purchasing management in firms' marketing and advertising strategies.

Consumers experience compared to advertising claims defines the advertising claim "reliability", namely the probability that an advertised claim is confirmed or not by the experienced purchaser. Of course, "confirmation" of an advertising claim by an experienced client contributes to repeat purchase while, "disappointment" contributes (in probability) to a consumer switching to some other firms. Since true products characteristics are necessarily random (due to the production process, use and misuse of the product) advertising claims truthfulness is inherently random as well. Thus, there is always a probability that advertising claims are not met. Advertising claims that underestimate product characteristics might be "reliable", namely mostly true, but then they might not entice first time purchasers, while overly optimistic advertising claims might entice first time purchasers but be perceived as unreliable by repeat purchasers who might switch to other competing products. In this sense, the decision to advertise is necessarily concurrent to the decision to "what to advertise". Such decisions are compounded by the fact that in prevalent marketing philosophy, a consumer is also a consumer of services (such as warranties, product servicing, etc.) and a firm profits not only from the revenues generated at the time of sale but also in derived revenues maintained as long as the customer remains a client of the firm. This paper provides some preliminary approaches to dealing with these issue by developing a sales/repeat purchase response stochastic model that defines these characteristics and by considering a decision model that raises a number of issues such as "how much to advertise" conjointly with "what to advertise". We begin by considering a long run

(average based) decision model and continue with a dynamic (optimal control) model.

## 2. Advertising Quantity and Advertising Reliability: Stationary Model

Let first time purchasers' sales  $S(p, x)$ , be determined by the price  $p(t)$  and a "Goodwill-Capital"  $\{x, t \geq 0\}$ . "Goodwill" is a random function, expressed by a non-homogenous non-stationary Poisson Process with mean  $\lambda(t)$  and generated by advertising efforts  $a(t)$  carried over time with an exponential memory and a forgetting rate parameter  $m$  (Nerlove and Arrow (1962); Tapiero (1975a); Tapiero (1977); Tapiero (1978)):

$$P(x(t) = i) = \frac{[\lambda(t)^i e^{-\lambda(t)}]}{i!} \quad (2.1)$$

$$d\lambda(t)/dt = -m\lambda + qa(t), \quad \lambda(0) = \lambda_0, q > 0$$

where  $q$  is a parameter expressing the advertising efficiency. For simplicity we set  $S(p, x) = x$ . Now let there be an advertising claim  $\alpha$ . The "better" this claim, the larger the effects of advertising on new purchasers. This means that  $q = q(\alpha)$ ,  $q'(\alpha) > 0$ , thereby generalizing the stochastic goodwill model in (2.1). In addition, an advertising claim confirmed by consumption experience will induce a repurchase and vice versa, non confirmation will induce disloyalty. Over the long run, assuming stationary policies (which are starred), we have:  $\lambda^* = q(\alpha^*) \alpha^* / m$ . Thus, in a stationary state, the rate of incoming first time purchasers has a Poisson distribution whose mean is  $\lambda^*$ , a function of both the advertising rate and the advertised claim.

An experienced unsatisfied customer may be a lost customer while a satisfied one may repeat purchase. Let  $\alpha^*$  be the advertising claim and let  $\tilde{\alpha}$  be the true product characteristic, a random variable, with cumulative distribution  $F(\tilde{\alpha})$ . If  $\tilde{\alpha} \geq \alpha^*$ , the probability of a repeat purchase after a unit consumption is  $1 - F(\alpha^*)$ , meaning that the customer experience is better than the advertised claim. If a client remains loyal for  $\tilde{k}_1$  units, till a unit is found to be non-conforming, then the total number of units purchased is  $1 + \tilde{k}_1$ . This can be specified by a geometric distribution, or:

$$g_1(\tilde{k}_1 : 1) = [1 - F_\alpha(\alpha^*)]^{\tilde{k}_1 - 1} F_\alpha(\alpha^*) \quad (2.2)$$

