## Chapter 2

# The constitutive relation error method for linear problems

### 2.1 Introduction

The reference problem considered here is the linear problem described by Equations [(1.6) - (1.8)]. It is discretized by the finite element displacement method. In order to evaluate the quality of a finite element calculation, one must:

define the notion of error;
define measures of this error.

The definition of the error is closely connected to the definition of the notion of approximate solution. Classically, in the framework of the finite element displacement method, the displacement field plays a special role and the error is defined as the difference between the exact displacement field and the finite element displacement field

$$\underline{e}_h = \underline{U}_{ex} - \underline{U}_h \tag{2.1}$$

From this definition, one can obviously deduce an error in the stresses

$$\sigma_{ex} - \sigma_h = \mathbf{K}\varepsilon(\underline{e}_h) \tag{2.2}$$

The major drawback of this approach from a mechanical point of view is that one considers as an approximation of the exact pair $(\underline{U}_{ex}, \sigma_{ex})$ a pair $(\underline{U}_h, \sigma_h)$ whose stress part does not verify the equilibrium equations – the very equations which constitute one of the foundations of the formulation. This observation is at the root of the introduction of the notion of error in constitutive relation presented for the first time in [LADEVEZE, 1975]. The works related to this early period have been collected in [LADEVEZE, 1995]. This approach is based on a partitioning of the equations of the problem to be solved into two groups:

the admissibility conditions: kinematic constraint equations, equilibrium equations and initial conditions;

the constitutive relation.

Indeed, the constitutive relation has a special status; in practice, it is often the least reliable equation. Therefore, it is natural to set this equation apart and to ensure that an approximate displacement-stress solution over the time interval of interest verifies the group of the most reliable equations (i.e. the admissibility conditions) exactly. This approximate solution verifies a certain constitutive relation which, in general, is different from the constitutive relation of the material. Thus, the quality of the approximation can be evaluated by comparing this constitutive relation with that of the material. In order to quantify this error, energy-type norms or other norms which have a strong physical meaning are used.

The difficult part, which is a priori an obstacle, is to develop admissible approximate solutions, i.e. solutions which satisfy the conditions of admissibility exactly. Indeed, the usual approximations, including those obtained by the finite element method, do not satisfy these conditions because the stress $\sigma_h$ is not in equilibrium with the given loads. For the finite element method, the completely new technique proposed in [LADEVEZE, 1975, 1977] and subsequently expanded enables one to circumvent this difficulty. For linear problems, one can refer particularly to [LADEVEZE - LEGUILLON, 1983], [LADEVEZE - PELLE - ROUGEOT, 1991], [LADEVEZE - ROUGEOT, 1997]. This technique is very general; it enables the explicit (therefore inexpensive) construction of an admissible solution starting from the solution calculated by the finite element method and taking advantage of its properties. In particular, this method does not depend on a finite element computation by the dual approach developed by FRAEIJS DE VEUBEKE [FRAEIJS DE VEUBEKE, 1965]. In [DEBONGNIE - BECKERS, 1999], the approach described here is called the *non-conventional dual approach*.

The global error in constitutive relation is related to the classical error between the exact solution and the finite element solution by the well-known PRAGER - SYNGE theorem [PRAGER - SYNGE, 1947]. Its potential in the domain of error evaluation has been known for a long time [TOTTENHAM, 1970], [AUBIN - BURCHARD, 1970]. Of course, the concept of error in constitutive relation is much more general and mechanically very different. Therefore, it should not be confused with the approach underlying the PRAGER - SYNGE hypercircle theorem; for example, the concept of error in constitutive relation is definitely not limited to linear problems.

Several review articles have presented the state of progress of the general concept [LADEVEZE, 1990, 1992], [PELLE, 1995]. This chapter outlines the approach which leads to estimators of the error in constitutive relation. The more technical aspects will be detailed in Chapter 8.

## 2.2 Error in constitutive relation

### 2.2.1 Admissible approximate solution

In accordance with what was said in the Introduction, an approximate solution of the reference problem is, by definition, an admissible displacement-stress pair $(\underline{\hat{U}}, \hat{\sigma})$, i.e. a pair such that:

$\underline{\hat{U}}$ is kinematically admissible

$$\underline{\hat{U}} \in \boldsymbol{\mathcal{U}} \quad \text{and} \quad \underline{\hat{U}}_{|\partial_1\Omega} = \underline{U}_d \tag{2.3}$$

$\hat{\underline{\underline{\sigma}}}$ is statically admissible

$$\hat{\underline{\underline{\sigma}}} \in \boldsymbol{\mathcal{S}} \quad \text{and} \quad \forall \underline{U}^* \in \boldsymbol{\mathcal{U}}_0$$
$$\int_\Omega \mathrm{Tr}\left[\hat{\underline{\underline{\sigma}}}\,\underline{\underline{\varepsilon}}(\underline{U}^*)\right] d\Omega = \int_\Omega \underline{f}_d \cdot \underline{U}^* \, d\Omega + \int_{\partial_2\Omega} \underline{F}_d \cdot \underline{U}^* \, dS \tag{2.4}$$

Under these conditions, the approximate character of an admissible displacement-stress pair $(\hat{\underline{U}}, \hat{\underline{\underline{\sigma}}})$ resides in the nonsatisfaction of the constitutive relation.

