Preface: Algebra and Geometry

Syzygy [from] Gr. συζυγία
yoke, pair, copulation, conjunction

— Oxford English Dictionary (etymology)

Implicit in the name "algebraic geometry" is the relation between geometry and equations. The qualitative study of systems of polynomial equations is the chief subject of commutative algebra as well. But when we actually study a ring or a variety, we often have to know a great deal about it before understanding its equations. Conversely, given a system of equations, it can be extremely difficult to analyze its qualitative properties, such as the geometry of the corresponding variety. The theory of syzygies offers a microscope for looking at systems of equations, and helps to make their subtle properties visible.

This book is concerned with the qualitative geometric theory of syzygies. It describes geometric properties of a projective variety that correspond to the numbers and degrees of its syzygies or to its having some structural property — such as being determinantal, or having a free resolution with some particularly simple structure. It is intended as a second course in algebraic geometry and commutative algebra, such as I have taught at Brandeis University, the Institut Poincaré in Paris, and the University of California at Berkeley.

What Are Syzygies?

In algebraic geometry over a field $\mathbb{K}$ we study the geometry of varieties through properties of the polynomial ring

$$S = \mathbb{K}[x_0, \dots, x_r]$$

and its ideals. It turns out that to study ideals effectively we we also need to study more general graded modules over S. The simplest way to describe a module is by generators and relations. We may think of a set $A \subset M$ of generators for an S-module M as a map from a free S-module $F = S^A$ onto M, sending the basis element of F corresponding to a generator $m \in A$ to the element $m \in M$.

Let M_1 be the kernel of the map $F \to M$; it is called the *module of syzygies* of M corresponding to the given choice of generators, and a *syzygy* of M is an element of M_1 — a linear relation, with coefficients in S, on the chosen generators. When we give M by generators and relations, we are choosing generators for M and generators for the module of syzygies of M.

The use of "syzygy" in this context seems to go back to Sylvester [1853]. The word entered the language of modern science in the seventeenth century, with the same astronomical meaning it had in ancient Greek: the conjunction or opposition of heavenly bodies. Its literal derivation is a yoking together, just like "conjunction", with which it is cognate.

If $r = 0$, so that we are working over the polynomial ring in one variable, the module of syzygies is itself a free module, since over a principal ideal domain every submodule of a free module is free. But when $r > 0$ it may be the case that any set of generators of the module of syzygies has relations. To understand them, we proceed as before: we choose a generating set of syzygies and use them to define a map from a new free module, say F_1, onto M_1; equivalently, we give a map $\phi_1 : F_1 \to F$ whose image is M_1. Continuing in this way we get a *free resolution* of M, that is, a sequence of maps

$$\cdots \longrightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F \longrightarrow M \longrightarrow 0,$$

where all the modules F_i are free and each map is a surjection onto the kernel of the following map. The image M_i of ϕ_i is called the *i-th module of syzygies* of M.

In projective geometry we treat S as a graded ring by giving each variable x_i degree 1, and we will be interested in the case where M is a finitely generated graded S-module. In this case we can choose a *minimal* set of homogeneous generators for M (that is, one with as few elements as possible), and we choose the degrees of the generators of F so that the map $F \to M$ preserves degrees. The syzygy module M_1 is then a graded submodule of F, and Hilbert's Basis Theorem tells us that M_1 is again finitely generated, so we may repeat the procedure. Hilbert's Syzygy Theorem tells us that the modules M_i are free as soon as $i \geq r$.

The free resolution of M appears to depend strongly on our initial choice of generators for M, as well as the subsequent choices of generators of M_1, and so on. But if M is a finitely generated graded module and we choose a minimal set of generators for M, then M_1 is, up to isomorphism, independent of the minimal set of generators chosen. It follows that if we choose minimal sets of generators at each stage in the construction of a free resolution we get a *minimal free resolution of* M that is, up to isomorphism, independent of all the choices made. Since, by the Hilbert Syzygy Theorem, M_i is free for $i > r$, we see that in the minimal free resolution $F_i = 0$ for $i > r+1$. In this sense the minimal free resolution is finite: it has length at most $r+1$. Moreover, any free resolution of M can be derived from the minimal one in a simple way (see Section 1B).