If the customer remains loyal until  $r$  units are found to be non-conforming, the number of units acquired by a customer is  $1 + \tilde{k}_r$ , given by the negative binomial distribution:

$$g_r(\tilde{k}_r : r) = C_{r-1}^{\tilde{k}_r - 1} [1 - F_\alpha(\alpha^*)]^{\tilde{k}_r - r} F_\alpha(\alpha^*)^r \quad (2.3)$$

with mean and variance:

$$\begin{aligned} E(k_r) &= r\theta^* \\ \text{var}(k_r) &= \frac{r\theta^*}{1 - F_\alpha(\alpha^*)} \\ \theta^* &= \frac{F_\alpha(\alpha^*)}{1 - F_\alpha(\alpha^*)} \end{aligned} \quad (2.4)$$

where  $\theta^*$  is the odd that the advertising claim turns out to be false. Estimate for these parameters can be determined based on historical data of the number of units bought by individual customers while they were loyal customers. Such data also reveals customers' "impatience". For example, "an impatient customer" will not repeat purchase as soon as a non-conforming unit is experienced. In this case, the total number of units bought by the customer would be only  $1 + \tilde{k}_1$  as stated earlier. A patient customer "might give a second a chance" to the firm and remain loyal if the quantity  $k_1$  consumed thus far is greater than some parameter  $k_1^*$  etc. Therefore, customers impatience can be denoted in terms of probability parameters  $\chi_j$ , expressing the proportion willing to accept  $j$  non-conforming claims before resigning and not repeat purchase. In this case, the probability distribution for the number of units bought by a customer is given by the mixture distribution:

$$\tilde{k} = \tilde{k}_i \text{ wp } \chi_i, \quad \tilde{k}_i = g_i(\cdot : i), \quad i = 1, 2, \dots, n \quad (2.5)$$

where  $\tilde{k}_i$ ,  $i = 1, 2, \dots$  are denoted by the negative binomial distribution (2.2). For example, say that in a heterogeneous population a proportion  $\chi_1$  is impatient while the remaining is "patient" willing to give a second chance to non-conforming experienced units. Then, the probability distribution of the number units sold in repeat purchase is:

$$\tilde{k} = \tilde{k}_i \text{ wp } \chi_i, \quad \tilde{k}_i = g_i(\cdot : i), \quad i = 1, 2 \quad (2.6)$$

where  $\chi_2 = 1 - \chi_1$ . As a result, the mean number of units that are repeat purchased and their variance are given by:

$$\begin{aligned} E(\tilde{k}) &= \sum_{j=1}^{\infty} E(\tilde{k}_j) \chi_j \\ \text{var}(\tilde{k}) &= \sum_{j=1}^{\infty} E(\tilde{k}_j^2) \chi_j - \left( \sum_{j=1}^{\infty} E(\tilde{k}_j) \chi_j \right)^2 \end{aligned} \quad (2.7)$$

If the underlying characteristic of the product advertised is its lifetime, a random variable  $\tilde{\tau}_i$ , identically and independently distributed, and if the

customer repeat purchases  $\left(\tilde{k}\right)$  units, then the time a customer remains loyal to the firm is given by the Compound random variable  $\tilde{T}$

$$\tilde{T} = \sum_{i=1}^{1+\tilde{k}} \tilde{\tau}_i \quad (2.8)$$

whose mean and variance can be calculated explicitly by:

$$\begin{aligned} E(\tilde{T}) &= E(1 + \tilde{k}) E(\tilde{\tau}_i); \\ var(\tilde{T}) &= E(1 + \tilde{k}) var(\tilde{\tau}_i) + var(1 + \tilde{k}) E(\tilde{\tau}_i)^2 \end{aligned} \quad (2.9)$$

In this sense, equations (2.7) and (2.9) provide an estimate for the number of units bought by an individual customer and the amount of time the customer remains loyal (and thereby, consuming the firm services associated to the product).

Further, since the number of first time purchasers over a given period of time has a Poisson distribution with mean rate  $\lambda^*$  while “loyalty time” is a random variable  $\tilde{T}$  whose mean and variance are given by (2.9), we recognize this type of model as an M/G/Infinity queue (Gross and Harris (1985)). Thus, in a stationary state, the number of customers, paying a service fee of  $w$  per unit time has also a Poisson distribution with mean:

$$\rho = \lambda^* E(\tilde{T}) = \frac{q(\alpha^*) a^*}{m} E(1 + \tilde{k}) E(\tilde{\tau}_i) \quad (2.10)$$

