CHAPTER 17

OPTIMAL CONTRACTS

We now introduce a model of a two person "partnership" known as the principal-agent model. It introduces incentive issues by assuming that *actions are unobservable* and the contracting parties may have *direct preferences with respect to actions*, as opposed to the standard partnership in which actions are observable and preferences are defined over monetary outcomes (see Volume I, Chapter 4). The basic principal-agent model assumes that the principal owns a production technology. In order for the technology to be productive he must hire an agent to perform a task. How the agent performs the task is unobservable to the principal, but it affects the probability distribution of the monetary outcome of the production technology. The incentive problem is caused (in part) by assuming that the agent has direct preferences with respect to what he does in the task (usually interpreted as the agent's effort), as well as his compensation (i.e., his share of the monetary outcome), while the principal is only concerned about the monetary outcome (net of the compensation paid to the agent). If the monetary outcome is the only contractible information available, then the sharing rule between the principal and the agent can only depend on the monetary outcome. Furthermore, the sharing rule based on the monetary outcome is the only mechanism available to the principal for inducing the agent to make action choices that are consistent with the principal's preferences. More generally, other performance measures may exist, and the monetary outcome may not be reported within the time frame of the contract, but we leave exploration of such settings until Chapter 18.

In this chapter we assume the principal and agent share the outcome x from the production technology operated by the agent, and cannot share the risks associated with that outcome with any other parties. The principal can represent a sole proprietor or a set of partners who own and finance the production technology, and hire the agent. Alternatively, as explored in Chapter 18, the agent can own and operate the production technology, and the principal can represent a set of investors who contract to share the agent's risk and provide investment capital. The capital market is not explicitly considered. However, the results obtained here are consistent with those obtained when the agency operates in a capital market, provided all risks are firm-specific, and therefore cannot be mitigated by appropriate investments in other firms (e.g., the market portfolio). The impact of economy-wide risk within a market setting is examined in Chapter 18.

The model examined in this chapter has an initial date at which the contract is signed and the agent exerts effort in a single task, and a terminal date at which the outcome x is realized and shared by the principal and the agent. The principal and the agent have the same information prior to signing the contract, and there is no additional information until the outcome is realized. In later chapters we extend the basic model to settings in which there are other performance measures at the contract termination date, the agent allocates effort among a number of tasks, the agent receives private information prior to taking his action and possibly prior to accepting the contract, and there is a sequence of action and consumption dates.

In this chapter we first (Section 17.1) introduce the basic principal-agent model, and provide a general discussion of the optimal contract when the agent has a finite number of alternative actions. In Section 17.2 we characterize first-best contracts, which, for example, apply if the principal can observe the agent's action. Section 17.3 explores the impact of the agent's risk and effort aversion on the characteristics of second-best contracts, which apply if the principal cannot observe the agent's action. Finally, Section 17.4 explores the characteristics of the second-best contract if the agent is risk neutral, but has limited liability constraints. Brief concluding remarks are provided in Section 17.5.

17.1 BASIC PRINCIPAL-AGENT MODEL

17.1.1 Basic Model Elements

As in the partnership model (see Volume I, Chapter 4), the outcome $x \in X \subseteq \mathbb{R}$ is determined by the action $a \in A$ (which, in this case, is an unobserved choice by the agent) and the outcome adequate events $\theta \in \Theta$. The principal and the agent have homogeneous beliefs about θ and those beliefs are denoted by a generalized probability density function $\varphi(\theta)$. However, it is useful in this analysis to suppress θ and focus on x as a random variable whose distribution depends on a. For example, if Θ is finite, then the generalized probability density function for x given a is

$$\varphi(x|a) = \sum_{\Theta(x,a)} \varphi(\theta),$$

where $$\Theta(x,a) = \{ \theta \mid \boldsymbol{x}(\theta,a) = x, \theta \in \Theta \}.$$

The principal's share of x is denoted π and the agent's share is c, so that $\pi = x - c$. We generally assume that the principal has unlimited resources so that $\Pi = \mathbb{R}$ is the set of possible values of π, but we assume (unless stated otherwise) that the agent cannot be paid less than a finite lower bound $\underline{c}$, so that $C = [\underline{c},\infty)$ is the set of possible compensation levels for the agent.

A compensation scheme (contract) specifies the amount to be paid to the agent at the contract settlement date. To be enforceable, the payment specified by the contract must be either fixed or at most vary with the *contractible information* available at the contract settlement date. To be contractible, the information must be acceptable to the court, or whatever institution is used to enforce the contract. In the basic model, it is assumed that the outcome x is *the only contractible information*. Hence, the agent's compensation scheme in this setting is $\boldsymbol{c}: X \to C$ and the set of possible compensation functions is denoted $\boldsymbol{C}$.

The principal's preferences are assumed to be a function of only his share of x, i.e., it is represented by a utility function u^p: $\Pi \to \mathbb{R}$, where Π is the set of possible values of π. The principal has no direct preferences with respect to the agent's action a. However, the agent's preferences may depend on both his consumption c and his action a, i.e., his preferences are represented by a utility function u^a: $C \times A \to \mathbb{R}$.

The *agent's utility function* is generally assumed to be *separable*, by which we mean it can be expressed as $u^a(c,a) = u(c)k(a) - v(a)$, with $k(a) > 0$. We consider three basic forms of separability:

(a) *Additive separability:* $u^a(c,a) = u(c) - v(a)$ (i.e., $k(a) = 1$);

(b) *Multiplicative separability:* $u^a(c,a) = u(c)k(a)$ (i.e., $v(a) = 0$);

(c) *Effort neutrality:* $u^a(c,a) = u(c)$ (i.e., $k(a) = 1$ and $v(a) = 0$).

The principal's and agent's preferences with respect to consumption are assumed to be increasing and concave, i.e., $u^{p\prime} > 0$, $u^{p\prime\prime} \le 0$, $u' > 0$ and $u'' \le 0$. If $k(a)$ is not constant, we assume $u(c)$ is non-positive, so that increases in both $k(a)$ and $v(a)$ reduce the agent's utility, thereby representing more costly effort.

The exponential utility function with a monetary cost of effort $\kappa(a)$ is an important example of a multiplicatively separable utility function.

Lemma 17.1

If the agent has a negative exponential utility for consumption and effort imposes a personal cost $\kappa(a)$ in the form of a reduction of consumption, then the utility function is multiplicatively separable, i.e.,

$$u^a(c,a) = -\exp[-r(c - \kappa(a))] = u(c)k(a),$$

where $\quad u(c) = -\exp[-rc]$ and $k(a) = \exp[r\kappa(a)]$, and r is a parameter representing the agent's risk aversion.

17.1.2 Principal's Decision Problem

In our discussion of partnerships we provided a general characterization of Pareto efficient sharing rules. In the analysis presented here we adopt a slightly different perspective. The principal is assumed to "own" the production technology that generates x, and he hires an agent from a market for agents. To entice an agent to accept his contract, the principal must offer a contract that provides the agent with an expected utility at least as great as the agent's "reservation utility" $\bar{U}$, which is the expected utility the agent could obtain from his next best alternative.

Observe that this approach assumes that the principal has all the bargaining power. In Section 17.4 and in Chapter 18 we consider settings in which the agent owns the technology and has the bargaining power, and he contracts with the principal to share risk (and possibly obtain capital). The principal's expected utility from sharing the risk (and providing investment capital) must be at least as great as from his next best alternative. Interestingly, the basic character of the optimal contract is the same in both settings.

In specifying the principal's decision problem, we view him as selecting both the contract $\boldsymbol{c}$ that he offers to the agent, and the action a he will induce the agent to select. Of course, it is the agent who selects the action. Hence, the contract must be such that it induces the agent to accept the contract and to select the specified action a. In agency theory we typically assume the agent will select the action a specified by the principal *if, and only if,* the agent cannot increase his expected utility by doing otherwise. Hence, the principal's decision problem is

Principal's Decision Problem:

$$\underset{\boldsymbol{c}\in\boldsymbol{C},a\in A}{\text{maximize}} \quad U^p(\boldsymbol{c},a) \equiv \int_X u^p(x-\boldsymbol{c}(x))\, d\Phi(x|a), \tag{17.1}$$

$$\text{subject to} \quad U^a(\boldsymbol{c},a) \equiv \int_X u^a(\boldsymbol{c}(x),a)\, d\Phi(x|a) \geq \bar{U}, \qquad \text{(contract acceptance)} \tag{17.2}$$

$$U^a(\boldsymbol{c},a) \geq U^a(\boldsymbol{c},\hat{a}), \quad \forall\, \hat{a}\in A, \ \text{(incentive compatibility)} \tag{17.3}$$

$$\boldsymbol{c}(x) \geq \underline{c}, \quad \forall\, x\in X. \quad \text{(feasible consumption)} \tag{17.4}$$

In (17.1) the principal maximizes his expected utility of his share of the outcome $\pi(x) = x - \boldsymbol{c}(x)$ that will result from his choice of compensation scheme $\boldsymbol{c} \in \boldsymbol{C}$ and induced action $a \in A$. His choice of $\boldsymbol{c}$ and a must satisfy the constraints (17.2)-(17.4). Constraint (17.2) is often referred to as the agent's *participation* or *individual rationality constraint*, and it ensures that the agent has no incentive not to accept the contract (in which case we assume he accepts). Constraints (17.3) (one for each $\hat{a}$) are usually referred to as the agent's *incentive compatibility constraints*. They ensure that given the compensation scheme $\boldsymbol{c}$, the agent has no incentive not to take the action a specified by the principal. That is, the action specified by the principal must be at least weakly preferred by the agent over all other actions, i.e., the induced action maximizes the agent's expected utility given the accepted compensation scheme, which can be expressed equivalently as[1]

$$a \in \underset{\hat{a} \in A}{\operatorname{argmax}} \; U^a(\boldsymbol{c}, \hat{a}).$$

Finally, constraint (17.4) ensures that the agent gets his minimum wage for all outcomes.

The principal's decision problem represents a subgame perfect Nash equilibrium to the sequential game shown in Figure 17.1.[2] The game starts with the principal proposing a contract $z = (\boldsymbol{c}, a)$, which the agent then accepts or rejects. If the agent accepts the proposed contract, he then chooses his action. Constraints (17.2) and (17.3) represent the sequential equilibrium conditions stating that it is incentive compatible for the agent to accept the contract and take the action specified by the principal.

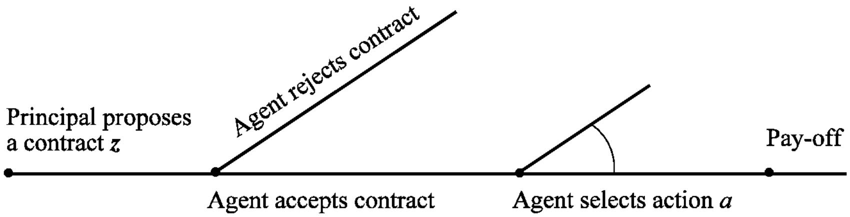

Figure 17.1: Principal's decision problem as a sequential game.

[1] *Argmax* is the set (of actions in this case) that maximizes the following objective function. If there is a unique optimum, then the set is a singleton, but the notation allows for the possibility of multiple optima so that the set contains more than one element.

[2] The concept of sequential equilibria is discussed in Volume I, Chapter 13.

17.1.3 Optimal Contract with a Finite Action and Outcome Space

We now consider settings in which A and X are finite sets. With A finite, the incentive compatibility constraints can be written as a set of $|A| - 1$ incentive constraints,[3]

$$U^a(\boldsymbol{c},a) \ \geq \ U^a(\boldsymbol{c},\hat{a}), \qquad \forall\, \hat{a} \in A\backslash\{a\}, \tag{17.3f}$$

and, similarly, with X finite, the consumption feasibility constraints are a set of $|X|$ constraints,

$$\boldsymbol{c}(x) \geq \underline{c}, \qquad \forall\, x \in X. \tag{17.4f}$$

Given this formulation, the Lagrangian for the principal's decision problem is:

$$\mathcal{L} = U^p(\boldsymbol{c},a) + \lambda\,[\,U^a(\boldsymbol{c},a) - \bar{U}\,]$$

$$+ \sum_{\hat{a} \in A\backslash\{a\}} \mu(\hat{a})\,[U^a(\boldsymbol{c},a) - U^a(\boldsymbol{c},\hat{a})] \ + \ \sum_{x \in X} \xi(x)\,[\boldsymbol{c}(x) - \underline{c}], \tag{17.5}$$

where λ, $\{\mu(\hat{a})\}_{\hat{a}\in A\backslash\{a\}}$, and $\{\xi(x)\}_{x\in X}$ are the multipliers associated with the constraints (17.2), (17.3f) and (17.4f), respectively.

For a given action a to be induced, the principal's compensation scheme choice can be viewed as consisting of $|X|$ choice variables, i.e., the compensation level $\boldsymbol{c}(x)$ for each possible outcome $x \in X$. Differentiating with respect to each $\boldsymbol{c}(x)$, $x \in X$, provides the following first-order conditions characterizing the optimal compensation scheme:

$$- \ u^{p\prime}(x - \boldsymbol{c}(x))\,\varphi(x|a) \ + \ \lambda\, u'(\boldsymbol{c}(x))\,k(a)\,\varphi(x|a)$$

$$+ \sum_{\hat{a} \in A\backslash\{a\}} \mu(\hat{a})\,u'(\boldsymbol{c}(x))\,[\,k(a)\,\varphi(x|a) \ - \ k(\hat{a})\,\varphi(x|\hat{a})\,] \ + \ \xi(x) \ = \ 0.$$

The multiplier $\xi(x)$ is zero if $\boldsymbol{c}(x) > \underline{c}$, in which case the compensation $\boldsymbol{c}(x)$ satisfies

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[3] $|A|$ is referred to as the cardinality of the set and represents the number of elements, i.e., in this case, the number of alternative actions in the set.

$$M(x,\boldsymbol{c}(x)) \equiv \frac{u^{p\prime}(x - \boldsymbol{c}(x))}{u'(\boldsymbol{c}(x))} = k(a)\left[\lambda + \sum_{\hat{a} \in A \backslash \{a\}} \mu(\hat{a})\, L(x|\hat{a},a)\right] > 0, \quad (17.6)$$

where
$$L(x|\hat{a},a) \equiv 1 - \frac{k(\hat{a})\varphi(x|\hat{a})}{k(a)\varphi(x|a)}.$$

The left-hand side expression, $M(x,c)$, is the ratio of the principal's and agent's marginal utilities. If the agent has no direct preference for actions, as in the partnerships examined in Volume I, Chapter 4, the principal is only concerned with efficient risk sharing, and the ratio is a constant. However, if the agent has direct preferences for his actions (and some of the incentive constraints are binding), then the right-hand side of (17.6) varies with the outcome x.

The $L(x|\,\hat{a},a)$ function on the right-hand side reflects the relative likelihood that outcome x will occur given the "desired" action a versus the undesired action $\hat{a}$. Since probabilities must sum to one for all actions, it follows that if there is an outcome x' that is more likely with a than with $\hat{a}$, then there must be another outcome x'' for which the reverse holds. Consequently, $L(x|\,\hat{a},a)$ is likely to be positive for some outcomes, but negative for others, and this is definitely the case if $k(a) = 1$ for all $a \in A$.

Since the principal's and agent's marginal utilities for their outcome shares are positive, the ratio $M(x,c)$ is non-negative. Hence, the left-hand side of (17.6) is always non-negative. However, the preceding comment implies that the right-hand side can be positive or negative. This creates the possibility of a corner solution. In particular, if for any $x \in X$ the multipliers λ and $\mu(\hat{a})$, $\hat{a} \in A\backslash\{a\}$ are such that

$$k(a)\left[\lambda + \sum_{\hat{a} \in A \backslash \{a\}} \mu(\hat{a}) L(x|\hat{a},a)\right] < M(x,\underline{c}),$$

then $\xi(x) > 0$ and $\boldsymbol{c}(x) = \underline{c}$, i.e., the agent is paid his minimum compensation.