### 2.2.2 Error in constitutive relation in the linear case

Let $(\hat{\underline{U}}, \hat{\underline{\underline{\sigma}}})$ be an admissible pair. If it verifies the constitutive relation $\hat{\underline{\underline{\sigma}}} - \mathbf{K}\underline{\underline{\varepsilon}}(\hat{\underline{U}}) = 0$, then this pair is the exact solution of the reference problem

$$\hat{\underline{U}} = \underline{U}_{ex} \quad \text{and} \quad \hat{\underline{\underline{\sigma}}} = \underline{\underline{\sigma}}_{ex}$$

If the pair does not verify the constitutive relation $\hat{\underline{\underline{\sigma}}} - \mathbf{K}\underline{\underline{\varepsilon}}(\hat{\underline{U}}) \neq 0$, then $(\hat{\underline{U}}, \hat{\underline{\underline{\sigma}}})$ is an approximation of $(\underline{U}_{ex}, \underline{\underline{\sigma}}_{ex})$, and the manner in which $(\hat{\underline{U}}, \hat{\underline{\underline{\sigma}}})$ verifies the constitutive relation makes it possible to assess its quality. One is thus led to introduce the following definition:

***Definition***

*One calls error in constitutive relation associated with the admissible pair $(\hat{\underline{U}}, \hat{\underline{\underline{\sigma}}})$ the quantity defined at all points of $\Omega$ by*

$$\underline{\underline{e}}_{RdC} = \hat{\underline{\underline{\sigma}}} - \mathbf{K}\underline{\underline{\varepsilon}}(\hat{\underline{U}}) \tag{2.5}$$

Thus, the error in constitutive relation $\underline{\underline{e}}_{RdC}$ is a quantity of the stress type.

This approach has certain advantages. The first advantage is that it directs the doubt to the constitutive relation, i.e. the equation of the reference problem which is the least reliable. The second advantage is that, in contrast with classical errors of the $\underline{e}_h$ type, the expression of $\underline{\underline{e}}_{RdC}$ does not contain the exact solution of the reference problem. Therefore, in order to calculate the error in constitutive relation, it is not necessary to know the exact solution of the problem: the knowledge of the approximate solution is sufficient! Finally, in the context of linear problems, the error measures show up very naturally.

However, the implementation of this approach requires the knowledge of an admissible displacement-stress pair. Therefore, in order to evaluate the error in constitutive relation, it is necessary to reconstruct an admissible pair by post-processing the solution of the approximate model, which is generally not an admissible pair; we will see that this step poses certain technical difficulties.

***Measures associated with the error***

Classically, the error in constitutive relation is measured using the energy norm on the structure being considered. We will therefore define the absolute global error as

$$\mathbf{e}_{RdC}(\hat{\underline{U}}, \hat{\sigma}) = \left\| \hat{\sigma} - \mathbf{K}\varepsilon(\hat{\underline{U}}) \right\|_{\sigma,\Omega} \tag{2.6}$$

with

$$\| \bullet \|_{\sigma,\Omega} = \left[ \int_{\Omega} \mathrm{Tr}(\bullet \mathbf{K}^{-1} \bullet)\, d\Omega \right]^{1/2} \tag{2.7}$$

When there is no risk of confusion, we will simply write $\mathbf{e}_{RdC}$ instead of $\mathbf{e}_{RdC}(\hat{\underline{U}}, \hat{\sigma})$.

To the above global error, one associates the relative error

$$\boldsymbol{\varepsilon} = \frac{\left\| \hat{\sigma} - \mathbf{K}\varepsilon(\hat{\underline{U}}) \right\|_{\sigma,\Omega}}{\left\| \hat{\sigma} + \mathbf{K}\varepsilon(\hat{\underline{U}}) \right\|_{\sigma,\Omega}} \tag{2.8}$$

Thus, $\boldsymbol{\varepsilon}$ is a global accuracy parameter which enables one to estimate the global quality of the approximation $(\hat{\underline{U}}, \hat{\sigma})$.

It is also easy to define the contribution of a part $E$ of (which, in practice, can be any element of the mesh) to the global accuracy by

$$\boldsymbol{\varepsilon}_E = \frac{\left\| \hat{\sigma} - \mathbf{K}\varepsilon(\hat{\underline{U}}) \right\|_{\sigma,E}}{\left\| \hat{\sigma} + \mathbf{K}\varepsilon(\hat{\underline{U}}) \right\|_{\sigma,\Omega}} \tag{2.9}$$

with

$$\| \bullet \|_{\sigma,E} = \left[ \int_{E} \mathrm{Tr}(\bullet \mathbf{K}^{-1} \bullet)\, dE \right]^{1/2} \tag{2.10}$$

These local contributions enable one to localize the discretization errors. From (2.8) and (2.9), it follows that

$$\varepsilon^2 = \sum_E \varepsilon_E^2 \tag{2.11}$$

REMARKS

**1.** Other measures of the error in constitutive relation can also be used. For example, one can use the following measure:

$$\varepsilon = \frac{\displaystyle\operatorname*{Sup}_{M\in\Omega}\left|\operatorname{Tr}[(\hat{\sigma} - \mathbf{K}\varepsilon(\hat{\underline{U}}))\mathbf{K}^{-1}(\hat{\sigma} - \mathbf{K}\varepsilon(\hat{\underline{U}}))\right|}{[\operatorname{mes}(\Omega)]^{-1/2}\left\|\hat{\sigma} + \mathbf{K}\varepsilon(\hat{\underline{U}})\right\|_{\sigma,\Omega}}$$

which is very close to an $L^\infty$ type norm [LADEVEZE - LEGUILLON, 1983].

**2.** Similarly, using the local contributions $\varepsilon_E$, one can define the quantities

$$\bar{\varepsilon}_E = \left[\frac{\operatorname{mes}(\Omega)}{\operatorname{mes}(E)}\right]^{1/2} \cdot \varepsilon_E = \frac{\dfrac{1}{[\operatorname{mes}(E)]^{1/2}}\left\|\hat{\sigma} - \mathbf{K}\varepsilon(\hat{\underline{U}})\right\|_{\sigma,E}}{\dfrac{1}{[\operatorname{mes}(\Omega)]^{1/2}}\left\|\hat{\sigma} + \mathbf{K}\varepsilon(\hat{\underline{U}})\right\|_{\sigma,\Omega}} \tag{2.12}$$

## 2.3 Properties of the error in constitutive relation

The objective of this section is to show that the error in constitutive relation – more precisely, the energy norm of this error – is a very convenient tool for introducing the classical potential energy and complementary energy theorems.