The expected number of sales per unit time is given by  $\psi$ , which is equated here to the total number of units bought in a time interval by customers arriving at a specific instant of time divided by this time interval, or

$$\psi = \lambda^* \frac{E(1 + \tilde{k})}{E(\tilde{T})} = \frac{\lambda^*}{E(\tilde{\tau}_i)} \quad (2.11)$$

Setting to  $C(a^*)$  the advertising cost per unit time, the firm profit per unit time is:

$$\begin{aligned} Max_{\alpha^*, a^*} \pi &= w\rho + p\psi - C(a^*) \\ &= \frac{q(\alpha^*) a^*}{m} \left( wE(1 + \tilde{k}) E(\tilde{\tau}_i) + \frac{p}{E(\tilde{\tau}_i)} \right) - C(a^*) \end{aligned} \quad (2.12)$$

An optimization of (2.12) with respect to the advertising budget and the advertising claim will provide a solution to this problem. Explicitly, a partial derivative with respect to advertising yields:

$$\pi_a = \frac{q(\alpha^*)}{m} \left( wE(1 + \tilde{k}) E(\tilde{\tau}_i) + p \frac{1}{E(\tilde{\tau}_i)} \right) - C'(a^*) \quad (2.13)$$

At the optimum however,  $\pi_a=0$  and therefore:

$$C'(a^*) = \frac{q(\alpha^*)}{m} \left( wE(1 + \tilde{k}) E(\tilde{\tau}_i) + p \frac{1}{E(\tilde{\tau}_i)} \right) \quad (2.13a)$$

And the marginal cost of advertising equals the marginal profit from both sales and derived profits as given by (2.13a). Similarly, a partial derivative with respect to the advertising claim yields:

$$\pi_{\alpha^*} = (\pi + C(a^*)) \frac{q_{\alpha}(\alpha^*)}{q(\alpha^*)} + \frac{q(\alpha^*) a^*}{m} \left( wE(\tilde{\tau}_i) \left( E(1 + \tilde{k}) \right)_{\alpha} \right) \quad (2.14)$$

At optimality  $\pi_{\alpha^*}=0$  we have:

$$\frac{m(\pi + C(a^*))}{a^* wE(\tilde{\tau}_i)} = \frac{\left( \left( E(1 + \tilde{k}) \right)_{\alpha} \right)}{(1/q(\alpha^*))_{\alpha}} \quad (2.15)$$

However, at optimality,  $\pi_a a + aC'(a^*) = \pi + C(a^*)$ ,  $\pi_a = 0$  and therefore,

$$\frac{mC'(a^*)}{wE(\tilde{\tau}_i)} = \frac{\left( \left( E(1 + \tilde{k}) \right)_{\alpha} \right)}{(1/q(\alpha^*))_{\alpha}} \quad (2.16)$$

Inserting, (2.13) in (2.16), we obtain at last an expression in the advertising claim only:

$$1 + \frac{p}{w(E(\tilde{\tau}_i))^2} (\ln(q(\alpha^*)))_{\alpha} = - \left( E(\tilde{k}) \right)_{\alpha} - E(\tilde{k}) \quad (2.17)$$

Explicitly, if we let  $q(\alpha^*) = e^{\varepsilon \alpha^*}$ , then we have  $q_{\alpha} = \varepsilon q$ , and:

$$1 + \frac{p\varepsilon}{w(E(\tilde{\tau}_i))^2} = - \left( E(\tilde{k}) \right)_{\alpha} - E(\tilde{k}) \quad (2.18)$$

Where  $\left( E(1 + \tilde{k}) \right)_{\alpha} < 0$  (since the more we claim, the smaller the number of units repurchased), and the optimal advertising claim is obtained by a solution of (2.18). Assume again for simplicity that a consumer remains loyal until a unit consumed does not conform to advertised claims, then the number of units repeat purchased is a geometric random variable given by:  $g_1(\tilde{k}_1 : 1) = [1 - F_{\alpha}(\alpha^*)]^{\tilde{k}-1} F_{\alpha}(\alpha^*)$  and  $E(\tilde{k}_1) = \theta^*$  where  $\theta^*$  is the odd that an advertised claim is not met, or