17.2 FIRST-BEST CONTRACTS

If none of the incentive constraints (17.3) are binding (so that $\mu(\hat{a}) = 0$ for all $\hat{a} \in A$), then we say the contract is *first-best*. In that setting, there is "no incentive problem" and the optimal contract achieves fully Pareto efficient action choice and risk sharing. If some of the incentive constraints (17.3) are binding, then there is a non-trivial incentive problem and we say the optimal incentive contract is *second-best* (with respect to risk sharing).

Definition *First-best Contracts*

A contract $z^* = (c^*, a^*)$ is first-best if it maximizes (17.1) subject to (17.2) and (17.4) in the principal's decision problem.

The following proposition characterizes the efficient risk sharing given the first-best action choice.

Proposition 17.1

If the contract $z = (c, a)$ is first-best, there exists a multiplier λ such that $c(x)$ satisfies

$$M(x, c(x)) = \lambda k(a), \quad \text{if } M(x, \underline{c}) \leq \lambda k(a), \text{ or}$$

$$c(x) = \underline{c}, \quad \text{if } M(x, \underline{c}) > \lambda k(a).$$

If the *principal is risk neutral*, then $u^p(\pi) = \pi$ and $u^{p\prime}(\pi) = 1$. In that case, $M(x,c)$ is independent of x and we write it as

$$M(c) \equiv \frac{1}{u'(c)} = w'(u(c)),$$

where $w(\cdot) \equiv u^{-1}(\cdot)$ denotes the inverse of the agent's utility for consumption.[4] Hence, $w(u)$ is the cost to the principal of providing the agent a utility of u, and $M(c)$ is the principal's marginal cost of increasing the agent's utility at the compensation c.

Proposition 17.2

If the principal is risk neutral and the agent is risk averse with a separable utility function, then the first-best compensation scheme is a constant wage for all outcomes that occur with positive probability given a^*, i.e.,

$$c^*(x) = w\left(\frac{\bar{U} + v(a^*)}{k(a^*)}\right) \equiv c^*, \quad \forall\, x \in X \text{ for which } \varphi(x|a^*) > 0.$$

Of course, if the principal is risk neutral and the agent is risk averse, efficient risk sharing calls for the principal to carry all the risk.

Grossman and Hart (GH) (1983) identify some conditions under which the first-best result is achieved.

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[4] Observe that $w(u(c)) = c$. Differentiating both sides yields $w'(u(c))\,u'(c) = 1$, which implies $w' = 1/u'$.

Proposition 17.3 (GH, Prop. 3)

Assume the agent's utility function is separable and the outcome x is contractible. The first-best result can be achieved if one of the following conditions holds.

(a) The *agent is effort neutral* and the two utility functions u^p and u belong to the HARA class with *identical risk cautiousness.*

(b) The *principal is risk neutral* and the agent is either *effort neutral* or the *first-best action is his least cost action*, i.e,

$$a^* \text{ minimizes } w((\bar{U} + v(a))/k(a)).$$

(c) The *agent is risk neutral* and has sufficient wealth.

(d) Shirking is detected with a sufficiently large positive probability, i.e., if the agent takes an action that is less costly to him than a^*, there is a sufficiently large positive probability that x will reveal that he has not taken a^*.

Given effort neutrality, result (a) follows directly from our discussion of partnerships in Volume I, Chapter 4, and is the case examined by Ross (1973). Recall that if the partners have HARA utilities with identical risk cautiousness, then the Pareto efficient sharing rules are linear and they induce identical preferences over actions. Note also that given the first-best action, first-best risk sharing can be obtained without restricting the two utility functions, i.e., the restriction to the HARA class with identical risk cautiousness is to create identical preferences over actions.

Result (b) identifies two settings in which it is optimal to pay the agent a fixed wage. The principal is risk neutral, and hence efficiently bears all risk, and paying a constant wage to the agent does not cause an incentive problem either because the agent is effort neutral or the principal fortunately desires to induce the action the agent will select if he bears no incentive risk.

Result (c) establishes that agent risk neutrality (with or without effort neutrality) is sufficient to achieve first-best as long as $x - \pi^* \geq \underline{c}$ (for all x which have a strictly positive probability of occurring given a^*), where π^* is the fixed amount paid to the risk averse principal. In this case, the firm is sold (or leased) to the agent, who in effect bears all risk. He will then make the optimal effort selection a^*. Section 17.4 examines the impact of binding limitations on the agent's ability to bear risk.

Result (d) is generally referred to as a setting in which there is "moving support," where the support for the distribution given action a is the set of outcomes that have a strictly positive probability of occurring.

Definition

$X(a) \equiv \{ x \mid \varphi(x|a) > 0, x \in X \}$ is the *support* of $\varphi(x|a)$. The support is *constant* (nonmoving) if $X(a) = X, \forall\, a \in A$.

Penalties cannot be used to achieve the first-best result if the support is constant. On the other hand, it may be possible if we have moving support, i.e., if $X(a)$ varies with a. However, to achieve first-best using the threat of penalties, we must have the following:

(*i*) $X(a) \backslash X(a^*) \neq \emptyset$ for all $a \in A^\dagger \equiv \{ a \in A \mid U^a(\boldsymbol{c}^*, a^*) < U^a(\boldsymbol{c}^*, a) \}$, i.e., there are outcomes that have a positive probability of occurring if the agent selects an action a that is less costly to him than a^*, but have zero probability if he selects a^*;

(*ii*) $u^a(c^*, a)\,[1 - \Phi(a)] + u^a(\underline{c}, a)\,\Phi(a) < u^a(c^*, a^*), \quad \forall\, a \in A^\dagger,$

where
$$\Phi(a) = \sum_{x \in X(a) \backslash X(a^*)} \varphi(x|a).$$

If the above conditions hold, then the first-best result can be achieved by paying the first-best wage, $\boldsymbol{c}(x) = c^*$, for each outcome that has a positive probability with a^* (i.e., $x \in X(a^*)$) and threatening to pay the minimum wage, $\boldsymbol{c}(x) = \underline{c}$, for any outcome that has zero probability with a^* (i.e., $x \in X \backslash X(a^*)$). Observe that the payment of $\underline{c}$ is merely a threat – it will never be paid (given that the agent is induced to select a^*). Of course, this is only possible if $\Phi(a) \geq [u^a(c^*, a) - u^a(c^*, a^*)] \,/\, [u^a(c^*, a) - u^a(\underline{c}, a)]$ for all $a \in A^\dagger$. This will tend to hold if either the probability of detection, $\Phi(a)$, is relatively large, or the loss in utility when shirking is detected, $u^a(c^*, a) - u^a(\underline{c}, a)$, is relatively large.

17.3 RISK AND EFFORT AVERSION

We now focus on settings in which first-best contracts cannot be achieved, i.e., at least some of the incentive compatibility constraints are binding. Consequently, we assume the *agent is both risk averse and effort averse*, i.e., $u' > 0$, $u'' < 0$, and either $k' > 0$ with $u < 0$ or $v' > 0$. On the other hand, we do not view an agent's risk bearing ability as a significant part of most incentive contracts. Therefore, we generally assume that the *principal is risk neutral*, i.e., $u^p(\pi) = \pi$, so that he would bear all risk in a first-best contract. This ensures that any risk borne by the agent in a second-best contract is for incentive purposes. Incentive risk is costly to the agent, but he is compensated for that cost and, hence, it is indirectly costly to the principal.

To exclude the possibility of using moving support (in combination with sufficient penalties) to achieve the first-best result, we generally assume the support is constant across the alternative actions. Furthermore, we assume the optimal action is not the agent's least cost action.

Our maintained assumptions in this section can be summarized as follows:

(a) the principal is risk neutral;

(b) the agent is both risk and effort averse, with a separable utility function;

(c) there is constant support;

(d) the optimal action to be induced is not the agent's least cost action.

These assumptions are sufficient to ensure that the first-best result is not achievable.

17.3.1 Finite Action Space

In this section we further make the following regularity assumptions:

– $A = \{ a_1, ..., a_M \}$, with $\boldsymbol{c}^*(a_\ell) < \boldsymbol{c}^*(a_j)$ if $\ell < j$,

where $$\boldsymbol{c}^*(a) \equiv w\left(\frac{\bar{U} + v(a)}{k(a)}\right),$$

i.e., the set of actions is finite and the actions are ordered in terms of the fixed wage that would be required to compensate the agent for his effort;

– $X = \{x_1, ..., x_N\}$, with $x_h < x_i$ if $h < i$, i.e., the set of possible outcomes is finite and ordered in terms of increasing outcomes.

A Two-stage Optimization Approach

The assumption that the principal is risk neutral permits us to separate the principal's decision problem into two stages. In the *first stage*, we determine, for each action, the contract that induces the particular action at the lowest possible expected cost – the expected cost for action a is denoted $\bar{\boldsymbol{c}}^\dagger(a)$. In the *second stage* we determine the optimal action by maximizing the principal's expected profit.

The contract used to induce an action can be stated as $\mathbf{c} = \{c_1, ..., c_N\}$, where $c_i = \boldsymbol{c}(x_i)$. However, GH state the principal's decision problem for a given action in terms of $\mathbf{u} = \{u_1, ..., u_N\}$, where $u_i = u(c_i)$. The advantage of this approach is that the objective function is now convex (since $u(c)$ is concave and, hence, $w(u)$ is convex) and the constraints are linear. Hence, the Kuhn-Tucker conditions yield necessary and sufficient conditions for optimality.

The Principal's Decision Problem for Inducing Action a_j:

$$\bar{c}^\dagger(a_j) = \underset{\mathbf{u}}{\text{minimize}} \sum_{i=1}^{N} w(u_i)\,\varphi(x_i|a_j), \tag{17.1f′}$$

$$\text{subject to} \quad U^a(\mathbf{u},a_j) \equiv k(a_j)\sum_{i=1}^{N} u_i\,\varphi(x_i|a_j) - v(a_j) \geq \bar{U}, \tag{17.2f′}$$

$$U^a(\mathbf{u},a_j) \geq U^a(\mathbf{u},a_\ell), \quad \forall\ \ell = 1, ..., M,\ \ell \neq j, \tag{17.3f′}$$

$$u_i \geq u(\underline{c}), \quad \forall\ i = 1, ..., N. \tag{17.4f′}$$

In general, all actions may not be implementable, i.e., there may not exist a feasible solution to the program for inducing a_j in which case we set $\bar{c}^\dagger(a_j) = \infty$. However, note that there is at least one action that can be implemented at a finite expected cost, namely the least cost action which can be implemented at its first-best cost.

Also, we cannot rule out the possibility of a corner solution in which (17.4f′) is binding for some x_i. GH avoid this by assuming that $C = (\underline{c},\infty)$ with[5]

$$\lim_{c \to \underline{c}} u(c) = -\infty.$$

Given this assumption, (17.4f′) is redundant, and we can restrict our attention to interior solutions. Hence, the Lagrangian for the above constrained minimization problem is

$$\mathcal{L} = \sum_{i=1}^{N} w(u_i)\,\varphi(x_i|a_j) - \lambda\left(k(a_j)\sum_{i=1}^{N} u_i\,\varphi(x_i|a_j) - v(a_j) - \bar{U}\right)$$

$$- \sum_{\substack{\ell=1 \\ \ell \neq j}}^{M} \mu_\ell\left(\sum_{i=1}^{N} u_i[k(a_j)\,\varphi(x_i|a_j) - k(a_\ell)\,\varphi(x_i|a_\ell)] - [v(a_j) - v(a_\ell)]\right).$$

[5] Note that for efficient risk sharing in partnerships (with no personal costs), the weaker condition $\lim_{c \to \underline{c}} u'(c) = \infty$ is sufficient to preclude corner solutions (see Volume I, Chapter 4).

Differentiating with respect to u_i provides the following characterization of an interior solution

$$w'(u_i) = k(a_j)\left[\lambda + \sum_{\substack{\ell=1\\ \ell\neq j}}^{M} \mu_\ell L(x_i|a_\ell,a_j)\right]. \tag{17.6'}$$

Let $\mathbf{c}_j^\dagger = \{c_{1j}^\dagger, ..., c_{Nj}^\dagger\}$, where $c_{ij}^\dagger = w(u_i)$ represents the optimal second-best contract for implementing action a_j, as determined by the solution to the above problem.

The *second stage* is to identify the optimal second-best action $a^\dagger$ by comparing the cost of each possible action to the expected gross outcome it will generate, i.e.,

$$a^\dagger \in \underset{a_j \in A}{\text{argmax}}\ \mathrm{E}[x|a_j] - \bar{c}^\dagger(a_j). \tag{17.7'}$$

The optimal second-best contract for implementing $a^\dagger$ is denoted $\mathbf{c}^\dagger$.

Proposition 17.4 (GH, Prop. 1 and 2)

Given the above assumptions with either additively or multiplicatively separable agent utility, there *exists* a second-best optimal action $a^\dagger$ and compensation plan $\mathbf{c}^\dagger$, and that solution is such that the participation constraint is binding, i.e.,

$$U^a(\mathbf{c}^\dagger, a^\dagger) = \bar{U}.$$

The existence of a solution to the principal's cost minimization problem for a given action is ensured by the fact that the cost is bounded below (by the first-best cost of implementing the action), the set of constraints (17.2f′)-(17.3f′) form a closed set, and there are a finite number of alternative actions. The key to the participation constraint (17.2f′) being binding is the assumption that the agent's utility of consumption is unbounded from below. To see this, suppose, to the contrary, that there is a solution $\mathbf{u}$ to the principal's decision problem for inducing some action a_j for which the participation constraint is not binding. Since the agent's utility of consumption is unbounded, there is another contract $\mathbf{u}'$ in the *additively separable case*, defined as

$$u_i' = u_i - \varepsilon, \quad i = 1, ..., N, \quad \varepsilon > 0,$$

which satisfies the participation constraint and is less costly to the principal. The contract $\mathbf{u}'$ clearly satisfies the incentive compatibility constraint since

$$U^a(\mathbf{u}',a_j) = \sum_{i=1}^{M} u_i'\,\varphi(x_i|a_j) - v(a_j) = U^a(\mathbf{u},a_j) - \varepsilon.$$

Similarly, in the *multiplicatively separable case* a feasible less costly contract $\mathbf{u}'$ can be found as

$$u_i' = u_i\,(1+\varepsilon), \quad i = 1, ..., N, \quad \varepsilon > 0,$$

with
$$U^a(\mathbf{u}',a_j) = k(a_j)\sum_{i=1}^{M} u_i'\,\varphi(x_i|a_j) = U^a(\mathbf{u},a_j)\,(1+\varepsilon).$$

Hence, a contract $\mathbf{u}$ for which the participation constraint is not binding cannot be an expected cost minimizing contract for inducing a_j. However, note that if the agent's utility of consumption is bounded below, for example, by zero for the square-root utility function, the participation constraint may be a non-binding constraint. Intuitively, the reason is that inducing a given action is based on the difference in utility associated with "rewards" for good outcomes and penalties for "bad" outcomes. If the lower bound constrains the utility for "bad" outcomes, then the utility for "good" outcomes necessary to induce the desired action can result in an expected utility greater than the agent's reservation utility. On the other hand, if the optimal contract is such that the utility for "bad" outcomes strictly exceeds the lower bound, then the participation constraint will be binding, based on the same reasoning as for the case with unbounded utility of consumption.

Characteristics of Optimal Second-best Compensation Contracts

The optimal contract for inducing an action, including the second-best action, is characterized by (17.6′). The parameters λ, μ_1, ..., μ_{j-1}, μ_{j+1}, ..., μ_N are non-negative Lagrange multipliers, and $\mu_\ell > 0$ only if the agent is indifferent between a_j and a_ℓ at the optimum.

