

8

Reproductive Value from the Life Table

When a woman of reproductive age is sterilized and so has no further children, the community's subsequent births are reduced. When a woman dies or otherwise leaves the community, all subsequent times are again affected. Our formal argument need make no distinction between emigration and death, between leaving the country under study for life and leaving this world altogether. A single theory answers questions about the numerical effect of sterilization, of mortality, and of emigration, all supposed to be taking place at a particular age x . By means of the theory we will be able to compare the demographic results of eradicating a disease that affects the death rate at young ages, say malaria, as against another that affects the death rate at older ages, say heart disease.

A seemingly different question is what would happen to a rapidly increasing population if its couples reduced their childbearing to bare replacement immediately. The period net reproduction rate R_0 , the number of girl children expected to be born to a girl child just born, would equal 1 from then on, and ultimately the population would be stationary. But the history of high fertility has built up an age distribution favorable to childbearing, and the ultimate stationary total will be much higher than the total at the time when the birth rate dropped to bare replacement. The amount by which it will be higher is calculable, and by the same function—reproductive value—that is used for problems of migration and changed mortality.

8.1 Concept of Reproductive Value

Without having these particular problems in mind, Fisher (1930, p. 27) developed a fanciful image of population dynamics that turns out to provide solutions to them. He regarded the birth of a child as the lending to him of a life, and the birth of that child's offspring as the subsequent repayment of the debt. Apply this to the female part of the population, in which the chance of a girl living to age a is $l(a)$, and the chance of her having a girl between ages a and $a + da$ is $m(a) da$, so that the expected number of children in the small interval of age specified is $l(a)m(a) da$. This quantity added through the whole of life is what was defined as the net reproduction rate R_0 in Section 6.1:

$$R_0 = \int_{\alpha}^{\beta} l(a)m(a) da,$$

where α is the youngest age of childbearing and β the oldest. The quantity R_0 is the expected number of girl children by which a girl child will be replaced; for the population it is the ratio of the number in one generation to the number in the preceding generation, according to the given $l(a)$ and $m(a)$ (see Chapter 9 for the generalization to stage-classified models).

Fisher's image discounts the future, at a rate of interest equal to the intrinsic rate r of Section 6.1. The value of 1 dollar, or one child, discounted back through a years at annual rate r compounded momentarily is e^{-ra} ; therefore the value of $l(a)m(a) da$ children is $e^{-ra}l(a)m(a) da$, as in the financial calculations of Section 2.5. The present value of the repayment of the debt is the integral of this last quantity through the ages to the end of reproduction. Thus the debt that the girl incurs at birth is 1, and the discounted repayment is the integral $\int_{\alpha}^{\beta} e^{-ra}l(a)m(a) da$. If loan and discounted repayment are to be equal, we must have

$$1 = \int_{\alpha}^{\beta} e^{-ra}l(a)m(a) da,$$

and this is the same as the characteristic equation (Lotka 1939, p. 65, and (6.1.2)), from which the r implied by a net maternity function $l(a)m(a)$ is calculated. The equation can now be seen in a new light: the equating of loan and discounted repayment is what determines r , r being interpretable either as the rate of interest on a loan or as Lotka's intrinsic rate of natural increase.

The loan-and-repayment interpretation of the characteristic equation suggests calculating how much of the debt is outstanding by the time the girl has reached age $x < \beta$. This is the same as the expected number of subsequent children discounted back to age x . Her expected births in the interval a to $a + da$, $a > x$, are $[l(a)/l(x)]m(a)$; and if these births are

discounted back $a - x$ years, her debt outstanding at age x is

$$v(x) = \int_x^\beta e^{-r(a-x)} \frac{l(a)}{l(x)} m(a) da$$

or, as Fisher (1930) wrote,

$$v(x) = \frac{1}{e^{-rx}l(x)} \int_x^\beta e^{-ra}l(a)m(a) da, \quad (8.1.1)$$

where $v(x)$ will be called reproductive value at age x . Evidently $v(0) = 1$, and, for $x > \beta$, $v(x) = 0$.

For his studies in genetics Fisher needed to know the extent to which persons of given age (say x), on the average contribute to the births of future generations. This seemingly different question is answered by a function proportional to $v(x)$; its value can be established at $v(x)/\kappa$, where, as in Section 6.1,

$$\kappa = \int_\alpha^\beta ae^{-ra}l(a)m(a) da; \quad (8.1.2)$$

that is, κ is the mean age of childbearing in the stable population. The basic proposition is that the addition of a girl or woman aged x to the population at time zero adds an expected $v(x)e^{rt}/\kappa$ baby girls at time t , always supposing the continuance of the same regime of fertility and mortality. The simplest derivation of this takes off from the real term of solution 7.5.2 to the Lotka renewal equation. A self-contained version is provided in Section 8.9.

8.1.1 Reproductive Value from the Lotka Integral Equation

One Woman Aged x . The continuous model of Section 7.5 provides the curve of descendants of an arbitrary initial age distribution, and its asymptotic trajectory is the real term $Q_1e^{r_1t}$ of (7.5.2), the value of Q_1 being given by (7.5.4). For a distribution consisting of one woman aged x , disregarding questions of continuity and of random variation, we find that the children expected at time t to $t + dt$ are $[l(x+t)/l(x)]m(x+t) dt$, which is therefore the function $G(t)$. Entering it in (7.5.4), that is, in $Q_1 = \int_0^\beta e^{-rt}G(t) dt/\kappa$, we have

$$Q_1 = \frac{\int_0^{\beta-x} e^{-rt}[l(x+t)/l(x)]m(x+t) dt}{\kappa}, \quad (8.1.3)$$

which except for the divisor κ is identical to $v(x)$ of (8.1.1), giving the discounted value of the expected future births to a woman aged x . In the special case of a baby just born, $x = 0$; and, by virtue of (8.1.3) and the characteristic equation (6.1.2), $Q_1 = 1/\kappa$.

Stable Age Distribution. The same constant Q_1 can be readily evaluated for a population of unity having the stable age distribution $be^{-ra}l(a)$. We can guess in advance from the nature of stability that the asymptotic population will be e^{rt} and the births be^{rt} , so Q_1 must equal b for this case.

The proof seems simplest if we start by calculating the total expected reproductive value of a stable population:

$$\int_0^\beta be^{-rx}l(x)v(x) dx,$$

then cancel the $e^{-rx}l(x)$ with the denominator of $v(x)$ of (8.1.1) to find

$$b \int_0^\beta \int_x^\beta e^{-ra}l(a)m(a) da dx,$$

and finally integrate by parts to obtain

$$b \int_0^\beta ae^{-ra}l(a)m(a) da = b\kappa$$

as the reproductive value of a population of unity having a stable age distribution. The constant Q_1 is this total reproductive value divided by κ , that is, $b\kappa/\kappa = b$, as suggested by intuition.

Arbitrary Age Distribution. A more general statement can be made. Let $p(x)$ be the age distribution as a density function; i.e., so that the number of individuals between ages a and $a+5$, say, is ${}_5N_a = \int_a^{a+5} p(x)dx$. Whatever the initial age distribution $p(x)$ of a closed population acted on by fixed rates of birth and death, its births have an asymptotic trajectory $Q_1e^{r_1t}$ where Q_1 , defined by (7.5.4), is equal to $\int_0^\beta p(x)v(x)dx/\kappa$, that is, the sum of reproductive value in the population divided by the mean age of childbearing.

