2

Linear Difference Equations of Higher Order

In this chapter we examine linear difference equations of high order, namely, those involving a single dependent variable.[1] Such equations arise in almost every field of scientific inquiry, from population dynamics (the study of a single species) to economics (the study of a single commodity) to physics (the study of the motion of a single body). We will become acquainted with some of these applications in this chapter. We start this chapter by introducing some rudiments of difference calculus that are essential in the study of linear equations.

2.1 Difference Calculus

Difference calculus is the discrete analogue of the familiar differential and integral calculus. In this section we introduce some very basic properties of two operators that are essential in the study of difference equations. These are the *difference operator* (Section 1.2)

$$\Delta x(n) = x(n+1) - x(n)$$

and the *shift operator*

$$Ex(n) = x(n+1).$$

[1]Difference equations that involve more than one dependent variable are called systems of difference equations; we will inspect these equations in Chapter 3.

It is easy to see that

$$E^k x(n) = x(n+k).$$

However, $\Delta^k x(n)$ is not so apparent. Let I be the *identity operator*, i.e., $Ix = x$. Then, one may write $\Delta = E - I$ and $E = \Delta + I$.

Hence,

$$\begin{aligned}\Delta^k x(n) &= (E-I)^k x(n)\\ &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} E^{k-i} x(n),\end{aligned}$$

$$\boxed{\Delta^k x(n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} x(n+k-i).} \tag{2.1.1}$$

Similarly, one may show that

$$E^k x(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} x(n). \tag{2.1.2}$$

We should point out here that the operator Δ is the counterpart of the derivative operator D in calculus. Both operators E and Δ share one of the helpful features of the derivative operator D, namely, the property of *linearity*.

"Linearity" simply means that $\Delta[ax(n)+by(n)] = a\Delta x(n) + b\Delta y(n)$ and $E[ax(n)+by(n)] = aEx(n) + bEy(n)$, for all a and $b \in \mathbb{R}$. In Exercises 2.1, Problem 1, the reader is allowed to show that both Δ and E are linear operators.

Another interesting difference, parallel to differential calculus, is the discrete analogue of the fundamental theorem of calculus. [2]

Lemma 2.1. *The following statements hold:*

(i)

$$\boxed{\sum_{k=n_0}^{n-1} \Delta x(k) = x(n) - x(n_0),} \tag{2.1.3}$$

[2] The fundamental theorem of calculus states that:

(i) $\int_a^b df(x) = f(b) - f(a)$,

(ii) $d\left(\int_a^x f(t)\,dt\right) = f(x)$.

(ii)

$$\boxed{\Delta\left(\sum_{k=n_0}^{n-1} x(k)\right) = x(n).} \tag{2.1.4}$$

PROOF. The proof remains as Exercises 2.1, Problem 3. □

We would now like to introduce a third property that the operator Δ has in common with the derivative operator D.

Let

$$p(n) = a_0 n^k + a_1 n^{k-1} + \cdots + a_k$$

be a polynomial of degree k. Then

$$\begin{aligned}\Delta p(n) &= \left[a_0(n+1)^k + a_1(n+1)^{k-1} + \cdots + a_k\right] \\ &\quad - \left[a_0 n^k + a_1 n^{k-1} + \cdots + a_k\right] \\ &= a_0 k n^{k-1} + \text{ terms of degree lower than } (k-1).\end{aligned}$$

Similarly, one may show that

$$\Delta^2 p(n) = a_0 k(k-1) n^{k-2} + \text{ terms of degree lower than } (k-2).$$

Carrying out this process k times, one obtains

$$\Delta^k p(n) = a_0 k!. \tag{2.1.5}$$

Thus,

$$\Delta^{k+i} p(n) = 0 \text{ for } i \geq 1. \tag{2.1.6}$$

2.1.1 The Power Shift

We now discuss the action of a polynomial of degree k in the shift operator E on the term b^n, for any constant b.

Let

$$p(E) = a_0 E^k + a_1 E^{k-1} + \cdots + a_k I \tag{2.1.7}$$

be a polynomial of degree k in E.

Then

$$\begin{aligned}p(E)b^n &= a_0 b^{n+k} + a_1 b^{n+k-1} + \cdots + a_k b^n \\ &= (a_0 b^k + a_1 b^{k-1} + \cdots + a_k) b^n \\ &= p(b) b^n. \end{aligned} \tag{2.1.8}$$

A generalization of formula (2.1.8) now follows.

Lemma 2.2. *Let $p(E)$ be the polynomial in* (2.1.7) *and let $g(n)$ be any discrete function. Then*

$$\boxed{p(E)(b^n g(n)) = b^n p(bE)g(n).} \tag{2.1.9}$$

Proof. This is left to the reader as Exercises 2.1, Problem 4. □

2.1.2 Factorial Polynomials

One of the most interesting functions in difference calculus is the *factorial polynomial* $x^{(k)}$ defined as follows. Let $x \in \mathbb{R}$. Then the kth factorial of x is given by

$$x^{(k)} = x(x-1)\cdots(x-k+1), \qquad k \in \mathbb{Z}^+.$$

Thus if $x = n \in \mathbb{Z}^+$ and $n \geq k$, then

$$\boxed{n^{(k)} = \frac{n!}{(n-k)!}}$$

and

$$n^{(n)} = n!.$$

The function $x^{(k)}$ plays the same role here as that played by the polynomial x^k in differential calculus. The following Lemma 2.3 demonstrates this fact.

So far, we have defined the operators Δ and E on sequences $f(n)$. One may extend the definitions of Δ and E to continuous functions $f(t)$, $t \in \mathbb{R}$, by simply letting $\Delta f(t) = f(t+1) - f(t)$ and $Ef(t) = f(t+1)$. This extension enables us to define $\Delta f(x)$ and $Ef(x)$ where $f(x) = x^{(k)}$ by

$$\Delta x^{(k)} = (x+1)^{(k)} - x^{(k)} \quad \text{and} \quad Ex^{(k)} = (x+1)^{(k)}.$$

Using this definition one may establish the following result.

Lemma 2.3. *For fixed $k \in \mathbb{Z}^+$ and $x \in \mathbb{R}$, the following statements hold:*

(i)

$$\boxed{\Delta x^{(k)} = kx^{(k-1)};} \tag{2.1.10}$$

(ii)

$$\boxed{\Delta^n x^{(k)} = k(k-1), \ldots, (k-n+1)x^{(k-n)};} \tag{2.1.11}$$

(iii)

$$\boxed{\Delta^k x^{(k)} = k!.} \tag{2.1.12}$$

PROOF. (i)

$$\begin{aligned}\Delta x^{(k)} &= (x+1)^{(k)} - x^{(k)}\\ &= (x+1)x(x-1)\cdots(x-k+2) - x(x-1)\\ &\quad\cdots(x-k+2)(x-k+1)\\ &= x(x-1)\cdots(x-k+2)\cdot k\\ &= kx^{(k-1)}.\end{aligned}$$

The proofs of parts (ii) and (iii) are left to the reader as Exercises 2.1, Problem 5. □

If we define, for $k \in \mathbb{Z}^+$,

$$x^{(-k)} = \frac{1}{x(x+1)\cdots(x+k-1)} \tag{2.1.13}$$

and $x^{(0)} = 1$, then one may extend Lemma 2.3 to hold for all $k \in \mathbb{Z}$. In other words, parts (i), (ii), and (iii) of Lemma 2.3 hold for all $k \in \mathbb{Z}$ (Exercises 2.1, Problem 6).

