2

First Examples of Free Resolutions

In this chapter we introduce a fundamental construction of resolutions based on simplicial complexes. This construction gives free resolutions of monomial ideals, but does not always yield minimal resolutions. It includes the Koszul complexes, which we use to establish basic bounds on syzygies of all modules, including the Hilbert Syzygy Theorem. We conclude the chapter with an example of a different kind, showing how free resolutions capture the geometry of sets of seven points in $\mathbb{P}^3$.

2A Monomial Ideals and Simplicial Complexes

We now introduce a beautiful method of writing down graded free resolutions of monomial ideals due to Bayer, Peeva and Sturmfels [Bayer et al. 1998]. So far we have used $\mathbb{Z}$-gradings only, but we can think of the polynomial ring S as $\mathbb{Z}^{r+1}$-graded, with $x_0^{a_0} \cdots x_r^{a_r}$ having degree $(a_0, \ldots, a_r) \in \mathbb{Z}^{r+1}$, and the free resolutions we write down will also be $\mathbb{Z}^{r+1}$-graded. We begin by reviewing the basics of the theory of finite simplicial complexes. For a more complete treatment, see [Bruns and Herzog 1998].

Simplicial Complexes

A *finite simplicial complex* Δ is a finite set N, called the set of *vertices* (or *nodes*) of Δ, and a collection F of subsets of N, called the *faces of* Δ, such that if $A \in F$ is a face and $B \subset A$ then B is also in F. Maximal faces are called facets.

A *simplex* is a simplicial complex in which every subset of N is a face. For any vertex set N we may form the *void* simplicial complex, which has no faces at all. But if Δ has any faces at all, then the empty set $\varnothing$ is necessarily a face of Δ. By contrast, we call the simplicial complex whose only face is $\varnothing$ the *irrelevant* simplicial complex on N. (The name comes from the Stanley–Reisner correspondence, which associates to any simplicial complex Δ with vertex set $N = \{x_0, \dots, x_n\}$ the square-free monomial ideal in $S = \mathbb{K}[x_0, \dots, x_r]$ whose elements are the monomials with support equal to a non-face of Δ. Under this correspondence the irrelevant simplicial complex corresponds to the irrelevant ideal $(x_0, \dots, x_r)$, while the void simplicial complex corresponds to the ideal (1).)

Any simplicial complex Δ has a *geometric realization*, that is, a topological space that is a union of simplices corresponding to the faces of Δ. It may be constructed by realizing the set of vertices of Δ as a linearly independent set in a sufficiently large real vector space, and realizing each face of Δ as the convex hull of its vertex points; the realization of Δ is then the union of these faces.

An *orientation* of a simplicial complex consists of an ordering of the vertices of Δ. Thus a simplicial complex may have many orientations — this is not the same as an orientation of the underlying topological space.

Labeling by Monomials

We will say that Δ is *labeled* (by monomials of S) if there is a monomial of S associated to each vertex of Δ. We then label each face A of Δ by the least common multiple of the labels of the vertices in A. We write m_A for the monomial that is the label of A. By convention the label of the empty face is $m_\varnothing = 1$.

Let Δ be an oriented labeled simplicial complex, and write $I \subset S$ for the ideal generated by the monomials $m_j = x^{\alpha_j}$ labeling the vertices of Δ. We will associate to Δ a graded complex of free S-modules

$$\mathscr{C}(\Delta) = \mathscr{C}(\Delta; S): \ \cdots \longrightarrow F_i \xrightarrow{\delta} F_{i-1} \longrightarrow \cdots \xrightarrow{\delta} F_0,$$

where F_i is the free S-module whose basis consists of the set of faces of Δ having i elements, which is sometimes a resolution of S/I. The differential δ is given by the formula

$$\delta A = \sum_{n \in A} (-1)^{\mathrm{pos}(n,A)} \frac{m_A}{m_{A \setminus n}} (A \setminus n),$$

where $\mathrm{pos}(n, A)$, the *position of vertex n in A*, is the number of elements preceding n in the ordering of A, and $A \setminus n$ denotes the face obtained from A by removing n.

If Δ is not void then $F_0 = S$; the generator is the face of Δ which is the empty set. Further, the generators of F_1 correspond to the vertices of Δ, and each generator maps by δ to its labeling monomial, so

$$\mathrm{H}_0(\mathscr{C}(\Delta)) = \mathrm{coker}\big(F_1 \xrightarrow{\delta} S\big) = S/I.$$

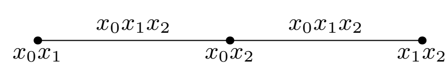
We set the degree of the basis element corresponding to the face A equal to the exponent vector of the monomial that is the label of A. With respect to this grading, the differential δ has degree 0, and $\mathscr{C}(\Delta)$ is a $\mathbb{Z}^{r+1}$-graded free complex.

For example we might take $S = \mathbb{K}$ and label all the vertices of Δ with $1 \in \mathbb{K}$; then $\mathscr{C}(\Delta;\mathbb{K})$ is, up to a shift in homological degree, the usual *reduced chain complex of* Δ *with coefficients in* S. Its homology is written $\mathrm{H}_i(\Delta;\mathbb{K})$ and is called the *reduced homology of* Δ *with coefficients in* S. The shift in homological degree comes about as follows: the homological degree of a simplex in $\mathscr{C}(\Delta)$ is the number of vertices in the simplex, which is one more than the dimension of the simplex, so that $\mathrm{H}_i(\Delta;\mathbb{K})$ is the $(i+1)$-st homology of $\mathscr{C}(\Delta;\mathbb{K})$. If $\mathrm{H}_i(\Delta;\mathbb{K}) = 0$ for $i \geq -1$, we say that Δ is $\mathbb{K}$-*acyclic*. (Since S is a free module over $\mathbb{K}$, this is the same as saying that $\mathrm{H}_i(\Delta;S) = 0$ for $i \geq -1$.)