Proposition 17.5 (GH, Prop. 6)

If the second-best action $a^\dagger = a_j$, with $j > 1$, then there is at least one less costly action a_ℓ, $\ell < j$, such that $\mu_\ell > 0$ and $U^a(\mathbf{c}^\dagger, a^\dagger) = U^a(\mathbf{c}^\dagger, a_\ell)$.

The key for this result is that if all the incentive constraints for less costly actions a_ℓ, $\ell < j$, are redundant in the optimal solution, those actions can be omitted from the principal's decision problem, i.e., we can set the action space to be $A' = \{a_j, ..., a_M\}$ without changing the optimal solution. Since the second-best action a_j is now the least costly action for the agent, it is optimally implemented by a fixed wage compensation scheme. However, a fixed wage will induce a_1

(and not a_j) in the original problem and, therefore, some of the incentive constraints for the less costly actions must be binding in the optimal solution.

An important point to recognize from (17.6′) is that the amount paid to the agent for a given outcome does not depend on the value of the outcome to the principal. Instead, the amount paid depends on the inverse of the likelihood of the outcome given the action to be induced relative to other actions between which the agent is indifferent (i.e., actions for which the incentive constraints are binding). That is, while the value of the outcomes to the principal affect which action he chooses to induce, the use of the outcome to induce the desired action reflects its *information content* about the agent's unobservable action, not its value to the principal. However, while the concept of information content is similar to its use in discussing inferences about random variables, it is important to recall that the principal at the compensation date is not trying to draw inferences from the outcomes about what action the agent selected. At that date, he knows precisely (i.e., has a rational conjecture with respect to) the action the agent selected given the contract that was offered to the agent at the contracting date. Instead, the interpretation is that an optimal contract that induces a given action and shares the risk efficiently is related to the information content of the outcomes about the agent's action.

In Volume I, Chapter 2, we identified the relation between the *monotone likelihood ratio property* (MLRP) and first-order stochastic dominance for an arbitrary set of parameters $\omega \in \Omega$. We now introduce MLRP into the agency context.

Definition *Monotone Likelihood Ratio Property*

The probability distribution for x given a satisfies the monotone likelihood ratio property (MLRP) if for any pair of outcomes (x_h, x_i), $h < i$, and any pair of actions (a_ℓ, a_j), $\ell < j$, it holds that

$$\frac{\varphi(x_h|a_\ell)}{\varphi(x_h|a_j)} > \frac{\varphi(x_i|a_\ell)}{\varphi(x_i|a_j)}.$$

That is, lower outcomes are more likely with lower cost actions. Recall that MLRP implies first-order stochastic dominance (FSD), which implies $\mathrm{E}[x|a_\ell] \leq \mathrm{E}[x|a_j]$. Given our previous assumption about the ordering of the action indices, both the expected outcome $\mathrm{E}[x|a_j]$ and the first-best cost $c^*(a_j)$ are increasing in j.

MLRP implies that the likelihood ratio $\varphi(x_i|a_\ell)/\varphi(x_i|a_j)$ is decreasing in i if $\ell < j$, so that $L(x_i|a_\ell, a_j)$ is increasing in i. Hence, in implementing any action a_j it follows immediately from (17.6′) that the agent receives higher compensation for higher outcomes *if* the binding incentive constraints consist only of actions that are less costly than a_j.

Proposition 17.6

Consider the optimal second-best contract $\mathbf{c}_j^\dagger$ for implementing a_j. MLRP implies that $c_{ij}^\dagger$ is nondecreasing in x_i if $\mu_\ell = 0$ for all $\ell > j$.

Is MLRP sufficient for $\mu_\ell = 0$ for all $\ell > j$? NO! GH provide an example satisfying MLRP in which $M = 3, j = 2$ is the second-best optimum, the constraints for both $\ell = 1$ and $\ell = 3$ are binding, and the resulting compensation contract is nonmonotonic and we provide a similar example at the end of this section. This example illustrates that while MLRP implies that a higher outcome is "good news" when comparing a more costly action to a less costly action, it does not imply that the optimal second-best compensation contract pays more for a higher outcome. The reason, of course, is that a compensation scheme that always pays more for higher outcomes may induce the agent to exert more effort than the principal prefers. In so doing, the agent would "earn" an expected utility higher than $\underline{U}$ and not provide sufficient return to the principal to pay for this additional compensation.

The following condition is sufficient to ensure that the optimal second-best compensation contract never pays less for higher outcomes.

Definition *Spanning Condition*

The spanning condition (SC) is satisfied if there exists a pair of probability functions $\boldsymbol{\varphi}^L$ and $\boldsymbol{\varphi}^H$ on X such that

(a) $\varphi^L(x_i), \varphi^H(x_i) \geq 0, \quad \forall\, i = 1, ..., N;$

(b) for each $a_j \in A$ there exists a weight $\zeta(a_j) \in [0,1]$, for each action j, such that

$$\varphi(x_i|a_j) = \zeta(a_j)\varphi^H(x_i) + (1 - \zeta(a_j))\varphi^L(x_i), \quad \forall\, i = 1, ..., N;$$

(c) $\boldsymbol{\varphi}^L$ and $\boldsymbol{\varphi}^H$ satisfy MLRP (i.e., $\varphi^L(x_i)/\varphi^H(x_i)$ is nonincreasing in i).

Observe that if there are only two outcomes (i.e., $N = 2$), then spanning is always satisfied (i.e., we can let $\varphi^L(x_1) = \varphi^H(x_2) = 1$ and $\zeta(a_j) = \varphi(x_2|a_j)$). Furthermore, if $N \geq 2$ and the spanning condition is satisfied, then we can view a_j as determining the probability of obtaining one of two fixed gambles, which makes it effectively equivalent to a two-outcome setting. In fact, many results that are easy to prove in the two-outcome setting can be readily extended to the N (or even infinite) outcome settings with SC. Observe that SC implies the MLRP for the distributions induced by the alternative actions.

Proposition 17.7 (GH, Prop. 7)

If the agent is strictly risk averse, then SC implies that the second-best optimal contract is nondecreasing in i, i.e., $c_{1j}^{\dagger} \leq c_{2j}^{\dagger} \leq \ldots \leq c_{Nj}^{\dagger}$.

Proof: Let $k_j = k(a_j)$, $\varphi_{ij} = \varphi(x_i|a_j)$, and $\zeta_j = \zeta(a_j)$. The first-order condition (17.6′) for an expected cost minimizing compensation contract can be expressed as

$$w'(u_i) = k_j\lambda + k_j\sum_{\ell \in J} \mu_\ell - \left[\sum_{\ell \in J} \omega_\ell \frac{\varphi_{i\ell}}{\varphi_{ij}}\right]\left[\sum_{h \in J} k_h\,\mu_h\right],$$

where
$$\omega_\ell = \frac{\mu_\ell k_\ell}{\sum_{h \in J} \mu_h k_h},$$

and J is the set of actions for which the incentive constraints are binding. The first two expressions on the right-hand side are constant, while the third varies with x_i.

SC (which includes MLRP) implies that for any set of actions $J \subset \{1,...,M\}$ and any normalized, non-negative weights ω_ℓ, $\Sigma_{\ell \in J}\, \omega_\ell = 1$,

$$\sum_{\ell \in J} \omega_\ell \frac{\varphi_{i\ell}}{\varphi_{ij}} = \frac{\bar{\zeta}\varphi_i^H + (1 - \bar{\zeta})\varphi_i^L}{\zeta_j\varphi_i^H + (1 - \zeta_j)\varphi_i^L}$$

is *either* nondecreasing ($\bar{\zeta} \geq \zeta_j$) *or* nonincreasing ($\bar{\zeta} \leq \zeta_j$) in x_i, where

$$\bar{\zeta} = \sum_{\ell \in J} \omega_\ell \zeta_\ell.$$

Of course, a cost minimizing compensation contract cannot be nonincreasing (unless it induces the least cost action in which case it is a fixed wage). Hence, it must be nondecreasing. **Q.E.D.**

Proposition 17.5 establishes that the incentive constraint for at least one less costly action is binding. If there are only two alternatives (i.e., $M = 2$) and a_2 is to be implemented, there is one binding incentive constraint. If there are more than two alternatives (i.e., $M > 2$) and $a_j, j \geq 2$, is implemented, then there may be multiple binding constraints. However, there are settings in which only the incentive constraint for a_{j-1} is binding. If that is the case, then MLRP implies that the optimal contract is nondecreasing in i.

Appendix 17A considers a condition known as the *concavity of distribution condition* (CDFC) which, with MLRP, is sufficient for the incentive constraint for a_{j-1} to be the only binding constraint and the compensation contract to be nondecreasing in i. Unfortunately, the examples provided in the literature of distributions that satisfy the CDFC condition seem very contrived. Based on Jewitt (1988), Appendix 17A also considers an alternative set of conditions that are satisfied by most "standard" distributions, but requires the agent's utility of compensation, $u \circ c(\cdot)$, to be a concave function of x_i for the second-best compensation contract. However, in any case, those conditions are sufficient, and not necessary, conditions for a single, adjacent incentive constraint to be binding.

A Finite Action/Outcome Example

We now illustrate the above analysis using a simple numerical example with three possible outcomes and three possible actions. The three outcomes are good, moderate and bad, represented by $x_g > x_m > x_b$, and the agent's compensation for the corresponding outcomes are c_g, c_m, c_b. The three actions are high, medium, and low effort, represented by a_H, a_M, and a_L, with $v_H > v_M > v_L$ representing the corresponding disutility levels. Panel A of Table 17.1 specifies the outcome probabilities for each action. Consistent with the outcome and action labels, a_H *FS*-dominates a_M, which in turn *FS*-dominates a_L.

We assume the agent has additively separable preferences and we use the following data for our numerical example.

$$u(c) = c^{\frac{1}{2}}; \; v_H = 55, \; v_M = 40, \; v_L = 0; \; \bar{U} = 200.$$

The magnitudes of the outcomes affect which action the principal chooses to induce, but they are immaterial to the determination of the optimal incentive contract for inducing a given level of effort.

If it is optimal for the principal to induce only a low level of effort, then it is optimal to pay the agent a fixed wage of $u^{-1}(\bar{U} + v_L) = 200^2 = 40{,}000$. If he chooses to induce either high or medium effort, then the principal must impose incentive risk on the agent. The principal's problem for determining the optimal incentive contract for inducing medium effort is

$$\underset{c_b, c_m, c_g}{\text{minimize}} \quad .20\, c_b + .60\, c_m + .20\, c_g,$$

$$\text{subject to} \quad .20\, c_b^{\frac{1}{2}} + .60\, c_m^{\frac{1}{2}} + .20\, c_g^{\frac{1}{2}} - 40 \geq 200,$$

$$.20\, c_b^{\frac{1}{2}} + .60\, c_m^{\frac{1}{2}} + .20\, c_g^{\frac{1}{2}} - 40 \geq .54\, c_b^{\frac{1}{2}} + .40\, c_m^{\frac{1}{2}} + .06\, c_g^{\frac{1}{2}},$$

$$.20\, c_b^{\frac{1}{2}} + .60\, c_m^{\frac{1}{2}} + .20\, c_g^{\frac{1}{2}} - 40 \geq .06\, c_b^{\frac{1}{2}} + .40\, c_m^{\frac{1}{2}} + .54\, c_g^{\frac{1}{2}} - 55.$$

This problem can be readily solved using a program like "Solver" in Excel. To do so, we transform the problem by using the utility levels u_b, u_m, u_g as the decision variables so that the objective function is convex and the constraints are linear. In the constraints we also collect terms so that all decision variables are on the left-hand side and all constants are on the right.

$$\begin{aligned}
\underset{u_g, u_m, u_b}{\text{minimize}} \quad & .20\, u_b^2 + .60\, u_m^2 + .20\, u_g^2, \\
\text{subject to} \quad & .20\, u_b + .60\, u_m + .20\, u_g \geq 240, \\
& -.34\, u_b + .20\, u_m + .14\, u_g \geq 40, \\
& .14\, u_b + .20\, u_m - .34\, u_g \geq -15.
\end{aligned}$$

		x_b	x_m	x_g	
Panel A: Probabilities $\varphi(x_i\|a_j)$					
	a_H	.06	.40	.54	
	a_M	.20	.60	.20	
	a_L	.54	.40	.06	
Panel B: Optimal compensation $c(x_i)$ to induce a_j					$\bar{c}^\dagger(a_j)$
	a_H	23,066	65,025	71,000	65,734
	a_M	21,805	70,225	67,492	59,850
	a_L	40,000	40,000	40,000	40,000
Panel C: Likelihood ratios $L(x_i\|a_\ell, a_M)$			$\lambda = 480$		μ_ℓ
	a_H	.7	1/3	– 1.7	27.257
	a_L	– 1.7	1/3	.7	122.743
Panel D: Likelihood ratios $L(x_i\|a_\ell, a_H)$			$\lambda = 510$		μ_ℓ
	a_M	– 7/3	– .5	17/27	0
	a_L	– 8	0	8/9	25.781

Table 17.1: Probabilities, optimal contracts, and likelihoods for finite action/outcome example.

The solution to this problem is presented in Panel B of Table 17.1, along with the optimal contract for inducing the high level of effort. Insight into the shape of the compensation functions for inducing a_M and a_L can be obtained by considering the likelihood ratios reported in Panels C and D in Table 17.1. The optimal contract for inducing a high level of effort is relatively straightforward. The fact the multiplier μ_M equals zero while μ_L is positive tells us that the incentive constraint for moderate effort is not binding. That is, if the incentives are sufficient to deter low effort, then they also deter moderate effort. With only the incentive constraint for a_L binding and $w'(u_i) = 2u_i = 2c_i^{1/2}$, we have, for example,

$$c_g = [\tfrac{1}{2}(\lambda + \mu_L L(x_g|a_L,a_H))]^2 = [\tfrac{1}{2}(510 + 8/9\times 25.781)]^2 = 71{,}000.$$

The fact that the likelihood function is increasing with x_i implies that the compensation is increasing in x_i. Again it is important to point out that, given that the principal is risk neutral, the compensation increases with x_i because large outcomes are more likely with high effort than with low effort, not because the amount available is larger.

This latter point is highlighted by the optimal compensation contract for inducing moderate effort. Observe that both incentive constraints are binding, which results in positive values for both μ_L and μ_H. The latter implies that if the principal offers a contract that focuses on inducing the agent to choose a_M instead of a_L, then the contract will induce the agent to work "too hard", i.e., to choose a_H. If the principal does not want the agent to work too hard, then he must, in a sense, penalize the agent for getting a high outcome instead of a moderate outcome. This is illustrated as follows:

$$\begin{aligned} c_g &= [\tfrac{1}{2}(\lambda + \mu_L L(x_g|a_L,a_M) + \mu_H L(x_g|a_H,a_M))]^2 \\ &= [\tfrac{1}{2}(480 + 122.743\times 0.7 - 27.257\times 1.7)]^2 = 67{,}492. \end{aligned}$$

The deviation from the base pay of $(\tfrac{1}{2}\times 480)^2 = 57{,}600$ reflects a bonus because this outcome is more likely with a_M than a_L less a penalty since it is less likely with a_M than with a_H (the likelihoods are $+0.7$ and -1.7, respectively).

Observe that the monotone likelihood property is satisfied by the example, but not the spanning condition. Hence, due to the lack of spanning, we can have two binding incentive constraints, and this can lead to a non-monotonic compensation for inducing a_M (which is less than maximum effort). Furthermore, even if there is a single binding incentive constraint, it need not be the adjacent constraint (as in the contract for inducing a_H). To illustrate the result with spanning, see Table 17.2 in which we have changed the probabilities for moderate effort to $\varphi(x_i|a_M) = \varphi(x_i|a_L)\times 1/6 + \varphi(x_i|a_H)\times 5/6$.