The reader may wonder whether the product and quotient rules of the differential calculus have discrete counterparts. The answer is affirmative, as may be shown by the following two formulas, where proofs are left to the reader as Exercises 2.1, Problem 7.

Product Rule:

$$\boxed{\Delta[x(n)y(n)] = Ex(n)\Delta y(n) + y(n)\Delta x(n).} \tag{2.1.14}$$

Quotient Rule:

$$\boxed{\Delta\left[\frac{x(n)}{y(n)}\right] = \frac{y(n)\Delta x(n) - x(n)\Delta y(n)}{y(n)Ey(n)}.} \tag{2.1.15}$$

2.1.3 The Antidifference Operator

The discrete analogue of the indefinite integral in calculus is the antidifference operator Δ^{-1}, defined as follows. If $\Delta F(n) = 0$, then $\Delta^{-1}(0) = F(n) = c$ for some arbitrary constant c. Moreover, if $\Delta F(n) = f(n)$, then $\Delta^{-1}f(n) = F(n) + c$, for some arbitrary constant c. Hence

$$\Delta\Delta^{-1}f(n) = f(n),$$
$$\Delta^{-1}\Delta F(n) = F(n) + c,$$

and

$$\Delta\Delta^{-1} = I \quad \text{but} \quad \Delta^{-1}\Delta \neq I.$$

Using formula (2.1.4) one may readily obtain

$$\boxed{\Delta^{-1} f(n) = \sum_{i=0}^{n-1} f(i) + c.} \tag{2.1.16}$$

Formula (2.1.16) is very useful in proving that the operator Δ^{-1} is linear.

Theorem 2.4. *The operator Δ^{-1} is linear.*

PROOF. We need to show that for $a, b \in \mathbb{R}$, $\Delta^{-1}[ax(n) + by(n)] = a\Delta^{-1}x(n) + b\Delta^{-1}y(n)$. Now, from formula (2.1.16) we have

$$\begin{aligned}
\Delta^{-1}[ax(n) + by(n)] &= \sum_{i=0}^{n-1} ax(i) + by(i) + c \\
&= a\sum_{i=0}^{n-1} x(i) + b\sum_{i=0}^{n-1} y(i) + c \\
&= a\Delta^{-1}x(n) + b\Delta^{-1}y(n).
\end{aligned}$$

□

Next we derive the antidifference of some basic functions.

Lemma 2.5. *The following statements hold:*

(i)

$$\Delta^{-k} 0 = c_1 n^{k-1} + c_2 n^{k-2} + \cdots + c_k. \tag{2.1.17}$$

(ii)

$$\Delta^{-k} 1 = \frac{n^k}{k!} + c_1 n^{k-1} + c_2 n^{k-2} + \cdots + c_k. \tag{2.1.18}$$

(iii)

$$\boxed{\Delta^{-1} n^{(k)} = \frac{n^{(k+1)}}{k+1} + c, \qquad k \neq -1.} \tag{2.1.19}$$

PROOF. The proofs of parts (i) and (ii) follow by applying Δ^k to the right-hand side of formulas (2.1.17) and (2.1.18) and then applying formulas (2.1.6) and (2.1.5), respectively. The proof of part (iii) follows from formula (2.1.10).

Finally, we give the discrete analogue of the integration by parts formula, namely, the summation by parts formula:

$$\boxed{\sum_{k=0}^{n-1} y(k)\Delta x(k) = x(n)y(n) - \sum_{k=0}^{n-1} x(k+1)\Delta y(k) + c.} \tag{2.1.20}$$

To prove formula (2.1.20) we use formula (2.1.14) to obtain

$$y(n)\Delta x(n) = \Delta(x(n)y(n)) - x(n+1)\Delta y(n).$$

Applying Δ^{-1} to both sides and using formula (2.1.16), we get

$$\sum_{k=0}^{n-1} y(k)\Delta x(k) = x(n)y(n) - \sum_{k=0}^{n-1} x(k+1)\Delta y(k) + c. \qquad \square$$

Exercises 2.1

1. Show that the operators Δ and E are linear.
2. Show that $E^k x(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} x(n)$.
3. Verify formulas (2.1.3) and (2.1.4).
4. Verify formula (2.1.9).
5. Verify formulas (2.1.11) and (2.1.12).
6. Show that Lemma 2.3 holds for $k \in \mathbb{Z}$.
7. Verify the product and quotient rules (2.1.14) and (2.1.15).
8. (Abel's Summation Formula). Prove that
$$\boxed{\sum_{k=1}^{n} x(k)y(k) = x(n+1)\sum_{k=1}^{n} y(k) - \sum_{k=1}^{n}\left(\Delta x(k)\sum_{r=1}^{k} y(r)\right).}$$
9. (Newton's Theorem). If $f(n)$ is a polynomial of degree k, show that
$$f(n) = f(0) + \frac{n^{(1)}}{1!}\Delta f(0) + \frac{n^{(2)}}{2!}\Delta^2 f(0) + \cdots + \frac{n^{(k)}}{k!}\Delta^{(k)} f(0).$$
10. (The Discrete Taylor Formula). Verify that
$$\boxed{f(n) = \sum_{i=0}^{k-1}\binom{n}{i}\Delta^i f(0) + \sum_{s=0}^{n-k}\binom{n-s-1}{k-1}\Delta^k f(s).}$$
11. (The Stirling Numbers). The Stirling numbers of the second kind $s_i(k)$ are defined by the difference equation $s_i(m+1) = s_{i-1}(m) + is_i(m)$ with $s_i(i) = s_1(i) = 1$ and $1 \le i \le m, s_1(k) = 0$ for $1 > k$. Prove that
$$x^m = \sum_{i=1}^{m} s_i(m)x^{(i)}. \tag{2.1.21}$$
12. Use (2.1.21) to verify Table 2.1 which gives the Stirling numbers $s_i(k)$ for $1 \le i, k \le 7$.
13. Use Table 2.1 and formula (2.1.21) to write $x^3, x^4,$ and x^5 in terms of the factorial polynomials $x^{(k)}$ (e.g., $x^2 = x^{(1)} + x^{(2)}$).
14. Use Problem 13 to find

TABLE 2.1. Stirling numbers $s_i(k)$.

$i\backslash k$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2		1	3	7	15	31	63
3			1	6	25	90	301
4				1	10	65	350
5					1	15	140
6						1	21
7							1

(i) $\Delta^{-1}(n^3+1)$.

(ii) $\Delta^{-1}\left(\dfrac{5}{n(n+3)}\right)$.

15. Use Problem 13 to solve the difference equation $y(n+1) = y(n) + n^3$.
16. Use Problem 13 to solve the difference equation $y(n+1) = y(n) - 5n^2$.
17. Consider the difference equation[3]

$$y(n+1) = a(n)y(n) + g(n). \tag{2.1.22}$$

(a) Put $y(n) = \left(\prod_{i=0}^{n-1} a(i)\right) u(n)$ in (2.1.22). Then show that $\Delta u(n) = g(n)/\prod_{i=0}^{n} a(i)$.

(b) Prove that

$$y(n) = \left(\prod_{i=0}^{n-1} a(i)\right) y_0 + \sum_{r=0}^{n-1} \left(\prod_{i=r+1}^{n-1} a(i)\right) g(r), \qquad y_0 = y(0).$$

(Compare with Section 1.2.)