### 2.3.1 New formulation for an elasticity problem

The reference problem [(1.6) - (1.8)] can be formulated in the following manner:

*Find* $(\underline{U}, \sigma)$ *such that:*

$\underline{U}$ *is KA (Kinematically Admissible);*

$\sigma$ *is SA (Statically Admissible);*

$\underline{U}$ *and* $\sigma$ *verify the constitutive relation:*

*at every point* M *of* $\Omega$, $\sigma = \mathbf{K}\varepsilon(\underline{U})$.

Let

$$\Psi(\underline{U}\,,\,\sigma) = \frac{1}{2}\,\|\,\sigma - \mathbf{K}\varepsilon(\underline{U})\,\|^2_{\sigma,\Omega}$$
$$= \frac{1}{2}\int_\Omega \mathrm{Tr}(\,[\sigma - \mathbf{K}\varepsilon(\underline{U})]\mathbf{K}^{-1}\,[\sigma - \mathbf{K}\varepsilon(\underline{U})]\,)\;d\Omega \tag{2.13}$$

The constitutive relation is verified everywhere if and only if $\Psi(\underline{U}\,,\,\sigma) = 0$. Therefore, the exact solution $(\underline{U}_{ex}\,,\,\sigma_{ex})$ of the reference problem verifies

$$\Psi(\underline{U}_{ex}\,,\,\sigma_{ex}) = 0$$

Since one always has $\Psi(\underline{U}\,,\,\sigma) \geq 0$, one can deduce that $(\underline{U}_{ex}\,,\,\sigma_{ex})$ is the solution of the minimization problem

$$\underset{\hat{\underline{U}}\,KA \text{ and } \hat{\sigma}\,SA}{\mathrm{Min}} \quad \Psi(\hat{\underline{U}}\,,\,\hat{\sigma}) \tag{2.14}$$

Under the hypotheses of Chapter 1, which guarantee the existence of a unique solution of the reference problem, Formulation (2.14) is equivalent to the classical formulation [(1.6) - (1.8)].

REMARK. In some situations, particularly for problems described as "ill-posed", Formulation [(1.6) - (1.8)] may have no solution; this is the case, for example, in identification problems in which both the forces and the displacements are prescribed simultaneously on a part (with nonzero measure) of $\partial\Omega$.

In this case, Problem (2.14) still has solutions, but if $(\tilde{\underline{U}}\,,\,\tilde{\sigma})$ is one of these solutions one has $\Psi(\tilde{\underline{U}}\,,\,\tilde{\sigma}) > 0$.

For these identification problems and, more generally, for inverse problems, one can refer to [BUI, 1992].

### 2.3.2 Potential energy and complementary energy

Let us consider an admissible pair $(\hat{\underline{U}}\,,\,\hat{\sigma})$. One has the following decoupling property:

*Property*

$$\Psi(\underline{\hat{U}}, \hat{\sigma}) = \mathrm{E}_p(\underline{\hat{U}}) + \mathrm{E}_c(\hat{\sigma}) \tag{2.15}$$

*with*

$$\mathrm{E}_p(\underline{\hat{U}}) = \frac{1}{2}\int_\Omega \mathrm{Tr}\left[\varepsilon(\underline{\hat{U}})\mathbf{K}\varepsilon(\underline{\hat{U}})\right] d\Omega - \left[\int_\Omega \underline{f}_d \cdot \underline{\hat{U}}\, d\Omega + \int_{\partial_2\Omega} \underline{F}_d \cdot \underline{\hat{U}}\, dS\right]$$

*and*

$$\mathrm{E}_c(\hat{\sigma}) = \frac{1}{2}\int_\Omega \mathrm{Tr}\left[\hat{\sigma}\,\mathbf{K}^{-1}\hat{\sigma}\right] d\Omega - \int_{\partial_1\Omega} \hat{\sigma}\,\underline{n} \cdot \underline{U}_d\, dS$$

*Proof*

By expanding $\Psi(\underline{\hat{U}}, \hat{\sigma})$, one obtains

$$\Psi(\underline{\hat{U}}, \hat{\sigma}) = \frac{1}{2}\int_\Omega \mathrm{Tr}(\hat{\sigma}\mathbf{K}^{-1}\hat{\sigma})\, d\Omega + \frac{1}{2}\int_\Omega \mathrm{Tr}(\varepsilon(\underline{\hat{U}})\mathbf{K}\varepsilon(\underline{\hat{U}}))\, d\Omega - \int_\Omega \mathrm{Tr}(\hat{\sigma}\varepsilon(\underline{\hat{U}}))\, d\Omega$$

Since $(\underline{\hat{U}}, \hat{\sigma})$ is admissible, one has

$$\begin{aligned}\int_\Omega \mathrm{Tr}(\hat{\sigma}\varepsilon(\underline{\hat{U}}))\, d\Omega &= \int_\Omega \left[\mathrm{div}(\hat{\sigma}\underline{\hat{U}}) - \underline{\mathrm{div}}\ \hat{\sigma} \cdot \underline{\hat{U}}\right] d\Omega \\ &= \int_{\partial\Omega} \hat{\sigma}\underline{n} \cdot \underline{\hat{U}}\, d\Omega - \int_\Omega \underline{\mathrm{div}}\ \hat{\sigma} \cdot \underline{\hat{U}}\, d\Omega \\ &= \int_\Omega \underline{f}_d \cdot \underline{\hat{U}}\, d\Omega + \int_{\partial_1\Omega} \hat{\sigma}\underline{n} \cdot \underline{U}_d\, d\Omega + \int_{\partial_2\Omega} \underline{F}_d \cdot \underline{\hat{U}}\, d\Omega\end{aligned}$$

The combination of these results yields Property (2.15).