$\theta^* = F_\alpha(\alpha^*) / [1 - F_\alpha(\alpha^*)]$ . As a result,  $\left(E(1 + \tilde{k})\right)_\alpha = \theta_\alpha$  and therefore,

$$1 + \frac{p\varepsilon}{w(E(\tilde{\tau}_i))^2} = -\theta_\alpha - \theta > 0 \quad (2.19)$$

which can be solved for the advertised claim. For simplicity say that the odds of an advertised claim is met is given by a logistic regression of the type  $\ln \theta^* = a_0 - a_1 \alpha$ , then  $\theta_\alpha = -a_1 \theta$ ,  $a_1 > 1$  and therefore:

$$1 + \frac{p\varepsilon}{w(E(\tilde{\tau}_i))^2} = (a_1 - 1) e^{a_0 - a_1 \alpha} \quad (2.20)$$

We note in particular:

$$\begin{aligned} \frac{d\vartheta}{d\alpha} &= \frac{\varepsilon}{a_1 (a_1 - 1) \theta (E(\tilde{\tau}_i))^2} \\ 1 + \vartheta \frac{\varepsilon}{(E(\tilde{\tau}_i))^2} - (a_1 - 1) e^{a_0 - a_1 \alpha} &= 0 \end{aligned} \quad (2.21)$$

$$\vartheta = p/w$$

Thus, the larger the advertising efficiency, the greater the claim and the greater the relative price  $p/w$ . And vice versa, when the odds that an advertised claim is not met by a consumption experience the price  $p/w$  is smaller. Interestingly, the larger the product expected life, the smaller the relative price  $p/w$ . The optimal claim, in this particular case is then specified by (2.20), or:

$$\alpha^* = \ln \left( \frac{(a_1 - 1) w (E(\tilde{\tau}_i))^2}{w (E(\tilde{\tau}_i))^2 + p\varepsilon} \right)^{\frac{a_0}{a_1}} \quad (2.22)$$

The corresponding advertising budget is then determined by equation (2.16). If an advertising claim is set to a conservative low figure,  $q(\alpha^*)$  will be smaller reducing the advertising efficiency but maintaining customers as repeat purchasers. Further, the higher the advertising claim, the higher the advertising budget (i.e. the firm has an essentially first purchaser dominant strategy). We can also note that the larger the advertising forgetting rate and the smaller the service payment fee, the more we advertise and as a result, the more we shall claim. Our conclusions, based on a simple and theoretical analysis confirms some hypotheses set by Lodish et al. (1995) stating that in well established brands (where a major part of sales are generated by loyal repeat purchasers), the tendency will be to advertise less.



### 3. The Non-stationary Advertising-Claims Model

Let  $y(t)$  be the number of first purchasers at time  $t$  with  $y(s) = x$ . Now assume that at new purchase events occur at the random (Poisson) times  $\tau_i, s < \tau_1 < \tau_2 < \dots < \tau_i < \dots$ , with a non-stationary Poisson process given by:

$$\frac{d\lambda(t)}{dt} = -m\lambda(t) + q(\alpha) a(t), \quad \lambda(0) = \lambda_0 \quad (2.23)$$

For generality, we let  $\xi_i$  be the number of new first purchasers, a *random quantity*. This means that at time  $\tau_i$ , at which time new purchasers come, we have:

$$\begin{aligned} y(t + dt) &= y(t) \text{ at } t \in (\tau_i, \tau_{i+1}) \\ y(\tau_i^+) &= y(\tau_i^-) + \xi_i \text{ at } t = \tau_i \\ i &= 1, 2, 3, \dots \end{aligned} \quad (2.24)$$

Or,

$$y(t) = y(0) + \int_0^t \int_{\mathbb{R}_n} z \mu(dx, dz) \quad (2.25)$$

where  $\mu(dx, dz)$  is a function denoting the number of jumps of first purchases process  $y(t)$  in the time interval  $(0, t]$  defined as follows:

$$\mu(\Delta, A) = \mu(t + \Delta t, A) - \mu(t, A) \quad (2.26)$$

and  $A$  is a Borel subset on  $\mathbb{R}^n$ . The measure  $\mu(\Delta, A)$  is called the jump measure of the process  $\{y(t), t \geq 0\}$ . If first time events occur one at a time, then we have a simple nonstationary Poisson process with,

$$\begin{aligned} Prob[\tau_{i+1} \geq s | \tau_i] &= \exp \left[ - \int_{\tau_i \wedge s}^{\infty} \lambda(x) dx \right] \text{ and} \\ \mu(dx, dz) &= \mu(dx) \delta(z - 1) \end{aligned} \quad (2.27)$$

as indicated in this paper where  $\mu(t)$  is a point process whose jumps equals one or such that:

$$\begin{aligned} P(\mu(dt) = 1) &= \lambda(t)dt + o(dt) \\ P(\mu(dt) = 0) &= 1 - \lambda(t)dt + o(dt) \\ P(\mu(dt) \geq 2) &= o(dt) \end{aligned} \quad (2.28)$$