The homology $\mathrm{H}_i(\Delta;\mathbb{K})$ and the homology $\mathrm{H}_i(\mathscr{C}(\Delta;S))$ are independent of the orientation of Δ — in fact they depend only on the homotopy type of the geometric realization of Δ and the ring $\mathbb{K}$ or S. Thus we will often ignore orientations.

Roughly speaking, we may say that the complex $\mathscr{C}(\Delta;S)$, for an arbitrary labeling, is obtained by extending scalars from $\mathbb{K}$ to S and "homogenizing" the formula for the differential of $\mathscr{C}(\Delta,\mathbb{K})$ with respect to the degrees of the generators of the F_i defined for the S-labeling of Δ.

Example 2.1. Suppose that Δ is the labeled simplicial complex



with the orientation obtained by ordering the vertices from left to right. The complex $\mathscr{C}(\Delta)$ is

$$0 \longrightarrow S^2(-3) \xrightarrow{\begin{pmatrix} -x_2 & 0 \\ x_1 & -x_1 \\ 0 & x_0 \end{pmatrix}} S^3(-2) \xrightarrow{\begin{pmatrix} x_0x_1 & x_0x_2 & x_1x_2 \end{pmatrix}} S.$$

This complex is represented by the Betti diagram

	0	1	2
0	1	–	–
1	–	3	2

As we shall soon see, the only homology of this complex is at the right-hand end, where $\mathrm{H}_0(\mathscr{C}(\Delta)) = S/(x_0x_1, x_0x_2, x_1x_2)$, so the complex is a free resolution of this S-module.

If we took the same simplicial complex, but with the trivial labeling by 1's, we would get the complex

$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} S,$$

represented by the Betti diagram

	0	1	2
-2	–	–	2
-1	–	3	–
0	1	–	–

which has reduced homology 0 (with any coefficients), as the reader may easily check.

We want a criterion that will tell us when $\mathscr{C}(\Delta)$ is a resolution of S/I; that is, when $\mathrm{H}_i(\mathscr{C}(\Delta)) = 0$ for $i > 0$. To state it we need one more definition. If m is any monomial, we write Δ_m for the subcomplex consisting of those faces of Δ whose labels divide m. For example, if m is not divisible by any of the vertex labels, then Δ_m is the empty simplicial complex, with no vertices and the single face $\varnothing$. On the other hand, if m is divisible by all the labels of Δ, then $\Delta_m = \Delta$. Moreover, Δ_m is equal to $\Delta_{\mathrm{LCM}\{m_i \mid i \in I\}}$ for some subset Δ' of the vertex set of Δ.

A *full subcomplex* of Δ is a subcomplex of all the faces of Δ that involve a particular set of vertices. Note that all the subcomplexes Δ_m are full.

Syzygies of Monomial Ideals

Theorem 2.2 (Bayer, Peeva, and Sturmfels). *Let Δ be a simplicial complex labeled by monomials $m_1, \ldots, m_t \in S$, and let $I = (m_1, \ldots, m_t) \subset S$ be the ideal in S generated by the vertex labels. The complex $\mathscr{C}(\Delta) = \mathscr{C}(\Delta; S)$ is a free resolution of S/I if and only if the reduced simplicial homology $\mathrm{H}_i(\Delta_m; \mathbb{K})$ vanishes for every monomial m and every $i \geq 0$. Moreover, $\mathscr{C}(\Delta)$ is a minimal complex if and only if $m_A \neq m_{A'}$ for every proper subface A' of a face A.*

By the remarks above, we can determine whether $\mathscr{C}(\Delta)$ is a resolution just by checking the vanishing condition for monomials that are least common multiples of sets of vertex labels.

Proof. Let $\mathscr{C}(\Delta)$ be the complex

$$\mathscr{C}(\Delta):\ \cdots \longrightarrow F_i \xrightarrow{\delta} F_{i-1} \longrightarrow \cdots \xrightarrow{\delta} F_0.$$

It is clear that S/I is the cokernel of $\delta : F_1 \to F_0$. We will identify the homology of $\mathscr{C}(\Delta)$ at F_i with a direct sum of copies of the vector spaces $\mathrm{H}_i(\Delta_m; \mathbb{K})$.

For each $\alpha \in \mathbb{Z}^{r+1}$ we will compute the homology of the complex of vector spaces

$$\mathscr{C}(\Delta)_\alpha : \ \cdots \longrightarrow (F_i)_\alpha \xrightarrow{\ \delta\ } (F_{i-1})_\alpha \longrightarrow \ \cdots \ \xrightarrow{\ \delta\ } (F_0)_\alpha,$$

formed from the degree-α components of each free module F_i in $\mathscr{C}(\Delta)$. If any of the components of α are negative then $\mathscr{C}(\Delta)_\alpha = 0$, so of course the homology vanishes in this degree.