This decoupling property enables one to split the minimization problem (2.14) into two minimization problems, one in terms of displacements and the other in terms of stresses:

The potential energy theorem:

*The displacement field $\underline{U}_{ex}$ is the solution of the problem*

$$\underset{\underline{\hat{U}}\ CA}{\mathrm{Min}}\ \mathrm{E}_p(\underline{\hat{U}}) \tag{2.16}$$

The complementary energy theorem:

*The stress field* $\sigma_{ex}$ *is the solution of the problem*

$$\underset{\hat{\sigma}\, SA}{\text{Min}}\; \text{E}_c(\hat{\sigma}) \tag{2.17}$$

### 2.3.3 Extensions

Many constitutive relations can be defined starting from a pair of convex dual functions $\varphi$ and $\varphi^*$ such that

$$\varphi(\varepsilon) + \varphi^*(\sigma) - \text{Tr}[\sigma\varepsilon] \geq 0$$

Besides, at a point $M$, the equality $\varphi(\varepsilon) + \varphi^*(\sigma) - \text{Tr}[\sigma\varepsilon] = 0$ is equivalent to the satisfaction of the constitutive relation at that point. The global error in constitutive relation is then defined by

$$\Psi(\underline{U}, \sigma) = \int_\Omega \varphi[\varepsilon(\underline{U})]\, d\Omega + \int_\Omega \varphi^*(\sigma)\, d\Omega - \int_\Omega \text{Tr}[\sigma\varepsilon(\underline{U})]\, d\Omega$$

The dual convex functions defined by

$$\Phi(\varepsilon) = \int_\Omega \varphi(\varepsilon)\, d\Omega \quad \text{and} \quad \Phi^*(\sigma) = \int_\Omega \varphi^*(\sigma)\, d\Omega$$

are the overpotentials of the constitutive relation.

The minimization of the error in constitutive relation leads to two theorems similar to the energy theorems:

*The displacement field* $\underline{U}_{ex}$ *is the solution of the problem*

$$\underset{\hat{\underline{U}}\, CA}{\text{Min}}\; \text{E}_p(\hat{\underline{U}})$$

with

$$\text{E}_p(\hat{\underline{U}}) = \Phi[\varepsilon(\hat{\underline{U}})] - \left[\int_\Omega \underline{f}_d \cdot \hat{\underline{U}}\, d\Omega + \int_{\partial_2\Omega} \underline{F}_d \cdot \hat{\underline{U}}\, dS\right]$$

*The stress field* $\sigma_{ex}$ *is the solution of the problem*

$$\underset{\hat{\sigma}\, SA}{\text{Min}}\; \text{E}_c(\hat{\sigma})$$

with

$$E_c(\hat{\sigma}) = \Phi^*(\hat{\sigma}) - \int_{\partial_1\Omega} \hat{\sigma}\,\underline{n} \cdot \underline{U}_d \, dS$$

REMARKS

**1.** In elasticity, one has

$$\Phi(\varepsilon) = \frac{1}{2}\int_\Omega \mathrm{Tr}[\varepsilon \mathbf{K} \varepsilon]\, d\Omega \quad \text{and} \quad \Phi^*(\sigma) = \frac{1}{2}\int_\Omega \mathrm{Tr}[\sigma\, \mathbf{K}^{-1} \sigma]\, d\Omega$$

**2.** The denomination "energy theorems" is not really appropriate; these theorems are related to the global formulation of the constitutive relation, not to the notion of energy.

**3.** This formalism owes much to the works of J.-J. MOREAU [MOREAU, 1966, 1974] and to the further developments by NAYROLES [NAYROLES, 1973].

## 2.4 Utilization in finite element calculations

### 2.4.1 Construction of an admissible pair

The calculation of the error in constitutive relation cannot be made directly on the pair $(\underline{U}_h, \sigma_h)$ because this pair is not admissible.

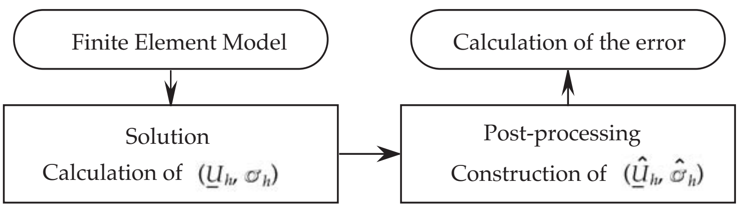


**Figure 2.1.** *Principle of utilization.*

Our method consists in post-processing the finite element solution in order to construct an admissible displacement-stress pair $(\hat{\underline{U}}_h, \hat{\sigma}_h)$, in the sense of the relations

(2.3) and (2.4), starting from (Figure 2.1):

the data of the reference problem;

the finite element solution $(\underline{U}_h, \sigma_h)$.

From a technical standpoint, this is a crucial aspect of the method. Indeed, on the one hand, the quality of the constructed pair $(\hat{\underline{U}}_h, \hat{\sigma}_h)$ determines the quality of the error measure and, on the other hand, the numerical cost of this construction should be small compared to the cost of the finite element analysis.

### 2.4.2 Displacement field

In the context of finite element displacement methods of the conforming type, the displacement field $\underline{U}_h$ is admissible. Therefore, for the sake of simplicity, one generally chooses

$$\hat{\underline{U}}_h = \underline{U}_h \tag{2.18}$$

REMARKS

**1.** Of course, $\underline{U}_h$ is admissible only with the provision that the displacement boundary conditions on $\partial_1\Omega$ be compatible with the interpolation chosen for the discretization.