2.2 General Theory of Linear Difference Equations

The normal form of a kth-order *nonhomogeneous linear* difference equation is given by

$$y(n+k) + p_1(n)\, y(n+k-1) + \cdots + p_k(n)\, y(n) = g(n), \tag{2.2.1}$$

where $p_i(n)$ and $g(n)$ are real-valued functions defined for $n \geq n_0$ and $p_k(n) \neq 0$ for all $n \geq n_0$. If $g(n)$ is identically zero, then (2.2.1) is said to be a homogeneous equation. Equation (2.2.1) may be written in the form

$$y(n+k) = -p_1(n)\, y(n+k-1) - \cdots - p_k(n)\, y(n) + g(n). \tag{2.2.2}$$

[3]This method of solving a nonhomogeneous equation is called the method of variation of constants.

By letting $n = 0$ in (2.2.2), we obtain $y(k)$ in terms of $y(k-1), y(k-2), \cdots, y(0)$. Explicitly, we have

$$y(k) = -p_1(0)y(k-1) - p_2(0)y(k-2) - \cdots - p_k(0)y(0) + g(0).$$

Once $y(k)$ is computed, we can go to the next step and evaluate $y(k+1)$ by letting $n = 1$ in (2.2.2). This yields

$$y(k+1) = -p_1(1)y(k) - p_2(1)y(k-1) - \cdots - p_k(1)y(1) + g(1).$$

By repeating the above process, it is possible to evaluate all $y(n)$ for $n \geq k$. Let us now illustrate the above procedure by an example.

Example 2.6. Consider the third-order difference equation

$$y(n+3) - \frac{n}{n+1}y(n+2) + ny(n+1) - 3y(n) = n, \tag{2.2.3}$$

where $y(1) = 0, y(2) = -1$, and $y(3) = 1$. Find the values of $y(4)$, $y(5)$, $y(6)$, and $y(7)$.

Solution First we rewrite (2.2.3) in the convenient form

$$y(n+3) = \frac{n}{n+1}y(n+2) - ny(n+1) + 3y(n) + n. \tag{2.2.4}$$

Letting $n = 1$ in (2.2.4), we have

$$y(4) = \frac{1}{2}y(3) - y(2) + 3y(1) + 1 = \frac{5}{2}.$$

For $n = 2$,

$$y(5) = \frac{2}{3}y(4) - 2y(3) + 3y(2) + 2 = -\frac{4}{3}.$$

For $n = 3$,

$$y(6) = \frac{3}{4}y(5) - 3y(4) + 3y(3) + 3 = -\frac{3}{2}.$$

For $n = 4$,

$$y(7) = \frac{4}{5}y(6) - 4y(5) + 3y(4) + 4 = 20.9.$$

Now let us go back to (2.2.1) and formally define its solution. A sequence $\{y(n)\}_{n_0}^{\infty}$ or simply $y(n)$ is said to be a *solution* of (2.2.1) if it satisfies the equation. Observe that if we specify the initial data of the equation, we are led to the corresponding initial value problem

$$y(k+n) + p_1(n)y(n+k-1) + \cdots + p_k(n)y(n) = g(n), \tag{2.2.5}$$

$$y(n_0) = a_0, \quad y(n_0+1) = a_1, \ldots, y(n_0+k-1) = a_{k-1}, \tag{2.2.6}$$

where the a_i's are real numbers. In view of the above discussion, we conclude with the following result.

Theorem 2.7. *The initial value problems* (2.2.5) *and* (2.2.6) *have a* unique *solution* $y(n)$.

Proof. The proof follows by using (2.2.5) for $n = n_0, n_0 + 1, n_0 + 2, \ldots$. Notice that any $n \geq n_0 + k$ may be written in the form $n = n_0 + k + (n - n_0 - k)$. By *uniqueness* of the solution $y(n)$ we mean that if there is another solution $\tilde{y}(n)$ of the initial value problems (2.2.5) and (2.2.6), then $\tilde{y}(n)$ must be identical to $y(n)$. This is again easy to see from (2.2.5). □

The question still remains whether we can find a closed-form solution for (2.2.1) or (2.2.5) and (2.2.6). Unlike our amiable first-order equations, obtaining a closed-form solution of (2.2.1) is a formidable task. However, if the coefficients p_i in (2.2.1) are constants, then a solution of the equation may be easily obtained, as we see in the next section.

In this section we are going to develop the general theory of kth-order linear *homogeneous* difference equations of the form

$$x(n+k) + p_1(n)x(n+k-1) + \cdots + p_k(n)x(n) = 0. \tag{2.2.7}$$

We start our exposition by introducing three important definitions.

Definition 2.8. The functions $f_1(n), f_2(n), \ldots, f_r(n)$ are said to be *linearly dependent* for $n \geq n_0$ if there are constants $a_1, a_2, \ldots, a_r$, not all zero, such that

$$a_1 f_1(n) + a_2 f_2(n) + \cdots + a_r f_r(n) = 0, \qquad n \geq n_0.$$

If $a_j \neq 0$, then we may divide (2.2.7) by a_j to obtain

$$\begin{aligned} f_j(n) &= -\frac{a_1}{a_j} f_1(n) - \frac{a_2}{a_j} f_2(n) \cdots - \frac{a_r}{a_j} f_r(n) \\ &= -\sum_{i \neq j} \frac{a_i}{a_j} f_i(n). \end{aligned} \tag{2.2.8}$$

Equation (2.2.8) simply says that each f_j with nonzero coefficient is a *linear combination* of the other f_i's. Thus two functions $f_1(n)$ and $f_2(n)$ are linearly dependent if one is a multiple of the other, i.e., $f_1(n) = af_2(n)$, for some constant a.

The negation of linear dependence is *linear independence*. Explicitly put, the functions $f_1(n), f_2(n), \ldots, f_r(n)$ are said to be *linearly independent* for $n \geq n_0$ if whenever

$$a_1 f_1(n) + a_2 f_2(n) + \cdots + a_r f_r(n) = 0$$

for all $n \geq n_0$, then we must have $a_1 = a_2 = \cdots = a_r = 0$.

Let us illustrate this new concept by an example.

Example 2.9. Show that the functions 3^n, $n3^n$, and $n^2 3^n$ are linearly independent on $n \geq 1$.

Solution Suppose that for constants a_1, a_2, and a_3 we have

$$a_1 3^n + a_2 n 3^n + a_3 n^2 3^n = 0, \qquad \text{for all } n \geq 1.$$

Then by dividing by 3^n we get

$$a_1 + a_2 n + a_3 n^2 = 0, \qquad \text{for all } n \geq 1.$$

This is impossible unless $a_3 = 0$, since a second-degree equation in n possesses at most two solutions $n \geq 1$. Hence $a_1 = a_2 = a_3 = 0$. Similarly, $a_2 = 0$, whence $a_1 = 0$, which establishes the linear independence of our functions.

Definition 2.10. A set of k linearly independent solutions of (2.2.7) is called a *fundamental set* of solutions.

As you may have noticed from Example 2.9, it is not practical to check the linear independence of a set of solutions using the definition. Fortunately, there is a simple method to check the linear independence of solutions using the so-called Casoratian $W(n)$, which we now define for the eager reader.

Definition 2.11. The Casoratian[4] $W(n)$ of the solutions $x_1(n), x_2(n), \ldots,$ $x_r(n)$ is given by

$$W(n) = \det \begin{pmatrix} x_1(n) & x_2(n) & \ldots & x_r(n) \\ x_1(n+1) & x_2(n+1) & \ldots & x_r(n+1) \\ \vdots & & & \\ x_1(n+r-1) & x_2(n+r-1) & \ldots & x_r(n+r-1) \end{pmatrix}. \tag{2.2.9}$$

Example 2.12. Consider the difference equation

$$x(n+3) - 7x(n+1) + 6x(n) = 0.$$

(a) Show that the sequences 1, $(-3)^n$, and 2^n are solutions of the equation.

