5.9 The connection with L-functions

Each modular form $f \in \mathcal{M}_k(\Gamma_1(N))$ has an associated Dirichlet series, its *L-function.* Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$, let $s \in \mathbf{C}$ be a complex variable, and write formally

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Convergence of $L(s, f)$ in a half plane of s-values follows from estimating the Fourier coefficients of f.

Proposition 5.9.1. *If $f \in \mathcal{M}_k(\Gamma_1(N))$ is a cusp form then $L(s, f)$ converges absolutely for all s with* $\mathrm{Re}(s) > k/2 + 1$*. If f is not a cusp form then $L(s, f)$ converges absolutely for all s with* $\mathrm{Re}(s) > k$*.*

Proof. First assume f is a cusp form. Let $g(q) = \sum_{n=1}^{\infty} a_n q^n$, a holomorphic function on the unit disk $\{q : |q| < 1\}$. Then by Cauchy's formula,

$$\begin{aligned}
a_n &= \frac{1}{2\pi i} \int_{|q|=r} g(q) q^{-n} dq/q && \text{for any } r \in (0, 1) \\
&= \int_{x=0}^{1} f(x + iy) e^{-2\pi i n(x+iy)} dx && \text{for any } y > 0, \text{ where } q = e^{2\pi i(x+iy)} \\
&= e^{2\pi} \int_{x=0}^{1} f(x + i/n) e^{-2\pi i n x} dx && \text{letting } y = 1/n.
\end{aligned}$$

Since f is a cusp form, $\mathrm{Im}(\tau)^{k/2}|f(\tau)|$ is bounded on the upper half plane $\mathcal{H}$ (Exercise 5.9.1(a)), and so estimating this last integral shows that $|a_n| \le Cn^{k/2}$. The result for a cusp form f now follows since $|a_n n^{-s}| = \mathcal{O}(n^{k/2-\mathrm{Re}(s)})$.

If E is an Eisenstein series in $\mathcal{M}_k(\Gamma_1(N))$ then by direct inspection its Fourier coefficients satisfy $|a_n| \le Cn^{k-1}$ (Exercise 5.9.1(b)) and now $|a_n n^{-s}| = \mathcal{O}(n^{k-1-\mathrm{Re}(s)})$. Since any modular form is the sum of a cusp form and an Eisenstein series the rest of the proposition follows. □

The estimate $|a_n(f)| \le Cn^{k/2}$ for $f \in \mathcal{S}_k(\Gamma_1(N))$ readily extends to $\mathcal{S}_k(\Gamma(N))$ and therefore to $\mathcal{S}_k(\Gamma)$ for any congruence subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$. Similarly for the estimate $|a_n(E)| \le Cn^{k-1}$ for Eisenstein series $E \in \mathcal{M}_k(\Gamma(N))$. The upshot is that every modular form with respect to a congruence subgroup satisfies condition (3′) in Proposition 1.2.4,

(3′) In the Fourier expansion $f(\tau) = \sum_{n=0}^{\infty} a_n q_N^n$, the coefficients satisfy the condition

$$|a_n| \le Cn^r \quad \text{for some positive constants } C \text{ and } r.$$

So finally the converse to that proposition holds as well: if f is holomorphic and weight-k invariant under Γ then f is a modular form if and only if it satisfies condition (3′).

The condition of f being a normalized eigenform is equivalent to its L-function series having an *Euler product.*

Theorem 5.9.2. *Let $f \in \mathcal{M}_k(N,\chi)$, $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$. The following are equivalent:*

- *f is a normalized eigenform.*
- *$L(s,f)$ has an Euler product expansion*

$$L(s,f) = \prod_p (1 - a_p p^{-s} + \chi(p) p^{k-1-2s})^{-1},$$

  where the product is taken over all primes.

Note that the Euler product here is the Hecke operator generating function product (5.12).

Proof. By Proposition 5.8.5, the first item here is equivalent to three conditions on the coefficients a_n, so it suffices to show that those conditions are equivalent to the second item here.