**2.** An example of the construction of a field $\hat{\underline{U}}_h$ different from $\underline{U}_h$ can be found, in the framework of thermal problems, in [LADEVEZE, 1977], [LADEVEZE - LEGUILLON, 1983].

**3.** For calculations in incompressible elasticity, the choice $\hat{\underline{U}}_h = \underline{U}_h$ is unsuitable. A specific construction of fields $\hat{\underline{U}}_h$ which verify the incompressibility condition $\mathrm{div}(\hat{\underline{U}}_h) = 0$ was developed in [GASTINE - LADEVEZE - MARIN - PELLE, 1992]. We will return to this point in Chapter 8.

### 2.4.3 Stress field

Conversely, the stress field $\sigma_h$ calculated by finite elements is not statically admissible. It is therefore necessary to reconstruct a field $\hat{\sigma}_h$ which verifies the equilibrium equations exactly. A priori, it is possible to construct such a field using a finite element method of the equilibrium type [FRAEIJS DE VEUBEKE, 1965], [STEIN - AHMAD, 1977]. In spite of their remarkable effectiveness, these methods have been used very rarely because of the difficulty of their implementation. In addition, the use of this dual method along with the finite element displacement method is not realistic today. Let us also note that the codes which are presently used in industry do not include equilibrium elements.

The method presented below was introduced in [LADEVEZE, 1975, 1977]. It relies on strong extension conditions. Today, we use a variant of this method, based on weak extension conditions, which is more effective, but also more costly (Chapter 8). It is described in [LADEVEZE, 1994] and developed in [LADEVEZE - ROUGEOT, 1997].

***Strong extension condition***

The basic idea consists in seeking a field $\hat{\sigma}_h$ as a prolongation (or an extension) of the finite element solution in the following sense:

*For any basis function $\varphi_i$ of the finite element type*
*and for any element E of the mesh*

$$\int_E (\hat{\sigma}_h - \sigma_h) \, \underline{\text{grad}} \, \varphi_i \, dE = 0 \tag{2.19}$$

Here, we merely outline the technique of construction of the stress field. This construction will be reintroduced in detail in Chapter 8.

REMARK. The extension condition (2.19) has a strong mechanical content. It requires, of the stress field $\hat{\sigma}_h$ that one is seeking, that it yield the same work in each element and for each finite element displacement field as the finite element stress field $\sigma_h$. In particular, for 3-node triangles, this condition imposes equality of the mean stresses in each element.

***Notations***

On the boundary $\partial E$ of each element $E$, one defines a function $\eta_E$, constant on each side of $E$ and with the value +1 or 1, such that on the side $\Gamma_{E_1E_2}$ common to two adjacent elements $E_1$ and $E_2$ one has $\eta_{E_1} + \eta_{E_2} = 0$; we will designate by $\underline{n}_E$ the outward unit normal on $\partial E$.

Let $i$ be a node of the mesh. We will denote by $E_1, E_2, \dots, E_N$ the $N$ elements connected to node $i$ and by $\Gamma_E^1, \Gamma_E^2, \dots, \Gamma_E^R$ the $R$ sides of an element $E$ which are connected to node $i$.

#### *Principles of the construction*

The construction of an equilibrated stress field $\hat{\sigma}_h$ is carried out in two stages. First, one constructs force densities $\hat{\underline{F}}$ on the element sides. The purpose of these densities is to represent the stress vectors on the element sides

$$\hat{\sigma}_{h|E} \, \underline{n}_E = \eta_E \, \hat{\underline{F}} \quad \text{on } \partial E \tag{2.20}$$

In this manner, the continuity of the stress vector in the $\underline{n}_E$ direction is automatically verified across an interface.

Naturally, on the sides which are part of $\partial_2\Omega$, one imposes

$$\eta_E \, \hat{\underline{F}} = \underline{F}_d \tag{2.21}$$

Finally, these force densities are constructed in such a way that the densities $\eta_E \, \hat{\underline{F}}$ are in equilibrium with the given interior loads $\underline{f}_d$ on each element $E$ of the mesh:

*For any field of rigid body displacement* $\underline{U}_S$,

$$\int_E \underline{f}_d \cdot \underline{U}_S \, dE + \int_{\partial E} \eta_E \, \hat{\underline{F}} \cdot \underline{U}_S \, dS = 0 \tag{2.22}$$

In the second stage, it suffices to construct a simple solution of the local problem on each element $E$,

$$\begin{aligned} &\underline{\mathrm{div}} \, \hat{\sigma}_h + \underline{f}_d = 0 \quad \text{in } E \\ &\hat{\sigma}_h \, \underline{n}_E = \eta_E \, \hat{\underline{F}} \quad \text{on } \partial E \end{aligned} \tag{2.23}$$

#### *Construction of the densities*

Let us consider a node $i$ of an element $E$. By applying the extension condition (2.19) to the shape function $\varphi_i$ of node $i$ and using the admissibility conditions (2.4) verified by $\hat{\sigma}_h$, one obtains

$$\int_{\partial E} \eta_E \hat{\underline{F}} \varphi_i \, d\Gamma = -\int_E \underline{f}_d \varphi_i \, dE + \int_E \sigma_h \, \underline{\mathrm{grad}} \, \varphi_i \, dE \tag{2.24}$$

i.e., observing that $\varphi_i$ is zero on the sides which do not contain $i$,

$$\sum_{r=1}^{R} \left| \int_{\Gamma_E^r} \eta_E \hat{\underline{F}} \varphi_i \, d\Gamma \right| = Q_E(i) \tag{2.25}$$

where $Q_E(i)$ denotes the right-hand side of (2.24), which is a known vector which depends on the data and on the finite element solution.