(b) Find the Casoratian of the sequences in part (a).

Solution

(a) Note that $x(n) = 1$ is a solution, since $1 - 7 + 6 = 0$. Furthermore, $x(n) = (-3)^n$ is a solution, since

$$(-3)^{n+3} - 7(-3)^{n+1} + 6(-3)^n = (-3)^n[-27 + 21 + 6] = 0.$$

Finally, $x(n) = 2^n$ is a solution, since

$$(2)^{n+3} - 7(2)^{n+1} + 6(2)^n = 2^n[8 - 14 + 6] = 0.$$

[4]This is the discrete analogue of the Wronskian in differential equations.

(b) Now,

$$
\begin{aligned}
W(n) &= \det \begin{pmatrix} 1 & (-3)^n & 2^n \\ 1 & (-3)^{n+1} & 2^{n+1} \\ 1 & (-3)^{n+2} & 2^{n+2} \end{pmatrix} \\
&= \begin{vmatrix} (-3)^{n+1} & (2)^{n+1} \\ (-3)^{n+2} & (2)^{n+2} \end{vmatrix} - (-3)^n \begin{vmatrix} 1 & (2)^{n+1} \\ 1 & (2)^{n+2} \end{vmatrix} \\
&\quad + (2)^n \begin{vmatrix} 1 & (-3)^{n+1} \\ 1 & (-3)^{n+2} \end{vmatrix} \\
&= (2)^{n+2}(-3)^{n+1} - (2)^{n+1}(-3)^{n+2} - (-3)^n((2)^{n+2} - (2)^{n+1}) \\
&\quad + (2)^n((-3)^{n+2} - (-3)^{n+1}) \\
&= -12(2)^n(-3)^n - 18(2)^n(-3)^n - 4(2)^n(-3)^n \\
&\quad + 2(2)^n(-3)^n + 9(2)^n(-3)^n + 3(2)^n(-3)^n \\
&= -20(2)^n(-3)^n.
\end{aligned}
$$

Next we give a formula, called Abel's formula, to compute the Casoratian $W(n)$. The significance of Abel's formula is its effectiveness in the verification of the linear independence of solutions.

Lemma 2.13 (Abel's Lemma). *Let $x_1(n), x_2(n), \ldots, x_k(n)$ be solutions of* (2.2.7) *and let $W(n)$ be their Casoratian. Then, for $n \geq n_0$,*

$$\boxed{W(n) = (-1)^{k(n-n_0)} \left(\prod_{i=n_0}^{n-1} p_k(i) \right) W(n_0).} \tag{2.2.10}$$

Proof. We will prove the lemma for $k = 3$, since the general case may be established in a similar fashion. So let $x_1(n)$, $x_2(n)$, and $x_3(n)$ be three independent solutions of (2.2.7). Then from formula (2.2.9) we have

$$W(n+1) = \det \begin{pmatrix} x_1(n+1) & x_2(n+1) & x_3(n+1) \\ x_1(n+2) & x_2(n+2) & x_3(n+2) \\ x_1(n+3) & x_2(n+3) & x_3(n+3) \end{pmatrix}. \tag{2.2.11}$$

From (2.2.7) we have, for $1 \leq i \leq 3$,

$$x_i(n+3) = -p_3(n)x_i(n) - \left[p_1(n)x_i(n+2) + p_2(n)x_i(n+1)\right]. \tag{2.2.12}$$

Now, if we use formula (2.2.12) to substitute for $x_1(n+3)$, $x_2(n+3)$, and $x_3(n+3)$ in the last row of formula (2.2.11), we obtain

$$W(n+1) = \det \begin{pmatrix} x_1(n+1) & x_2(n+1) & x_3(n+1) \\ x_1(n+2) & x_2(n+2) & x_3(n+2) \\ -p_3x_1(n) & -p_3x_2(n) & -p_3x_3(n) \\ -\big(p_2x_1(n+1) & -\big(p_2x_2(n+1) & -\big(p_2x_3(n+1) \\ +p_1x_1(n+2)\big) & +p_1x_2(n+2)\big) & +p_1x_3(n+2)\big) \end{pmatrix}. \tag{2.2.13}$$

Using the properties of determinants, it follows from (2.2.13) that

$$W(n+1) = \det \begin{pmatrix} x_1(n+1) & x_2(n+1) & x_3(n+1) \\ x_1(n+2) & x_2(n+2) & x_3(n+2) \\ -p_3(n)x_1(n) & -p_3(n)x_2(n) & -p_3(n)x_3(n) \end{pmatrix} \tag{2.2.14}$$

$$= -p_3(n) \det \begin{pmatrix} x_1(n+1) & x_2(n+1) & x_3(n+1) \\ x_1(n+2) & x_2(n+2) & x_3(n_2) \\ x_1(n) & x_2(n) & x_3(n) \end{pmatrix}$$

$$= -p_3(n)(-1)^2 \det \begin{pmatrix} x_1(n) & x_2(n) & x_3(n) \\ x_1(n+2) & x_2(n+2) & x_3(n+2) \\ x_1(n+1) & x_2(n+1) & x_3(n+1) \end{pmatrix}.$$

Thus

$$W(n+1) = (-1)^3 p_3(n) W(n). \tag{2.2.15}$$

Using formula (1.2.3), the solution of (2.2.15) is given by

$$\boxed{W(n) = \left[\prod_{i=n_0}^{n-1} (-1)^3 p_3(i)\right] W(n_0) = (-1)^{3(n-n_0)} \prod_{i=n_0}^{n-1} p_3(i) W(n_0).}$$

□

This completes the proof of the lemma for $k = 3$. The general case is left to the reader as Exercises 2.2, Problem 6.

We now examine and treat one of the special cases that arises as we try to apply this Casoratian. For example, if (2.2.7) has constant coefficients $p_1, p_2, \ldots, p_k$, then we have

$$\boxed{W(n) = (-1)^{k(n-n_0)} p_k^{(n-n_0)} W(n_0).} \tag{2.2.16}$$

Formula (2.2.10) has the following important correspondence.

Corollary 2.14. *Suppose that $p_k(n) \neq 0$ for all $n \geq n_0$. Then the Casoratian $W(n) \neq 0$ for all $n \geq n_0$ if and only if $W(n_0) \neq 0$.*

Proof. This corollary follows immediately from formula (2.2.10) (Exercises 2.2, Problem 7). □

Let us have a close look at Corollary 2.14 and examine what it really says. The main point in the corollary is that either the Casoratian is identically zero (i.e., zero for all $n \geq n_0$, for some n_0) or never zero for any $n \geq n_0$. Thus to check whether $W(n) \neq 0$ for all $n \in \mathbb{Z}^+$, we need only to check whether $W(0) \neq 0$. Note that we can always choose the most suitable n_0 and compute $W(n_0)$ there.

Next we examine the relationship between the linear independence of solutions and their Casoratian. Basically, we will show that a set of k solutions is a *fundamental set* (i.e., linearly independent) if their Casoratian $W(n)$ is never zero.