Only the likelihood ratio for the adjacent incentive constraint is reported for each induced action, since only that incentive constraint is binding. This, plus the monotone likelihood property, then implies that the compensation is monotonically increasing.

		x_b	x_m	x_g	
Panel A: Probabilities $\varphi(x_i\|a_j)$					
	a_H	.06	.40	.54	
	a_M	.14	.40	.46	
	a_L	.54	.40	.06	
Panel B: Optimal compensation $\boldsymbol{c}(x_i)$ to induce a_j					$\bar{\boldsymbol{c}}^{\dagger}(a_j)$
	a_H	7,439	65,025	74,939	66,923
	a_M	26,678	57,600	69,344	58,673
	a_L	40,000	40,000	40,000	40,000
Panel C: Likelihood ratios $L(x_i\|a_\ell,a_M)$			λ = 480		μ_ℓ
	a_L	- 20/7	0	20/23	53.667
Panel D: Likelihood ratios $L(x_i\|a_\ell,a_H)$			λ = 510		μ_ℓ
	a_M	- 4/7	0	4/27	253.125

Table 17.2: Probabilities, optimal contracts, and likelihoods for finite action/outcome example with spanning.

17.3.2 Convex Action Space

The preceding analysis assumed that the set of actions A is finite. In this section we relax that assumption. To keep things simple, we assume that the action a is unidimensional, the agent's utility function is *additively separable*, X is finite, and the principal is risk neutral.

The key change is that the set of actions is now an interval on the real line, i.e.,

$$A = [\underline{a}, \bar{a}] \subseteq \mathbb{R},$$

and $\varphi(x_i|a)$ has constant support and is twice differentiable with respect to $a \in A$, $\forall\, x_i \in X$:

$$\varphi_a(x_i|a) \equiv \partial\varphi(x_i|a)/\partial a \;\text{ and }\; \varphi_{aa}(x_i|a) \equiv \partial\varphi_a(x_i|a)/\partial a.$$

While multiplicatively separable functions can be readily handled, much of the initial literature focused on additively separable utility functions. Hence, we assume

$$u^a(c,a) = u(c) - v(a), \quad \text{with } u' > 0,\, u'' < 0,\, v' > 0,\, v'' \geq 0,$$

with $C = [\underline{c},\infty)$ representing the set of feasible consumption levels.[6]

The principal's and agent's expected utilities are now differentiable with respect to a, and we introduce the following notation:

$$U_a^p(\boldsymbol{c},a) = \sum_{i=1}^{N} [x_i - c_i]\, \varphi_a(x_i|a),$$

$$U_a^a(\boldsymbol{c},a) = \sum_{i=1}^{N} u(c_i)\, \varphi_a(x_i|a) - v'(a),$$

$$U_{aa}^a(\boldsymbol{c},a) = \sum_{i=1}^{N} u(c_i)\, \varphi_{aa}(x_i|a) - v''(a).$$

Principal's Decision Problem

In the basic formulation of the principal's problem in the introduction to this chapter, incentive constraint (17.3) is stated in its generic form. Note that in this formulation (17.3) represents an infinite (and even an uncountable) number of constraints. However, if A is convex, then a *necessary condition* for inducing action a is that it be a local optimum for the agent given the contract $\boldsymbol{c}$. Given that $\varphi(x_i|a)$ is twice differentiable, this implies that to satisfy incentive constraint (17.3), $\boldsymbol{c}$ must satisfy the following two conditions:

$$U_a^a(\boldsymbol{c},a) = 0 \text{ and } U_{aa}^a(\boldsymbol{c},a) \leq 0.$$

While these two conditions are necessary, they are not sufficient to ensure that the agent will select a (since there may be other local optima). Of course, if $\boldsymbol{c}$

[6] The measure used to represent the level of effort is inherently arbitrary. For example, we can always define it to be the level of disutility v, so that the agent's utility function is written as $u^a(c,v) = u(c) - v$ and the probability of outcome x_i is expressed as $\hat{\varphi}(x_i|v)$. The relation of this revised model to the initial representation can be characterized as follows: $\hat{\varphi}(x_i|v) = \varphi(x_i|v^{-1}(v))$ and the set of alternative "actions" is $[\underline{v},\bar{v}]$ with $\underline{v} = v^{-1}(v(\underline{a}))$ and $\bar{v} = v^{-1}(v(\bar{a}))$. Of course, other representations are possible as well, e.g., in some settings it is useful to assume a is the probability of the good outcome in a binary outcome model or the expected outcome.

is such that $U_a^a(\boldsymbol{c},a) = 0$ and $U_{aa}^a(\boldsymbol{c},a') \leq 0., \forall\, a' \in A$, then the agent's decision problem is globally concave and he will implement action a.

In the analysis that follows, we adopt the approach that was common in most of the early agency theory literature. In particular, we assume that the single first-order condition for the agent's incentive constraint is a sufficient representation of the infinite number of incentive constraints (17.3), i.e.,

$$U_a^a(\boldsymbol{c},a) = 0. \tag{17.3c}$$

In that case, the Lagrangian for the principal's decision problem can be restated as

$$\mathcal{L} = U^p(\boldsymbol{c},a) + \lambda\,[U^a(\boldsymbol{c},a) - \bar{U}] + \mu\, U_a^a(\boldsymbol{c},a) + \sum_{i=1}^{N} \xi_i\,[c_i - \underline{c}], \qquad (17.5'')$$

and the associated first-order conditions are:

$$c_i\!: \; -\varphi(x_i|a) + \lambda\, u'(c_i)\,\varphi(x_i|a) + \mu\, u'(c_i)\,\varphi_a(x_i|a) + \xi_i = 0,$$

$$a\!: \; U_a^p(\boldsymbol{c},a) + \lambda\, U_a^a(\boldsymbol{c},a) + \mu\, U_{aa}^a(\boldsymbol{c},a) = U_a^p(\boldsymbol{c},a) + \mu\, U_{aa}^a(\boldsymbol{c},a) = 0,$$

since $$U_a^a(\boldsymbol{c},a) = 0.$$

Let the "local" likelihood ratio be defined as

$$L(x_i|a) \equiv \frac{\varphi_a(x_i|a)}{\varphi(x_i|a)}.$$

This brings us to a key expression that characterizes the optimal incentive contract. If $a \in (\underline{a},\bar{a})$ and $c_i > \underline{c}$, i.e., we have a strictly interior solution, then $\xi_i = 0$ and the above first-order conditions imply:[7, 8]

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[7] If the principal is risk averse, then $M(c)$ is replaced by $M(x,c) = u^{p\prime}(x-c)/u'(c)$. In this setting we could also consider a lower bound on $x - c$ (reflecting the principal's limited wealth or limited liability considerations).

[8] With multiplicative separability of the agent's utility function the first-order condition for the optimal incentive contract is given by

$$M(c_i) = k(a)\big[\lambda + \mu\,[k'(a)/k(a) + L(x_i|a)]\big].$$

$$M(c_i) = \lambda + \mu L(x_i|a), \tag{17.6''}$$

and

$$\mu = -\frac{U_a^p(\mathbf{c},a)}{U_{aa}^a(\mathbf{c},a)}. \tag{17.7''}$$

Note that (17.6″) is applicable to the contract used to implement any action, whereas condition (17.7″) applies only to the second-best optimal action. Observe that $M(c_i) \geq 0$ for all $c_i \geq \underline{c}$, and

$$\lim_{c_i \to \underline{c}} M(c_i) = 0, \quad \text{if, and only if,} \quad \lim_{c_i \to \underline{c}} u'(c_i) = +\infty.$$

The likelihood ratio $L(x_i|a)$ must be positive for some outcomes and negative for others (since its expected value is equal to zero). If for some x_i, $L(x_i|a)$ is sufficiently negative (and μ is positive) that

$$\lambda + \mu L(x_i|a) < M(\underline{c}),$$

then consumption constraint (17.4) is binding and $c_i = \underline{c}$.

Since the incentive constraint is an equality constraint, it is conceivable that μ could be positive, negative, or zero. Many papers assume that μ is positive, or use indirect arguments to establish that it is positive. Jewitt (1988) provides a "direct" argument for the case in which the principal is risk neutral.

Proposition 17.8 (Jewitt 1988, Lemma 1)
If the principal is risk neutral, then μ satisfying $U_a^a(\mathbf{c},a) = 0$ and (17.6″) is positive.

Proof: Solve (17.6″) for $\varphi_a(x_i|a)$,

$$\varphi_a(x_i|a) = \frac{1}{\mu}\varphi(x_i|a)\big[M(c_i) - \lambda\big]. \tag{17.8}$$

Substituting into $U_a^a(\mathbf{c},a) = 0$ gives

$$\sum_{i=1}^{N} u(c_i)\big[M(c_i) - \lambda\big]\varphi(x_i|a) = \mu v'(a). \tag{17.9}$$

Summing both sides of (17.8) across all $i = 1, ..., N$ and recognizing that $\Sigma_i \varphi_a(x_i|a) = 0$, establishes that

$$\mathrm{E}[M(c_i)] = \lambda.$$

Hence, (17.9) can be interpreted as stating that the covariance of $u(c_i)$ and $M(c_i)$ is equal to $\mu v'(a)$.[9] Since $M(c_i)$ and $u(c_i)$ "move" in the same direction, they have nonnegative covariance, and since $v'(a) > 0$ by assumption, it follows that $\mu \geq 0$. We can rule out $\mu = 0$, since, with a risk neutral principal, (17.6″) would imply that c_i is a constant and a constant wage cannot satisfy $U_a^a(c,a) = 0$ if $v'(a) > 0$. **Q.E.D.**

A Hurdle Model Example

We now introduce the basic agency version of what we call the "hurdle" model.[10] It is a simple model with two possible outcomes for the principal and a convex action space for the agent. This model is extended and used several times throughout the book to illustrate some of the reported results.

The agent's action is depicted as jumping over a hurdle of random height h, which is uniformly distributed over the interval [0,1]. The agent's action is $a \in [0,1]$, which represents the height he jumps and is equal to the *ex ante* probability he will clear the hurdle. If he clears the hurdle, there is a high probability (represented by $1 - \varepsilon$) he will generate a good outcome x_g. On the other hand, if he fails to clear the hurdle, there is a high probability he will generate a bad outcome $x_b < x_g$. More specifically, the outcome probabilities given the agent's action a and hurdle height h is

$$\varphi(x_g|a,h) = \begin{cases} 1 - \varepsilon & \text{if } a \geq h, \\ \varepsilon & \text{if } a < h, \end{cases} \qquad \text{where } \varepsilon \in [0,½).$$

Hence, the prior probabilities and their derivatives for the two outcomes given a are

$$\varphi(x_g|a) = a(1 - 2\varepsilon) + \varepsilon, \qquad \varphi_a(x_g|a) = (1 - 2\varepsilon),$$

$$\varphi(x_b|a) = a(2\varepsilon - 1) + 1 - \varepsilon, \qquad \varphi_a(x_b|a) = -(1 - 2\varepsilon),$$

and the expected outcome given a and its sensitivities to a and ε are

$$\mathrm{E}[x|a] = (x_g - x_b)\varphi(x_g|a) + x_b,$$

$$\mathrm{E}_a[x|a] = (x_g - x_b)(1 - 2\varepsilon) > 0,$$

---

[9] $\mathrm{Cov}[u,M] = \mathrm{E}[(u - \mathrm{E}[u])(M - \mathrm{E}[M])] = \mathrm{E}[u(M - \lambda)] - \mathrm{E}[u]\times\mathrm{E}[M - \lambda] = \mathrm{E}[u(M - \lambda)]$, since $\mathrm{E}[M - \lambda] = \mu\mathrm{E}[L] = 0$.

[10] The hurdle model was introduced in Volume I, where it was used to illustrate decision making under uncertainty (Chapter 2) and the value of decision-facilitating information (Chapter 3).

$$E_\varepsilon[x|a] = (x_g - x_b)(1 - 2a) \geq 0, \quad \text{for } a \leq ½,$$

$$E_{a\varepsilon}[x|a] = -2(x_g - x_b) < 0.$$

Observe that the expected outcome is a linear increasing function of a with a marginal productivity decreasing in ε. The expected outcome is a linear increasing function of ε with a slope decreasing in a for $a \leq ½$. With $a = ½$, the expected outcome is independent of ε.

The likelihood ratios for the two outcomes are

$$L(x_g|a) = \frac{1-2\varepsilon}{a(1-2\varepsilon) + \varepsilon} > 0, \quad L(x_b|a) = \frac{2\varepsilon - 1}{a(2\varepsilon-1) + 1 - \varepsilon} < 0,$$

since $\varepsilon < ½$, and $a \leq 1$. Note that as ε increases, the likelihood ratio for the good outcome decreases, whereas the likelihood ratio for the bad outcome increases. Hence, both the marginal productivity of the agent's action and the information content of the outcomes about the agent's action decrease in ε.

In the following numerical example we assume the agent has additively separable preferences and use the following data:

$$u(c) = \ln(c); \;\; v(a) = a/(1 - a); \;\; \bar{U} = 0; \;\; x_g = 20; \;\; x_b = 10.$$

The optimal contracts are shown in Table 17.3 for varying values of ε. The optimal jump size is decreasing in ε as both the marginal productivity of the agent's action and the information content of the outcomes about the agent's action decrease in ε. Although the agent jumps lower for higher values of ε, the expected profit to the principal is higher due to the expected outcome being an increasing function of ε.

	$U^p(c,a)$	c_g	c_b	a
$\varepsilon = 0.00$	10.526	5.775	0.868	0.274
$\varepsilon = 0.05$	10.560	5.606	0.823	0.239
$\varepsilon = 0.10$	10.612	5.405	0.775	0.198
$\varepsilon = 0.15$	10.686	5.168	0.722	0.148
$\varepsilon = 0.20$	10.788	4.887	0.665	0.086
$\varepsilon = 0.25$	10.925	4.551	0.603	0.005

Table 17.3: Optimal contracts for varying values of ε.

In order to focus on the impact of the information content of the outcomes about the agent's action we consider how the expected cost minimizing contract for inducing $a = ½$ varies with ε. Note that with $a = ½$, the expected outcome is not affected by ε. In particular, the principal's expected profit is solely determined in this case by the risk premium paid to the agent for the expected cost minimizing contract that induces him to select $a = ½$.[11] The expected compensation cost can be written as the sum of the risk premium, $\tau(\varepsilon)$, and the agent's certainty equivalent, $CE(a)$, i.e.,

$$\mathrm{E}[c(x)|a] = \tau(\varepsilon) + CE(a),$$

where the certainty equivalent satisfies the agent's participation constraint,

$$u(CE(a)) - v(a) = \bar{U}.$$

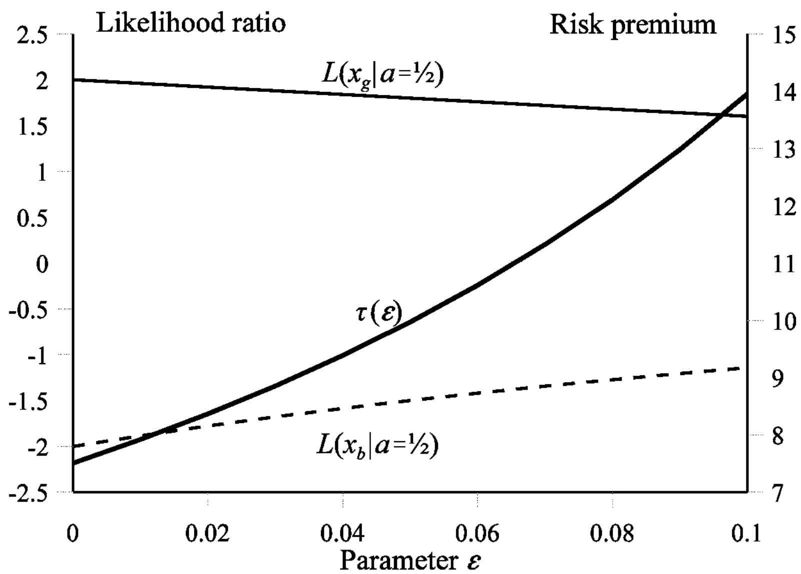

Figure 17.2: Likelihood ratios and risk premium for inducing $a = ½$ for varying values of ε.