The Geometric Content of Syzygies

The minimal free resolution of a module M is a good tool for extracting information about M. For example, Hilbert's motivation for his results just quoted was to devise a simple formula for the dimension of the d-th graded component of M as a function of d. He showed that the function $d \mapsto \dim_{\mathbb{K}} M_d$, now called the *Hilbert function* of M, agrees for large d with a polynomial function of d. The coefficients of this polynomial are among the most important invariants of the module. If $X \subset \mathbb{P}^r$ is a curve, the Hilbert polynomial of the homogeneous coordinate ring S_X of X is

$$(\deg X)\, d + (1 - \operatorname{genus} X),$$

whose coefficients $\deg X$ and $1 - \operatorname{genus} X$ give a topological classification of the embedded curve. Hilbert originally studied free resolutions because their discrete invariants, the *graded Betti numbers*, determine the Hilbert function (see Chapter 1).

But the graded Betti numbers contain more information than the Hilbert function. A typical example is the case of seven points in $\mathbb{P}^3$, described in Section 2C: every set of 7 points in $\mathbb{P}^3$ in linearly general position has the same Hilbert function, but the graded Betti numbers of the ideal of the points tell us whether the points lie on a rational normal curve.

Most of this book is concerned with examples one dimension higher: we study the graded Betti numbers of the ideals of a projective curve, and relate them to the geometric properties of the curve. To take just one example from those we will explore, Green's Conjecture (still open) says that the graded Betti numbers of the ideal of a canonically embedded curve tell us the curve's Clifford index (most of the time this index is 2 less than the minimal degree of a map from the curve to $\mathbb{P}^1$). This circle of ideas is described in Chapter 9.

Some work has been done on syzygies of higher-dimensional varieties too, though this subject is less well-developed. Syzygies are important in the study of embeddings of abelian varieties, and thus in the study of moduli of abelian varieties (for example [Gross and Popescu 2001]). They currently play a part in the study of surfaces of low codimension (for example [Decker and Schreyer 2000]), and other questions about surfaces (for example [Gallego and Purnaprajna 1999]). They have also been used in the study of Calabi–Yau varieties (for example [Gallego and Purnaprajna 1998]).

What Does Solving Linear Equations Mean?

A free resolution may be thought of as the result of fully solving a system of linear equations with polynomial coefficients. To set the stage, consider a system of linear equations $AX = 0$, where A is a $p \times q$ matrix of elements of $\mathbb{K}$, which we may think of as a linear transformation

$$F_1 = \mathbb{K}^q \xrightarrow{A} \mathbb{K}^p = F_0.$$

Suppose we find some solution vectors $X_1, \dots, X_n$. These vectors constitute a complete solution to the equations if every solution vector can be expressed as a linear combination of them. Elementary linear algebra shows that there are complete solutions consisting of $q - \operatorname{rank} A$ independent vectors. Moreover, there is a powerful test for completeness: A given set of solutions $\{X_i\}$ is complete if and only if it contains $q - \operatorname{rank} A$ independent vectors.

A set of solutions can be interpreted as the columns of a matrix X defining a map $X : F_2 \to F_1$ such that

$$F_2 \xrightarrow{X} F_1 \xrightarrow{A} F_0$$

is a complex. The test for completeness says that this complex is exact if and only if $\operatorname{rank} A + \operatorname{rank} X = \operatorname{rank} F_1$. If the solutions are linearly independent as well as forming a complete system, we get an exact sequence

$$0 \to F_2 \xrightarrow{X} F_1 \xrightarrow{A} F_0.$$

Suppose now that the elements of A vary as polynomial functions of some parameters $x_0, \dots, x_r$, and we need to find solution vectors whose entries also vary as polynomial functions. Given a set $X_1, \dots, X_n$ of vectors of polynomials that are solutions to the equations $AX = 0$, we ask whether every solution can be written as a linear combination of the X_i with polynomial coefficients. If so we say that the set of solutions is complete. The solutions are once again elements of the kernel of the map $A : F_1 = S^q \to F_0 = S^p$, and a complete set of solutions is a set of generators of the kernel. Thus Hilbert's Basis Theorem implies that there do exist finite complete sets of solutions. However, it might be that every complete set of solutions is linearly dependent: the syzygy module $M_1 = \ker A$ is not free. Thus to understand the solutions we must compute the dependency relations on them, and then the dependency relations on these. This is precisely a free resolution of the cokernel of A. When we think of solving a system of linear equations, we should think of the whole free resolution.