To determine the preceding statement we contemplate k solutions $x_1(n), x_2(n), \ldots, x_k(n)$ of (2.2.7). Suppose that for some constants a_1, $a_2, \ldots, a_k$ and $n_0 \in \mathbb{Z}^+$,

$$a_1 x_1(n) + a_2 x_2(n) + \cdots + a_k(n)\ x_k(n) = 0, \qquad \text{for all } n \geq n_0.$$

Then we can generate the following $k-1$ equations:

$$\begin{aligned} a_1 x_1(n+1) + a_2 x_2(n+1) + \cdots + a_k x_k(n+1) &= 0, \\ &\vdots \\ a_1 x_1(n+k-1) + a_2 x_2(n+k-1) + \cdots + a_k x_k(n+k-1) &= 0. \end{aligned}$$

This assemblage may be transcribed as

$$X(n)\xi = 0, \tag{2.2.17}$$

where

$$X(n) = \begin{pmatrix} x_1(n) & x_2(n) & \ldots & x_k(n) \\ x_1(n+1) & x_2(n+1) & \ldots & x_k(n+1) \\ \vdots & \vdots & & \vdots \\ x_1(n+k-1) & x_2(n+k-1) & \ldots & x_k(n+k-1) \end{pmatrix},$$

$$\xi = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}.$$

Observe that $W(n) = \det X(n)$.

Linear algebra tells us that the vector (2.2.17) has only the trivial (or zero) solution (i.e., $a_1 = a_2 = \cdots = a_k = 0$) if and only if the matrix $X(n)$ is nonsingular (invertible) (i.e., $\det X(n) = W(n) \neq 0$ for all $n \geq n_0$). This deduction leads us to the following conclusion.

Theorem 2.15. *The set of solutions $x_1(n), x_2(n), \ldots, x_k(n)$ of* (2.2.7) *is a fundamental set if and only if for some $n_0 \in Z^+$, the Casoratian $W(n_0) \neq 0$.*

Proof. Exercises 2.2, Problem 8. □

Example 2.16. Verify that $\{n, 2^n\}$ is a fundamental set of solutions of the equation

$$x(n+2) - \frac{3n-2}{n-1}x(n+1) + \frac{2n}{n-1}x(n) = 0.$$

Solution We leave it to the reader to verify that n and 2^n are solutions of the equation. Now, the Casoratian of the solutions n, 2^n is given by

$$W(n) = \det \begin{pmatrix} n & 2^n \\ n+1 & 2^{n+1} \end{pmatrix}.$$

Thus

$$W(0) = \det \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = -1 \neq 0.$$

Hence by Theorem 2.15, the solutions n, 2^n are linearly independent and thus form a fundamental set.

Example 2.17. Consider the third-order difference equation

$$x(n+3) + 3x(n+2) - 4x(n+1) - 12x(n) = 0.$$

Show that the functions 2^n, $(-2)^n$, and $(-3)^n$ form a fundamental set of solutions of the equation.

Solution

(i) Let us verify that 2^n is a legitimate solution by substituting $x(n) = 2^n$ into the equation:

$$2^{n+3} + (3)(2^{n+1}) - (4)(2^{n+1}) - (12)(2^n) = 2^n[8 + 12 - 8 - 12] = 0.$$

We leave it to the reader to verify that $(-2)^n$ and $(-3)^n$ are solutions of the equation.

(ii) To affirm the linear independence of these solutions we construct the Casoratian

$$W(n) = \det \begin{pmatrix} 2^n & (-2)^n & (-3)^n \\ 2^{n+1} & (-2)^{n+1} & (-3)^{n+1} \\ 2^{n+2} & (-2)^{n+2} & (-3)^{n+2} \end{pmatrix}.$$

Thus

$$W(0) = \det \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 3 \\ 4 & 4 & 9 \end{pmatrix} = -20 \neq 0.$$

By Theorem 2.15, the solutions 2^n, $(-2)^n$, and 3^n are linearly independent, and thus form a fundamental set.

We are now ready to discuss the fundamental theorem of homogeneous linear difference equations.

Theorem 2.18 (The Fundamental Theorem). *If $p_k(n) \neq 0$ for all $n \geq n_0$, then* (2.2.7) *has a fundamental set of solutions for $n \geq n_0$.*

Proof. By Theorem 2.7, there are solutions $x_1(n), x_2(n), \ldots, x_k(n)$ such that $x_i(n_0+i-1) = 1$, $x_i(n_0) = x_i(n_0+1) = \cdots = x_i(n_0+i-2) = x_i(n_0+i) = \cdots = x_i(n_0+k-1) = 0, 1 \leq i \leq k$. Hence $x_1(n_0) = 1, x_2(n_0+1) = 1, x_3(n_0+2) = 1, \ldots, x_k(n_0+k-1) = 1$. It follows that $W(n_0) = \det I = 1$. This implies by Theorem 2.15 that the set $\{x_1(n), x_2(n), \ldots, x_k(n)\}$ is a fundamental set of solutions of (2.2.7). □

We remark that there are infinitely many fundamental sets of solutions of (2.2.7). The next result presents a method of generating fundamental sets starting from a known set.

Lemma 2.19. *Let $x_1(n)$ and $x_2(n)$ be two solutions of* (2.2.7)*. Then the following statements hold:*

(i) *$x(n) = x_1(n) + x_2(n)$ is a solution of* (2.2.7).

(ii) *$\tilde{x}(n) = ax_1(n)$ is a solution of* (2.2.7) *for any constant a.*

Proof. (Exercises 2.2, Problem 9.) □

From the preceding lemma we conclude the following principle.

Superposition Principle. If $x_1(n), x_2(n), \ldots, x_r(n)$ are solutions of (2.2.7), then

$$x(n) = a_1x_1(n) + a_2x_2(n) + \cdots + a_rx_r(n)$$

is also a solution of (2.2.7) (Exercises 2.2, Problem 12).

Now let $\{x_1(n), x_2(n), \ldots, x_k(n)\}$ be a fundamental set of solutions of (2.2.7) and let $x(n)$ be any given solution of (2.2.7). Then there are constants $a_1, a_2, \ldots, a_k$ such that $x(n) = \sum_{i=1}^{k} a_ix_i(n)$. To show this we use the notation (2.2.17) to write $X(n)\xi = \hat{x}(n)$, where

$$\hat{x}(n) = \begin{pmatrix} x(n) \\ x(n+1) \\ \vdots \\ x(n+k-1) \end{pmatrix}.$$

Since $X(n)$ is invertible (Why?), it follows that

$$\xi = X^{-1}(n)\hat{x}(n),$$

and, for $n = n_0$,

$$\xi = X^{-1}(n_0)\hat{x}(n_0).$$

The above discussion leads us to define the general solution of (2.2.7).

Definition 2.20. Let $\{x_1(n), x_2(n), \ldots, x_k(n)\}$ be a fundamental set of solutions of (2.2.7). Then the *general solution* of (2.2.7) is given by $x(n) = \sum_{i=1}^{k} a_i x_i(n)$, for arbitrary constants a_i.

It is worth noting that any solution of (2.2.7) may be obtained from the general solution by a suitable choice of the constants a_i.

The preceding results may be restated using the elegant language of linear algebra as follows: Let S be the set of all solutions of (2.2.7) with the operations $+$, $\cdot$ defined as follows:

(i) $(x + y)(n) = x(n) + y(n), \qquad \text{for } x, y \in S, \quad n \in Z^+,$

(ii) $(ax)(n) = ax(n), \qquad \text{for } x \in S, a$ constant.

Equipped with linear algebra we now summarize the results of this section in a compact form.