The risk premium for inducing $a = ½$ and the likelihood ratios for the two outcomes are shown in Figure 17.2 for varying values of ε. Note that the risk

[11] Recall from Chapter 2, Volume I, that the risk premium is defined as the agent's expected compensation minus the certainty equivalent of his compensation.

premium and the variation in the likelihood ratios are inversely related, i.e., the more variation there is in the likelihood ratios, the lower is the risk premium. Of course, this is due to the fact that it is the variation in the likelihood ratios that determines the information content of the outcomes about the agent's unobserved action. We will return to this issue in Chapter 18.

Sufficient Conditions for Using a First-order Incentive Constraint

The preceding discussion assumed that imposing $U_a^a(\boldsymbol{c},a) = 0$ as the incentive compatibility constraint is sufficient to result in a contract that will induce the agent to select a. This is not always the case, since local incentive compatibility does not imply global incentive compatibility (see Appendix 17A). However, Jewitt (1988) identifies conditions that are sufficient for that to be the case.

Proposition 17.9 (Jewitt 1988, Theorem 1)

If the principal is risk neutral and $u^a(c,a) = u(c) - v(a)$ with $u' > 0$, $u'' < 0$, $v' > 0$, $v'' \geq 0$, then the first-order approach is valid if the following conditions (a)-(d) hold.

(a) (*i*) $G_i(a) \equiv \sum_{l=1}^{i} \Phi(x_l|a)$ is nonincreasing and convex in a for each value of $i = 1, ..., N$.

(*ii*) $\mathrm{E}[x|a]$ is nondecreasing and concave in a.

(b) $L(x_i|a)$ is nondecreasing and concave in x_i for each value of a.

(c) The function $u \circ M^{-1}(m)$ is concave.

(d) The optimal incentive contract in the first-order problem is interior, i.e., $c_i > \underline{c}$.

Proof: Let $\boldsymbol{c}$ solve the associated first-order problem. By Proposition 17.8, $\mu > 0$. (17.6″), conditions (b) and (d) imply that $M(c_i) = \lambda + \mu L(x_i|a)$ is nondecreasing and concave for all i.

Condition (c) implies $u(c)$ is a concave transformation of $M(c_i)$. Hence, the above implies that $u(c_i)$ is nondecreasing and concave in x_i for all i.

The final step is to prove that $U^a(\boldsymbol{c},a)$ is concave preserving (to ensure global concavity), and Jewitt (1988) claims that condition (a) is necessary and sufficient for

$$\Omega(a) \equiv \sum_{i=1}^{N} \omega(x_i)\, \varphi(x_i|a)$$

to be a nondecreasing concave function of action a for any nondecreasing, concave function $\omega(x_i)$, such as $u(c(x_i))$. **Q.E.D.**

Condition (a) ensures that an action a second-order stochastically dominates a randomized action strategy with the same expected action. The conditions (b)-(d) ensure that the agent's utility is a concave function of x_i. Convexity of $v(a)$, then implies that the agent prefers not to randomize between actions. Hence, the incentive constraints cannot be binding for several distinctly different actions, since the agent then could select a randomized strategy over these actions and obtain the same expected utility.[12]

Jewitt demonstrates that a sufficient condition for (a) is that the production technology $x = f(a,\theta)$ is a concave function of a for each state of nature θ, which is a very natural assumption in a production context. Jewitt suggests that condition (b), i.e., $L(x_i|a)$ is nondecreasing concave in x_i for each value of a, can be interpreted as the variations in output at higher levels being relatively less useful in providing "information" on the agent's effort than they are at lower levels of output. For many "standard" distributions the likelihood ratio is a linear increasing (and thus concave) function of x_i (see below and Appendix 2B). As demonstrated in Appendix 17C, condition (c) is satisfied for all HARA utility functions with risk cautiousness less than or equal to 2 (which includes the square-root, the negative exponential, and the logarithmic utility functions). Jewitt does not include condition (d) because he does not impose a lower bound on the compensation in the statement of the principal's decision problem.

Exponential Family of Distributions

Jewitt (1988) states that any member of the *exponential family of distributions* satisfies his condition (a) (he actually uses a stronger condition) in an *appropriate parameterization*, provided the expected outcome is concave in a. In particular, any density which can be written in the form[13]

[12] Note the similarities between these conditions and the sufficient conditions for the local incentive constraint being the only binding incentive constraint with a finite action space.

[13] These densities are an important class since they are those possessing sufficient statistics (see Appendix 18A). Appendix 2B characterizes a number of the classical members of the one-parameter exponential family. Observe that it includes distributions with X finite (binomial), X countably infinite (Poisson), and absolutely continuous distributions over $X = [0, \infty)$ (exponential and gamma) and over $(-\infty, +\infty)$ (Normal).

$$\varphi(x|a) = \theta(x)\beta(a)\exp[\alpha(a)\psi(x)], \tag{17.10}$$

with α and β nondecreasing, satisfies condition (a(i)) of Proposition 17.9.

Observe that for this class of distributions

$$L(x|a) = \alpha'(a)\psi(x) + \frac{\beta'(a)}{\beta(a)}.$$

Hence, satisfaction of condition (b) of Proposition 17.9 requires $\psi(\cdot)$ to be concave.

Corollary (Jewitt 1988, Corollary 1)

Let the outcome density satisfy (17.10) with $\psi(x)$ concave. Then conditions (a) and (b) of Proposition 17.9 are satisfied, provided only that $\mathrm{E}[x|a]$ is concave in a.

Appendix 17B provides examples that satisfy the above conditions and demonstrates that they satisfy conditions (a) and (b) of Proposition 17.9.

17.3.3 Convex Outcome Space – The Mirrlees Problem

The prior analysis has assumed that X is finite, although our discussion of the exponential family introduced distributions that were absolutely continuous on an interval in the real line. We now focus on absolutely continuous distributions and assume $X = (\underline{x},\overline{x})$, with the possibility that the lower bound can be $-\infty$ and the upper bound $+\infty$.

Much of the prior analysis, where A can be either finite or convex, can be extended to the case in which $X \subseteq \mathbb{R}$ is convex. However, Mirrlees (1975) has identified a potential problem in this case.

We know that if there is moving support, so that $X(a)\backslash X(a^*) \neq \emptyset$, and sufficiently severe penalties can be imposed, then the first-best solution can be obtained by paying a fixed wage for $x \in X(a^*)$ and threatening to impose a severe penalty if an "unacceptable" outcome occurs. The key here is that the penalties need never be imposed, provided the agent takes first-best action a^*.

To ensure that there is an "incentive problem," i.e., the first-best solution cannot be achieved, we usually assume constant support, i.e., $X(a) = X$, $\forall\, a \in A$. However, under some conditions there may be no solution to the second-best problem. Instead, it may be possible to get "arbitrarily close" to the first-best solution by imposing "severe penalties" on a "small" set of "bad" outcomes.

To provide insight into this issue, consider the following distributions and utility functions:

Distribution:

Exponential: $\varphi(x|a) = \frac{1}{a}\exp\left[-\frac{x}{a}\right],$ $X = [0, \infty),$

$L(x|a) = \frac{1}{a^2}(x - a),$ $\Rightarrow L \in [-\frac{1}{a}, \infty).$

Normal: $\varphi(x|a) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}(x - a)^2\right],$ $X = (-\infty, +\infty),$

$L(x|a) = \frac{1}{\sigma^2}(x - a),$ $\Rightarrow L \in (-\infty, +\infty).$

Utility Function:

Log: $u(c) = \ln(c),$ $C = (0, \infty),$

$M(c) = c,$ $\Rightarrow M \in (0, \infty).$

Square-root: $u(c) = \sqrt{c},$ $C = [0, \infty),$

$M(c) = 2\sqrt{c},$ $\Rightarrow M \in [0, \infty).$

Observe that with the exponential distribution $\lambda + \mu L$ is positive for all $x \in (0,\infty)$ if, and only if, $a > \mu/\lambda$, which would result in an interior solution for $c(x)$ for all x with either the log or the square-root utility functions. Since $u'(c) \to \infty$ as $c \to 0$, it is likely that this condition will be satisfied. It will certainly be satisfied with the log utility function since $u(c) \to -\infty$ as $c \to 0$. With the square-root utility function, we have a corner solution if $a < \mu/\lambda$, i.e.,

$$c(x) = 0 \text{ for } x \in \left(0, \frac{a}{\mu}[\mu - \lambda a]\right).$$

On the other hand, with the normal distribution, $\lambda + \mu L$ is negative for all $x < a - \lambda\sigma^2/\mu$. This can be handled with a square-root utility function by letting let $c(x) = 0$ for those values of x, but that is not possible with the log utility function. Hence, we have a problem with normal distributions and the log utility function, since $\mu > 0$ implies $\lambda + \mu L < 0$ for some values of x and M must be positive. In fact, a solution to the second-best problem does not exist unless we impose a positive lower bound on consumption, i.e., $C = [\underline{c}, \infty)$ with $\underline{c} > 0$.

The following theorem (due to Mirrlees, 1975) characterizes the nonexistence of a second-best solution when $L \to -\infty$ as $x \to \underline{x}$ and $u(c) \to -\infty$ as $c \to \underline{c}$.

Proposition 17.10 (Mirrlees 1975, Theorem 1)
Assume MLRP holds with

(a) $\lim_{x \to \underline{x}} L(x|a) = -\infty$,

(b) $u^a(c,a) = u(c) - v(a),\; v'(a) > 0,\; v''(a) \geq 0$,

(c) $\lim_{c \to \underline{c}} u(c) = -\infty, \quad u'(c) > 0,\; u''(c) < 0$,

(d) $u^p(x - c) = x - c$, i.e., risk neutral principal.

Under these assumptions it is possible to approximate arbitrarily closely, but not attain, the first-best optimum.

Proof: Let (c^*,a^*) denote the first-best contract, where c^* is the first-best fixed wage, and consider an $x^p > \underline{x}$ such that $\Phi_a(x^p|a) < 0$. Given x^p, consider a contract that pays a fixed penalty c^p for outcomes below x^p, and another fixed wage $\bar{c}$ for outcomes above x^p (with $\underline{c} < c^p < c^* < \bar{c}$). The two wages are such that the agent gets the same expected utility as with the first-best contract, and such that the first-best action a^* is incentive compatible. That is, c^p and $\bar{c}$ satisfy the following two conditions:

$$\text{agent expected utility:}\quad u(\bar{c}) - [u(\bar{c}) - u(c^p)]\,\Phi(x^p|a^*) = u(c^*), \tag{17.11}$$

$$\text{agent action choice:}\quad -[u(\bar{c}) - u(c^p)]\,\Phi_a(x^p|a^*) - v'(a^*) = 0, \tag{17.12}$$

i.e.,
$$\bar{c} = u^{-1}\left(u(c^*) - v'(a^*)\frac{\Phi(x^p|a^*)}{\Phi_a(x^p|a^*)} \right) > c^*,$$

and
$$c^p = u^{-1}\left(u(c^*) + v'(a^*)\frac{1 - \Phi(x^p|a^*)}{\Phi_a(x^p|a^*)} \right) < c^*,$$

which is possible since $u(c^p)$ can range between $u(\bar{c})$ and $-\infty$. Observe that (17.11) and Jensen's inequality imply

$$\bar{c} > \mathrm{E}[c|a^*] = \bar{c} - (\bar{c} - c^p)\Phi(x^p|a^*) > c^*.$$

For any large number $K > 0$, we can choose x^p so small that $L(x|a^*) < -K, \forall x < x^p$. Hence,

$$\frac{1}{K} > -\frac{\Phi(x^p|a^*)}{\Phi_a(x^p|a^*)},$$

$$\Rightarrow u^{-1}\left(u(c^*) + \frac{1}{K}v'(a^*)\right) > \bar{c},$$

$$\Rightarrow \lim_{K\to\infty} \bar{c} = c^* \text{ and } \lim_{K\to\infty} \mathrm{E}[c|a^*] = c^*.$$ **Q.E.D.**

17.3.4 Randomized Contracts

The vast majority of the principal-agent literature ignores the possibility of randomized strategies. However, there are a few papers, most notably Fellingham, Kwon, and Newmann (1984) and Arya, Young, and Fellingham (AYF) (1993), that have considered randomized contracts. In a randomized contract, the principal offers a pair of contracts with a stipulation of a randomization process that will choose between the two contracts after the agent accepts the randomized contract but before he selects his action.[14] Why might randomization be valuable to the principal? The agent's *ex ante* expected utility will equal his reservation utility. However, if the maximum expected utility the principal can achieve with non-random contracts for alternative agent reservation utility levels is convex in the region of the agent's reservation utility,[15] then it will be optimal for the principal to offer a pair of contracts such that the agent's expected utility is greater than his reservation utility if he gets the "good contract" and less than his reservation utility if he gets the "bad contract." This type of situation is depicted in Figure 17.3(a).

AYF focus on additive and multiplicatively separable utility functions in which the utility for compensation is negative exponential, i.e., we have either

(a) Additive separability: $u^a(c,a) = u(c) - v(a),$

---

[14] This type of randomization is termed *ex-ante* randomization. Gjesdal (1982) considers *ex-post* randomization between incentive contracts after the agent has selected his action. He shows that separability of the utility function is sufficient to ensure that *ex-post* randomization is not optimal.

[15] Stated alternatively, the set of possible principal and agent utilities that can be achieved with alternative non-random contracts is not convex in the region of the agent's reservation utility level.

(b) Multiplicative separability: $u^a(c,a) = u(c)v(a)$,

with $u(c) = -\exp[-c/\rho]$, $v'(c) > 0$, and $v''(c) \geq 0$.

AYF assume that the incentive constraint is characterized by the first-order condition for the agent's choice problem. If $\boldsymbol{c}(x)$ induces the agent to select action a, then

(a) $\mathrm{E}_a[u(\boldsymbol{c}(x))|a] = v'(a)$,

(b) $\mathrm{E}_a[u(\boldsymbol{c}(x))|a]v(a) = \mathrm{E}[u(\boldsymbol{c}(x))|a]v'(a)$.

Now observe that if $\boldsymbol{c}(x)$ is increased by a fixed amount $k > 0$, then $u(\boldsymbol{c}(x) + k) = -\exp[-\boldsymbol{c}(x)/\rho]\exp[-k/\rho]$ with $\exp[-k/\rho] < 1$, from which it follows that

(a) $\exp[-k/\rho]\mathrm{E}_a[u(\boldsymbol{c}(x))|a] < v'(a)$,

(b) $\exp[-k/\rho]\mathrm{E}_a[u(\boldsymbol{c}(x))|a]v(a) = \exp[-k/\rho]\mathrm{E}[u(\boldsymbol{c}(x))|a]v'(a)$.

From (b) we observe that the compensation level has no impact on the action choice when there is multiplicative separability. This implies that the second-best action is independent of the reservation utility level and that $\mathrm{E}[x - \boldsymbol{c}(x,\bar{U})|a(\bar{U})]$ is a decreasing, concave function of the reservation utility. The latter implies that there are no gains to randomization (AYF, Prop. 2).

From (a) we observe that the compensation level affects the action choice when there is additive separability – the larger k the less the effort induced by $\boldsymbol{c}(x)$. This implies that the larger the reservation utility, the more expensive it is to induce a given effort level, and hence the less the effort that will be induced.