Hence,  $\mu(t)$  is a random variable distributed according to a Poisson law whose mean (parameter) is:

$$\Lambda(t) = \int_0^t \lambda(x)dx; \frac{d\Lambda(t)}{dt} = \lambda(t); \Lambda(0) = 0 \quad (2.29)$$

In summary, first time purchases are generated by a non-homogenous Poisson process with mean  $\lambda(t)$  –the goodwill equation, a function of advertising  $a(t)$  at time  $t$  and the advertising claim  $\alpha$ . The implications of this observation is that at time  $t$ , the total number of cumulative first purchases is given by  $N(t)$  which has a Poisson distribution with parameter  $\Lambda(t)$  given by equation (2.29). If the amount of time a customer remains loyal to the firm (namely remains a repeat purchaser) has a probability density function  $b(\tilde{T})$ , then the total number of customers that have experienced the firm's product and are no longer customers is given by the event  $(M(t)|N(t)=n)$ . Thus, the current number of customers the firm has (and paying for derived services and charges associated to the product consumption) is  $n-(M(t)|N(t)=n)$ . First, note the probability distribution of the conditional event is given by the binomial distribution which is specified as follows:

$$P[M(t)|N(t) = n] = \binom{n}{M(t)} \pi(t)^{M(t)} [1 - \pi(t)]^{n-M(t)}, \quad (2.30)$$

$$\pi(t) = \int_0^t \frac{\lambda(u)b(t-u)}{\Lambda(t)} du$$

Since  $n$  has a Poisson distribution,  $M(t)$  has also a Poisson distribution with mean  $\Phi(t) = \Lambda(t)\pi(t)$ , explicitly given by equation (2.31),

$$P(M(t) = j) = \frac{\Phi(t)^j \exp(-\Phi(t))}{j!} \quad (2.31)$$

$$\Phi(t) = \int_0^t \lambda(u)b(t-u)du$$

As a result, the number of clients that have left the firm is given by the difference of two non-homogenous Poisson processes, one with mean given by equation (2.29) and the other by (2.31). Of course, when the first purchase rate is constant, and for  $t$  large, the amount of time a customer remains loyal is independent of time with  $M(t)$  a Poisson distribution with mean:

$$\Lambda^\infty = \lambda \int_0^\infty b(t-u)du = \lambda \int_0^\infty dB(u) = \lambda E(\tilde{T}) \quad (2.32)$$

where  $E(\tilde{T})$  is the mean loyalty time. From a modeling point of view, it is seen that  $a(t)$ , the advertising budget, determines the mean rate of new sales while the density function  $B(\cdot)$  is defined by the advertising claim. First note that revenues derived from current customers has a mean given by  $w(\Lambda(t) - \Phi(t))$ . Product sales revenues from both first and repeat purchasers are calculated as follows. Consider a new client arriving at time  $u$  and buying at price  $p$ . Such a client will repeat purchase  $\tilde{k}$  units at times  $u + \tilde{\tau}_1, u + \tilde{\tau}_1 + \tilde{\tau}_2, \dots, u + \sum_{i=1}^{\tilde{k}} \tilde{\tau}_i$  and since the arrival rate is Poisson and given by  $\lambda(u)$  at this time, the discounted value of all future purchases (to time  $u$ ) by the client arriving at time  $u$  and discounted at the discount rate  $r$  is:

$$E \left\{ p\lambda(u)E \left( 1 + \sum_{j=1}^{\tilde{k}} e^{-r \sum_{i=1}^j \tau_i} \right) \right\} \quad (2.33)$$