Theorem 2.21. *The space $(S, +, \cdot)$ is a linear (vector) space of dimension k.*

Proof. Use Lemma 2.19. To construct a basis of S we can use the fundamental set in Theorem 2.18 (Exercises 2.2, Problem 11). □

Exercises 2.2

1. Find the Casoratian of the following functions and determine whether they are linearly dependent or independent:

   (a) $5^n, 3 \cdot 5^{n+2}, e^n$.

   (b) $5^n, n\,5^n, n^2\,5^n$.

   (c) $(-2)^n, 2^n, 3$.

   (d) $0, 3^n, 7^n$.

2. Find the Casoratian $W(n)$ of the solutions of the difference equations:

   (a) $x(n+3) - 10x(n+2) + 31x(n+1) - 30x(n) = 0$, if $W(0) = 6$.

   (b) $x(n+3) - 3x(n+2) + 4x(n+1) - 12x(n) = 0$, if $W(0) = 26$.

3. For the following difference equations and their accompanied solutions:

   (i) determine whether these solutions are linearly independent, and

(ii) find, if possible, using only the given solutions, the general solution:
(a) $x(n+3) - 3x(n+2) + 3x(n+1) - x(n) = 0; \quad 1, n, n^2,$
(b) $x(n+2) + x(n) = 0; \quad \cos\left(\frac{n\pi}{2}\right), \sin\left(\frac{n\pi}{2}\right),$
(c) $x(n+3)+x(n+2)-8x(n+1)-12x(n) = 0; \quad 3^n, (-2)^n, (-2)^{n+3},$
(d) $x(n+4) - 16x(n) = 0; \quad 2^n, n2^n, n^2 2^n.$

4. Verify formula (2.2.10) for the general case.

5. Show that the Casoratian $W(n)$ in formula (2.2.9) may be given by the formula

$$W(n) = \det \begin{pmatrix} x_1(n) & x_2(n) & \dots & x_k(n) \\ \Delta x_1(n) & \Delta x_2(n) & \dots & \Delta x_k(n) \\ \vdots & \vdots & & \vdots \\ \Delta^{k-1} x_1(n) & \Delta^{k-1} x_2(n) & \dots & \Delta^{k-1} x_k(n) \end{pmatrix}.$$

6. Verify formula (2.2.16).

7. Prove Corollary 2.14.

8. Prove Theorem 2.15.

9. Prove Lemma 2.19.

10. Prove the superposition principle: If $x_1(n), x_2(n), \dots, x_r$ are solutions of (2.2.7), then any linear combination of them is also a solution of (2.2.7).

11. Prove Theorem 2.21.

12. Suppose that for some integer $m \geq n_0, p_k(m) = 0$ in (2.2.1).

    (a) What is the value of the Casoratian for $n \geq m$?

    (b) Does Corollary 2.14 still hold? (Why?)

*13. Show that the equation $\Delta^2 y(n) = p(n)y(n+1)$ has a fundamental set of solutions whose Casoratian $W(n) = -1$.

14. Contemplate the second-order difference equation $u(n+2)+p_1(n)u(n+1) + p_2(n)u(n) = 0$. If $u_1(n)$ and $u_2(n)$ are solutions of the equation and $W(n)$ is their Casoratian, prove that

$$\boxed{u_2(n) = u_1(n) \left[\sum_{r=0}^{n-1} W(r)/u_1(r)u_1(r+1) \right].} \tag{2.2.18}$$

15. Contemplate the second-order difference equation $u(n+2) - \frac{(n+3)}{(n+2)} u(n+1) + \frac{2}{(n+2)} u(n) = 0$.

(a) Verify that $u_1(n) = \frac{2^n}{n!}$ is a solution of the equation.

(b) Use formula (2.2.18) to find another solution $u_2(n)$ of the equation.

16. Show that $u(n) = (n+1)$ is a solution of the equation $u(n+2) - u(n+1) - 1/(n+1)u(n) = 0$ and then find a second solution of the equation by using the method of Exercises 2.2, Problem 15.

2.3 Linear Homogeneous Equations with Constant Coefficients

Consider the kth-order difference equation

$$x(n+k) + p_1 x(n+k-1) + p_2 x(n+k-2) + \cdots + p_k x(n) = 0, \quad (2.3.1)$$

where the p_i's are constants and $p_k \neq 0$. Our objective now is to find a fundamental set of solutions and, consequently, the general solution of (2.3.1). The procedure is rather simple. We suppose that solutions of (2.3.1) are in the form λ^n, where λ is a complex number. Substituting this value into (2.3.1), we obtain

$$\lambda^k + p_1 \lambda^{k-1} + \cdots + p_k = 0. \quad (2.3.2)$$

This is called the *characteristic equation* of (2.3.1), and its roots λ are called the *characteristic roots*. Notice that since $p_k \neq 0$, none of the characteristic roots is equal to zero. (Why?) (Exercises 2.3, Problem 19.)

We have two situations to contemplate:

Case (a). Suppose that the characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct. We are now going to show that the set $\{\lambda_1^n, \lambda_2^n, \ldots, \lambda_k^n\}$ is a fundamental set of solutions. To prove this, by virtue of Theorem 2.15 it suffices to show that $W(0) \neq 0$, where $W(n)$ is the Casoratian of the solutions. That is,

$$W(0) = \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_k^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \ldots & \lambda_k^{k-1} \end{pmatrix}. \quad (2.3.3)$$

This determinant is called the *Vandermonde determinant.*

It may be shown by mathematical induction that

$$\boxed{W(0) = \prod_{1 \le i < j \le k} (\lambda_j - \lambda_i).} \quad (2.3.4)$$

The reader will prove this conclusion in Exercises 2.3, Problem 20.

Since all the λ_i's are distinct, it follows from (2.3.4) that $W(0) \neq 0$. This fact proves that $\{\lambda_1^n, \lambda_2^n, \ldots, \lambda_k^n\}$ is a fundamental set of solutions of (2.3.1). Consequently, the general solution of (2.3.1) is

$$x(n) = \sum_{i=1}^{k} a_i \lambda_i^n, \quad a_i \ \text{ a complex number.} \tag{2.3.5}$$

Case (b). Suppose that the distinct characteristic roots are $\lambda_1, \lambda_2, \ldots, \lambda_r$ with multiplicities $m_1, m_2, \ldots, m_r$ with $\sum_{i=1}^{r} m_i = k$, respectively. In this case, (2.3.1) may be written as

$$(E-\lambda_1)^{m_1}(E-\lambda_2)^{m_2}\cdots(E-\lambda_r)^{m_r}x(n) = 0. \tag{2.3.6}$$

A vital observation here is that if $\psi_1(n), \psi_2(n), \ldots, \psi_{m_i}(n)$ are solutions of

$$(E-\lambda_i)^{m_i}x(n) = 0, \tag{2.3.7}$$

then they are also solutions of (2.3.6). For if $\Psi_s(n)$ is a solution of (2.3.7), then $(E-\lambda_i)^{m_i}\Psi_s(n) = 0$. Now

$$\begin{aligned}&(E-\lambda_1)^{m_1}\cdots(E-\lambda_i)^{m_i}\cdots(E-\lambda_r)^{m_r}\Psi_s(n)\\ &= (E-\lambda_1)^{m_1}\cdots(E-\lambda_{i-1})^{m_{i-1}}(E-\lambda_{i+1})^{m_{i+1}}\cdots\\ &(E-\lambda_r)^{m_r}(E-\lambda_i)^{m_i}\Psi_s(n) = 0.\end{aligned}$$

Suppose we are able to find a fundamental set of solutions for each (2.3.7), $1 \le i \le r$. It is not unreasonable to expect, then, that the union of these r fundamental sets would be a fundamental set of solutions of (2.3.6). In the following lemma we will show that this is indeed the case.