AYF (Prop. 1) prove that randomization is beneficial in case (a) if $\varphi_{aa}(x|a) = 0$ (e.g., if $\varphi(x|a) = a\varphi^1(x) + (1-a)\varphi^2(x)$), and

$$[v'(a)]^2 > -v''(a)[\bar{U} + v(a)].$$

The key here is that under the assumed conditions, while $\mathrm{E}[x - \boldsymbol{c}(x,\bar{U})|a(\bar{U})]$ is decreasing in the reservation utility, it is convex around the specified U (resulting in a non-convex set).

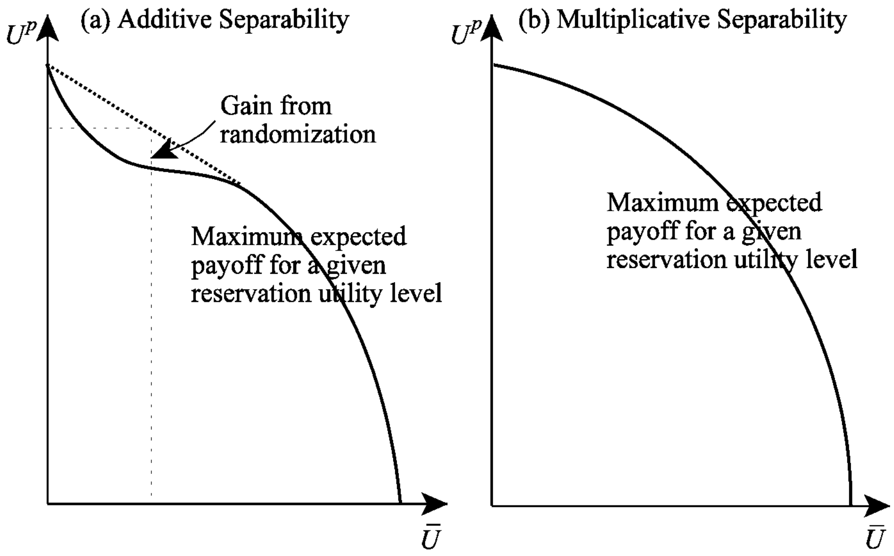

Figure 17.3: Expected utility frontiers with additive and multiplicative separable negative exponential utility.

17.4 AGENT RISK NEUTRALITY AND LIMITED LIABILITY

If the agent is risk neutral, the first-best result can be attained provided all risk can be shifted to the agent (see Proposition 17.3(c)). There are essentially two mechanisms for shifting the risk to the agent. First, the agent can "purchase" the firm, i.e., the agent makes a lump-sum payment to the principal in return for ownership of the outcome x. Of course, this can only be achieved if the agent has sufficient capital to purchase the firm. Second, the agent can "rent" the firm, i.e., the agent agrees to pay the principal a fixed amount after the outcome has been realized. This requires that the agent has other resources that he can use to make up the shortfall between a low value of x and the amount of the rent.

This result was recognized early on, so that virtually all of the initial principal-agent models assumed the agent is risk averse. This has shifted somewhat in recent times. A model is much easier to analyze if the agent is risk neutral. Hence, a researcher will typically make that assumption as long as there is something else in the model assumptions that precludes implementation of the first-best result. There are two such factors: the agent does not have sufficient resources to implement the first-best result or he has private information at the time of contracting (a setting we will consider in Chapter 23).

Innes (1990) provides an analysis in which the agent is assumed to be risk neutral and does not have sufficient resources to implement the first-best result. He makes the following assumptions:

- the agent is an entrepreneur who owns a production technology, but has no investment capital;
- implementation of the production technology requires the agent's effort a and investment of q units of capital;
- the principal (investors) will provide the amount q if offered a contract in which the expected payment equals q (given the assumption of investor risk neutrality and a zero interest rate);
- limited liability precludes contracts in which the principal makes payments to the agent at the end of the period;[16]
- $\varphi(x|a)$ satisfies MLRP, $A = [0,\bar{a}]$, and $X = [0,\infty)$;
- $u^a(c,a) = c - v(a)$, with $v'(a) > 0$, $v''(a) > 0$, i.e., the agent is risk neutral and effort averse.[17]

Unlike the previously discussed models, the agent owns the production technology and has the bargaining power. He offers a contract to the principal (investors) that provides an expected return on the capital invested that is equivalent to the return that could be obtained in the market. Let $\boldsymbol{\pi}(x)$ represent the amount paid to the principal. Hence, the agent's consumption is $\boldsymbol{c}(x) = x - \boldsymbol{\pi}(x)$. The agent's decision problem is

$$\underset{\boldsymbol{\pi},\, a \in A}{\text{maximize}} \quad \mathrm{E}[x - \boldsymbol{\pi}(x)|a] - v(a),$$

$$\text{subject to} \quad \mathrm{E}[\boldsymbol{\pi}(x)|a] \geq q,$$

$$0 \leq \boldsymbol{\pi}(x) \leq x, \quad \forall\, x \in X,$$

$$\mathrm{E}[x - \boldsymbol{\pi}(x)|a] - v(a) \geq \mathrm{E}[x - \boldsymbol{\pi}(x)|a'] - v(a'), \quad \forall\, a' \in A,$$

[16] Both debt and equity financing generally have limited liability in the sense that the holders of these claims cannot be required to pay for the firm's liabilities.

[17] Innes allows for a slightly more general form of utility function, $u^a(c,a) = k(a)c - v(a)$.

where the last constraint is the incentive compatibility constraint that ensures that the agent is at least weakly motivated to provide the effort a given the payoff function offered to investors.

Monotonic Contracts

Innes introduces the following *monotonicity constraint*:

$$\boldsymbol{\pi}(x+\varepsilon) \geq \boldsymbol{\pi}(x), \quad \forall\,(x,\varepsilon) \in \mathbb{R}_+^2.$$

He argues that this constraint can be viewed as the result of the principal's and agent's ability to "sabotage" non-monotonic contracts. For example, after observing a perfect signal about the firm's profits, investors may be in a position to reduce the firm's actual profits, or the agent may supplement the profits (by borrowing on a personal account).

In a *debt contract*, $\boldsymbol{\pi}(x) = \min\{x, D\}$ where D is the designated nominal amount to be paid to the principal in return for q. That is, if the outcome x is insufficient to meet the obligation to pay D, then the outcome x is paid to the principal.

Consider a monotonic contract $\boldsymbol{\pi}(x)$ that induces action a, i.e., $\mathrm{E}_a[x \mid a] = \mathrm{E}_a[\boldsymbol{\pi}(x) \mid a] + v'(a)$, and identify the debt contract that provides the principal with the same expected return, i.e.,

$$\int_0^D x\,\varphi(x \mid a)\,dx + D\,\Phi(D \mid a) = \mathrm{E}[\boldsymbol{\pi}(x) \mid a].$$

Innes' Lemma 1 proves that

$$\int_0^D x\,\varphi_a(x \mid a)\,dx + D\,\Phi_a(x \mid a) < \mathrm{E}_a[\boldsymbol{\pi}(x) \mid a],$$

which implies $a(D) > a$ (Innes' Lemma 2), i.e., the debt contract will induce a higher action than any arbitrary monotonic contract. As depicted in Figure 17.4, the key here is that moving to a debt contract reduces the amount the agent receives for low values of x and increases what he receives for high values, i.e., the agent has a call option on x with strike price D. This gives him stronger incentives to achieve the high outcomes.

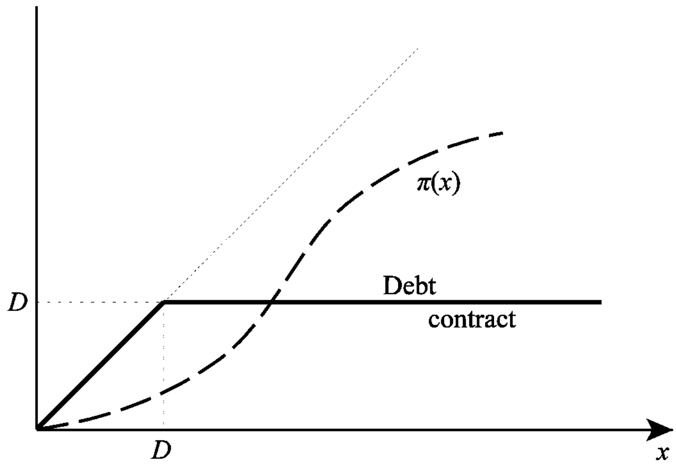

Figure 17.4: Debt contract and general monotonic contract.

Let $a(D)$ represent the action induced by debt contract D. Innes' Corollary 2 demonstrates that $a(D)$ is a continuous function and he notes that increasing the action from a to the induced action $a(D)$ will make both the principal and the agent better off. Hence, it immediately follows that the optimal monotonic contract is a debt contract.

Proposition 17.11 (Innes 1990, Prop. 1)
A solution to the agent's problem (with a monotonicity constraint) exists and has the following properties:

(a) $\boldsymbol{\pi}(x) = \min\{x,D\}, \quad \forall\, x \in X,$

(b) $\mathrm{E}[\boldsymbol{\pi}(x) \,|\, a] = q,$

(c) $a < a^* \equiv$ first-best effort choice.

That is, while a debt contract is the best monotonic contract, the optimal debt contract will induce less than the first-best level of effort.

Non-monotonic Contracts

If the monotonicity constraint is dropped, then there is a greater range of feasible contracts. Innes identifies two possibilities here. First, it may be possible to obtain the first-best result. Second, if the first-best result cannot be attained, then it is optimal to use what Innes calls a *live-or-die contract*. In that contract, there is an outcome cut-off $x^\dagger$ such that $x < x^\dagger$ goes to the principal and $x > x^\dagger$ goes to the agent, i.e.,

$$\boldsymbol{\pi}(x) = \begin{cases} x & \forall\ x \le x^\dagger, \\ 0 & \forall\ x > x^\dagger. \end{cases}$$

The argument for the optimality of the live-or-die contract is similar to the argument for a debt contract when there is a monotonicity constraint.

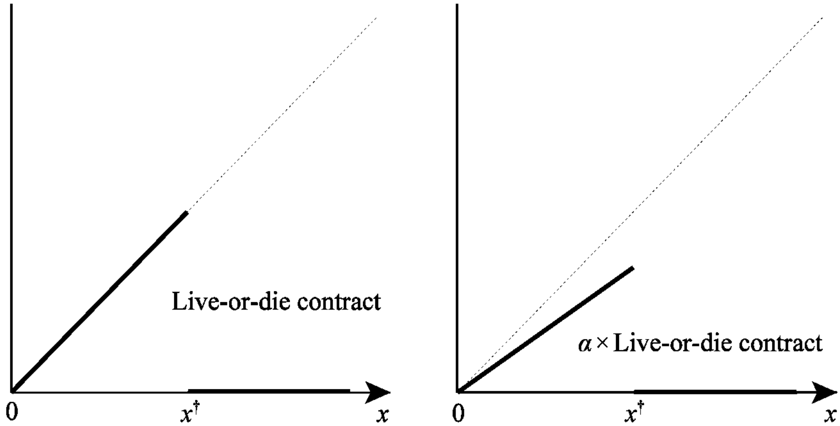

Figure 17.5: Live-or-die contracts.

A live-or-die contract is used as a starting point for assessing whether the first-best result can be achieved. Observe that the first-best action a^* is such that $\mathrm{E}_a[x|a^*] = v'(a^*)$. The first step is to identify an outcome x^* such that

$$\int_0^{x^*} x\, d\Phi_a(x|a^*) = 0,$$

i.e., a^* maximizes the expected payment to the principal in a live-or-die contract in which x^* is the cutoff. If x^* is also such that

$$\int_0^{x^*} x\, d\Phi(x|a^*) \ge q,$$

the first-best can be achieved (see Innes' Propositions 3 and 4) with the following contract:

$$\boldsymbol{\pi}^*(x) = \alpha^* \begin{cases} x & \forall\ x \le x^*, \\ 0 & \forall\ x > x^*, \end{cases}$$

where $$\alpha^* = q \div \left[\int_0^{x^*} x\, d\Phi(x|a^*)\, dx \right].$$

The key here is that

$$\mathrm{E}_a[x|a^*] = \mathrm{E}_a[\pi^*(x)|a^*] + v'(a^*) = v'(a^*),$$

because π^* has been constructed so that $\mathrm{E}_a[\pi^*(x)|a^*] = 0$ and $\mathrm{E}[\pi^*(x)|a^*] = q$.

Now consider the second case, which occurs if x^* is such that

$$\int_0^{x^*} x\, d\Phi(x|a^*) < q.$$

The first-best result cannot be achieved here and the optimal second-best contract is a live-or-die contract in which the induced effort is less than a^* (see Innes' Proposition 2) and the principal receives an expected return of q. The argument for the optimality of the live-or-die contract is the same as the argument for the optimality of a debt contract when there is a monotonicity constraint. The live-or-die contract provides the maximum incentive for the agent to expend effort.

17.5 CONCLUDING REMARKS

This chapter lays a foundation for the analysis in subsequent chapters. The agency model in this chapter is simple in that there is a single agent, who performs a single task for a single period, and the outcome from his effort is the only contractible information. For most of the analysis, the principal is risk neutral and the agent is risk and effort averse. The outcome from the agent's action belongs to the principal, and he must compensate the agent for his reservation wage, the cost of his effort, and a risk premium associated with the incentive risk used to motivate the desired intensity of effort.

The following can be viewed as the key results of the chapter. First, Proposition 17.1 characterizes a first-best contract and Proposition 17.3 identifies conditions under which it can be achieved. Second, Sections 17.3.1 and 17.3.2 identify several characteristics of second-best contracts, including identification of conditions under which the agent's compensation is an increasing function of our likelihood measure, which is an increasing function of the firm's outcome (see Propositions 17.6, 17.7, 17.8, and 17.9).

APPENDIX 17A: CONTRACT MONOTONICITY AND LOCAL INCENTIVE CONSTRAINTS

Several agency papers introduce a *concavity of distribution function condition* (CDFC) to facilitate the analysis by providing a set of conditions that are sufficient for the optimal incentive contract to be characterized by an increasing compensation scheme and a single binding incentive constraint which only considers the next most costly action relative to the second-best action. Unfortunately, this condition is not intuitively appealing because it is not satisfied by "standard" distributions. Based on Jewitt (1988), we provide a less restrictive condition on the distribution function which is satisfied by most standard distributions. For example, it is satisfied if the production function exhibits decreasing marginal productivity of effort for each state of nature (see Jewitt, 1988). On the other hand, since the Jewitt condition is based on second-order stochastic dominance (as opposed to first-order stochastic dominance for CDFC) additional conditions are needed on the utility function and the likelihood ratio to ensure that the agent's equilibrium utility of compensation is a concave function of the outcome.

Monotonicity and Local Incentive Constraints with a Finite Action Set

The following analysis examines a setting in which the set of actions is finite and the agent's preferences are additively separable. We let $\varphi_{ij} = \varphi(x_i|a_j)$ and $v_j = v(a_j)$ with $v_1 < v_2 < ... < v_M$, i.e., the actions are strictly increasing in their "cost" to the agent (and, hence, $c_j^* < c_\ell^*$ for $j < \ell$).

The following sufficient conditions for the optimal incentive contract to be nondecreasing are also sufficient conditions for a single binding incentive constraint which only considers the next most costly action relative to the second-best action. Therefore, we consider a relaxed version of the principal's decision problem for inducing an action a_j in which the incentive constraints (17.3f′) are substituted with a single "local" incentive constraint that only considers the next most costly action a_{j-1}, i.e., $\hat{A} = \{a_{j-1}, a_j\}$ and

$$U^a(\mathbf{u},a_j) \geq U^a(\mathbf{u},a_{j-1}). \qquad (17.3\text{f}'')$$

The optimal incentive contract in the relaxed program is denoted $\hat{\mathbf{c}}_j$.