*Determination of projections of the densities*

By applying Equations (2.25) to all the elements connected to node $i$, one finds that the projections of the densities $\eta_E \hat{\underline{F}}$,

$$\underline{b}_n^r(i) = \int_{\Gamma_{E_n}^r} \eta_E \hat{\underline{F}} \varphi_i \, d\Gamma \tag{2.26}$$

are solutions of the linear system

$$\sum_{r=1}^{R_n} \underline{b}_n^r(i) = Q_{E_n}(i) \quad \text{for } n \in \{1, 2, \dots, N\} \tag{2.27}$$

To make things clear, let us assume that $i$ is an interior node of  .

The number of equations of System (2.27) is less than or equal to the number of unknowns. Moreover, these equations are not independent. Indeed, let us consider one of the unknowns $\underline{b}_n^r(i)$ which appear in the $n$th equation. There is only one element $E_m$, distinct from $E_n$, which has $\Gamma_{E_n}^r$ as a side. It follows that this unknown $\underline{b}_n^r(i)$ also appears in the $m$th equation (and only in that equation other than the $n$th equation). But because of the property $\eta_{E_n} + \eta_{E_m} = 0$ it appears in that equation with the opposite sign. Consequently, the sum of the rows of the matrix of System (2.27) is zero. Hence, System (2.27) has solutions only if the following compatibility condition is verified:

$$\sum_{n=1}^{N} Q_{E_n}(i) = 0 \tag{2.28}$$

By summing the $Q_{E_n}(i)$, one gets

$$\sum_{n=1}^{N} Q_{E_n}(i) = \sum_{n=1}^{N} \left| -\int_{E_n} \underline{f}_d \varphi_i \, dE + \int_{E_n} \sigma_h \, \underline{\text{grad}} \, \varphi_i \, dE \right|$$
$$= -\int_{\Omega} \underline{f}_d \varphi_i \, dE + \int_{\Omega} \sigma_h \, \underline{\text{grad}} \, \varphi_i \, dE \tag{2.29}$$

and, due to the equilibrium in the finite element sense (1.15),

$$-\int_{\Omega} \sigma_h \, \underline{\text{grad}} \, \varphi_i \, dE + \int_{\Omega} \underline{f}_d \, \varphi_i \, dE = 0 \tag{2.30}$$

the compatibility condition (2.28) is always verified and System (2.27) always has an infinite number of solutions.

REMARKS

**1.** Different techniques for choosing a particular solution of (2.27) will be studied in Chapter 8.

**2.** We will show in Chapter 8 that situations with nodes located on the boundary $\partial\Omega$ can be treated in an similar way.

#### *Determination of the densities*

Therefore, we can assume that a solution of (2.27) is known for each node of the mesh being considered. Thus, if $\Gamma$ is a side of the mesh, for each node $i_1, i_2, \ldots, i_M$ of $\Gamma$ (where *M* depends on the degree and type of the elements), one knows a vector

$$\int_{\Gamma} \hat{\underline{F}} \, \varphi_{i_m} \, d\Gamma \quad \text{for } m \in \{1, 2, \ldots, M\} \tag{2.31}$$

By choosing for $\hat{\underline{F}}$ an interpolation of the same type as that used for the displacement field on $\Gamma$, one can determine $\hat{\underline{F}}$ uniquely on $\Gamma$.

#### *Equilibrium with the interior loads*

Let us consider Equations (2.24) for a particular element *E* and for all the shape functions $\varphi_i$ of this element. Any rigid body displacement field $\underline{U}_S$ on *E* can be written as a linear combination of the shape functions of the element

$$\underline{U}_S = \sum_i \underline{V}_i \, \varphi_i \tag{2.32}$$

By carrying out the same linear combination on Equations (2.24), one obtains the equilibrium conditions (2.22). This important property is a direct consequence of the extension condition.

### *Construction of an admissible field*

The objective is to find a solution of (2.23), the local problem on the element *E*.

#### *Analytical construction*

When the given interior loads $\underline{f}_d$ are polynomials, the basic idea is to look for a solution $\hat{\sigma}_{h|E}$ which is also a polynomial.

However, in the context of elasticity problems, for such a solution to exist the densities $\hat{\underline{F}}$ must verify at the vertices of the element compatibility conditions which result from the symmetry of the operator $\hat{\sigma}_{h|E}$. For example, for triangular elements in 2D, $\hat{\underline{F}}$ must verify at each vertex the conditions

$$\eta_E \, \hat{\underline{F}}_{|\Gamma} \cdot \tilde{\underline{n}}_E = \eta_E \, \hat{\underline{F}}_{|\tilde{\Gamma}} \cdot \underline{n}_E$$

with the notations described in Figure 2.2.

However, the densities $\hat{\underline{F}}$ constructed by the above method usually do not verify these compatibility conditions.

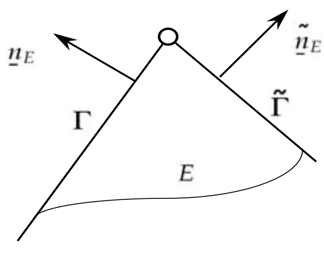


**Figure 2.2.** *Notations.*

This leads one into decomposing each triangle into three subtriangles (Figure 2.3) and seeking $\hat{\sigma}_{h|E}$ as a polynomial in each of the subtriangles.

Similar decomposition techniques are used for quadrilaterals, tetrahedra and bricks (see Chapter 8).