Fix a prime p. Multiplying condition (2) in Proposition 5.8.5 by p^{-rs} and summing over $r \geq 2$ shows, after a little algebra, that it is equivalent to

$$\sum_{r=0}^{\infty} a_{p^r} p^{-rs} \cdot (1 - a_p p^{-s} + \chi(p) p^{k-1-2s}) = a_1 + (1 - a_1) p^{-s}. \tag{5.23}$$

If also condition (1) in Proposition 5.8.5 holds then this becomes

$$\sum_{r=0}^{\infty} a_{p^r} p^{-rs} \cdot (1 - a_p p^{-s} + \chi(p) p^{k-1-2s}) = 1. \tag{5.24}$$

Conversely, suppose (5.24) holds. Letting $s \to +\infty$ shows $a_1 = 1$ so condition (1) in Proposition 5.8.5 holds, and so does (5.23), implying condition (2) in Proposition 5.8.5. So conditions (1) and (2) in Proposition 5.8.5 are equivalent to

$$\sum_{r=0}^{\infty} a_{p^r} p^{-rs} = (1 - a_p p^{-s} + \chi(p) p^{k-1-2s})^{-1} \quad \text{for } p \text{ prime}. \tag{5.25}$$

Before continuing, note that the Fundamental Theorem of Arithmetic (positive integers factor uniquely into prime powers) implies that for a function g of prime powers (Exercise 5.9.2),

$$\prod_p \sum_{r=0}^{\infty} g(p^r) = \sum_{n=1}^{\infty} \prod_{p^r \| n} g(p^r). \tag{5.26}$$

The notation $p^r \| n$ means that p^r is the highest power of p that divides n, and we are assuming that g is small enough to justify formal rearrangements.

Now, if (5.25) holds along with condition (3) of Proposition 5.8.5 then compute

$$\begin{aligned}
L(s,f) &= \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} \left(\prod_{p^r \| n} a_{p^r} \right) n^{-s} && \text{by the third condition} \\
&= \sum_{n=1}^{\infty} \prod_{p^r \| n} a_{p^r} p^{-rs} = \prod_p \sum_{r=0}^{\infty} a_{p^r} p^{-rs} && \text{by (5.26)} \\
&= \prod_p (1 - a_p p^{-s} + \chi(p) p^{1-k-2s})^{-1} && \text{by (5.25),}
\end{aligned}$$

giving the Euler product expansion.

Conversely, given the Euler product expansion, compute (using the geometric series formula and (5.26))

$$\begin{aligned}
L(s,f) &= \prod_p (1 - a_p p^{-s} + \chi(p) p^{1-k-2s})^{-1} \\
&= \prod_p \sum_{r=0}^{\infty} b_{p,r} p^{-rs} \quad \text{for some } \{b_{p,r}\} \\
&= \sum_{n=1}^{\infty} \prod_{p^r \| n} b_{p,r} p^{-rs} = \sum_{n=1}^{\infty} \left(\prod_{p^r \| n} b_{p,r} \right) n^{-s}.
\end{aligned}$$

So $a_n = \prod_{p^r \| n} b_{p,r}$, giving condition (3) of Proposition 5.8.5 and showing in particular that $b_{p,r} = a_{p^r}$. This in turn implies (5.25), implying conditions (1) and (2) of Proposition 5.8.5. □

As an example, the L-function of the Eisenstein series $E_k^{\psi,\varphi}/2$ works out to (Exercise 5.9.3)

$$L(s, E_k^{\psi,\varphi}/2) = L(s,\psi) L(s-k+1, \varphi) \tag{5.27}$$

where the L-functions on the right side are as defined in Chapter 4. For another example see Exercise 5.9.4.