Thus we may suppose $\alpha \in \mathbb{N}^{r+1}$. Set $m = x^\alpha = x_0^{\alpha_0} \cdots x_r^{\alpha_r} \in S$. For each face A of Δ, the complex $\mathscr{C}(\Delta)$ has a rank-one free summand $S \cdot A$ which, as a vector space, has basis $\{n \cdot A \mid n \in S \text{ is a monomial}\}$. The degree of $n \cdot A$ is the exponent of nm_A, where m_A is the label of the face A. Thus for the degree α part of $S \cdot A$ we have

$$S \cdot A_\alpha = \begin{cases} \mathbb{K} \cdot (x^\alpha / m_A) \cdot A & \text{if } m_A | m, \\ 0 & \text{otherwise.} \end{cases}$$

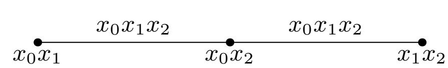
It follows that the complex $\mathscr{C}(\Delta)_\alpha$ has a $\mathbb{K}$-basis corresponding bijectively to the faces of Δ_m. Using this correspondence we identify the terms of the complex $\mathscr{C}(\Delta)_\alpha$ with the terms of the reduced chain complex of Δ_m having coefficients in $\mathbb{K}$ (up to a shift in homological degree as for the case, described above, where the vertex labels are all 1). A moment's consideration shows that the differentials of these complexes agree.

Having identified $\mathscr{C}(\Delta)_\alpha$ with the reduced chain complex of Δ_m, we see that the complex $\mathscr{C}(\Delta)$ is a resolution of S/I if and only if $\mathrm{H}_i(\Delta_m; \mathbb{K}) = 0$ for all $i \geq 0$, as required for the first statement.

For minimality, note that if A is an $(i{+}1)$-face and A' an i-face of Δ, then the component of the differential of $\mathscr{C}(\Delta)$ that maps $S \cdot A$ to $S \cdot A'$ is 0 unless $A' \subset A$, in which case it is $\pm m_A / m_{A'}$. Thus $\mathscr{C}(\Delta)$ is minimal if and only if $m_A \neq m_{A'}$ for all $A' \subset A$, as required. $\square$

For more information about the complexes $\mathscr{C}(\Delta)$ and about a generalization in which cell complexes replace simplicial complexes, see [Bayer et al. 1998] and [Bayer and Sturmfels 1998].

Example 2.3. We continue with the ideal (x_0x_1, x_0x_2, x_1x_2) as above. For the labeled simplicial complex Δ



the distinct subcomplexes Δ' of the form Δ_m are the empty complex Δ_1, the complexes $\Delta_{x_0x_1}$, $\Delta_{x_0x_2}$, $\Delta_{x_1x_2}$, each of which consists of a single point, and the complex Δ itself. As each of these is contractible, they have no higher reduced homology, and we see that the complex $\mathscr{C}(\Delta)$ is the minimal free resolution of $S/(x_0x_1, x_0x_2, x_1x_2)$.

Any full subcomplex of a simplex is a simplex, and since the complexes Δ_1, $\Delta_{x_0x_1}$, $\Delta_{x_0x_2}$, $\Delta_{x_1x_2}$, and Δ are all contractible, they have no reduced homology (with any coefficients). This idea gives a result first proved, in a different way, by Diana Taylor [Eisenbud 1995, Exercise 17.11].

Corollary 2.4. *Let $I = (m_1, \dots, m_n) \subset S$ be any monomial ideal, and let Δ be a simplex with n vertices, labeled $m_1, \dots, m_n$. The complex $\mathscr{C}(\Delta)$, called the Taylor complex of $m_1, \dots, m_n$, is a free resolution of S/I.* □

For an interesting consequence see Exercise 2.1.

Example 2.5. The Taylor complex is rarely minimal. For instance, taking

$$(m_1, m_2, m_3) = (x_0x_1, x_0x_2, x_1x_2)$$

as in the example above, the Taylor complex is a nonminimal resolution with Betti diagram

	0	1	2	3
0	1	–	–	1
1	–	3	3	–

Example 2.6. We may define the *Koszul complex* $\mathbf{K}(x_0, \dots, x_r)$ of $x_0, \dots, x_r$ to be the Taylor complex in the special case where the $m_i = x_i$ are variables. We have exhibited the smallest examples on page 4. By Theorem 2.2 the Koszul complex is a minimal free resolution of the residue class field $\mathbb{K} = S/(x_0, \dots, x_r)$.

We can replace the variables $x_0, \dots, x_r$ by any polynomials $f_0, \dots, f_r$ to obtain a complex we will write as $\mathbf{K}(f_0, \dots, f_r)$, the Koszul complex of the sequence $f_0, \dots, f_r$. In fact, since the differentials have only $\mathbb{Z}$ coefficients, we could even take the f_i to be elements of an arbitrary commutative ring.

Under nice circumstances, for example when the f_i are homogeneous elements of positive degree in a graded ring, this complex is a resolution if and only if the f_i form a *regular sequence*. See Section A2F or [Eisenbud 1995, Theorem 17.6].

2B Bounds on Betti Numbers and Proof of Hilbert's Syzygy Theorem

We can use the Koszul complex and Theorem 2.2 to prove a sharpening of Hilbert's Syzygy Theorem 1.1, which is the vanishing statement in the following proposition. We also get an alternate way to compute the graded Betti numbers.