One reward for this point of view is a criterion analogous to the rank criterion given above for the completeness of a set of solutions. We know no simple criterion for the completeness of a given set of solutions to a system of linear equations over S, that is, for the exactness of a complex of free S-modules $F_2 \to F_1 \to F_0$. However, if we consider a whole free resolution, the situation is better: a complex

$$0 \longrightarrow F_m \xrightarrow{\phi_m} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

of matrices of polynomial functions is exact if and only if the ranks r_i of the ϕ_i satisfy the conditions $r_i + r_{i-1} = \operatorname{rank} F_i$, as in the case where S is a field, and the set of points

$$\{p \in \mathbb{K}^{r+1} \mid \text{the evaluated matrix } \phi_i|_{x=p} \text{ has rank } < r_i\}$$

has codimension $\geq i$ for each i. (See Theorem 3.4.)

This criterion, from joint work with David Buchsbaum, was my first real result about free resolutions. I've been hooked ever since.

Experiment and Computation

A qualitative understanding of equations makes algebraic geometry more accessible to experiment: when it is possible to test geometric properties using their equations, it becomes possible to make constructions and decide their structure by computer. Sometimes unexpected patterns and regularities emerge and lead to surprising conjectures. The experimental method is a useful addition to the method of guessing new theorems by extrapolating from old ones. I personally owe to experiment some of the theorems of which I'm proudest. Number theory provides a good example of how this principle can operate: experiment is much easier in number theory than in algebraic geometry, and this is one of the reasons that number theory is so richly endowed with marvelous and difficult conjectures. The conjectures discovered by experiment can be trivial or very difficult; they usually come with no pedigree suggesting methods for proof. As in physics, chemistry or biology, there is art involved in inventing feasible experiments that have useful answers.

A good example where experiments with syzygies were useful in algebraic geometry is the study of surfaces of low degree in projective 4-space, as in work of Aure, Decker, Hulek, Popescu and Ranestad [Aure et al. 1997]. Another is the work on Fano manifolds such as that of of Schreyer [2001], or the applications surveyed in [Decker and Schreyer 2001, Decker and Eisenbud 2002]. The idea, roughly, is to deduce the form of the equations from the geometric properties that the varieties are supposed to possess, guess at sets of equations with this structure, and then prove that the guessed equations represent actual varieties. Syzygies were also crucial in my work with Joe Harris on algebraic curves. Many further examples of this sort could be given within algebraic geometry, and there are still more examples in commutative algebra and other related areas, such as those described in the *Macaulay 2 Book* [Decker and Eisenbud 2002].

Computation in algebraic geometry is itself an interesting field of study, not covered in this book. It has developed a great deal in recent years, and there are now at least three powerful programs devoted to computation in commutative algebra, algebraic geometry and singularities that are freely available: CoCoA, Macaulay 2, and Singular.[1] Despite these advances, it will always be easy to give sets of equations that render our best algorithms and biggest machines useless, so the qualitative theory remains essential.

A useful adjunct to this book would be a study of the construction of Gröbner bases which underlies these tools, perhaps from [Eisenbud 1995, Chapter 15], and the use of one of these computing platforms. The books [Greuel and Pfister 2002, Kreuzer and Robbiano 2000] and, for projective geometry, the forthcoming book [Decker and Schreyer $\geq$ 2004], will be very helpful.

What's In This Book?

The first chapter of this book is introductory: it explains the ideas of Hilbert that give the definitive link between syzygies and the *Hilbert function*. This is the origin of the modern theory of syzygies. This chapter also introduces the basic discrete invariants of resolution, the *graded Betti numbers*, and the convenient Betti diagrams for displaying them.

At this stage we still have no tools for showing that a given complex is a resolution, and in Chapter 2 we remedy this lack with a simple but very effective idea of Bayer, Peeva, and Sturmfels for describing some resolutions in terms of *labeled simplicial complexes*. With this tool we prove the Hilbert Syzygy Theorem and we also introduce Koszul homology. We then spend some time on the example of seven points in $\mathbb{P}^3$, where we see a deep connection between syzygies and an important invariant of the positions of the seven points.

In the next chapter we explore a case where we can say a great deal: sets of points in $\mathbb{P}^2$. Here we characterize all possible resolutions and derive some invariants of point sets from the structure of syzygies.

The following Chapter 4 introduces a basic invariant of the resolution, coarser than the graded Betti numbers: the *Castelnuovo–Mumford regularity*. This is a topic of central importance for the rest of the book, and a very active one for research. The goal of Chapter 4, however, is modest: we show that in the setting of sets of points in $\mathbb{P}^r$ the Castelnuovo–Mumford regularity is the degree needed to interpolate any function as a polynomial function. We also explore different characterizations of regularity, in terms of local or Zariski cohomology, and use them to prove some basic results used later.