To see this, note that the total reproductive value of $p(x)$ is

$$V = \int_0^\beta p(x)v(x) dx = \int_0^\beta p(x) \int_0^{\beta-x} e^{-rt} \frac{l(x+t)}{l(x)} m(x+t) dt dx. \quad (8.1.4)$$

But this is the same as the numerator of Q_1 in (7.5.4), where $G(t)$ is the number of children expected to be generated by the initial population $p(x)$. For the number of those children born at time t will be

$$G(t) = \int_0^\beta p(x) \frac{l(x+t)}{l(x)} m(x+t) dx, \quad (8.1.5)$$

and multiplying by e^{-rt} and then integrating over t gives the double integral in (8.1.4). This demonstrates that $Q_1 = V/\kappa$, where V is the number of women, each weighted by the $v(x)$ for her age x . Once again, $1/\kappa$ of the present value of the balance outstanding by age x in the hypothetical loan is equal to the contribution of a woman aged x to the ultimate trajectory,

and both the loan and the trajectory are additive for a group of women of arbitrary ages.

The foregoing proof depends on the solution of the integral equation. The proof in Section 8.9, on the other hand, stands on its own feet. The same result can be derived using matrix formulations, without assuming age-classification (Chapter 9).

Once we know the effect on the birth trajectory of adding one girl and assume a fixed birth rate b , we can obtain the effect on the population trajectory by dividing by b . This is obvious, for since the birth rate b is B/N , births divided by population, the population must be $N = B/b$, births divided by the birth rate. Hence the effect of adding a girl or woman aged x is to add $v(x)e^{rt}/\kappa$ to ultimate births and $v(x)e^{rt}/\kappa b$ to ultimate population.

To obtain some intuitive feeling for the reason why the effect of one child just born on the ultimate birth trajectory is to raise it by $v_0e^{rt}/\kappa = e^{rt}/\kappa$, rather than just e^{rt} or some other value, suppose that all children are born at the same maternal age and that this age is κ . Then the birth of an additional girl child now will result in R_0 girl children in κ years, R_0^2 children in 2κ years, and R_0^n in the n th generation, where R_0 is, as before, the net reproduction rate; that is to say, a child born now outlines a birth curve (Fig. 8.1) rising in the ratio of R_0 every κ years, but with births occurring only at κ -year intervals. In other words, the curve outlined gives the number of births per κ years resulting from one birth at the outset; it is reduced to births per year by dividing by $\kappa : e^{rt}/\kappa$. This argument is at best heuristic; the result applies much more generally than to the primitive model in which all births occur at the same maternal age.

8.1.2 Numerical Calculation

The expression for $v(x)$ in (8.1.1) applies to exact age x , and an approximation analogous to that customarily made for the stable age distribution is

$$v_x \approx \frac{e^{-2\frac{1}{2}r} {}_5L_x F_x + e^{-7\frac{1}{2}r} {}_5L_{x+5} F_{x+5} + \dots}{l_x}. \quad (8.1.6)$$

This, with numerator and denominator multiplied by e^{-rx} , is shown in Table 8.1 for Mauritius females, 1966. Figure 8.2 shows the curves of $v(x)$ for Mauritius, the United States, and Hungary, taken from Keyfitz and Flieger (1971, pp. 315, 361, and 443).

For the average reproductive value for the age interval x to $x+4$ at last birthday the recurrence formula

$${}_5V_x = \frac{{}_5}{2}F_x + \frac{e^{-5r} {}_5L_{x+5}}{{}_5L_x} \left(\frac{{}_5}{2}F_{x+5} + {}_5V_{x+5} \right) \quad (8.1.7)$$

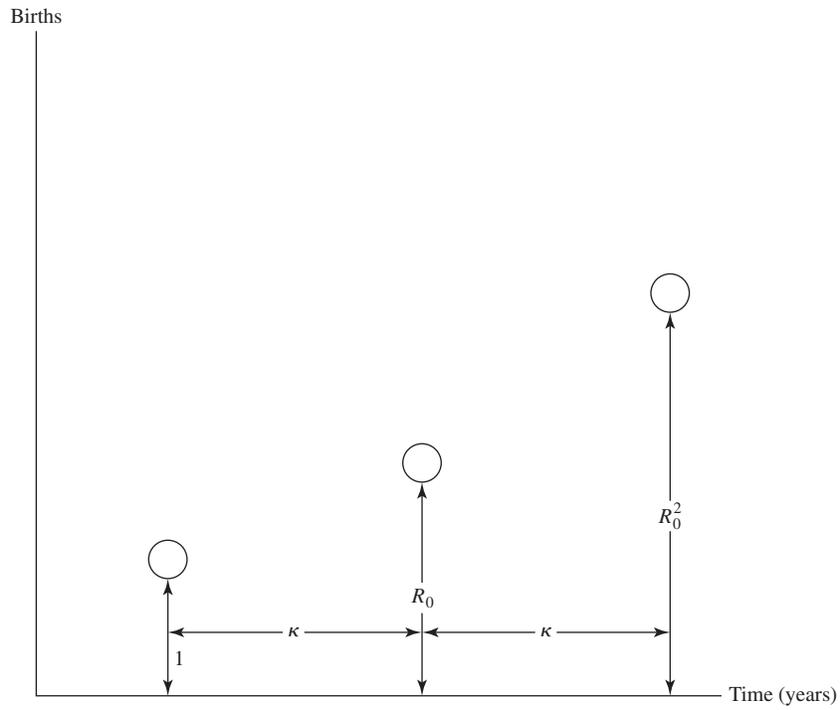


Figure 8.1. Effect of one birth if all children are born at age κ of mother.

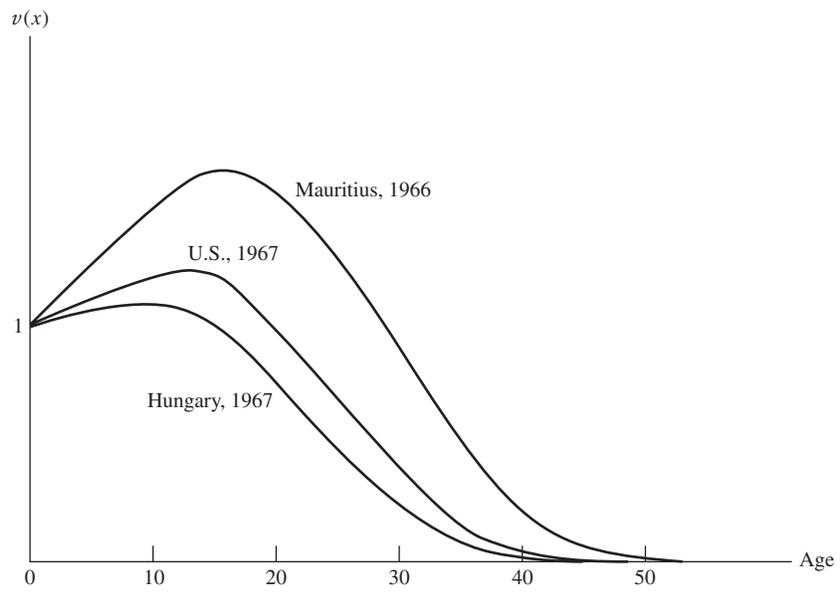


Figure 8.2. Curves of reproductive value for females of three countries.