Proposition 2.7. *Let M be a graded module over $S = \mathbb{K}[x_0, \dots, x_r]$. The graded Betti number $\beta_{i,j}(M)$ is the dimension of the homology, at the term $M_{j-i} \otimes \bigwedge^i \mathbb{K}^{r+1}$, of the complex*

$$\begin{aligned} 0 \to M_{j-(r+1)} \otimes {\textstyle\bigwedge^{r+1}} \mathbb{K}^{r+1} \to \cdots \\ \to M_{j-i-1} \otimes {\textstyle\bigwedge^{i+1}} \mathbb{K}^{r+1} \to M_{j-i} \otimes {\textstyle\bigwedge^{i}} \mathbb{K}^{r+1} \to M_{j-i+1} \otimes {\textstyle\bigwedge^{i-1}} \mathbb{K}^{r+1} \to \\ \cdots \to M_j \otimes {\textstyle\bigwedge^{0}} \mathbb{K}^{r+1} \to 0. \end{aligned}$$

In particular we have $\beta_{i,j}(M) \leq H_M(j-i)\binom{r+1}{i}$, *so* $\beta_{i,j}(M) = 0$ *if* $i > r+1$.

See Exercise 2.5 for the relation of this to Corollary 1.10.

Proof. To simplify the notation, let $\beta_{i,j} = \beta_{i,j}(M)$. By Proposition 1.7,

$$\beta_{i,j} = \dim_{\mathbb{K}} \operatorname{Tor}_i(M, \mathbb{K})_j.$$

Since $\mathbf{K}(x_0, \ldots, x_r)$ is a free resolution of $\mathbb{K}$, we may compute $\operatorname{Tor}_i^S(M, \mathbb{K})_j$ as the degree-j part of the homology of $M \otimes_S \mathbf{K}(x_0, \ldots, x_r)$ at the term

$$M \otimes_S \bigwedge^i S^{r+1}(-i) = M \otimes_{\mathbb{K}} \bigwedge^i \mathbb{K}^{r+1}(-i).$$

Decomposing M into its homogeneous components $M = \oplus M_k$, we see that the degree-j part of $M \otimes_{\mathbb{K}} \bigwedge^i \mathbb{K}^{r+1}(-i)$ is $M_{j-i} \otimes_{\mathbb{K}} \bigwedge^i \mathbb{K}^{r+1}$. The differentials of $M \otimes_S \mathbf{K}(x_0, \ldots, x_r)$ preserve degrees, so the complex decomposes as a direct sum of complexes of vector spaces of the form

$$M_{j-i-1} \otimes_{\mathbb{K}} \bigwedge^{i+1} \mathbb{K}^{r+1} \longrightarrow M_{j-i} \otimes_{\mathbb{K}} \bigwedge^{i} \mathbb{K}^{r+1} \longrightarrow M_{j-i+1} \otimes_{\mathbb{K}} \bigwedge^{i-1} \mathbb{K}^{r+1}.$$

This proves the first statement. The inequality on $\beta_{i,j}$ follows at once. $\square$

The upper bound given in Proposition 2.7 is achieved when $\mathfrak{m}M = 0$ (and conversely — see Exercise 2.6). It is not hard to deduce a weak lower bound, too (Exercise 2.7), but is often a very difficult problem, to determine the actual range of possibilities, especially when the module M is supposed to come from some geometric construction.

An example will illustrate some of the possible considerations. A true geometric example, related to this one, will be given in the next section. Suppose that $r = 2$ and the Hilbert function of M has values

$$H_M(j) = \begin{cases} 0 & \text{if } j < 0, \\ 1 & \text{if } j = 0, \\ 3 & \text{if } j = 1, \\ 3 & \text{if } j = 2, \\ 0 & \text{if } j > 2. \end{cases}$$

To fit with the way we write Betti diagrams, we represent the complexes in Proposition 2.7 with maps going from right to left, and put the term $M_j \otimes \bigwedge^i \mathbb{K}^{r+1}(-i) = M_j(-i)^{\binom{r+1}{i}}$ (the term of degree $i+j$) in row j and column i. Because the differential has degree 0, it goes diagonally down and to the left.

$$\begin{array}{c|ccccccc}
M & & & & M \otimes_{\mathbb{K}} \wedge \mathbb{K}^r(-1) & & & \\
\hline
M_0 & \mathbb{K}^1 & & \mathbb{K}^3 & & \mathbb{K}^3 & & \mathbb{K}^1 \\
 & & \swarrow & & \swarrow & & \swarrow & \\
M_1 & \mathbb{K}^3 & & \mathbb{K}^9 & & \mathbb{K}^9 & & \mathbb{K}^3 \\
 & & \swarrow & & \swarrow & & \swarrow & \\
M_2 & \mathbb{K}^3 & & \mathbb{K}^9 & & \mathbb{K}^9 & & \mathbb{K}^3
\end{array}$$

From this we see that the termwise maximal Betti diagram of a module with the given Hilbert function, valid if the module structure of M is trivial, is

	0	1	2	3
0	1	3	3	1
1	3	9	9	3
2	3	9	9	3

On the other hand, if the differential

$$d_{i,j}: \ M_{j-i} \otimes \bigwedge^i \mathbb{K}^3 \to M_{j-i+1} \otimes \bigwedge^{i-1} \mathbb{K}^3$$

has rank k, both $\beta_{i,j}$ and $\beta_{i-1,j}$ drop from this maximal value by k.