In the relaxed version of the principal's decision problem (and in any setting in which only the adjacent incentive constraint is binding), the optimal contract is characterized by the likelihood ratio $L(x_i|a_{j-1},a_j)$. The distribution is defined to satisfy the "local" MLRP, if $L(x_i|a_{j-1},a_j)$ is nondecreasing in *i*.

Definition *Concavity of Distribution Function Condition*

If the agent's preferences are additively separable, the *concavity of distribution function condition* (CDFC) is satisfied if for any $h,j,\ell \in \{1, ..., M\}$ there exists a $\xi \in [0,1]$ such that[18]

$$v_j = \xi v_h + (1-\xi) v_\ell \quad \Rightarrow \quad \Phi_{ij} \le \xi \Phi_{ih} + (1-\xi)\,\Phi_{i\ell}, \quad \forall\, i = 1, ..., N,$$

where $$\Phi_{ij} = \sum_{\iota=1}^{i} \varphi_{\iota j}.$$

That is, CDFC is satisfied if the "utility" cost of a_j is expressed as a weighted average of the utility cost of a_h and a_ℓ, and a_j FS-dominates a gamble between a_h and a_ℓ, with probabilities equal to the utility cost weights. Hence, if the compensation function is nondecreasing, the agent (weakly) prefers a_j to a randomized strategy over a_h and a_ℓ with an equal expected utility cost.

The following proposition identifies sufficient conditions for the second-best incentive contract for inducing a_j to be such that it is nondecreasing in the outcome, and such that only the incentive constraint for a_{j-1} is binding.

Proposition 17A.1 (GH, Prop. 8)

If the agent is strictly risk averse with additively separable preferences, then "local" MLRP and CDFC imply that the optimal second-best contract $\mathbf{c}_j^\dagger$ for inducing a_j satisfies

$$M(c_{ij}^\dagger) = \lambda + \mu\, L(x_i|a_{j-1},a_j), \quad i = 1, ..., N,$$

with λ, $\mu = \mu_{j-1} > 0$, and is nondecreasing.

Proof: The first-order condition in (a) characterizes the second-best contract for inducing a_j with $\hat{A}$ (by Proposition 17.6), and $U^a(\hat{\mathbf{c}}_j,a_{j-1}) = U^a(\hat{\mathbf{c}}_j,a_j) = \underline{U}$ by Proposition 17.4. Now show that there does not exist an lower cost action a_ℓ, $\ell < j-1$, such that $U^a(\hat{\mathbf{c}}_j,a_\ell) > U^a(\hat{\mathbf{c}}_j,a_j)$. Assume the contrary.

Let ξ be such that $v_{j-1} = \xi v_\ell + (1-\xi) v_j$. Local MLRP and CDFC imply that

[18] Note that the condition is more appropriately termed a "convexity of distribution function condition" since the distribution function at any given outcome is a convex function of the "utility" cost of effort.

$$U^a(\hat{\mathbf{c}}_j, a_{j-1}) = \sum_{i=1}^{N} u_i \varphi_{i,j-1} - v_{j-1} \geq \sum_{i=1}^{N} u_i \big(\xi \varphi_{i\ell} + (1-\xi)\varphi_{ij}\big) - v_{j-1}$$

$$= \xi\, U^a(\hat{\mathbf{c}}_j, a_\ell) + (1-\xi)\, U^a(\hat{\mathbf{c}}_j, a_j)$$

$$= \xi\, U^a(\hat{\mathbf{c}}_j, a_\ell) + (1-\xi)\, U^a(\hat{\mathbf{c}}_j, a_{j-1}),$$

which violates the assumed condition.

Now show that there does not exist a more costly action a_h, i.e., $c_h^* > c_j^*$, that would be preferred to a_j for the chosen contract, i.e., $U^a(\hat{\mathbf{c}}_j, a_h) > U^a(\hat{\mathbf{c}}_j, a_j)$. Assume the contrary.

Proposition 17.4 implies that $U^a(\hat{\mathbf{c}}_j, a_{j-1}) = U^a(\hat{\mathbf{c}}_j, a_j) = \bar{U}$. Let ξ be such that $v_j = \xi v_{j-1} + (1-\xi) v_h$. Local MLRP and CDFC imply that

$$U^a(\hat{\mathbf{c}}_j, a_j) = \sum_{i=1}^{N} u_i \varphi_{ij} - v_j \geq \sum_{i=1}^{N} u_i \big(\xi \varphi_{i,j-1} + (1-\xi)\varphi_{ih}\big) - v_j$$

$$= \xi\, U^a(\hat{\mathbf{c}}_j, a_{j-1}) + (1-\xi)\, U^a(\hat{\mathbf{c}}_j, a_h)$$

$$= \xi\, U^a(\hat{\mathbf{c}}_j, a_j) + (1-\xi)\, U^a(\hat{\mathbf{c}}_j, a_h),$$

which violates the assumed condition.

Hence, $\hat{\mathbf{c}}_j$ implements a_j not only with the action set $\hat{A}$, but also with the full action set A, and is no more costly than the optimal contract for inducing a_j with the full action set A, so $\hat{\mathbf{c}}_j = \mathbf{c}_j^\dagger$.

Monotonicity of $\mathbf{c}_j^\dagger$ follows immediately from the first-order condition in (a) and local MLRP with $\mu > 0$. **Q.E.D.**

As noted, the CDFC is not satisfied by "standard" distributions, and may therefore be difficult to justify. However, note that the proof of the proposition is based on an FSD argument. Since the agent is risk averse, it is natural to consider weaker conditions on the distribution function combined with an SSD argument. However, even though the agent is risk averse, the second-best compensation scheme may be sufficiently "convex" so that the composite function $u \circ c(x)$ may not be a concave function of x. Hence, in addition to a condition on the distribution function, we also need conditions to ensure that $u \circ c(x)$ is concave in order to use a second-order stochastic dominance argument.

Let

$$G_{ij} \equiv \sum_{\iota=1}^{i} \Phi(x_\iota | a_j), \quad i = 1, ..., N; j = 1, ..., M,$$

denote the accumulated distribution function.

Definition *Convexity of Accumulated Distribution Function Condition*
If the agent's preferences are additively separable, the *convexity of accumulated distribution function condition* (CADFC) is satisfied if for any $h,j,\ell \in \{1, ..., M\}$ there exists a $\xi \in [0,1]$ such that

$$v_j = \xi v_h + (1-\xi)v_\ell \quad \Rightarrow \quad G_{ij} \le \xi G_{ih} + (1-\xi)G_{i\ell}, \quad \forall\, i = 1, ..., N.$$

That is, CADFC is satisfied if the "utility" cost of a_j is expressed as a weighted average of the utility cost of a_h and a_ℓ, and a_j SS-dominates a gamble between a_h and a_ℓ, with probabilities equal to the utility cost weights. Hence, if the agent's utility of compensation $u \circ \mathbf{c}(x)$ is nondecreasing and concave, the agent (weakly) prefers a_j to a randomized strategy over a_h and a_ℓ with equal expected utility cost.

Note that CDFC implies CADFC, whereas the converse does not necessarily hold, i.e., CADFC is a weaker condition on the distribution than CDFC.

Definition *Concavity of Utility of Compensation Condition*
If the agent's preferences are additively separable, the *concavity of utility of compensation condition* (CUCC) is satisfied if $u \circ \hat{\mathbf{c}}_j(x)$ is a concave function of x.

Note that the condition is on the optimal incentive contract in the relaxed program and, thus, satisfaction of the condition depends on endogenously determined Lagrange multipliers. However, the following lemma provides sufficient conditions for CUCC in terms of (almost) exogenous characteristics of the problem.

Lemma 17A.1
If the agent's preferences is additively separable, CUCC is satisfied if the following conditions hold:

(a) The local likelihood ratio $L(x_i|a_{j-1}, a_j)$ is increasing and concave in x_i.

(b) The function $u \circ M^{-1}(m)$ is concave.

(c) The optimal incentive contract in the relaxed program is interior, i.e., $\hat{c}_{ij} > \underline{c}$.

Proof: The optimal incentive contract in the relaxed program is characterized by

$$M(\hat{c}_{ij}) = \lambda + \mu L(x_i|a_{j-1}, a_j),$$

with $\mu > 0$ (by Proposition 17.6) whenever $\hat{c}_{ij} > \underline{c}$. The agent's utility of compensation can therefore be written as

$$u \circ \hat{\mathbf{c}}_j(x) = u \circ M^{-1}(\lambda + \mu L(x|a_{j-1}, a_j)),$$

which is concave in x by (a) and (b). (c) ensures that there is no "flat" part of the utility of compensation for low outcomes (in which case it would be impossible to satisfy CUCC). **Q.E.D.**

Condition (a) is satisfied by most standard distributions (see Appendix 2B), while condition (b) is satisfied for all HARA utility functions with risk cautiousness less than or equal to 2 (see Appendix 17C – this includes the square-root, the negative exponential, and the logarithmic utility functions). If the risk cautiousness is above 2, CUCC may still be satisfied if the local likelihood ratio is sufficiently concave and μ is sufficiently high. Whether condition (c) is satisfied depends on the optimal solution, but in most analyses one wants to make sure that it is satisfied.

Proposition 17A.2

If the agent is strictly risk averse with additively separable preferences, then "local" MLRP, CADFC, and CUCC imply that the optimal second-best contract $\mathbf{c}_j^\dagger$ for inducing a_j satisfies

$$M(c_{ij}^\dagger) = \lambda + \mu L(x_i|a_{j-1}, a_j), \quad i = 1, ..., N,$$

with λ, $\mu = \mu_{j-1} > 0$, and is nondecreasing.

Proof: The proof is the same as for Proposition 17A.1 except that the SSD argument is used instead of an FSD argument. Local MLRP, i.e., $L(x_i|a_{j-1}, a_j)$ is nondecreasing in i, and CUCC imply that u_i is nondecreasing and concave in i for $\hat{\mathbf{c}}_j$. Hence, CADFC implies for any $\ell < j < h$ and $\xi \in [0,1]$ that

$$\sum_{i=1}^{N} u_i \varphi_{ij} \geq \sum_{i=1}^{N} u_i \left(\xi \varphi_{i\ell} + (1-\xi)\varphi_{ih}\right).$$

The proof then proceeds as in the proof of Proposition 17A.1. **Q.E.D.**

Observe that in the convex action space case, Jewitt assumes $\mathrm{E}[x\,|\,a]$ is nondecreasing and concave in a (see Proposition 17.9). This along with SSD ensures that $U^a(c,a)$ is globally concave, which is sufficient for the first derivative to identify the action a that is induced. Proposition 17A.2 is based on conditions that ensure that a local optimum is a global optimum without ensuring global concavity.

Monotonicity and Local Incentive Constraints with a Convex Action Set
The following analysis now considers settings in which the set of actions is a convex interval on the real line. The conditions for contract monotonicity are basically the same as in the case with a finite action space, i.e., the local incentive constraint is a sufficient representation of the full set of incentive constraints.

Recall from Volume I, Chapter 2, that MLRP implies that $L_x(x_i|a) \equiv \partial L(x_i|a)/\partial x_i \geq 0$, with strict inequality for some x_i, and $\Phi_{ia}(a) \leq 0$ (i.e., FSD), where

$$\Phi_i(a) = \sum_{\iota=1}^{i} \varphi_\iota(a) \quad \text{and} \quad \varphi_\iota(a) = \varphi(x_\iota|a).$$

Lemma 17A.2
Assume MLRP and $\mu \geq 0$, then c_i is a nondecreasing function of x_i.

Proof: If $\mu = 0$, then we have Pareto efficiency, which implies c_i is nondecreasing in x_i. If $\mu > 0$, then $\lambda + \mu L(x_i|a)$ is increasing in x_i, which in turn implies that c_i is nondecreasing in x_i, since $M(c_i)$ can only be increased for larger x_i if c_i is increased. **Q.E.D.**

Lemma 17A.3
Assume MLRP and $a \in (\underline{a}, \overline{a})$, then $\mu \neq 0$.

Proof: Assume $\mu = 0$. This implies $x_i - c_i$ is strictly increasing in x_i. The latter implies that $U_a^P(\mathbf{c}, a) > 0$ for all $a \in A$, due to FSD implied by MLRP. However, for (17.7″) to hold with $\mu = 0$, we require $U_a^P(\mathbf{c}, a) = 0$. A contradiction.
Q.E.D.

Definition
If the agent's preferences are additively separable with $v'(a) > 0$ and $v''(a) \geq 0$, the probability function $\varphi_i(a)$ satisfies the *concave distribution function condition* (CDFC) if

$$\Phi_{iaa}(a) \geq 0, \quad \forall\, x_i \in X,\, a \in A.$$

While MLRP requires the distribution function $\Phi_i(a)$ to decrease as a increases, CDFC requires it to decrease at a decreasing rate. The MLRP condition is satisfied by most "standard" distributions if we view disutility as an increasing function of the mean of those distributions. However, those "standard" distributions do not satisfy CDFC. Rogerson (1985) provides a "contrived" distribution that does satisfy these two conditions:

$$\Phi_i(a) = \left(\frac{x_i}{x_N}\right)^{a-\underline{a}}.$$

In using MLRP and CDFC to characterize the second-best optimal contract, Rogerson introduces three specifications of the principal's decision problem:

(*i*) *Unrelaxed:*	max (17.1)	subject to (17.2), (17.3: $a \in \text{argmax } U^a(\mathbf{c},a')$), and (17.4);
(*ii*) *Relaxed:*	max (17.1)	subject to (17.2), (17.3c: $U_a^a(\mathbf{c},a) = 0$), and (17.4);
(*iii*) *Double Relaxed:*	max (17.1)	subject to (17.2), (17.3r: $U_a^a(\mathbf{c},a) \geq 0$), and (17.4).

Proposition 17A.3 (Rogerson 1985, Prop. 1)

If a solution to (*iii*) exists and a solution to (*i*) exists with $a^\dagger > \underline{a}$, then MLRP and CDFC imply that if $(\mathbf{c}^\dagger, a^\dagger)$ is a solution to (*iii*), then

(a) it is also a solution to (*i*), with $c_i^\dagger$ nondecreasing in x_i,

(b) if $a^\dagger < \bar{a}$, it is also a solution to (*ii*), and the principal would prefer the agent to provide more "effort," i.e., $U_a^p(\mathbf{c}^\dagger, a^\dagger) \geq 0$.

Recall that in the finite action case we used MLRP and CDFC (from GH) to establish that only one incentive constraint is binding – the constraint for the action that is the next most costly action to the agent. Thus, it is not surprising that these conditions are also sufficient to permit us to replace the set of incentive constraints in (*i*) with the "local" first-order condition in (*ii*).

The alternative set of conditions based on Jewitt (1988) is considered in the text.

APPENDIX 17B: EXAMPLES THAT SATISFY JEWITT'S CONDITIONS FOR THE SUFFICIENCY OF A FIRST-ORDER INCENTIVE CONSTRAINT

Jewitt (1988) provides the following examples which satisfy his sufficient conditions (see Proposition 17.9) for the use of a first-order incentive constraint. In the first example, the set of possible outcomes is binary, whereas in the second that set is a convex set of the real line. In both examples, effort is represented as the expected outcome from the agent's actions, i.e., $\mathrm{E}[x|a]$. This representation is always possible, and it ensures that Jewitt's condition (a(*ii*)) is satisfied. Of course, he also requires that this definition of a results in $v(a)$ such that $v'(a) > 0$ and $v''(a) > 0$, which is a restrictive assumption.

A Binary Outcome Example

In the binary outcome example, $X = \{x_1,x_2\}$ and $a \in A = [x_1,x_2]$, with

$$\varphi(x_1|a) = \frac{x_2 - a}{x_2 - x_1} \quad \text{and} \quad \varphi(x_2|a) = 1 - \varphi(x_1|a) = \frac{a - x_1}{x_2 - x_1}.$$

This formulation can be used for any two-outcome example in which $\varphi(x_1|a)$ is a decreasing function. Interestingly, as the following demonstrates, this representation satisfies Jewitt's conditions (a(*i*)) and (b).