Table 8.1. Calculation of reproductive values for females of exact ages 0, 5, 10, . . . , 50, Mauritius, 1966; $r = 0.0305371$

Age	${}_5L_xF_x$	$e^{-(x+2\frac{1}{2})r}$	$\sum_{y=x}^{\beta-5} e^{-(y+2\frac{1}{2})r} {}_5L_yF_y$	$e^{-rx}l_x$	v_x
x	(1)	(2)	(3)	(4)	(5) = (3)/(4)
0		0.92650	1	1	1
5		0.79531	1	0.78463	1.2745
10	0.0014	0.68269	1	0.66822	1.4965
15	0.1858	0.58602	0.99907	0.57152	1.7481
20	0.6236	0.50304	0.89019	0.48700	1.8279
25	0.6061	0.43181	0.57649	0.41212	1.3988
30	0.4730	0.37067	0.31477	0.34841	0.9034
35	0.3239	0.31818	0.13944	0.29392	0.4744
40	0.1201	0.27312	0.03639	0.24719	0.1472
45	0.0146	0.23445	0.00358	0.20816	0.0172
50	0.0008	0.20125	0.00016	0.17341	0.0009

Source: Net maternity function from Keyfitz and Flieger (1971, p. 315).

Table 8.2. Values of ${}_5V_x$, the Fisher reproductive value of females aged x to $x + 4$ at last birthday, and ${}_5V_x/(b\kappa)$, the coefficient of the amount by which population at time t is raised by one added person aged x to $x + 4$ at time zero, Mauritius, 1966

Age	${}_5V_x$	${}_5V_x/(b\kappa)$
0-4	1.159	1.092
5-9	1.381	1.301
10-14	1.618	1.524
15-19	1.783	1.679
20-24	1.611	1.517
25-29	1.151	1.084
30-34	0.690	0.650
35-39	0.312	0.294
40-44	0.083	0.078
45-49	0.009	0.008
$v_0 = 1$		
$v_0/(b\kappa) = \frac{1}{(0.03889)(27.30)} = 0.942$		

Source: Keyfitz and Flieger (1971), p. 315.

provides a reasonable approximation. However, the ${}_5V_x$ of Table 8.2 was calculated, not in this way, but by the method (easier if a computer is available) of finding the left eigenvector of the projection matrix in Sections 7.1 and 7.5; see Section 9.1 for the connection between reproductive value and eigenvectors.

Evidently total reproductive value of a population of arbitrary age distribution acted on by a fixed regime increases at rate r in the short as

well as the long run. Such a statement is conspicuously not true for the size of the total population, whose increase in the short run depends on its age distribution. Both births and population acted on by a fixed regime *ultimately* go into an exponential trajectory with parameter r ; the total of reproductive values *immediately* follows an exponential trajectory (Fisher 1930, p. 30).

The above like other pieces of theory in this book can be justified only by its ability to answer demographic questions. The following section deals with the first of a series of such questions.

8.2 Ultimate Effect of Small Out-Migration Occurring in a Given Year

When people leave a crowded island like Barbados or Java, they make life somewhat easier for those who remain behind, assuming that the rates of mortality and fertility do not change as a result of their departure.

The age at which they leave determines the effect. Departures of persons who are already past the ages of reproduction cannot influence the ultimate population trajectory; the effect of their leaving is only the subtraction of the person-years they themselves will live from the time of departure to death.

A one-time departure of a person of reproductive age or below will lower the expected population trajectory, but cannot change its rate of climb as long as the age-specific rates of birth and death remain unchanged. In symbols, if the ultimate trajectory is Qe^{rt} , a one-time departure of an individual or a group under age β can lower Q but will not alter r . It follows from the theory of Section 8.1 that a female of age x leaving reduces the female births at time t by $v(x)e^{rt}/\kappa$ and the female population at time t by $v(x)e^{rt}/(b\kappa)$, where we take t to be large. Thus the change in Q for population due to the departure of one female aged x is $\Delta Q = -v(x)/(b\kappa)$.

We are still on the one-sex model and suppose female dominance, that is, that births are determined by the number of females at the several ages and not by the number of males. This would be true if males were in the majority or polygyny prevailed or artificial insemination were applied. The extension of the ideas of the present chapter to a genuine two-sex model depends on behavioral variables not readily incorporable in demographic theory.

The effect of a one-time bulge in births follows readily. With ΔB extra births in a given year the birth trajectory would be raised $e^{rt} \Delta B/\kappa$, and the population trajectory would be raised this amount divided by the birth rate b .

Does a female of random age affect the ultimate population more or less than a girl baby? The former, entering at time zero, raises the population

at time t an expected e^{rt} , while the latter raises it by $e^{rt}/(b\kappa)$. The mean age of childbearing κ is never very far from 27, and the reciprocal of 27 is 0.037. For low-fertility populations b is considerably less than $1/\kappa \approx 0.037$; hence a baby has more effect than a female between zero and ω randomly chosen from the stable age distribution. For high-fertility populations, on the other hand, b is greater than $1/\kappa$ and a baby has less expected effect than a randomly selected female. Thus for Mexico the departure of a woman of random age has more effect than averting one birth; for the United States averting a birth has more effect.

The same technique can be used to find a variety of equivalents. By what amount, for example, would births have to drop in a particular year to offset an immigration of 1000 women aged 15 to 19 in the same year? The population at distant time t resulting from 1000 women aged 15 to 19 is $1000 {}_5V_{15} e^{rt}/(b\kappa)$. The population from B births at time t is $Bv_0 e^{rt}/(b\kappa)$. Equating these two expressions, we obtain

$$B = \frac{1000 {}_5V_{15}}{v_0}$$

as the required equivalent number of births. From the Mauritius information in Table 8.2 we have, since ${}_5V_x$ is normed to $v_0 = 1$,

$$B = 1000 {}_5V_{15} = 1783.$$

In any one year (or other period) a drop of 1783 female births would be required to offset the immigration of 1000 women aged 15 to 19 at last birthday.

8.3 Effect of Continuing Birth Control and Sterilization

Suppose that a few women each year resort to birth control when they are of age a , and this occurs year after year, so that the birth rate $m(a)$ is permanently lowered for age a , but all other age-specific birth rates remain unaltered. If the change in the age-specific birth rate in the single year of age a is $\Delta m(a)$, a quantity that will carry a minus sign for decrease in $m(a)$, the change in the intrinsic rate of the population is determined by finding the derivative $dr/dm(a)$ in the characteristic equation $\int_{\alpha}^{\beta} e^{-rx} l(x) m(x) dx = 1$ as in Section 6.3, and for finite increments Δr and $\Delta m(a)$ is approximately

$$\Delta r \approx - \frac{e^{-ra} l(a) \Delta m(a)}{\kappa}, \quad (8.3.1)$$

the same as (6.3.8). The result depends on $\Delta m(a)$ being small enough so that e^{-ra} , as well as κ , is substantially unaffected. Subject to this same condition, we can find the combined effect of small increments at two different ages, say a and $a + 1$. The effect on r will be approximately the sum

of the Δr for $\Delta m(a)$ and that for $\Delta m(a+1)$, and similarly for any other groups of ages. This type of perturbation analysis of the rate of increase is expanded on in Chapter 13.