Other considerations come into play as well. For example, suppose that M is a cyclic module (a module requiring only one generator), generated by M_0. Equivalently, $\beta_{0,j} = 0$ for $j \neq 0$. It follows that the differentials $d_{1,1}$ and $d_{1,2}$ have rank 3, so $\beta_{1,1} = 0$ and $\beta_{1,2} \leq 6$. Since $\beta_{1,1} = 0$, Proposition 1.9 implies that $\beta_{i,i} = 0$ for all $i \geq 1$. This means that the differential $d_{2,2}$ has rank 3 and the differential $d_{3,3}$ has rank 1, so the maximal possible Betti numbers are

	0	1	2	3
0	1	–	–	–
1	–	3	8	3
2	–	9	9	3

Whatever the ranks of the remaining differentials, we see that any Betti diagram of a cyclic module with the given Hilbert function has the form

	0	1	2	3
0	1	–	–	–
1	–	3	$\beta_{2,3}$	$\beta_{3,4}$
2	–	$1+\beta_{2,3}$	$6+\beta_{3,4}$	3

for some $0 \leq \beta_{2,3} \leq 8$ and $0 \leq \beta_{3,4} \leq 3$. For example, if all the remaining differentials have maximal rank, the Betti diagram would be

	0	1	2	3
0	1	–	–	–
1	–	3	–	–
2	–	1	6	3

We will see in the next section that this diagram is realized as the Betti diagram of the homogeneous coordinate ring of a general set of 7 points in $\mathbb{P}^3$ modulo a nonzerodivisor of degree 1.

2C Geometry from Syzygies: Seven Points in $\mathbb{P}^3$

We have seen above that if we know the graded Betti numbers of a graded S-module, then we can compute the Hilbert function. In geometric situations, the graded Betti numbers often carry information beyond that of the Hilbert function. Perhaps the most interesting current results in this direction center on *Green's Conjecture* described in Section 9B.

For a simpler example we consider the graded Betti numbers of the homogeneous coordinate ring of a set of 7 points in "linearly general position" (defined below) in $\mathbb{P}^3$. We will meet a number of the ideas that occupy the next few chapters. To save time we will allow ourselves to quote freely from material developed (independently of this discussion!) later in the text. The inexperienced reader should feel free to look at the statements and skip the proofs in the rest of this section until after having read through Chapter 6.

The Hilbert Polynomial and Function...

Any set X of 7 distinct points in $\mathbb{P}^3$ has Hilbert polynomial equal to the constant 7 (such things are discussed at the beginning of Chapter 4). However, not all sets of 7 points in $\mathbb{P}^3$ have the same Hilbert function. For example, if X is not contained in a plane then the Hilbert function $H = H_{S_X}(d)$ begins with the values $H(0) = 1$, $H(1) = 4$, but if X is contained in a plane then $H(1) < 4$.

To avoid such degeneracy we will restrict our attention in the rest of this section to 7-tuples of points that are in *linearly general position.* We say that a set of points $Y \subset \mathbb{P}^r$ is in linearly general position if there are no more than 2 points of Y on any line, no more than 3 points on any 2-plane, ..., no more than r points in an $r-1$ plane. Thinking of the points as coming from vectors in $\mathbb{K}^{r+1}$, this means that every subset of at most $r+1$ of the vectors is linearly independent. Of course if there are at least $r+1$ points, this is equivalent to say simply that every subset of exactly $r+1$ of the vectors is linearly independent.

The condition that a set of points is in linearly general position arises frequently. For example, the general hyperplane section of any irreducible curve over a field of characteristic 0 is a set of points in linearly general position [Harris 1980] and this is usually, though not always, true in characteristic p as well [Rathmann 1987]. See Exercises 8.17–8.20.

It is not hard to show — the reader is invited to prove a more general fact in Exercise 2.9 — that the Hilbert function of any set X of 7 points in linearly general position in $\mathbb{P}^3$ is given by the table

d	0	1	2	3	...
$H_{S_X}(d)$	1	4	7	7	...

In particular, any set X of 7 points in linearly general position lies on exactly $3 = \binom{3+2}{2} - 7$ independent quadrics. These three quadrics cannot generate the ideal: since $S = \mathbb{K}[x_0, \dots, x_3]$ has only four linear forms, the dimension of the space of cubics in the ideal generated by the three quadrics is at most $4 \times 3 = 12$, whereas there are $\binom{3+3}{3} - 7 = 13$ independent cubics in the ideal of X. Thus the ideal of X requires at least one cubic generator in addition to the three quadrics.

One might worry that higher degree generators might be needed as well. The ideal of 7 points on a line in $\mathbb{P}^3$, for example, is minimally generated by the two linear forms that generate the ideal of the line, together with any form of degree 7 vanishing on the points but not on the line. But Theorem 4.2(c) tells us that since the 7 points of X are in linearly general position the Castelnuovo–Mumford regularity of S_X (defined in Chapter 4) is 2, or equivalently, that the Betti diagram of S_X fits into 3 rows. Moreover, the ring S_X is reduced and of dimension 1 so it has depth 1. The Auslander–Buchsbaum Formula A2.15 shows that the resolution will have length 3. Putting this together, and using Corollary 1.9 we see that the minimal free resolution of S_X must have Betti diagram of the form

	0	1	2	3
0	1	–	–	–
1	–	$\beta_{1,2}$	$\beta_{2,3}$	$\beta_{3,4}$
2	–	$\beta_{1,3}$	$\beta_{2,4}$	$\beta_{3,5}$

where the $\beta_{i,j}$ that are not shown are zero. In particular, the ideal of X is generated by quadrics and cubics.