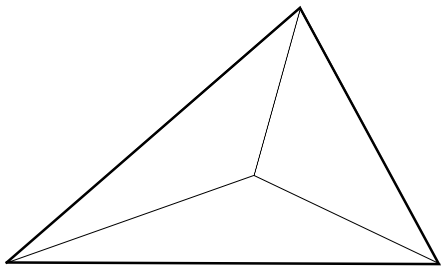

**Figure 2.3.** *Decomposition of an element.*

*Approximate numerical construction*

Once a set of interface force densities $\hat{\underline{F}}$ has been chosen, the best field $\hat{\sigma}_{h|E}$ is the solution of the problem

$$\underset{\sigma_{|E}}{\text{Min}} \left\| \sigma_{|E} - \sigma_h \right\|_{\sigma,E} \tag{2.33}$$

for $\sigma_{|E}$ such that

$$\begin{aligned} &\underline{\text{div}}\, \sigma_{|E} + \underline{f}_d = 0 \quad && \text{in } E \\ &\sigma_{|E}\, \underline{n}_E = \eta_E \hat{\underline{F}} && \text{on } \partial\text{E} \end{aligned} \tag{2.34}$$

One has

$$\left\| \sigma_{|E} - \sigma_h \right\|^2_{\sigma,E} = \left\| \sigma_{|E} \right\|^2_{\sigma,E} + \left\| \sigma_h \right\|^2_{\sigma,E} + 2\int_\Omega \text{Tr}\left[\sigma_{|E} \mathbf{K}^{-1} \sigma_h\right] d\Omega$$

and

$$\begin{aligned} \int_E \text{Tr}\left[\sigma_{|E} \mathbf{K}^{-1} \sigma_h\right] dE &= \int_E \text{Tr}\left[\sigma_{|E}\, \varepsilon(\underline{U}_h)\right] dE \\ &= \int_E \underline{f}_d \cdot \underline{U}_h \, d\Omega + \int_{\partial E} \eta_E \hat{\underline{F}} \cdot \underline{U}_h \, d\Gamma \end{aligned}$$

It follows that (2.33) is equivalent to the minimization problem

$$\underset{\sigma_{|E}}{\text{Min}} \frac{1}{2}\int_E \text{Tr}\left[\sigma_{|E}\,\mathbf{K}^{-1}\sigma_{|E}\right] dE \tag{2.35}$$

where $\sigma_{|E}$ still verifies (2.34).

By duality, this is the same as seeking a displacement field $\underline{V}_E$ defined on $E$ such that

$$\begin{aligned}&\underline{V}_E \in \boldsymbol{\mathcal{U}}(E) \quad \text{and} \quad \forall \underline{V}^* \in \boldsymbol{\mathcal{U}}(E)\\ &\int_E \text{Tr}(\,\varepsilon(\underline{V}_E)\mathbf{K}\varepsilon(\underline{V}^*)\,)\, dE = \int_E \underline{f}_d \cdot \underline{V}^* dE + \int_{\partial E} \eta_E \hat{\underline{F}} \cdot \underline{V}^* d\Gamma\end{aligned} \tag{2.36}$$

where $\boldsymbol{\mathcal{U}}(E)$ denotes the space of the restrictions to $E$ of the fields in $\boldsymbol{\mathcal{U}}$.

Then, $\sigma_{|E}$ is given by

$$\sigma_{|E} = \mathbf{K}\varepsilon(\underline{V}_E) \tag{2.37}$$

One can therefore obtain an approximation of $\sigma_{|E}$ by solving Problem (2.36) by a standard finite element method in $E$.

In practice, in order to obtain a good approximation, it is sufficient to consider a discretization of $E$ by a single element with an interpolation of degree $p+k$, where $p$ is the degree of interpolation used in the finite element analysis and $k$ is a positive integer.

Numerical experiments showed that with $k \geq 3$ one obtains a good quality approximate field.

REMARKS

**1.** Of course, the analytical construction yields strictly admissible fields only if $\underline{f}_d$ is a polynomial with a degree compatible with the polynomial form chosen for the field $\hat{\sigma}_{h|E}$. For example, for 3-node triangles, the load $\underline{f}_d$ must be constant within each element. In practice, this limitation is hardly restrictive. Nevertheless, if the load $\underline{f}_d$ is a higher-degree polynomial, one can construct a strictly admissible field $\hat{\sigma}_{h|E}$ by seeking this field in a space of higher-order polynomials. Thus, it is theoretically possible to get a very good approximation of any type of load $\underline{f}_d$ which is not a polynomial.

**2.** The technique for the approximate construction of $\hat{\sigma}_{h|E}$ works in the same manner regardless of the type of load $\underline{f}_d$. But, of course, the field thus constructed is only a good approximation of an admissible field.

**3.** This approximate construction technique can also be used with no modification for curved isoparametric elements [COOREVITS - DUMEAU - PELLE, 1999].

**4.** The use of elements of degree $p + k$, where $p$ is the degree of the finite element interpolation and $k$ is a positive integer for the approximate construction of the stress $\hat{\sigma}_{h|E}$, was proposed by BABUSKA and STROUBOULIS. In their terminology, they talk about "equilibration" of degree $k$. In particular, they showed that the error indicator associated with $k = 1$ is extremely effective.

### 2.4.4 Relationship with the errors in the solution

In the context of linear problems, it is easy to connect the error in constitutive relation with the classically used errors in the solution:

error in the solution for the displacement

$$\underline{e}_h = \underline{U}_{ex} - \underline{U}_h$$

error in the solution for the stress

$$\mathbf{K}\varepsilon(\underline{e}_h) = \sigma_{ex} - \sigma_h$$

Let $\hat{\sigma}_h$ be a statically admissible stress field. The relationship between the error in the solution and the error in constitutive relation $\hat{\sigma}_h - \mathbf{K}\varepsilon(\underline{U}_h)$ is given by the popular hypercircle theorem [PRAGER - SYNGE, 1947].