Now suppose a permanent change in $m(a)$ for all a from age x onward, so that the new birth function is $m(a)$, $a < x$, and $(1-f)m(a)$, $a \geq x$, f being a small positive or negative fraction. This could be the result of sterilization becoming the custom at age x , or of the fraction f of women at age x turning to conventional birth control in order to avoid all further children. If f is small we can enter $-fm(a)$ for $\Delta m(a)$ in the preceding display, and find the total effect Δr by adding the Δr 's for the several ages:

$$\Delta r = -\frac{f \int_x^\beta e^{-ra} l(a) m(a) da}{\kappa}. \quad (8.3.2)$$

The integral here will be recognized as the same one that turned up in $v(x)$ of (8.1.1). Entering $v(x)$ makes this

$$\Delta r = -\frac{f e^{-rx} l(x) v(x)}{\kappa}.$$

In words, the decrease by the fraction f of fertility rates for all ages above x lowers the intrinsic rate by $v(x)$ multiplied by $f e^{-rx} l(x)/\kappa$. Remembering that $b e^{-rx} l(x)$ is the fraction of the population at age x , where at this point it is convenient to make x integral and have it represent exact ages $x - \frac{1}{2}$ to $x + \frac{1}{2}$, we can say that the decrease in r is $f/(b\kappa)$ times the fraction of the population aged x , times the reproductive value at age x . More simply, the integral in (8.3.2) is the fraction of current mothers aged x and over, so (8.3.2) tells us that the effect on r is equal to the fraction f dropping out of childbearing, times the fraction of babies born to women aged x and over, divided by the mean age of childbearing. Designating by b_x the fraction of births occurring to mothers aged x and over, (8.3.2) can be written

$$\Delta r = -\frac{f b_x}{\kappa}.$$

Conventional birth control, sterilization, or mortality, if they take place year after year can lower births to women over age x by a small fraction f , and if they do the rate of increase r is reduced by f times the fraction b_x of children born to women aged x and older, divided by the mean age of childbearing.

The preceding discussion also covers the consequences of a fall in the death rate. Suppose that the rate at ages $x - \frac{1}{2}$ to $x + \frac{1}{2}$ goes from $\mu(x)$ to $\mu(x) + \Delta\mu(x)$ and remains at that level, or (what is practically the same) that $\Delta\mu(x)/\delta$ is permanently added to the density $\mu(x)$ over a narrow age interval δ . Then all the results of this section apply. The derivation first finds the effect of $\Delta\mu(x)/\delta$ on $l(x)$, using the approximate formula

$e^{-\Delta\mu(x)/\delta} \approx 1 - \Delta\mu(x)/\delta$. Thereafter the derivation is the same as for (8.3.1), since $l(a)$ and $m(a)$ enter symmetrically into the characteristic equation.

8.4 Large Change in Regime

So far only small changes have been discussed. We now ask the same question in reference to an arbitrary, possibly large, change: if birth control is applied by women aged x and above, what fraction of births must they avoid in order to change the rate of increase from r to $r + \Delta r$?

Suppose that in every cohort women aged x and higher apply birth control to the point where they reduce their age-specific rates by the fraction f of what they were before; sterilization of f of the women reaching age x would have this effect. The original intrinsic rate of increase was found by solving for r in the characteristic equation. The equation for the new rate of increase $r + \Delta r$ breaks down into two parts:

$$\int_{\alpha}^{\beta} \exp[-(r + \Delta r)a] l(a)m(a) da - f \int_x^{\beta} \exp[-(r + \Delta r)a] l(a)m(a) da = 1, \quad (8.4.1)$$

where we suppose $a \leq x \leq \beta$. Equation 8.4.1 could be solved for x if f and $r + \Delta r$ were given, or for $r + \Delta r$ if x and f were given. A simple explicit solution is available for f , the fraction of decrease above the given age x that will suffice to change the intrinsic rate from r to $r + \Delta r$:

$$f = \frac{\int_{\alpha}^{\beta} \exp[-(r + \Delta r)a] l(a)m(a) da - 1}{\int_x^{\beta} \exp[-(r + \Delta r)a] l(a)m(a) da}. \quad (8.4.2)$$

Result 8.4.2 depends in no way on Δr being small. [Find its limiting value when Δr is small.]

The numerator of (8.4.2) is bound to be positive for $\Delta r < 0$, corresponding to the birth control formulation in which f is defined as positive and birth rates go from $m(a)$ to $(1 - f)m(a)$. In the special case where the desired $r + \Delta r = 0$ we would have the simpler form

$$f = \frac{\int_{\alpha}^{\beta} l(a)m(a) da - 1}{\int_x^{\beta} l(a)m(a) da} = \frac{R_0 - 1}{\int_x^{\beta} l(a)m(a) da}. \quad (8.4.3)$$

The f of (8.4.3) is the fraction by which women aged x and over must reduce fertility to bring the rate of population increase r down to zero. The

age x is arbitrary but is required to stay within certain limits if $0 < f < 1$. For data for Colombia, 1965, one observes that no reduction of fertility in women 30 and over could bring stationarity if ages under 30 retained existing rates, for we have $R_0 = 2.267$ and $\int_{30}^{50} l(a)m(a) da = 1.001$, and hence a drop to $R_0 = 1$ would not occur even if all fertility above age 30 disappeared.*

One would have thought that a girl child would contribute the same amount to the ultimate trajectory irrespective of the age of her mother; all babies start at age zero, after all. The expression $\Delta r = e^{-ra}l(a)\Delta m(a)/\kappa$ in (8.3.1) is consistent with this view, for it says that the effect of a small change $\Delta m(a)$ in the age-specific birth rate is proportional to $e^{-ra}l(a)$, that is, proportional to the number of women of that age in the stable population; this has to be right, in that a given change in the birth rate will alter the number of babies in proportion to the number of women to whom the change is applied. The expression for Δr in (8.3.1) supposes that $\Delta m(a)$ is small enough not to affect κ , the mean age of childbearing.

But for the ultimate effect of a large change that takes place generation after generation, it does make a difference whether women are young or old when they have their children. Avoiding births at age 40 is not as effective as avoiding them at age 20, because of the more rapid turnover of a population in which births occur to younger mothers. This is taken into account in (8.4.2) and (8.4.3).

8.5 Emigration as a Policy Applied Year After Year

Each year some inhabitants of Java go to Sumatra under an official transmigration program that has been government policy for two-thirds of a century. The authorities have always recognized that the amount of relief provided to Java depends on the age of the migrants at the time of their out-migration, and that young couples are the ideal ones to go, but they have tended to exaggerate the effect. Widjojo (1970) shows realistic population projections under alternative assumptions about the rate of movement, from which the consequences of different policies can be seen.

*The net reproductive rate has come to play a central role in modelling epidemic diseases, treated as a problem in pathogen demography. In this context, R_0 is the expected number of secondary cases caused by a single infected individual over its entire infectious period. Whether R_0 is greater or less than 1, when the population consists entirely of susceptible hosts, determines whether the disease will spread or die out. Calculations essentially identical to those used here to determine the amount by which fertility must be reduced in order to stop population growth are used to calculate the level of vaccination that must be imposed to stop the spread of a disease. See Diekmann et al. (1990), Anderson and May (1991), and Diekmann and Heesterbeek (2000).

In this section we will examine one aspect of policy only: the effect of the age of the migrants on the ultimate rate of increase of the population.