Using Corollary 1.10 we compute successively $\beta_{1,2} = 3$, $\beta_{1,3} - \beta_{2,3} = 1$, $\beta_{2,4} - \beta_{3,4} = 6$, $\beta_{3,5} = 3$, and the Betti diagram has the form

	0	1	2	3
0	1	–	–	–
1	–	3	$\beta_{2,3}$	$\beta_{3,4}$
2	–	$1+\beta_{2,3}$	$6+\beta_{3,4}$	3

(This is the same diagram as at the end of the previous section. Here is the connection: Extending the ground field if necessary to make it infinite, we could use Lemma A2.3 and choose a linear form $x \in S$ that is a nonzerodivisor on S_X. By Lemma 3.15 the graded Betti numbers of S_X/xS_X as an S/xS-module are the same as those of S_X as an S-module. Using our knowledge of the Hilbert function of S_X and the exactness of the sequence

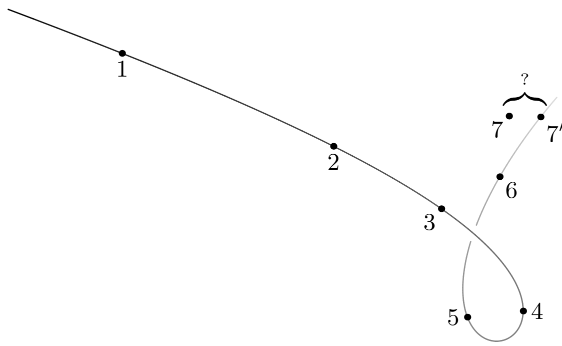
$$0 \longrightarrow S_X(-1) \xrightarrow{x} S_X \longrightarrow S_X/xS_x \longrightarrow 0,$$

we see that the cyclic (S/xS)-module S_X/xS_x has Hilbert function with values $1, 3, 3$. This is what we used in Section 2B.)

. . . and Other Information in the Resolution

We see that even in this simple case the Hilbert function does not determine the $\beta_{i,j}$, and indeed they can take different values. It turns out that the difference reflects a fundamental geometric distinction between different sets X of 7 points in linearly general position in $\mathbb{P}^3$: whether or not X lies on a curve of degree 3.

Up to linear automorphisms of $\mathbb{P}^3$ there is only one irreducible curve of degree 3 not contained in a plane. This *twisted cubic* is one of the *rational normal curves* studied in Chapter 6. Any 6 points in linearly general position in $\mathbb{P}^3$ lie on a unique twisted cubic (see Exercise 6.5). But for a twisted cubic to pass through 7 points, the seventh must lie on the twisted cubic determined by the first 6. Thus most sets of seven points do not lie on any twisted cubic.



Theorem 2.8. *Let X be a set of 7 points in linearly general position in $\mathbb{P}^3$. There are just two distinct Betti diagrams possible for the homogeneous coordinate ring S_X:*

	0	1	2	3
0	1	–	–	–
1	–	3	–	–
2	–	1	6	3

and

	0	1	2	3
0	1	–	–	–
1	–	3	2	–
2	–	3	6	3

In the first case the points do not lie on any curve of degree 3*. In the second case, the ideal J generated by the quadrics containing X is the ideal of the unique curve of degree* 3 *containing X, which is irreducible.*

Proof. Let q_0, q_1, q_2 be three quadratic forms that span the degree 2 part of $I := I_X$. A *linear syzygy* of the q_i is a vector (a_0, a_1, a_2) of linear forms with $\sum_{i=0}^{2} a_i q_i = 0$. We will focus on the number of independent linear syzygies, which is $\beta_{2,3}$.

If $\beta_{2,3} = 0$, Proposition 1.9 implies that $\beta_{3,4} = 0$ and the computation of the differences of the $\beta_{i,j}$ above shows that the Betti diagram of $S_X = S/I$ is the first of the two given tables. As we shall see in Chapter 6, any irreducible curve of degree ≤ 2 lies in a plane. Since the points of X are in linearly general position, they are not contained in the union of a line and a plane, or the union of 3 lines, so any degree 3 curve containing X is irreducible. Further, if C is an irreducible degree 3 curve in $\mathbb{P}^3$, not contained in a plane, then the C is a twisted cubic, and the ideal of C is generated by three quadrics, which have 2 linear syzygies. Thus in the case where X is contained in a degree 3 curve we have $\beta_{2,3} \geq 2$.

Now suppose $\beta_{2,3} > 0$, so that there is a nonzero linear syzygy $\sum_{i=0}^{2} a_i q_i = 0$. If the a_i were linearly dependent then we could rewrite this relation as $a'_1 q'_1 + a'_2 q'_2 = 0$ for some independendent quadrics q'_1 and q'_2 in I. By unique factorization, the linear form a'_1 would divide q'_2; say $q'_2 = a'_1 b$. Thus X would be contained in the union of the planes $a_1' = 0$ and $b = 0$, and one of these planes would contain four points of X, contradicting our hypothesis. Therefore a_0, a_1, a_2 are linearly independent linear forms.

Changing coordinates on $\mathbb{P}^3$ we can harmlessly assume that $a_i = x_i$. We can then read the relation $\sum x_i q_i = 0$ as a syzygy on the x_i. But from the exactness of the Koszul complex (see for example Theorem 2.2 as applied in Example 2.6), we know that all the syzygies of x_0, x_1, x_2 are given by the columns of the matrix

$$\begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix},$$

and thus we must have

$$\begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

for some linear forms b_i. Another way to express this equation is to say that q_i is $(-1)^i$ times the determinant of the 2×2 matrix formed by omitting the i-th column of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ b_0 & b_1 & b_2 \end{pmatrix},$$

where the columns are numbered $0, 1, 2$. The two rows of M are independent because the q_i, the minors, are nonzero. (Throughout this book we will follow the convention that a *minor* of a matrix is a subdeterminant times an appropriate sign.)