Chapter 5 is devoted to the most important result on Castelnuovo–Mumford regularity to date: the theorem by Castelnuovo, Mattuck, Mumford, Gruson, Lazarsfeld, and Peskine bounding the regularity of projective curves. The techniques introduced here reappear many times later in the book.

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[1] These software packages are freely available for many platforms, at cocoa.dima.unige.it, www.math.uiuc.edu/Macaulay2 and www.singular.uni-kl.de, respectively. These web sites are good sources of further information and references.

The next chapter returns to examples. We develop enough material about linear series to explain the free resolutions of all the curves of genus 0 and 1 in complete embeddings. This material can be generalized to deal with nice embeddings of any hyperelliptic curve.

Chapter 7 is again devoted to a major result: Green's Linear Syzygy theorem. The proof involves us with exterior algebra constructions that can be organized around the Bernstein–Gelfand–Gelfand correspondence, and we spend a section at the end of Chapter 7 exploring this tool.

Chapter 8 is in many ways the culmination of the book. In it we describe (and in most cases prove) the results that are the current state of knowledge of the syzygies of the ideal of a curve embedded by a complete linear series of *high degree* — that is, degree greater than twice the genus of the curve. Many new techniques are needed, and many old ones resurface from earlier in the book. The results directly generalize the picture, worked out much more explicitly, of the embeddings of curves of genus 0 and 1. We also present the conjectures of Green and Green–Lazarsfeld extending what we can prove.

No book on syzygies written at this time could omit a description of Green's conjecture, which has been a wellspring of ideas and motivation for the whole area. This is treated in Chapter 9. However, in another sense the time is the worst possible for writing about the conjecture, since major new results, recently proven, are still unpublished. These results will leave the state of the problem greatly advanced but still far from complete. It's clear that another book will have to be written some day...

Finally, I have included two appendices to help the reader: Appendix 1 explains local cohomology and its relation to sheaf cohomology, and Appendix 2 surveys, without proofs, the relevant commutative algebra. I can perhaps claim (for the moment) to have written the longest exposition of commutative algebra in [Eisenbud 1995]; with this second appendix I claim also to have written the shortest!

Prerequisites

The ideal preparation for reading this book is a first course on algebraic geometry (a little bit about curves and about the cohomology of sheaves on projective space is plenty) and a first course on commutative algebra, with an emphasis on the homological side of the field. Appendix 1 proves all that is needed about local cohomology and a little more, while Appendix 2 may help the reader cope with the commutative algebra required.

How Did This Book Come About?

This text originated in a course I gave at the Institut Poincaré in Paris, in 1994. The course was presented in my imperfect French, but this flaw was corrected by three of my auditors, Freddy Bonnin, Clément Caubel, and Hèléne Maugendre. They wrote up notes and added a lot of polish.

I have recently been working on a number of projects connected with the exterior algebra, partly motivated by the work of Green described in Chapter 7. This led me to offer a course on the subject again in the Fall of 2001, at the University of California, Berkeley. I rewrote the notes completely and added many topics and results, including material about exterior algebras and the Bernstein–Gelfand–Gelfand correspondence.

Other Books

Free resolutions appear in many places, and play an important role in books such as [Eisenbud 1995], [Bruns and Herzog 1998], and [Miller and Sturmfels 2004]. The last is also an excellent reference for the theory of monomial and toric ideals and their resolutions. There are at least two book-length treatments focusing on them specifically, [Northcott 1976] and [Evans and Griffith 1985]. The books [Cox et al. 1997] and [Schenck 2003] give gentle introductions to computational algebraic geometry, with lots of use of free resolutions, and many other topics. The notes [Eisenbud and Sidman 2004] could be used as an introduction to parts of this book.

Thanks

I've worked on the things presented here with some wonderful mathematicians, and I've had the good fortune to teach a group of PhD students and postdocs who have taught me as much as I've taught them. I'm particularly grateful to Dave Bayer, David Buchsbaum, Joe Harris, Jee Heub Koh, Mark Green, Irena Peeva, Sorin Popescu, Frank Schreyer, Mike Stillman, Bernd Sturmfels, Jerzy Weyman, and Sergey Yuzvinsky, for the fun we've shared while exploring this terrain.

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Notation

Throughout the text $\mathbb{K}$ denotes an arbitrary field; $S = \mathbb{K}[x_0, \dots, x_r]$ denotes a polynomial ring; and $\mathfrak{m} = (x_0, \dots, x_r) \subset S$ denotes its homogeneous maximal ideal. Sometimes when r is small we rename the variables and write, for example, $S = \mathbb{K}[x, y, z]$.

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