Lemma 2.22. *The set* $G_i = \left\{\lambda_i^n, \binom{n}{1}\lambda_i^{n-1}, \binom{n}{2}\lambda_2^{n-2}, \ldots, \binom{n}{m_i-1}\lambda_i^{n-m_i+1}\right\}$ *is a fundamental set of solutions of* (2.3.7) *where* $\binom{n}{1} = n, \binom{n}{2} = \frac{n(n-1)}{2!}, \ldots, \binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$.

PROOF. To show that G_i is a fundamental set of solutions of (2.3.7), it suffices, by virtue of Corollary 2.14, to show that $W(0) \neq 0$. But

$$W(0) = \begin{vmatrix} 1 & 0 & \ldots & 0 \\ \lambda_i & 1 & \ldots & 0 \\ \lambda_i^2 & 2\lambda_i & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ \lambda_i^{m_i-1} & \dfrac{(m_i-1)}{1!}\lambda_i^{m_i-2} & \ldots & \dfrac{1}{2!3!\cdots(m_i-2)!} \end{vmatrix}.$$

Hence

$$W(0) = \frac{1}{(2!3!\ldots(m_i-2)!} \neq 0.$$

It remains to show that $\binom{n}{r}\lambda_i^{n-r}$ is a solution of (2.3.7). From equation (2.1.9) it follows that

$$\begin{aligned}(E-\lambda_i)^{m_i}\binom{n}{r}\lambda_i^{n-r} &= \lambda_i^{n-r}(\lambda_i E-\lambda_i)^{m_i}\binom{n}{r}\\ &= \lambda_i^{n+m_i-r}(E-I)^{m_i}\binom{n}{r}\\ &= \lambda_i^{n+m_i-r}\Delta^{m_i}\binom{n}{r}\\ &= 0 \quad \text{using (2.1.6).}\end{aligned}$$ □

Now we are finally able to find a fundamental set of solutions.

Theorem 2.23. *The set $G = \bigcup_{i=1}^{r} G_i$ is a fundamental set of solutions of (2.3.6).*

Proof. By Lemma 2.22, the functions in G are solutions of (2.3.6). Now

$$W(0) = \det\begin{pmatrix} 1 & 0 & \dots & 1 & 0 & \dots\\ \lambda_1 & 1 & \dots & \lambda_r & 1 & \dots\\ \lambda_1^2 & 2\lambda_1 & \dots & \lambda_r^2 & 2\lambda_r & \dots\\ \vdots & \vdots & & \vdots & \vdots & \\ \lambda_1^{k-1} & (k-1)\lambda_1^{k-2} & \dots & \lambda_r^{k-1} & (k-1)\lambda_r^{k-2} & \dots \end{pmatrix}. \tag{2.3.8}$$

This determinant is called the generalized Vandermonde determinant. (See Appendix B.) It may be shown [76] that

$$\boxed{W(0) = \prod_{1\le i<j\le k}(\lambda_j-\lambda_i)^{m_j m_i}.} \tag{2.3.9}$$

As $\lambda_i \neq \lambda_j$, $W(0) \neq 0$. Hence by Corollary 2.14 the Casoratian $W(n) \neq 0$ for all $n \geq 0$. Thus by Theorem 2.15, G is a fundamental set of solutions. □

Corollary 2.24. *The general solution of (2.3.6) is given by*

$$x(n) = \sum_{i=1}^{r}\lambda_i^n\left(a_{i0}+a_{i1}n+a_{i2}n^2+\cdots+a_{i,m_i-1}n^{m_i-1}\right). \tag{2.3.10}$$

Proof. Use Lemma 2.22 and Theorem 2.23. □

Example 2.25. Solve the equation

$$\begin{gathered}x(n+3)-7x(n+2)+16x(n+1)-12x(n)=0,\\ x(0)=0,\quad x(1)=1,\quad x(2)=1.\end{gathered}$$

Solution The characteristic equation is

$$r^3 - 7r^2 + 16r - 12 = 0.$$

Thus, the characteristic roots are $\lambda_1 = 2 = \lambda_2, \lambda_3 = 3$.

The characteristic roots give us the general solution

$$x(n) = a_0 2^n + a_1 n 2^n + b_1 3^n.$$

To find the constants a_0, a_1, and b_1, we use the initial data

$$\begin{aligned} x(0) &= a_0 + b_1 = 0, \\ x(1) &= 2a_0 + 2a_1 + 3b_1 = 1, \\ x(2) &= 4a_0 + 8a_1 + 9b_1 = 1. \end{aligned}$$

Finally, after solving the above system of equations, we obtain

$$a_0 = 3, \quad a_1 = 2, \quad b_1 = -3.$$

Hence the solution of the equation is given by $x(n) = 3(2^n) + 2n(2^n) - 3^{n+1}$.

Example 2.26. Complex Characteristic Roots

Suppose that the equation $x(n+2) + p_1 x(n+1) + p_2 x(n) = 0$ has the complex roots $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$. Its general solution would then be

$$x(n) = c_1(\alpha + i\beta)^n + c_2(\alpha - i\beta)^n.$$

Recall that the point (α, β) in the complex plane corresponds to the complex number $\alpha + i\beta$. In polar coordinates,

$$\alpha = r\cos\theta, \qquad \beta = r\sin\theta, \qquad r = \sqrt{\alpha^2 + \beta^2}, \qquad \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right).$$

Hence,[5]

$$\begin{aligned} x(n) &= c_1(r\cos\theta + ir\sin\theta)^n + c_2(r\cos\theta - ir\sin\theta)^n \\ &= r^n[(c_1 + c_2)\cos(n\theta) + i(c_1 - c_2)\sin(n\theta)] \\ &= r^n[a_1\cos(n\theta) + a_2\sin(n\theta)], \end{aligned} \tag{2.3.11}$$

where $a_1 = c_1 + c_2$ and $a_2 = i(c_1 - c_2)$.

Let

$$\cos\omega = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}, \quad \sin\omega = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}, \quad \omega = \tan^{-1}\left(\frac{a_2}{a_1}\right).$$

[5] We used De Moivre's Theorem: $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$.

Then (2.3.11) becomes

$$\begin{aligned} x(n) &= r^n\sqrt{a_1^2+a_2^2}[\cos\omega\cos(n\theta)+\sin\omega\sin(n\theta)] \\ &= r^n\sqrt{a_1^2+a_2^2}\cos(n\theta-\omega), \\ x(n) &= Ar^n\cos(n\theta-\omega). \end{aligned} \tag{2.3.12}$$

Example 2.27. The Fibonacci Sequence (The Rabbit Problem)

This problem first appeared in 1202, in *Liber abaci*, a book about the abacus, written by the famous Italian mathematician Leonardo di Pisa, better known as Fibonacci. The problem may be stated as follows: How many pairs of rabbits will there be after one year if starting with one pair of mature rabbits, if each pair of rabbits gives birth to a new pair each month starting when it reaches its maturity age of two months? (See Figure 2.1.)