Let  $E(e^{-r\tilde{\tau}}) = L_{\tau}^*(r)$  be the Laplace Transform of an individual unit life time. Note that if the client repeats purchase one unit only ( $k = 1$ ), then  $E(e^{-r\tilde{\tau}}) = L_{\tau}^*(r)$ . For two units ( $k=2$ ), with independent product lifetimes,  $E(e^{-r\tilde{\tau}}) + E(e^{-r(\tilde{\tau}_1 + \tilde{\tau}_2)}) = L_{\tau}^*(r) + (L_{\tau}^*(r))^2$  etc. for a larger number of units. As a result, if the customer buys  $j$  units in repeat purchase, we have:

$$\begin{aligned} & L_{\tau}^*(r) + (L_{\tau}^*(r))^2 + \dots + (L_{\tau}^*(r))^j = \\ &= \sum_{i=0}^j (L_{\tau}^*(r))^i - 1 \\ &= \frac{1 - (L_{\tau}^*(r))^j}{1 - L_{\tau}^*(r)} - 1 \\ &= \frac{L_{\tau}^*(r)(1 - (L_{\tau}^*(r))^{j-1})}{1 - L_{\tau}^*(r)} \end{aligned}$$

As a result, the expected discounted value of all repeat purchases by clients that have had their first purchases at time  $u$  is:

$$\begin{aligned}
 p\lambda(u)E\left(\sum_{j=1}^{\tilde{k}} e^{-r\sum_{i=1}^j \tau_i}\right) &= p\lambda(u) \sum_{j=1}^{\infty} P_j \frac{L_{\tau}^*(r) \left(1 - (L_{\tau}^*(r))^{j-1}\right)}{1 - L_{\tau}^*(r)} \\
 &= p\lambda(u) \frac{L_{\tau}^*(r)}{1 - L_{\tau}^*(r)} \sum_{j=1}^{\infty} P_j \left(1 - (L_{\tau}^*(r))^{j-1}\right) \\
 &= p\lambda(u) \frac{L_{\tau}^*(r)}{1 - L_{\tau}^*(r)} \\
 &\quad \times \left(\sum_{j=1}^{\infty} P_j - \frac{1}{L_{\tau}^*(r)} \sum_{k=1}^{\infty} P_k (L_{\tau}^*(r))^k\right) \\
 &= p\lambda(u) \frac{L_{\tau}^*(r)}{1 - L_{\tau}^*(r)} \left(1 - \frac{\Pi_P(L_{\tau}^*(r))}{L_{\tau}^*(r)}\right)
 \end{aligned} \tag{2.34}$$

where  $P_j$  is the probability that there are  $j$  such repeat purchases and  $\Pi_P$  is the probability generating function of the number of units repeat purchased, then equation (2.34) can be summarized by:

$$p\lambda(u)E\left(\sum_{j=1}^{\tilde{k}} e^{-r\sum_{i=1}^j \tau_i}\right) = p\lambda(u) \frac{L_{\tau}^*(r)}{1 - L_{\tau}^*(r)} \left(1 - \frac{\Pi_P(L_{\tau}^*(r))}{L_{\tau}^*(r)}\right) \tag{2.35}$$

For example if a customer does not repeat purchase as soon as consumption experience does not conform to advertising claim, we have:

$$P_k = g_1(k_1 : 1) = [1 - F_{\alpha}(\alpha^*)]^{k-1} F_{\alpha}(\alpha^*) \tag{2.36}$$

whose probability generating function is:

$$\begin{aligned}
 \Pi_P &= \sum_{k=1}^{\infty} P_k z^k = \\
 &= z F_{\alpha}(\alpha^*) \sum_{k=1}^{\infty} [1 - F_{\alpha}(\alpha^*)]^{k-1} z^{k-1} = \\
 &= \frac{z F_{\alpha}(\alpha^*)}{1 - z [1 - F_{\alpha}(\alpha^*)]}
 \end{aligned} \tag{2.37}$$

and therefore,

$$\begin{aligned}
 p\lambda(u)E\left(\sum_{j=1}^{\tilde{k}} e^{-r\sum_{i=1}^j \tau_i}\right) &= \\
 &= p\lambda(u)\frac{L_{\tau}^*(r)}{1-L_{\tau}^*(r)}\left(\frac{1-L_{\tau}^*(r)[1-F_{\alpha}(\alpha^*)]-F_{\alpha}(\alpha^*)}{1-L_{\tau}^*(r)[1-F_{\alpha}(\alpha^*)]}\right)
 \end{aligned} \tag{2.38}$$