Let N be a positive integer and let A be the ring $\mathbf{Z}[\mu_3]$. For any character $\chi : (A/NA)^* \longrightarrow \mathbf{C}^*$, Section 4.11 constructed a modular form $\theta_\chi \in \mathcal{M}_1(3N^2, \psi)$ where $\psi(d) = \chi(d)(d/3)$. Recall that χ needs to be trivial on A^* for θ_χ to be nonzero, so assume this. The arithmetic of A and Theorem 5.9.2 show that θ_χ is a normalized eigenform. The relevant facts about A were invoked in the proof of Corollary 3.7.2 and in Section 4.11. To reiterate, A is a principal ideal domain. For each prime $p \equiv 1 \pmod 3$ there exists an element $\pi_p \in A$ such that $\pi_p \bar{\pi}_p = p$, but there is no such element if $p \equiv 2 \pmod 3$. The maximal ideals of A are

- for each prime $p \equiv 1 \pmod 3$, the two ideals $\langle \pi_p \rangle$ and $\langle \overline{\pi}_p \rangle$,
- for each prime $p \equiv 2 \pmod 3$, the ideal $\langle p \rangle$,
- for $p = 3$, the ideal $\langle \sqrt{-3} \rangle$.

Let $\pi_p = p$ for each prime $p \equiv 2 \pmod 3$, let $\pi_3 = \sqrt{-3}$, and take the set of generators of the maximal ideals,

$$\mathcal{S} = \{\pi_p, \overline{\pi}_p : p \equiv 1 \pmod 3\} \cup \{\pi_p : p \equiv 2 \pmod 3\} \cup \{\pi_3\}.$$

Then each nonzero $n \in A$ can be written uniquely as

$$n = u \prod_{\pi \in \mathcal{S}} \pi^{a_\pi}, \quad u \in A^*, \text{ each } a_\pi \in \mathbf{N},\ a_\pi = 0 \text{ for all but finitely many } \pi.$$

Correspondingly $\chi(n) = \prod_{\pi \in \mathcal{S}} \chi(\pi)^{a_\pi}$. The Fourier coefficients of θ_χ were given in (4.50),

$$a_m(\theta_\chi) = \tfrac{1}{6} \sum_{\substack{n \in A \\ |n|^2 = m}} \chi(n).$$

Compute that therefore

$$L(s, \theta_\chi) = \tfrac{1}{6} \sum_{\substack{n \in A \\ n \neq 0}} \chi(n)|n|^{-2s} = \prod_{\pi \in \mathcal{S}} (1 - \chi(\pi)|\pi|^{-2s})^{-1} = \prod_p L_p(s, \theta_\chi),$$

where (Exercise 5.9.5)

$$L_p(s, \theta_\chi)^{-1} = \begin{cases} 1 - (\chi(\pi_p) + \chi(\bar{\pi}_p))p^{-s} + \chi(p)p^{-2s} & \text{if } p \equiv 1 \pmod 3, \\ 1 - \chi(p)p^{-2s} & \text{if } p \equiv 2 \pmod 3, \\ 1 - \chi(\sqrt{-3})3^{-s} & \text{if } p = 3. \end{cases} \tag{5.28}$$

Since $L_p(s, \theta_\chi) = (1 - a_p(\theta_\chi)p^{-s} + \psi(p)p^{-2s})^{-1}$ in all cases, Theorem 5.9.2 shows that θ_χ is a normalized eigenform.

Exercises

5.9.1. (a) For any cusp form $f \in \mathcal{S}_k(\Gamma_1(N))$ show that the function $\varphi(\tau) = \mathrm{Im}(\tau)^{k/2}|f(\tau)|$ is bounded on the upper half plane $\mathcal{H}$. (A hint for this exercise is at the end of the book.)

(b) Establish the relation $1 \leq \sigma_{k-1}(n)/n^{k-1} < \zeta(k-1)$ where ζ is the Riemann zeta function. Show that the Fourier coefficients a_n of any Eisenstein series satisfy $|a_n| \leq Cn^{k-1}$.

5.9.2. Prove formula (5.26). (A hint for this exercise is at the end of the book.)

5.9.3. Prove formula (5.27). What is a half plane of convergence?

5.9.4. Recall the functions f, f_1, f_2, and f_3 from Exercise 5.8.3. The exercise showed that the 4-dimensional space spanned by these functions contains only three normalized eigenforms. How do the L-functions of the three eigenforms relate to $L(s, f)$?

5.9.5. Establish formula (5.28).

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