We claim that both rows of M give relations on the q_i. The vector (x_0, x_1, x_2) is a syzygy by virtue of our choice of coordinates. To see that (b_0, b_1, b_2) is also a syzygy, note that the Laplace expansion of

$$\det \begin{pmatrix} x_0 & x_1 & x_2 \\ b_0 & b_1 & b_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$$

is $\sum_i b_i q_i$. However, this 3×3 matrix has a repeated row, so the determinant is 0, showing that $\sum_i b_i q_i = 0$. Since the two rows of M are linearly independent, we see that the q_i have (at least) 2 independent syzygies with linear forms as coefficients.

The ideal $(q_0, q_1, q_2) \subset I$ that is generated by the minors of M is unchanged if we replace M by a matrix PMQ, where P and Q are invertible matrices of scalars. It follows that matrices of the form PMQ cannot have any entries equal to zero. This shows that M is 1-generic in the sense of Chapter 6, and it follows from Theorem 6.4 that the ideal $J = (q_0, q_1, q_2) \subset I$ is prime and of codimension 2—that is, J defines an irreducible curve C containing X in $\mathbb{P}^3$.

From Theorem 3.2 it follows that a free resolution of S_C may be written as

$$0 \to S^2(-3) \xrightarrow{\begin{pmatrix} x_0 & b_0 \\ x_1 & b_1 \\ x_2 & b_2 \end{pmatrix}} S^3(-2) \xrightarrow{\begin{pmatrix} q_0 & q_1 & q_2 \end{pmatrix}} S \longrightarrow S_C \longrightarrow 0.$$

From the resolution of S_C we can also compute its Hilbert function:

$$\begin{aligned} H_{S_C}(d) &= \binom{3+d}{3} - 3\binom{3+d-2}{3} + 2\binom{3+d-3}{3} \\ &= 3d+1 \quad \text{ for } d \geq 0. \end{aligned}$$

Thus the Hilbert polynomial of the curve is $3d+1$. It follows that C is a cubic curve — see [Hartshorne 1977, Prop. I.7.6], for example. ☐

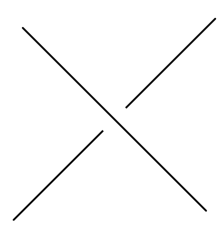
It may be surprising that in Theorem 2.8 the only possibilities for $\beta_{2,3}$ are 0 and 2, and that $\beta_{3,4}$ is always 0. These restrictions are removed, however, if one looks at sets of 7 points that are not in linearly general position though they have the same Hilbert function as a set of points in linearly general position; some examples are given in Exercises 2.11–2.12.

2D Exercises

1. Suppose that $m_1, \ldots, m_n$ are monomials in S. Show that the projective dimension of $S/(m_1, \ldots, m_n)$ is at most n. No such principle holds for arbitrary homogeneous polynomials; see Exercise 2.4.

2. Let $0 \leq n \leq r$. Show that if M is a graded S-module which contains a submodule isomorphic to $S/(x_0, \ldots, x_n)$ (so that $(x_0, \ldots, x_n)$ is an associated prime of M) then the projective dimension of M is at least $n+1$. If $n+1$ is equal to the number of variables in S, show that this condition is necessary as well as sufficient. (Hint: For the last statement, use the Auslander–Buchsbaum theorem, Theorem A2.15.)

3. Consider the ideal $I = (x_0, x_1) \cap (x_2, x_3)$ of two skew lines in $\mathbb{P}^3$:



Prove that $I = (x_0x_2,\, x_0x_3,\, x_1x_2,\, x_1x_3)$, and compute the minimal free resolution of S/I. In particular, show that S/I has projective dimension 3 even though its associated primes are precisely (x_0, x_1) and (x_2, x_3), which have height only 2. Thus the principle of Exercise 2.2 can't be extended to give the projective dimension in general.

4. Show that the ideal $J = (x_0x_2 - x_1x_3,\ x_0x_3, x_1x_2)$ defines the union of two (reduced) lines in $\mathbb{P}^3$, but is not equal to the saturated ideal of the two lines. Conclude that the projective dimension of S/J is 4 (you might use the Auslander–Buchsbaum formula, Theorem A2.15). In fact, three-generator ideals can have any projective dimension; see [Bruns 1976] or [Evans and Griffith 1985, Corollary 3.13].

5. Let M be a finitely generated graded S-module and let $B_j = \sum_i (-1)^i \beta_{i,j}(M)$. Show from Proposition 2.7 that

$$B_j = \sum_i (-1)^i H_M(j-i)\binom{r+1}{i}.$$

This is another form of the formula in Corollary 1.10.

6. Show that if M is a graded S module, then

$$\beta_{0,j}(M) = H_M(j) \quad \text{for all } j$$

if and only if $\mathfrak{m}M = 0$.

7. If M is a graded S-module, show that

$$\beta_{i,j}(M) \geq H_M(j-i)\binom{r+1}{i} - H_M(j-i+1)\binom{r+1}{i-1} - H_M(j-i-1)\binom{r+1}{i+1}.$$

8. Prove that the complex

$$0 \to S^2(-3) \xrightarrow{\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}} S^3(-2) \xrightarrow{\begin{pmatrix} x_1x_3 - x_2^2 & -x_0x_3 + x_1x_2 & x_0x_2 - x_1^2 \end{pmatrix}} S$$

is indeed a resolution of the homogeneous coordinate ring S_C of the twisted cubic curve C, by the following steps:

(a) Identify S_C with the subring of $\mathbb{K}[s,t]$ consisting of those graded components whose degree is divisible by 3. Show in this way that $H_{S_C}(d) = 3d+1$ for $d \geq 0$.