We can express (8.4.2) in terms of a generalization of reproductive value. In this general reproductive value, say $v_{x,\bar{r}}$, future children are discounted, not at the intrinsic rate r of the observed population, but at the rate \bar{r} at which the emigration policy is to aim:

$$v_{x,\bar{r}} = \int_0^{\beta-x} e^{-\bar{r}t} \frac{l(x+t)}{l(x)} m(x+t) dt.$$

Then the alternative form of (8.4.2) is

$$f_x = \frac{v_{0,\bar{r}} - 1}{e^{-\bar{r}x} l_x v_{x,\bar{r}}}. \quad (8.5.1)$$

The argument of this section pivots on the simple result 8.5.1. If $\bar{r} = 0$, we obtain the fraction f_x emigrating out of each cohort for stationarity. In general, (8.5.1) serves to show how much emigration is required to attain the demographic objective represented by a rate of increase \bar{r} , given the continuance of the life table $l(a)$ and the birth rates $m(a)$.

To apply (8.5.1) we need only the net maternity function $l(a)m(a)$. For Mauritius, 1966, this is given in Table 6.1 in 5-year age intervals. The intrinsic rate of Mauritius is estimated at 30.54 per thousand. How much emigration will be required for the modest goal of bringing it down to 20 per thousand? If the emigrants are $x = 25$ years of age, (8.5.1) tells us that with $\bar{r} = 0.020$ a fraction $f_{25} = 0.417$ of each cohort must leave on reaching this age. If the emigrants are $x = 20$ years of age, the proportion that will have to leave is smaller, 0.279.

Thus, to bring about a drop from the actual increase of 30.54 per thousand to one of 20.00 per thousand, the departure of 41.7 percent of each cohort will be required if the emigrants leave at age 25, and of 27.9 percent if they leave at age 20. Emigration is not the easiest means of population control.

To find the amount of emigration that will hold the ultimate rate of increase down to zero we need the value of f_x when \bar{r} is replaced by zero in (8.5.1). The integral in the numerator is then R_0 , the net reproduction rate, and the integral in the denominator is the part of R_0 beyond age x . Hence we have again (8.4.3),

$$f_x = \frac{R_0 - 1}{\int_x^\beta l(a)m(a) da} \quad (8.5.2)$$

as the fraction of the age x that must emigrate per year to hold the ultimate population stationary, x again being low enough for f_x not to exceed unity. To see (8.5.2) independently of its derivation as a special case of (8.5.1) we note that to bring the net reproduction rate down to 1 we need to lose $R_0 - 1$ births per woman from each birth cohort. The number of births per

woman lost by removing a proportion f of women at age x is

$$f_x \int_x^\beta l(a)m(a) da.$$

Equating this to $R_0 - 1$ yields (8.5.2).

8.6 The Momentum of Population Growth

The authorities of some underdeveloped countries fear that once birth control is introduced their populations will immediately stop increasing. Such fears are misplaced, partly because diffusion takes time, and even when birth control is available it is not immediately used. But let us leave aside this behavioral aspect, and consider only the momentum of population growth that arises because the age distribution of a rapidly increasing population is favorable to increase. The concept has been introduced in Section 7.3.4; here we take advantage of the age-classification to explore what determines population momentum.

Suppose that all couples adopt birth control immediately and drop their births to a level that permits bare replacement. With U. S. mortality rates fertile couples need on the average (Section 16.3) 2.36 children to give a net reproduction rate R_0 of unity. An average of 2.36 children covers the loss of those dying before maturity, the fact that not everyone finds a mate, and some sterility among couples.

We saw that without any change in birth rates the ultimate birth trajectory due to $p(x)dx$ persons at age x to $x + dx$ would be $e^{rt}p(x)v(x)dx/\kappa$, and for the whole population distributed as $p(x)$ would be $e^{rt}\int_0^\beta p(x)v(x)dx/\kappa$. For calculating the effect of the fall to bare replacement we want the trajectory based on the existing age distribution $p(x)$, but with a function $v^*(x)$, corresponding to an intrinsic rate $r = 0$. We can arrange this, without changing any other feature of the age incidence of childbearing, by replacing $m(x)$ by $m^*(x) = m(a)/R_0$, which will ensure that $R_0^* = 1$ and $r^* = 0$. Then the ultimate stationary number of births must be

$$\int_0^\beta p(x)v^*(x)dx/\kappa, \quad (8.6.1)$$

where κ becomes μ , the mean age of childbearing in the stationary population because $v^* = 0$:

$$\frac{v^*(x)}{\kappa} = \frac{1}{\mu l(x)} \int_x^\beta \frac{l(a)m(a) da}{R_0}.$$

Ascertaining the ultimate stationary total population requires dividing by b , the stationary birth rate, which is the same as multiplying by e_0 , the expectation of life at age zero.

Expression 8.6.1 is readily usable. If we have a table of the net maternity function in 5-year age intervals up to age 49 and the initial age distribution, then, by cumulating the net maternity function to obtain ${}_5V_x^*$ and multiplying 10 pairs of ${}_5N_x$ and ${}_5V_x^*$, we have the ultimate stationary population

$${}_0e_0 \Sigma_0^{\beta-5} {}_5N_x {}_5V_x^* / \mu, \quad (8.6.2)$$

where

$${}_5V_x^* = \frac{(5/{}_5L_x)(\frac{1}{2}{}_5L_x F_x + {}_5L_{x+5} F_{x+5} + \dots)}{R_0}.$$

This calculation will give the same result as a full population projection with the new $m^*(x)$.

If the initial age distribution $p(x)$ can be taken as stable, the result is even simpler. Entering $p(x) = p_0 b e^{-rx} l(x)$ in (8.6.1), where r is the intrinsic rate before the drop to zero increase, canceling out $l(x)$ in numerator and denominator, and multiplying by ${}_0e_0$ to produce the stationary population rather than stationary births, we obtain

$$(1/p_0) {}_0e_0 \int_0^\beta p(x) v^*(x) dx = \frac{b {}_0e_0}{\mu} \int_0^\beta \int_x^\beta e^{-rx} l(a) \frac{m(a)}{R_0} da dx \quad (8.6.3)$$

as the ratio of the ultimate stationary population to the population at the time when the fall occurs.

The double integral is evaluated by writing b_x for $\int_x^\beta l(a) m(a) da / R_0$ and integrating by parts in (8.6.3) to obtain

$$\frac{b {}_0e_0}{\mu} \int_0^\beta e^{-rx} b_x dx = \frac{b {}_0e_0}{\mu} \left[\frac{e^{-rx}}{-r} b_x \Big|_0^\beta - \frac{1}{r} \int_0^\beta e^{-rx} \frac{l(x) m(x)}{R_0} dx \right].$$

We find that the right-hand side reduces to

$$\frac{b {}_0e_0}{r\mu} \left(\frac{R_0 - 1}{R_0} \right) \quad (8.6.4)$$

on applying the fact that $b_0 = 1$ and $\int_0^\beta e^{-rx} l(x) m(x) dx = 1$. Expression 8.6.4 gives the ratio of the ultimate population to population just before the fall to zero increase and is the main result of this section.

For Ecuador, 1965, the data are $1000b = 44.82$, ${}_0e_0 = 60.16$, $1000r = 33.31$, $\mu = 29.41$, and $R_0 = 2.59$. These make expression 8.6.4 equal to 1.69. By simple projection or by (8.6.2), which does not depend on the stable assumption, we would have a ratio of the ultimate stationary to the present population of 1.67. This experiment and others show that the degree of stability in many underdeveloped countries makes (8.6.4) realistic.