If for simplicity we keep this expression to calculate the discounted value of a customer's repeat purchases, we have the following optimal control problem:

$$\underset{a_1(t) \geq 0, \alpha}{Max} \int_0^{\infty} e^{-ru} [p\lambda(u)G(\alpha) + w(\Lambda(u) - \Phi(u)) - C(a)] du$$

Subject to:

$$d\lambda(u)/dt = -m\lambda + q(\alpha)a(u), \quad \lambda(0) = \lambda_0$$

$$d\Lambda(u)/dt = \lambda(u), \quad \Lambda(t) = \Lambda_0$$

$$\Phi(u) = \int_0^u \lambda(v)b(u-v)dv$$

$$G(\alpha) = 1 + \frac{L_{\tau}^*(r)}{1-L_{\tau}^*(r)}\left(1 - \frac{\Pi_P(L_{\tau}^*(r))}{L_{\tau}^*(r)}\right) \tag{2.39}$$

The solution of this problem, a three states variables control problem, can be solved by an application of Pontryagin Maximum Principle. For discussion purposes, if we set the advertising claim constant and let the advertising policy be defined a-priori in terms of some function of time such as  $C(a) = C(\bar{a}, t)$  (or as a feedback function of the means of first time purchasers, repeat purchasers and lost clients for example), we have by simple Laplace transform techniques that:

$$s\lambda^*(s) - \lambda(0) = -m\lambda^*(s) + q(\alpha)a^*(s) \tag{2.40}$$

$$, \quad s\Lambda^*(s) - \Lambda(0) = \lambda^*(s), \quad \Phi^*(s) = \lambda^*(s)b^*(s) \tag{2.41}$$

where

$$\begin{aligned}
 \lambda^*(s) &= \int_0^{\infty} e^{-s\lambda(t)} dt, \\
 \Lambda^*(s) &= \int_0^{\infty} e^{-s\Lambda(t)} dt, \\
 \Phi^*(s) &= \int_0^{\infty} e^{-s\Phi(t)} dt
 \end{aligned} \tag{2.42}$$

As a result,

$$\begin{aligned}
 \lambda^*(s) &= \frac{q(\alpha) a^*(s) + \lambda(0)}{(s + m)}, \\
 \Lambda^*(s) &= \frac{q(\alpha) a^*(s) + \lambda(0)}{s(s + m)} + \frac{\Lambda(0)}{s}, \\
 \Phi^*(s) &= \frac{q(\alpha) a^*(s) + \lambda(0)}{(s + m)} b^*(s).
 \end{aligned} \tag{2.43}$$

where  $b(t) = \sum_{i=1}^{\tilde{k}} \tau_i$  and therefore  $b^*(s) = \prod_P (L_\tau(s))$  where  $L_\tau(s)$  is the transform of an individual unit life time while  $\prod_P(.)$  is the probability generating function of the number of units bought by a customer (and thereby a function of the advertising claim). Inserting in our objective function, we have:

$$\begin{aligned}
 J &= \int_0^{\infty} e^{-ru} \Pi du = \\
 &= p\lambda^*(r)G(\alpha) + w(\Lambda^*(r) - \Phi^*(r)) - C^*(\bar{a}, r)
 \end{aligned} \tag{2.44}$$

For a linear advertising cost, this is reduced to:

$$\begin{aligned}
 J &= \frac{a^*(r)}{(r + m)} \left\{ pq(\alpha)G(\alpha^*) + wq(\alpha)\left(\frac{1}{r} - \prod_P(L_\tau(r))\right) - 1 \right\} \\
 &+ p\frac{\lambda(0)}{(r + m)}G(\alpha^*) + w\left(\frac{\lambda(0)}{r(r + m)} + \frac{\Lambda(0)}{r} - \frac{\lambda(0)}{(r + m)}b^*(r)\right)
 \end{aligned} \tag{2.45}$$

which is linear in the advertising policy transform. For example, if the advertising policy consists of a fixed quantity  $\bar{a}$ , then  $a^*(r) = \bar{a}/r$  and the

optimal advertising allocation and claim policy if given by maximizing (2.44) with respect to  $(\bar{a}, \alpha)$ .