***Theorem***

$$\left[\mathbf{e}_{RdC}(\underline{U}_h, \hat{\sigma}_h)\right]^2 = \| \sigma_{ex} - \hat{\sigma}_h \|^2_{\sigma,\Omega} + \| \underline{U}_{ex} - \underline{U}_h \|^2_{u,\Omega} \tag{2.38}$$

*where*

$$\| \cdot \|^2_{u,\Omega} = \int_\Omega \mathrm{Tr}\left[\varepsilon(\cdot)\mathbf{K}\varepsilon(\cdot)\right] d\Omega$$

*Proof*

One has

$$\| \hat{\sigma}_h - \mathbf{K}\varepsilon(\underline{U}_h) \|^2_{\sigma,\Omega} = \| (\hat{\sigma}_h - \sigma_{ex}) + (\sigma_{ex} - \mathbf{K}\varepsilon(\underline{U}_h)) \|^2_{\sigma,\Omega}$$
$$= \| \hat{\sigma}_h - \sigma_{ex} \|^2_{\sigma,\Omega} + \| \underline{e}_h \|^2_{u,\Omega} + 2\int_\Omega \mathrm{Tr}\Big[ (\hat{\sigma}_h - \sigma_{ex})\varepsilon(\underline{e}_h) \Big]\, d\Omega$$

The displacement field $\underline{e}_h$, which is the difference between two *KA* fields, is a field *KA* to zero $\underline{e}_h \in \boldsymbol{\mathcal{U}}_0$. Similarly, the stress field $\hat{\sigma}_h - \sigma_{ex}$, which is the difference between two *SA* fields, is *SA* to zero. It follows that

$$\int_\Omega \mathrm{Tr}\Big[ (\hat{\sigma}_h - \sigma_{ex})\varepsilon(\underline{e}_h) \Big]\, d\Omega = 0$$

which proves Theorem (2.38).

REMARKS

**1.** The hypercircle theorem is nothing other than a form of the PYTHAGORAS theorem in the space of the stresses $\boldsymbol{\mathcal{S}}$ (Figure 2.4) endowed with the inner product

$$(\sigma_1 , \sigma_2) \mapsto \int_\Omega \mathrm{Tr}\Big[ \sigma_1 \mathbf{K}^{-1} \sigma_2 \Big]\, d\Omega$$

**2.** If one introduces as the approximate stress the average of the finite element stress and the admissible stress

$$\sigma^* = \frac{1}{2}(\hat{\sigma}_h + \sigma_h) \tag{2.39}$$

one gets

$$\mathbf{e}_{RdC} = \| \varepsilon_{RdC} \|_{\sigma,\Omega} = 2 \| \sigma_{ex} - \sigma^* \|_{\sigma,\Omega} \quad \text{and} \quad \varepsilon = \frac{\| \sigma_{ex} - \sigma^* \|_{\sigma,\Omega}}{\| \sigma^* \|_{\sigma,\Omega}} \tag{2.40}$$

Thus, the measure of the error in constitutive relation enables one to calculate the error between $\sigma_{ex}$ and $\sigma^*$ *exactly*, in the sense of the energy norm, without knowing the exact solution $\sigma_{ex}$.

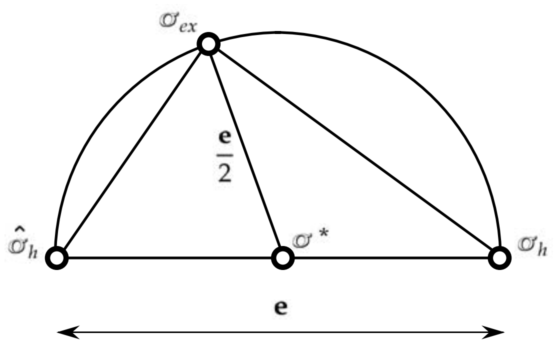


**Figure 2.4.** *The* PRAGER - SYNGE *hypercircle.*

**3.** An alternative and more mathematical presentation of the concept of error in constitutive relation starting from a mixed formulation is given in [DESTUYNDER - METIVET, 1996], [DESTUYNDER - COLLOT - SALAUN, 1997].

**4.** Several error estimators use the method of construction of admissible stress fields described in Section 2.4.3. This is the case, for example, of the variant proposed for thermal problems in [BABUSKA - STROUBOULIS - UPADHYAY - GANGARAJ - COPPS, 1994] and of the estimators given in [STEIN - OHNIMUS, 1999], [STEIN - BARTHOLD - OHNIMUS - SCHMIDT, 1998]. This is also the case of the estimators developed for electromagnetism in [DEMKOWICZ - KIM, 2000] and [REMACLE - DULAR - GENON - LEGROS, 1996].

**5.** A dose of equilibrium, i.e. of "mechanics", is introduced in most of today's error estimators – a tendency which seems to be increasing – in order to improve their effectiveness. We will clarify this point in Chapter 3.

### 2.4.5 Asymptotic behavior

The hypercircle theorem leads in particular to the following inequalities:

$$\| \sigma_{ex} - \sigma_h \|_{\sigma,\Omega} \leq \mathbf{e}_{RdC} \quad \text{and} \quad \| \underline{U}_{ex} - \underline{U}_h \|_{u,\Omega} \leq \mathbf{e}_{RdC} \tag{2.41}$$

Therefore, the measure of the error in constitutive relation is a reliable error measure; its effectivity index is always greater than 1, i.e. $\gamma \geq 1$.

Besides, we will show in Chapter 8 that there exists a constant $C$ independent of the size of the elements such that

$$\mathbf{e}_{RdC} \leq C \| \underline{U}_{ex} - \underline{U}_h \|_{u,\Omega}$$

In summary, one has

$$\| \underline{U}_{ex} - \underline{U}_h \|_{u,\Omega} \leq \mathbf{e}_{RdC} \leq C \| \underline{U}_{ex} - \underline{U}_h \|_{u,\Omega} \tag{2.42}$$

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Mastering Calculations in Linear and Nonlinear
Mechanics
Ladevèze, P.; Pelle, J.P.
2005, XI, 413 p., Hardcover
ISBN: 978-0-387-21294-4