Table 8.3. Values of $b\overset{\circ}{e}_0/\sqrt{R_0}$, the approximate ratio of the ultimate to the present population if the birth rate falls immediately from $b = 0.045$ to that needed for bare replacement, $1/\overset{\circ}{e}_0$

Initial R_0	$\overset{\circ}{e}_0$		
	40	50	60
1.5	1.47	1.84	2.20
2.0	1.27	1.59	1.91
2.5	1.14	1.42	1.71

James Frauenthal has pointed out to me that $(b\overset{\circ}{e}_0/r\mu)[(R_0 - 1)/R_0]$ of (8.6.4) is very nearly $b\overset{\circ}{e}_0/\sqrt{R_0}$. For R_0 is approximately $e^{r\mu}$, and hence

$$\begin{aligned} \frac{b\overset{\circ}{e}_0}{r\mu} \left(\frac{R_0 - 1}{R_0} \right) &= \frac{b\overset{\circ}{e}_0}{\sqrt{R_0}} \left[\frac{e^{r\mu/2} - e^{-r\mu/2}}{r\mu} \right] \\ &= \frac{b\overset{\circ}{e}_0}{\sqrt{R_0}} \left(1 + \frac{r^2\mu^2}{24} + \frac{r^4\mu^4}{960} + \dots \right) \end{aligned}$$

on expanding both the exponentials in powers of $r\mu$. The product $r\mu$ is of the order of unity, so that $r^2\mu^2/24$ must be close to 0.05. The example of Ecuador, 1965, gives $b\overset{\circ}{e}_0/\sqrt{R_0} = 1.68$ as compared with 1.69 for (8.6.4).

To obtain an intuitive meaning of this, note that the absolute number of births just after the fall must be $1/R_0$ times the births just before the fall. Births will subsequently rise and then drop in waves of diminishing amplitude, and it seems likely that the curve will oscillate about the mean of the absolute numbers before and after the fall. If the geometric mean of 1 and $1/R_0$ applies, the ultimate number of births will be $1/\sqrt{R_0}$ times the births before the fall. In that case the ultimate population will be $\overset{\circ}{e}_0/\sqrt{R_0}$ times the births before the fall, or $b\overset{\circ}{e}_0/\sqrt{R_0}$ times the population before the fall.

In words, the approximation $b\overset{\circ}{e}_0/\sqrt{R_0}$ says that the momentum factor is proportional to the birth rate and the expectation of life, and inversely proportional to the square root of the net reproduction rate. Table 8.3 suggests to what degree the factor depends on $\overset{\circ}{e}_0$ and to what degree on R_0 for an initial birth rate of $1000b = 45$. The conclusion is that with an immediate fall in fertility to bare replacement Ecuador and demographically similar countries would increase by about 50 percent or more before attaining stationarity. Note that (8.6.4) or $b\overset{\circ}{e}_0/\sqrt{R_0}$ is a good approximation to the degree in which the age distribution before the fall is stable. [Using model tables or otherwise, comment on the consistency of the pattern $b = 0.045$, $\overset{\circ}{e}_0 = 60$, $R_0 = 1.5$ that gives rise to the ratio 2.20 in Table 8.3.]

Table 8.4. Deaths from malaria and heart disease, Philippines, 1959 and 1960

Age x to $x + n$	Malaria, Cause B-16, 1959	Degenerative heart disease, Cause B-26, 1959	${}_nV_x$ Reproductive value for females, Philippines, 1960
All ages	913	918	
-5	251	12	1.21
5-14	156	7	1.64
15-24	133	37	2.00
25-44	186	198	0.76
45-64	138	322	0
65+	45	333	0
Unknown	4	9	
Total repro- ductive value for deaths of stated age	967	250	

Source: *United Nations Demographic Yearbook* (1961, p. 498); Keyfitz and Flieger (1971, p. 411).

8.7 Eliminating Heart Disease Would Make Very Little Difference to Population Increase, Whereas Eradication of Malaria Makes a Great Deal of Difference

Age distributions of deaths from malaria and heart disease are shown in Table 8.4 for the Philippines, 1959. Evidently malaria affects the young ages, whereas heart disease is negligible before middle life. Although the two causes are responsible for about equal numbers of deaths, malaria has a much greater effect on the chance that a child will survive to reproductive age and on the number of women living through reproduction.

Finding the effect on the population trajectory of eliminating deaths in any one year requires that each death at age x be evaluated as $v(x)$, that is to say, we need the sum $\int_0^\beta p(x)v(x) dx$, where now $p(x) dx$ is the population removed by death at ages x to $x+dx$. (The constants b and κ will not affect the relative positions of the two causes.) The broad age groups and lumping of the two sexes in Table 8.4 prevent us from attaining high accuracy. Table 8.4 shows unweighted arithmetic averages of $v(x)$ for the age groups required. The value of the malaria deaths, if they were female, would be $(251)(1.21) + (156)(1.64) + (133)(2.00) + (186)(0.76) = 967$; that of the heart disease deaths similarly calculated would be 250. In practice

men and women influence mortality in different degrees, and no easy way to allow for this suggests itself.

But the complexities that a two-sex model would introduce would not greatly affect the present conclusion: although absolute numbers of deaths from heart disease are about equal to those from malaria, malaria has nearly 4 times the effect on subsequent population.

8.8 The Stable Equivalent

The stable equivalent Q , associated with long-run projections, helps to interpret an observed past age distribution from the viewpoint of reproductive potential, and so bridges the present chapter and the preceding one dealing with reproductive value. It is the natural companion of the intrinsic rate of natural increase r . The rate r tells us *how fast* the population would ultimately increase at present age-specific rates; Q tells us *at what level* the ultimate population curve would stand.

8.8.1 Population Projection and the Stable Approximation Thereto

Most of this chapter has used the continuous renewal equation model for age-classified populations. Here we shift perspective to the discrete population projection matrix method. We are given an observed (from a mathematical viewpoint an arbitrary) age distribution for one sex, which is arranged as a vertical vector $\mathbf{n}(0)$, together with a set of age-specific birth and death rates arranged in the form of a matrix \mathbf{A} . If a 5-year projection interval, and 5-year age groups to age 85 to 89 at last birthday are recognized, \mathbf{A} has 18×18 elements and $\mathbf{n}(0)$ has 18×1 . The first row of \mathbf{A} provides for fertility, and the subdiagonal for survivorship; this is, in fact, the Leslie matrix of Section 3.1. The age distribution projected through $5t$ years is

$$\mathbf{n}(t) = \mathbf{A}^t \mathbf{n}(0). \quad (8.8.1)$$

An approximation to this projection, called asymptotic because it is approached as closely as one wishes with sufficiently large t , is

$$\mathbf{n}(t) \approx \mathbf{q}e^{5rt}, \quad (8.8.2)$$

where the vector \mathbf{q} is the stable equivalent of the age distribution.