(b) Compute the Hilbert functions of the terms S, $S^3(-2)$, and $S^2(-3)$. Show that their alternating sum $H_S - H_{S^3(-2)} + H_{S^2(-3)}$ is equal to the Hilbert function H_{S_C}.

(c) Show that the map

$$S^2(-3) \xrightarrow{\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}} S^3(-2)$$

is a monomorphism. As a first step you might prove that it becomes a monomorphism when the polynomial ring S is replaced by its quotient field, the field of rational functions.

(d) Show that the results in parts (b) and (c) together imply that the complex exhibited above is a free resolution of S_C.

9. Let X be a set of $n \leq 2r+1$ points in $\mathbb{P}^r$ in linearly general position. Show that X imposes independent conditions on quadrics: that is, show that the space of quadratic forms vanishing on X is $\binom{r+2}{2} - n$ dimensional. (It is enough to show that for each $p \in X$ there is a quadric not vanishing on p but vanishing at all the other points of X.) Use this to show that X imposes independent conditions on forms of degree ≥ 2. The same idea can be used to show that any $n \leq dr+1$ points in linearly general position impose independent conditions on forms of degree d.

Deduce the correctness of the Hilbert function for 7 points in linearly general position given by the table in Section 2C.

10. The sufficient condition of Exercise 2.9 is far from necessary. One way to sharpen it is to use Edmonds' Theorem [1965], which is the following beautiful and nontrivial theorem in linear algebra (see [Graham et al. 1995, Chapter 11, Theorem 3.9] for an exposition):

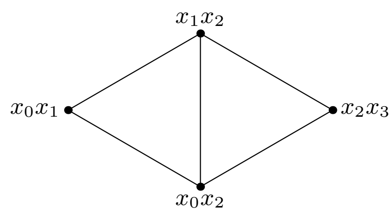
Theorem 2.9. *Let $v_1, \ldots, v_{ds}$ be vectors in an s-dimensional vector space. The list $(v_1, \ldots, v_{ds})$ can be written as the union of d bases if and only if no $dk+1$ of the vectors v_i lie in a k-dimensional subspace*, *for every k.* ☐

Now suppose that Γ is a set of at most $2r+1$ points in $\mathbb{P}^r$, and, for all $k < r$, each set of $2k+1$ points of Γ spans at least a $(k+1)$-plane. Use Edmonds' Theorem to show that Γ imposes independent conditions on quadrics in $\mathbb{P}^r$ (Hint: You can apply Edmonds' Theorem to the set obtained by counting one of the points of Γ twice.)

11. Show that if X is a set of 7 points in $\mathbb{P}^3$ with 6 points on a plane, but not on any conic curve in that plane, while the seventh point does not line in the plane, then X imposes independent conditions on forms of degree ≥ 2 and $\beta_{2,3} = 3$.

12. Let $\Lambda \subset \mathbb{P}^3$ be a plane, and let $D \subset \Lambda$ be an irreducible conic. Choose points $p_1, p_2 \notin \Lambda$ such that the line joining p_1 and p_2 does not meet D. Show that if X is a set of 7 points in $\mathbb{P}^3$ consisting of p_1, p_2 and 5 points on D, then X imposes independent conditions on forms of degree ≥ 2 and $\beta_{2,3} = 1$. (Hint: To show that $\beta_{2,3} \geq 1$, find a pair of reducible quadrics in the ideal having a common component. To show that $\beta_{2,3} \leq 1$, show that the quadrics through the points are the same as the quadrics containing D and the two points. There is, up to automorphisms of $\mathbb{P}^3$, only one configuration consisting of a conic and two points in $\mathbb{P}^3$ such that the line though the two points does not meet the conic. You might produce such a configuration explicitly and compute the quadrics and their syzygies.)

13. Show that the labeled simplicial complex



gives a nonminimal free resolution of the monomial ideal

$$(x_0x_1, x_0x_2, x_1x_2, x_2x_3).$$

Use this to prove that the Betti diagram of a minimal free resolution is

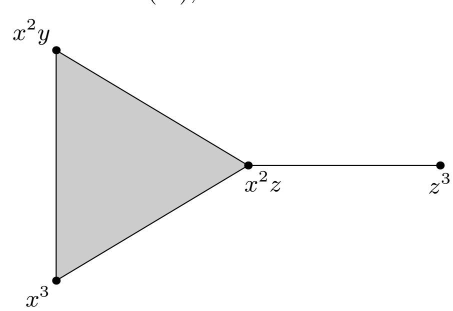
	0	1	2	3
0	1	–	–	–
1	–	4	4	1

14. Use the Betti diagram in Exercise 2.13 to show that the minimal free resolution of $(x_0x_1, x_0x_2, x_1x_2, x_2x_3)$ cannot be written as $\mathscr{C}(\Delta)$ for any labeled simplicial complex Δ. (It can be written as the free complex coming from a certain topological cell complex; for this generalization see [Bayer and Sturmfels 1998].)

15. Show the ideal

$$I = (x^3, x^2y, x^2z, y^3) \subset S = \mathbb{K}[x, y, z]$$

has minimal free resolution $\mathscr{C}(\Delta)$, where Δ is the labeled simplicial complex



Compute the Betti diagram, the Hilbert function, and the Hilbert polynomial of S/I, and show that in this case the bound given in Corollary 1.3 is not sharp. Can you see this from the Betti diagram?

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