Preface

This is a small book on algebra where the stress is laid on the structure of *fields*, hence its title.

You will hear about equations, both polynomial and differential, and about the algebraic structure of their solutions. For example, it has been known for centuries how to explicitely solve polynomial equations of degree 2 (Babylonians, many centuries ago), 3 (Scipione del Ferro, Tartaglia, Cardan, around 1500 A.D.), and even 4 (Cardan, Ferrari, XVI$^{\text{th}}$ century), using only algebraic operations and radicals (nth roots). However, the case of degree 5 remained unsolved until Abel showed in 1826 that a general equation of degree 5 cannot be solved that way.

Soon after that, Galois defined the group of a polynomial equation as the group of permutations of its roots (say, complex roots) that preserve all algebraic identities with rational coefficients satisfied by these roots. Examples of such identities are given by the elementary symmetric polynomials, for it is well known that the coefficients of a polynomial are (up to sign) elementary symmetric polynomials in the roots. In general, all relations are obtained by combining these, but sometimes there are new ones and the group of the equation is smaller than the whole permutation group.

Galois understood how this symmetry group can be used to characterize the solvability of the equation. He defined the notion of *solvable group* and showed that if the group of the equation is solvable, then one can express its roots with radicals, and conversely.

Telling this story will lead us along interesting paths. You will, for example, learn why certain problems of construction by ruler and compass which were posed by the ancient Greeks and remained unsolved for centuries have no solution. On the other hand, you will know why (and maybe discover *how*) one can construct certain regular polygons.

There is an analogous theory for linear differential equations, and we will introduce a similar group of matrices. You will also learn why the explicit computation of certain indefinite integrals, such as $\int \exp(x^2)$, is hopeless.

On the menu are also some theorems from analysis: the transcendance of the number π, the fact that the complex numbers form an algebraically closed field, and also Puiseux's theorem that shows how one can parametrize the roots of polynomial equations, the coefficients of which are allowed to vary.

There are some exercices at the end of each chapter. Please take some time to look at them. There is no better way to feel at ease with the topics in this book. Don't worry, some of them are even easy!

I downloaded the portraits of mathematicians from the *MacTutor History of Mathematics* site, *http://www-groups.dcs.st-andrews.ac.uk/~history/*. I encourage those of you who are interested in History of Mathematics to browse this archive. Reading the books in the bibliography, like the small [4], is also highly recommended. I found the the scans of mathematical stamps at the address *http://jeff560.tripod.com/* — those interested in that subject will be delighted to browse the book [13].

I taught most of this book at École polytechnique (Palaiseau, France). I would like to take the opportunity here to acknowledge all the advice and comments I received from my colleagues, namely, Jean-Michel Bony, Jean Lannes, David Renard and Claude Viterbo. I would also like to thank Sarah Carr for her help in polishing the English translation.

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