To calculate \mathbf{q} choose a large t and equate the right-hand sides of (8.8.1) and (8.8.2). If the population were of stable age distribution from the start, and contained \mathbf{q} individuals at the several ages, by time $5t$ it would grow to $\mathbf{q}e^{5rt}$. In fact, we know that it is of age distribution $\mathbf{n}(0)$, and when projected it grows to $\mathbf{A}^t \mathbf{n}(0)$ by time $5t$. The matrix equation for the

Table 8.5. Female population total by conventional projection and by contribution of dominant root, starting from United States, 1960* (000s)

Year	t	Leslie projection with fixed 1960 rates	Contribution of positive term Qe^{rt}
1960	0	91,348	76,840
1970	10	106,220	94,986
1980	20	125,669	117,416
1990	30	150,129	145,144
2000	40	181,464	179,419
2010	50	222,196	221,789
2020	60	273,949	274,164
2030	70	338,990	338,907
2040	80	418,996	418,939

*Right-hand column is $Qe^{rt} = 76,840e^{0.0212t}$.

calculation of \mathbf{q} is thus

$$\mathbf{q}e^{5rt} = \mathbf{A}^t \mathbf{n}(0)$$

or

$$\mathbf{q} = \frac{\mathbf{A}^t \mathbf{n}(0)}{e^{5rt}}. \quad (8.8.3)$$

One way of describing (8.8.3) is to say that $\mathbf{n}(0)$, the initial population is projected *forward* t periods by the matrix \mathbf{A} and *backward* an equal length of time by the real root r , that is, by dividing by e^{5rt} . The quantity $\mathbf{q}e^{rt}$ corresponds to the real term in the solution of the Lotka equation (7.5.2), but is more complete in providing the several ages of the population rather than births alone. The total of all ages, written as $Q = \sum q_i$, is shown in Table 8.5 for United States females, starting with the 1960 age distribution and projected by 1960 births and life table.

The intrinsic rate of natural increase for the regime of 1959–61 being $r = 0.0212$, and the stable equivalent of the initial population being $Q = 76,840,000$, the future female population t years after 1960, if age-specific rates remained fixed and the stable model applied, would be $76,840,000e^{0.0212t}$. Table 8.5 compares this at 10-year intervals with the full projection, which implicitly uses all terms in the right-hand side of (7.5.2). By the year 2000 the discrepancy is down to 1.1 percent.

However, between 1960 and 1965 some of the postwar cohorts moved into childbearing ages, and the age distribution became more favorable, to the point where the stable equivalent and the observed total practically coincided, both being just under 99 million (Table 8.6). At the same time a drastic decline in the birth rates occurred, so that the intrinsic rate fell

Table 8.6. Female population P_0 and stable equivalent number Q , United States, 1919–21 to 1965, adjusted births

Year	Observed female population (000s) $N(0)$	Stable equivalent (000s) Q
1919–21	52,283	55,519
1924–26	57,016	61,442
1929–31	60,757	72,304
1934–36	63,141	78,879
1939–41	65,811	77,279
1944–46	69,875	72,016
1949–51	76,216	68,376
1954–56	83,248	69,535
1960	91,348	76,840
1965	98,703	98,645

Source: Keyfitz and Flieger (1968).

to $r = 0.01267$. Hence the future from the 1965 vantage point was

$$98,645,000e^{0.01267(t-5)}, \quad (8.8.4)$$

if t is still measured from 1960.

8.8.2 Application of the Stable Equivalent Q

Table 8.6 shows Q to be considerably above the observed female population $N(0) = \sum n_i(0)$ for the United States during the 1930s, and below it in the 1950s. This reflects the tendency for there to be proportionately more women of the age of motherhood in the population for some years after a fall in the birth rate. The crude birth rate usually lags behind the intrinsic birth rate after an upturn or downturn in fertility. The stable equivalent Q measures the favorability of the age distribution to reproduction, given the current regime of mortality and fertility.

In Table 8.7 historical data on Q are presented for four other countries. Again a high Q relative to population after a fall in birth rates appears for England and Wales between 1901 and 1941, and for Australia and Canada before 1941. The Netherlands also shows this feature, but to a more moderate degree.

8.8.3 Relation Between Q and Reproductive Value V

Reproductive value, the discounted future girl children that will be born to a woman, has a close relation to Q . [Prove that Q , like V but unlike $N(t)$, has the property of increasing at a constant rate under a fixed regime of

Table 8.7. Observed female population and stable equivalent, historical data for four countries

Country and year	Female population (000s) $N(0)$	Stable equivalent (000s) Q	Ratio $Q/N(0)$
Australia			
1911	2,152	2,395	1.11
1921	2,683	3,013	1.12
1933	3,263	4,267	1.31
1947	3,782	3,501	0.93
1957	4,758	4,215	0.89
1960	5,083	4,494	0.88
1965	5,632	5,659	1.00
Canada			
1931	5,001	5,706	1.14
1941	5,608	6,356	1.13
1951	6,751	6,431	0.95
1961	8,794	8,120	0.92
1965	9,479	9,839	1.04
England and Wales			
1861	10,324	10,802	1.05
1871	11,695	11,966	1.02
1881	13,373	13,608	1.02
1891	14,989	15,805	1.05
1901	16,845	19,047	1.13
1911	18,655	22,014	1.18
1921	19,816	22,229	1.12
1931	20,839	27,321	1.31
1941	21,515	27,522	1.28
1946	21,979	20,511	0.93
1951	22,751	22,741	1.00
1956	23,150	21,577	0.93
1961	23,820	19,764	0.83
Netherlands			
1901	2,615	2,647	1.01
1910	2,960	3,064	1.03
1920	3,419	3,615	1.06
1930	3,954	4,386	1.11
1940	4,437	4,983	1.12
1945	4,619	4,551	0.99
1950	5,074	5,077	1.00
1955	5,395	5,405	1.00
1960	5,766	5,615	0.97
1965	6,081	5,942	0.98

Source: Keyfitz and Flieger (1968).

Table 8.8. Observed female population, stable equivalent, and reproductive value (000s)

Country and Year	Observed female population $N(0)$	Stable equivalent Q	Ratio of stable to observed $Q/N(0)$	Reproductive value in units of girl babies V
Austria, 1964	3,845	3,187	0.83	1,665
Czechoslovakia, 1964	7,198	7,312	1.02	3,253
Denmark, 1964	2,380	2,326	0.98	1,091
Fiji Islands, 1964	219	229	1.04	229
Finland, 1964	2,370	2,540	1.07	1,227
Germany (East), 1964	9,257	7,871	0.85	3,499
Germany (West), 1964	30,980	27,755	0.90	13,124
Netherlands, 1964	6,081	5,942	0.98	3,665
Norway, 1964	1,854	1,649	0.89	914
Puerto Rico, 1964	1,309	1,375	1.05	1,050
Roumania, 1964	9,665	13,250	1.37	4,088
Switzerland, 1964	2,940	2,861	0.97	1,431

Source: Keyfitz and Flieger (1968).

mortality and fertility. The proof involves the fact that $(\mathbf{A}^t/e^{5rt})\mathbf{n}(0)$ is invariant with respect to t as long as t is large; in particular, \mathbf{A}^t/e^{5rt} is the same when $t + 1$ is written for t (Section 8.1).]

In fact, V is a simple multiple of Q . In the continuous representation, V is exactly equal to Q multiplied by the intrinsic birth rate b and by the mean age of childbearing in the stable population κ , two constants obtainable from the age-specific rates and having nothing to do with the observed age distribution. The reader may prove this statement by rearranging the double integral contained in $\int_0^\beta N(x)v(x) dx$, where $v(x)$ is defined as in (8.1.1), and showing it to be the same as the numerator of the first coefficient Q in the solution (7.5.4) to the Lotka equation. In the present notation this will prove

$$Q = \frac{V}{b\kappa}. \quad (8.8.5)$$

Goodman (1968) shows this result to apply in the discrete case. Values of $N(0)$, Q , and V are given in Table 8.8 for a number of countries.