(a(*i*)) $$G_1(a) = \Phi(x_1|a) = \varphi(x_1|a)$$

$$\Downarrow$$

$$G_{1a}(a) = -\frac{1}{x_2 - x_1} < 0 \quad \Rightarrow \quad G_{1aa}(a) = 0,$$

$$G_2(a) = \Phi(x_1|a) + \Phi(x_2|a) = G_1(a) + 1,$$

which has the same properties as $G_1(a)$.

(b) $$\varphi_a(x_1|a) = -\frac{1}{x_2 - x_1} \quad \Rightarrow \quad L(x_1|a) = -\frac{1}{x_2 - a} < 0,$$

$$\varphi_a(x_2|a) = \frac{1}{x_2 - x_1} \quad \Rightarrow \quad L(x_2|a) = \frac{1}{a - x_1} > 0.$$

Therefore, $L(x_1|a) \le 0 \le L(x_2|a)$, $\forall\, a \in A$, and the concavity condition is automatically satisfied because there are only two possible values of x_i.

In addition to satisfying Jewitt's condition (a), this example satisfies MLRP (since $L(x_1|a) < L(x_2|a)$) and CDFC (since $\Phi_{1aa}(a) = 0$).

An Exponential Distribution Example

In this example, $x \in [0,\infty)$ and $\varphi(x|a)$ has an exponential distribution (which belongs to the exponential family with $\theta(x) = 1$, $\beta(a) = \alpha(a) = 1/a$, and $\psi(x) = x$):

$$\varphi(x|a) = \frac{1}{a}\exp\left[-\frac{x}{a}\right].$$

In this case,

$$L(x|a) = \frac{1}{a^2}(x - a) \qquad L_x(x|a) = \frac{1}{a^2} > 0 \qquad L_{xx}(x|a) = 0,$$

which satisfies MLRP and Jewitt's condition (b). Furthermore,

$$\Phi(x|a) = \int_0^x \frac{1}{a}\exp\left[-\frac{\tau}{a}\right]d\tau = -\exp\left[-\frac{\tau}{a}\right]\Bigg|_0^x = 1 - \exp\left[-\frac{x}{a}\right],$$

$$\Phi_a(x|a) = -\frac{x}{a^2}\exp\left[-\frac{x}{a}\right] < 0,$$

$$\Phi_{aa}(x|a) = \frac{x}{a^4}(2a - x)\exp\left[-\frac{x}{a}\right] \begin{cases} < 0 & \text{if } x \in (2a,\infty), \\ \ge 0 & \text{if } x \in (0,2a]. \end{cases}$$

Hence, the distribution satisfies FSD, but does *not* satisfy CDFC for $x > 2a$. To test Jewitt's condition (a(i)) we compute

$$G(x|a) \equiv \int_0^x \Phi(y|a)\,dy = x + a\exp\left[-\frac{x}{a}\right] - a,$$

$$G_a(x|a) = \left(1 + \frac{x}{a}\right)\exp\left[-\frac{x}{a}\right] - 1 < 0, \qquad \forall\, x \in (0,\infty),$$

$$G_{aa}(x|a) = \frac{x^2}{a^3}\exp\left[-\frac{x}{a}\right] > 0, \quad \forall\ x \in (0,\infty).$$

Hence, $G(x|a)$ is decreasing and convex.

APPENDIX 17C: CHARACTERISTICS OF OPTIMAL INCENTIVE CONTRACTS FOR HARA UTILITY FUNCTIONS

HARA utility functions were introduced in Volume I, Chapter 2. If the agent's utility for consumption is HARA, then

$$u(c) \sim \begin{cases} -\beta\, e^{-c/\beta} & \text{if } \alpha = 0,\ \beta > 0, \\ \ln(c+\beta) & \text{if } \alpha = 1,\ c+\beta > 0, \\ \dfrac{1}{\alpha-1}[\alpha c+\beta]^{1-1/\alpha} & \text{if } \alpha \neq 0{,}1,\ \alpha c+\beta \geq 0, \end{cases}$$

where α is the agent's risk cautiousness. The analysis in this chapter establishes that, if the principal is risk neutral, optimal contracts take the general form:

$$\boldsymbol{c}(m(x)) = \begin{cases} M^{-1}(m(x)) & \text{if } m(x) > M(\underline{c}), \\ \underline{c} & \text{otherwise}, \end{cases}$$

where $m(x)$ is a linear function of the likelihood ratios for x given the induced action a relative to the alternative actions for which the incentive constraints are binding (see, for example, (17.6), (17.6′), and (17.6″)).

Observe that with HARA utility functions:

$$M(c) = \frac{1}{u'(c)} = \begin{cases} e^{c/\beta} & \text{if } \alpha = 0,\ \beta > 0,\ c \geq \underline{c}, \\ c+\beta & \text{if } \alpha = 1,\ c \geq \underline{c} > -\beta, \\ [\alpha c+\beta]^{1/\alpha} & \text{if } \alpha \neq 0,1,\ c \geq \underline{c} \geq -\beta/\alpha. \end{cases}$$

Hence, for $m = m(x) \geq M(\underline{c})$,

$$M^{-1}(m) = \begin{cases} \beta \ln m & \text{if } \alpha = 0,\ \beta > 0,\ m \geq e^{\underline{c}/\beta} > 0, \\ m - \beta & \text{if } \alpha = 1,\ m \geq \underline{c} + \beta > 0, \\ \alpha^{-1}(m^{\alpha} - \beta) & \text{if } \alpha \neq 0,1,\ m \geq [\alpha \underline{c} + \beta]^{1/\alpha} \geq 0. \end{cases}$$

Furthermore, the relation between the agent's utility for consumption and the likelihood measure m is

$$u(M^{-1}(m)) = \begin{cases} -\beta m^{-1} & \text{if } \alpha = 0,\ \beta > 0,\ m \geq e^{\underline{c}/\beta}, \\ \ln m & \text{if } \alpha = 1,\ m \geq \underline{c} + \beta > 0, \\ \dfrac{1}{\alpha - 1} m^{\alpha - 1} & \text{if } \alpha \neq 0,1,\ m \geq [\alpha \underline{c} + \beta]^{1/\alpha} \geq 0. \end{cases}$$

From the above we can readily characterize how the agent's compensation and utility vary with the likelihood measure m for $m \geq M(\underline{c})$. Of course, for $m \leq M(\underline{c})$, the compensation is equal to $\underline{c}$.

Proposition 17C.1

If the agent has separable utility with HARA utility $u(c)$ for consumption, then for $m \geq M(\underline{c})$:

(a) the agent's compensation is a strictly concave (convex) function of the likelihood measure m if the agent's risk cautiousness α is less (more) than 1, and is linear if $\alpha = 1$;

(b) the agent's utility is a strictly concave (convex) function of the likelihood measure m if the agent's risk cautiousness α is less (more) than 2, and is linear if $\alpha = 2$.

Proof: In the proof we assume that the set of possible values of m is a convex set on the real line, so that $c(m)$ and $u \circ M^{-1}(m)$ are continuously differentiable functions. The results also hold if the set of possible values of m is finite.

(a): Recall that $c(m) = M^{-1}(m)$.

$$c'(m) = \begin{cases} \beta m^{-1} & \text{if } \alpha = 0,\ \beta > 0,\ m \geq e^{\underline{c}/\beta}, \\ 1 & \text{if } \alpha = 1,\ m \geq \underline{c} + \beta > 0, \\ m^{\alpha - 1} & \text{if } \alpha \neq 0, 1,\ m \geq [\alpha \underline{c} + \beta]^{1/\alpha} \geq 0. \end{cases}$$

$$c''(m) = \begin{cases} -\beta m^{-2} & \text{if } \alpha = 0,\ \beta > 0,\ m \geq e^{\underline{c}/\beta}, \\ 0 & \text{if } \alpha = 1,\ m \geq \underline{c} + \beta > 0, \\ (\alpha - 1) m^{\alpha - 2} & \text{if } \alpha \neq 0, 1,\ m \geq [\alpha \underline{c} + \beta]^{1/\alpha} \geq 0. \end{cases}$$

(b): $$\frac{du(M^{-1}(m))}{dm} = \begin{cases} \beta m^{-2} & \text{if } \alpha = 0,\ \beta > 0,\ m \geq e^{\underline{c}/\beta}, \\ m^{-1} & \text{if } \alpha = 1,\ m \geq \underline{c} + \beta > 0, \\ m^{\alpha - 2} & \text{if } \alpha \neq 0, 1,\ m \geq [\alpha \underline{c} + \beta]^{1/\alpha} \geq 0. \end{cases}$$

$$\frac{d^2u(M^{-1}(m))}{dm^2} = \begin{cases} -2\beta m^{-3} & \text{if } \alpha = 0,\ \beta > 0,\ m \geq e^{\underline{c}/\beta}, \\ -m^{-2} & \text{if } \alpha = 1,\ m \geq \underline{c} + \beta > 0, \\ (\alpha - 2) m^{\alpha - 3} & \text{if } \alpha \neq 0, 1,\ m \geq [\alpha \underline{c} + \beta]^{1/\alpha} \geq 0. \end{cases}$$

Q.E.D.

Observe that if there exist likelihood measures $m < M(\underline{c})$, then the compensation and utility levels are flat, with $c = \underline{c}$ and $u(c) = u(\underline{c})$ for those values of m. This does not disturb the convexity of either $c(m)$ or $u \circ M^{-1}(m)$. However, the linear cases become piecewise linear, and the concave functions are not concave over the entire range.

Most analytical research is based on a general concave utility function or assumes the utility function is either exponential or square-root. The exponential utility function has $\alpha = 0$, which implies that the optimal compensation and utility functions are strictly concave functions of the likelihood measure for $m \geq M(\underline{c})$. The square-root utility function, on the other hand, has $\alpha = 2$ (since $1 - ½ = ½$), which implies the optimal compensation is a strictly convex function of the likelihood measure, while the utility function is linear (or piecewise linear if there exists $m < M(\underline{c})$).

In the first-stage of the GH approach we minimize the expected compensation cost to induce a given action. This is equivalent to minimizing the risk premium paid to the agent, since the risk premium is given by

$$\pi(\boldsymbol{c},a) = \mathrm{E}[c|a] - CE(\boldsymbol{c},a),$$

where the certainty equivalent is given by the participation constraint as the first-best cost of implementing a (provided the participation constraint is binding), i.e.,

$$CE(\boldsymbol{c},a) = w\left(\frac{\bar{U} + v(a)}{k(a)}\right),$$

where $w(\cdot) = u^{-1}(\cdot)$ denotes the inverse of the agent's utility for consumption.

In subsequent analyses, *with additive separable utility functions* of the HARA class, we use properties of the change in risk premium that occurs when the level of utility is increased by the same amount for all outcomes. That is, for a given compensation contract $\boldsymbol{c}$ that implements a we consider another compensation contract $\boldsymbol{c}^{\lambda}$ defined by

$$u(\boldsymbol{c}^{\lambda}(x)) = u(\boldsymbol{c}(x)) + \lambda, \quad \forall\, x \in X.$$

Clearly, if $\boldsymbol{c}$ implements a, $\boldsymbol{c}^{\lambda}$ also implements a since[19]

$$\underset{\hat{a}\in A}{\operatorname{argmax}} \int_X u(\boldsymbol{c}(x))\, d\Phi(x|a) - v(\hat{a}) = \underset{\hat{a}\in A}{\operatorname{argmax}} \int_X [u(\boldsymbol{c}(x)) + \lambda]\, d\Phi(x|a) - v(\hat{a}).$$

The risk premium paid to the agent for contract $\boldsymbol{c}^{\lambda}$ is given by

$$\pi(\boldsymbol{c}^{\lambda},a) = \int_X w(u(\boldsymbol{c}(x)) + \lambda)\, d\Phi(x|a) - w\left(\int_X [u(\boldsymbol{c}(x)) + \lambda]\, d\Phi(x|a)\right).$$

Increasing the level of utility, increases the variance of the compensation and, therefore, one might think that the risk premium paid to the agent also increases. However, due to wealth effects on the agent's risk aversion, the relation between the utility level and the risk premium is more complicated than suggested by this intuition. The following proposition demonstrates that the risk premium increases with λ if the agent's utility is a concave function of the likelihood measure (or, equivalently, the derivative of the inverse utility function, i.e.,

[19] In this analysis we do not consider the impact on the participation constraint. In subsequent applications we consider cases in which the level of utility is increased for outcomes that are affected by the agent's action and decreased correspondingly for outcomes that are not affected by the agent's action. The variation is such that it leaves both incentives and the agent's expected utility unchanged.

$w'(\cdot)$,[20] is convex). On the other hand, if the agent's utility is a convex function of the likelihood measure, the risk premium decreases as the utility increases.

Proposition 17C.2

If the agent has an additively separable utility function, the risk premium $\pi(\mathbf{c}^\lambda, a)$ is increasing (decreasing) in λ if the agent's utility is a concave (convex) function of the likelihood measure m.

Proof: From the definition of the risk premium and Jensen's inequality we get that

$$\frac{\partial \pi(\mathbf{c}^\lambda, a)}{\partial \lambda} = \int_X w'(u(\mathbf{c}(x)) + \lambda)\, d\Phi(x|a) \; - \; w'\Big(\int_X [u(\mathbf{c}(x)) + \lambda]\, d\Phi(x|a)\Big) > (<) \; 0,$$

if, and only if, $w'(\cdot)$ is convex (concave). Now recall that $w'(u(\mathbf{c}(m))) = M(\mathbf{c}(m)) = m$. Hence, $w'(\cdot)$ is the inverse function of $u\circ\mathbf{c}(\cdot)$ so that $w'(\cdot)$ is convex (concave) if, and only if, $u\circ\mathbf{c}(\cdot)$ is concave (convex). **Q.E.D.**

Of course, if the agent's utility for consumption is HARA we can use Proposition 17C.1 to obtain the result that the risk premium is increasing (decreasing) in λ if the risk cautiousness is less (more) than 2, and independent of λ if $\alpha = 2$.

REFERENCES

Arya, A., R. A. Young, and J. C. Fellingham. (1993) "Preference Representation and Randomization in Principal-Agent Contracts," *Economic Letters* 42, 25-30.

Fellingham, J. C., Y. Kwon, and D. P. Newman. (1984) "Ex ante Randomization in Agency Models," *Rand Journal of Economics* 15, 290-301.

Gjesdal, F. (1982) "Information and Incentives: The Agency Information Problem," *Review of Economic Studies* 49, 373-390.

Grossman, S. J., and O. D. Hart. (1983) "An Analysis of the Principal-Agent Problem," *Econometrica* 51, 7-45.

Holmström, B. (1979) "Moral Hazard and Observability," *Bell Journal of Economics* 10, 74-91.

Innes, R. D. (1990) "Limited Liability and Incentive Contracting with Ex-ante Action Choices," *Journal of Economic Theory* 52, 45-67.

[20] Note that the derivative of the inverse utility function is equal to the marginal cost of providing utility to the agent.

Jewitt, I. (1988) "Justifying the First-Order Approach to Principal-Agent Problems," *Econometrica* 56, 1177-1190.

Mirrlees, J. A. (1999) "The Theory of Moral Hazard and Unobservable Behaviour: Part I," *Review of Economic Studies* 66, 3-21. (Working paper originally completed in 1975.)

Rogerson, W. (1985) "The First-Order Approach to Principal-Agent Problems," *Econometrica* 53, 1357-1368.

Ross, S. (1973) "The Economic Theory of Agency: The Principal's Problem," *American Economic Review* 63, 134-139.

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