$$\frac{\partial J}{\partial \alpha} = \frac{a^*(r)}{(r+m)} \left\{ \begin{array}{l} +pq'(\alpha)G(\alpha^*) + \\ +p \left( q(\alpha) + \frac{\lambda(0)}{(r+m)} \right) G'(\alpha^*) \\ +wq'(\alpha) \left( \frac{1}{r} - \prod_P(L_\tau(r)) \right) \\ -w \left( \frac{\lambda(0)}{(r+m)} + q(\alpha) \right) \frac{\partial \prod_P(L_\tau(r))}{\partial \alpha} \end{array} \right\} \quad (2.46)$$

For an optimal advertising claim  $\frac{\partial J}{\partial \alpha} = 0$ , leading to:

$$\begin{aligned} \frac{p}{w} &= \frac{A}{B} \\ A &= \left( \frac{\lambda(0)}{(r+m)} + q(\alpha) \right) \frac{\partial \prod_P(L_\tau(r))}{\partial \alpha} \\ &\quad - q'(\alpha) \left( \frac{1}{r} - \prod_P(L_\tau(r)) \right) \\ B &= q'(\alpha)G(\alpha^*) + \left( q(\alpha) + \frac{\lambda(0)}{(r+m)} \right) G'(\alpha^*) \end{aligned} \quad (2.47)$$

while optimization with the parameter  $0 \leq \bar{a} \leq a_{\max}$  yields

$$\bar{a} = \begin{cases} a_{\max} & \text{if } p > w \left( \frac{\prod_P(L_\tau(r)) - \frac{1}{r}}{G(\alpha^*)} \right) \\ 0 & \text{else} \end{cases} \quad (2.48)$$

If we consider a nonlinear advertising cost, while maintaining the constant advertising rate, we have an optimal marginal cost of advertising given by:

$$C'(\bar{a}) = \frac{q(\alpha)a^*(r)G(\alpha)}{(r+m)} \left[ p + w \frac{1/r - \prod_P(L_\tau(r))}{G(\alpha)} \right] \quad (2.49)$$

Similarly, if we assume that the advertising policy is proportional to the current means of new purchases and the number of clients lost by the firm, then we have:

$$a(t) = \bar{a} + a_1\lambda(t) + a_2(\Lambda(t) - \Phi(t)) \quad (2.50)$$

Or,

$$a^*(r) = \frac{\bar{a}}{r} + a_1\lambda^*(r) + a_2(\Lambda^*(r) - \Phi^*(r)) \quad (2.51)$$

and (after some elementary manipulations), we have:

$$\begin{aligned}
a^*(r) &= \frac{A(r)}{B(r)} \\
A(r) &= \bar{a}(r+m) + \\
&\quad + (a_1 r + a_2 - a_2 r \prod_P (L_\tau(r))) \lambda(0) \\
&\quad + a_2 (r+m) \Lambda(0) \\
B(r) &= r(r+m) - \\
&\quad - q(\alpha)(a_1 r + a_2 - a_2 \prod_P (L_\tau(r))r)
\end{aligned} \tag{2.52}$$

which is optimized with respect to its parameters, a function of the advertising claims.

#### 4. Conclusion and Discussion

Advertising claims once experienced by consumers may determine the propensity to repeat purchase. This paper has focused attention on the design of claims and advertising policies simultaneously based on a stochastic model of advertising efficiency and repeat purchase of experienced clients. First, we have considered a stationary stochastic model on the basis of which a number conclusions were drawn regarding the effects of advertising claims on advertising policies. Subsequently, we have considered an intertemporal for advertising and repeat purchase based on claims verification. The model thus constructed was reduced to a deterministic optimal control model which we have solved under specific assumptions. Explicitly, for infinite and discounted horizon problems, we have shown that the problem can be treated analytically if we assume that the optimal advertising policy is a linear function of the problem's state variables. The results we have obtained have established a clear relationship between optimal claims and optimal advertising policies as a function of the advertising efficiency, first purchasers response to advertising and the future benefits of derived consumption (such as associated service contracts, products warranties and their like). The simultaneous considerations of these issues in the context of advertising optimization model have not been considered previously.

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