A question arises of the degree to which Q , the stable equivalent, is sensitive to the particular set of age-specific birth and death rates used in its calculation. The first row of Table 8.9 shows Q for the age distribution of 1960, worked out according to the 1960 and 1965 patterns of mortality and fertility as embodied in the two \mathbf{A} 's; the second row shows the corresponding Q 's for the 1965 age distribution. The values obtained for Q depend greatly on the set of age-specific rates applied as \mathbf{A} . But if we study only the *change* in Q in the United States between 1960 and 1965, it turns out that the increase is 11.14 percent on the 1960 \mathbf{A} and 10.84 percent on the

Table 8.9. Stable equivalents Q for United States females in 1960 and 1965, each calculated with two different matrices \mathbf{A} (thousands)

	Calculated with matrix \mathbf{A} of	
	1960	1965
Age distribution:		
1960	76,912	89,001
1965	85,478	98,645
Percent increase	11.14	10.84

Table 8.10. Stable equivalents Q for the United States and Mexico (thousands)

	Calculated with matrix \mathbf{A} of	
	United States, 1962	Mexico, 1962
Age distribution:		
United States	82,933	63,395
Mexico	23,388	18,863
Ratio, Mexico to United States	0.282	0.272

1965 \mathbf{A} . This way of making a comparison (applied between France and Italy) is due to Vincent (1945), who noted the virtual invariance with respect to the mortality and fertility patterns used. We are entitled to say that the age distribution of women in the United States became about 11 percent more favorable to reproduction during the 5 years in question, and the statement is true almost without regard to the fertility and mortality patterns used in making this assessment.

As an example of a place comparison, Table 8.10 shows Q values for Mexico and the United States, both for 1962. The Q for Mexico is 0.282 of that for the United States when the \mathbf{A} of the latter is used; it is 0.272 when the \mathbf{A} of the former is used. Jeffrey Evans has programmed place comparisons among five countries which demonstrate the same invariance.

Section 12.3 below uses the stable equivalent to compare the effect on age distribution of eliminating cancer with that of eliminating heart disease.

8.8.4 A More General Stable Equivalent

Age is merely a special case of the stable equivalent. Any model that possesses the ergodic property, that is, that tends asymptotically to a distribution unaffected by the initial distribution, is equally capable of analysis by the methods given above. In fact (8.8.3) remains unchanged; only now the matrix \mathbf{A} and the vector $\mathbf{n}(0)$ provide for the two sexes, regions, mar-

ried and single populations, and any other groups recognized in the model. For details see Chapter 9 and some applications in Keyfitz (1969).

8.9 Reproductive Value as a Contribution to Future Births

Section 8.1 appeals to intuition to make it appear likely that the effect of adding one girl or woman aged x to the population is to raise the number of births t years hence, where t is large, in proportion to $v(x)e^{rt}$, $v(x)$ being defined as

$$v(x) = \frac{\int_x^\beta e^{-ra}l(a)m(a) da}{e^{-rx}l(x)}.$$

This result can be derived from the Lotka equation of Section 7.5, but here we examine a demonstration that is self contained, using the familiar device of calculating the situation at time t from two successive moments near the present. For purposes of this section $v(x)$ will be defined afresh, in terms of the ultimate birth trajectory.

Suppose that a woman aged x at time zero contributes $v(x)e^{rt}$ to the births at subsequent time t , where $v(x)$ is to be determined and t is large. This means that her disappearance would lower the ultimate birth trajectory by $v(x)e^{rt}$. We assume that age-specific birth and death rates are fixed, so that her descendants will ultimately increase in geometric proportion and be unaffected by other members of the population.

The woman aged x can, in the next short period of time and age, say Δ , have a child, and whether or not she has a child can survive to the next age, $x + \Delta$. The chance of her having a child is $m(x)\Delta$, and the chance of her surviving is $l(x + \Delta)/l(x)$. By having a child she would contribute $v(0)e^{r(t-\Delta)}$ to the births at time t , and by surviving she would convert herself into a woman of reproductive value $v(x + \Delta)$ and so contribute $v(x + \Delta)e^{r(t-\Delta)}$. If the progression of childbearing and aging at the given rates over the time Δ is not to affect the ultimate birth trajectory, we can equate the two expressions for later births:

$$v(x)e^{rt} = \left[m(x)v(0)\Delta + \frac{l(x + \Delta)}{l(x)}v(x + \Delta) \right] e^{r(t-\Delta)}. \quad (8.9.1)$$

If we multiply both sides of (8.9.1) by

$$\frac{1}{\Delta} \frac{l(x)}{v(0)} e^{-rx} e^{-rt},$$

we obtain

$$\begin{aligned} \frac{1}{\Delta} l(x) \frac{v(x)}{v(0)} e^{-rx} &= m(x) l(x) e^{-rx} e^{-r\Delta} \\ &+ \frac{1}{\Delta} l(x + \Delta) \frac{v(x + \Delta)}{v(0)} e^{-r(x+\Delta)}. \end{aligned} \quad (8.9.2)$$

Subtracting the rightmost term from both sides and letting $\Delta \rightarrow 0$, we have directly

$$-\frac{d}{dx} l(x) \frac{v(x)}{v(0)} e^{-rx} = m(x) l(x) e^{-rx},$$

and integrating gives

$$e^{-rx} l(x) \frac{v(x)}{v(0)} = \int_x^\beta e^{-ra} l(a) m(a) da, \quad (8.9.3)$$

so that, if $v(0)$ is set equal to unity, $v(x)$ again comes out as shown in (8.1.1). No constant of integration is needed, since both sides are unity for $x = 0$. Equation (8.9.3) establishes $v(x)$ to within a multiplicative constant.

Let us find the constant $v(0)$ that corresponds to the ultimate effect of adding one female to the population.

If the initial age distribution is stable, we know that the population at time t must be e^{rt} for each person initially present, and hence the births at time t are be^{rt} . Equating the two values for time t , we have

$$be^{rt} = \int_0^\beta be^{-rx} l(x) v(x) dx e^{rt}; \quad (8.9.4)$$

since from (8.9.3) $v(x)$ may be written as

$$\frac{v(0)}{e^{-rx} l(x)} \int_x^\beta e^{-ra} l(a) m(a) da,$$

the $be^{-rx} l(x)$ within the integral of (8.9.4), as well as e^{rt} outside the integral, cancels, and we obtain the following equation for $v(0)$:

$$\frac{1}{v(0)} = \int_0^\beta \int_x^\beta e^{-ra} l(a) m(a) da dx. \quad (8.9.5)$$

The integral on the right-hand side is evaluated by integration by parts and turns out to be κ , the mean age of childbearing in the stable population. This proves that for the $v(x)$ function of this section, $v(0) = 1/\kappa$, and that the $v(x)$ function of the main body of the chapter, defined in (8.1.1), gives the ultimate birth trajectory due to a woman aged x as $e^{rt} v(x)/\kappa$.



<http://www.springer.com/978-0-387-22537-1>

Applied Mathematical Demography

Keyfitz, N.; Caswell, H.

2005, XXVI, 558 p. 74 illus., Hardcover

ISBN: 978-0-387-22537-1