Table 2.2 shows the number of pairs of rabbits at the end of each month. The first pair has offspring at the end of the first month, and thus we have two pairs. At the end of the second month only the first pair has offspring, and thus we have three pairs. At the end of the third month, the first and second pairs will have offspring, and hence we have five pairs. Continuing this procedure, we arrive at Table 2.2. If $F(n)$ is the number of pairs of rabbits at the end of n months, then the recurrence relation that represents this model is given by the second-order linear difference equation

$$F(n+2)=F(n+1)+F(n), \qquad F(0)=1, \qquad F(1)=2, \qquad 0\le n\le 10.$$

This example is a special case of the Fibonacci sequence, given by

$$F(n+2)=F(n+1)+F(n), \qquad F(0)=0, \qquad F(1)=1, \qquad n\ge 0. \tag{2.3.13}$$

The first 14 terms are given by 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, and 377, as already noted in the rabbit problem.

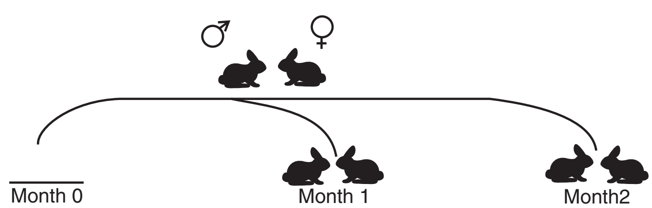

FIGURE 2.1.

TABLE 2.2. Rabbits' population size.

Month	0	1	2	3	4	5	6	7	8	9	10	11	12
Pairs	1	2	3	5	8	13	21	34	55	89	144	233	377

The characteristic equation of (2.3.13) is

$$\lambda^2 - \lambda - 1 = 0.$$

Hence the characteristic roots are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.
The general solution of (2.3.13) is

$$F(n) = a_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + a_2 \left(\frac{1-\sqrt{5}}{2}\right)^n, \qquad n \geq 1. \tag{2.3.14}$$

Using the initial values $F(1) = 1$ and $F(2) = 1$, one obtains

$$a_1 = \frac{1}{\sqrt{5}}, \qquad a_2 = -\frac{1}{\sqrt{5}}.$$

Consequently,

$$F(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n). \tag{2.3.15}$$

It is interesting to note that $\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \alpha \approx 1.618$ (Exercises 2.3, Problem 15). This number is called the *golden mean*, which supposedly represents the ratio of the sides of a rectangle that is most pleasing to the eye. This Fibonacci sequence is very interesting to mathematicians; in fact, an entire publication, *The Fibonacci Quarterly*, dwells on the intricacies of this fascinating sequence.

Exercises 2.3

1. Find homogeneous difference equations whose solutions are:
   (a) $2^{n-1} - 5^{n+1}$.
   (b) $3\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)$.
   (c) $(n+2)5^n \sin\left(\frac{n\pi}{4}\right)$.
   (d) $(c_1 + c_2 n + c_3 n^2)7^n$.
   (e) $1 + 3n - 5n^2 + 6n^3$.

2. Find a second-order linear homogeneous difference equation that generates the sequence 1, 2, 5, 12, 29, …; then write the solution of the obtained equation.

In each of Problems 3 through 8, write the general solution of the difference equation.

3. $x(n+2) - 16x(n) = 0$.
4. $x(n+2) + 16x(n) = 0$.
5. $(E-3)^2(E^2+4)x(n) = 0$.

6. $\Delta^3 x(n) = 0$.

7. $(E^2 + 2)^2 x(n) = 0$.

8. $x(n+2) - 6x(n+1) + 14x(n) = 0$.

9. Consider Example 2.26. Verify that $x_1(n) = r^n \cos n\theta$ and $x_2(n) = r^n \sin n\theta$ are two linearly independent solutions of the given equation.

10. Consider the integral defined by

$$I_k(\varphi) = \int_0^\pi \frac{\cos(k\theta) - \cos(k\varphi)}{\cos\theta - \cos\varphi} d\theta, \quad k = 0, 1, 2, \ldots, \quad \varphi \in \mathbb{R}.$$

(a) Show that $I_k(\varphi)$ satisfies the difference equation

$$I_{n+2}(\varphi) - 2\cos\varphi I_{n+1}(\varphi) + I_n(\varphi) = 0, \quad I_0(\varphi) = 0, \quad I_1(\varphi) = \pi.$$

(b) Solve the difference equation in part (a) to find $I_n(\varphi)$.

11. The Chebyshev polynomials of the first and second kinds are defined, respectively, as follows:

$$T_n(x) = \cos(n \cos^{-1}(x)), \qquad U_n(x) = \frac{1}{\sqrt{1-x^2}} \sin[(n+1)\cos^{-1}(x)],$$

for $|x| < 1$.

(a) Show that $T_n(x)$ obeys the difference equation

$$T_{n+2}(x) - 2xT_{n+1}(x) + T_n(x) = 0, \quad T_0(x) = 1, \quad T_1(x) = x.$$

(b) Solve the difference equation in part (a) to find $T_n(x)$.

(c) Show that $U_n(x)$ satisfies the difference equation

$$U_{n+2}(x) - 2xU_{n+1}(x) + U_n(x) = 0, \quad U_0(x) = 1, \quad U_1(x) = 2x.$$

(d) Write down the first three terms of $T_n(x)$ and $U_n(x)$.

(e) Show that $T_n(\cos\theta) = \cos n\theta$ and that

$$U_n(\cos\theta) = (\sin[(n+1)\theta])/\sin\theta.$$

12. Show that the general solution of

$$x(n+2) - 2sx(n+1) + x(n) = 0, \qquad |s| < 1,$$

is given by

$$x(n) = c_1 T_n(s) + c_2 U_n(s).$$

13. Show that the general solution of $x(n+2) + p_1 x(n+1) + p_2 x(n) = 0$, $p_2 > 0$, $p_1^2 < 4p_2$, is given by $x(n) = r^n[c_1 T_n(s) + c_2 U_{n-1}(s)]$, where $r = \sqrt{p_2}$ and $s = P_1/(2\sqrt{p_2})$.

14. The Lucas numbers L_n are defined by the difference equation

$$L_{n+2} = L_{n+1} + L_n, \quad n \geq 0, \quad L_0 = 2, \quad L_1 = 1.$$

Solve the difference equation to find L_n.

15. Show that $\lim_{n\to\infty}(F(n+1))/F(n) = \alpha$, where $\alpha = (1+\sqrt{5})/2$.

16. Prove that consecutive Fibonacci numbers $F(n)$ and $F(n+1)$ are relatively prime.

17. (a) Prove that $F(n)$ is the nearest integer to $1/\sqrt{5}((1+\sqrt{5})/2)^n$.

 (b) Find $F(17)$, $F(18)$, and $F(19)$, applying part (a).

*18. Define $x = a \bmod p$ if $x = mp + a$. Let p be a prime number with $p > 5$.

 (a) Show that $F(p) = 5^{(p-1)/2} \bmod p$.

 (b) Show that $F(p) = \pm 1 \bmod p$.

19. Show that if $p_k \neq 0$ in (2.3.1), then none of its characteristic roots is equal to zero.

20. Show that the Vandermonde determinant (2.3.3) is equal to

$$\prod_{1\leq i<j\leq k} (\lambda_j - \lambda_i).$$

21. Find the value of the $n \times n$ tridiagonal determinant

$$D(n) = \begin{vmatrix} b & a & 0 & \dots & 0 & 0 \\ a & b & a & \dots & 0 & 0 \\ 0 & a & b & \dots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & b & a \\ 0 & 0 & 0 & \dots & a & b \end{vmatrix}.$$

22. Find the value of the $n \times n$ tridiagonal determinant

$$D(n) = \begin{vmatrix} a & b & 0 & \dots & 0 & 0 \\ c & a & b & \dots & 0 & 0 \\ 0 & c & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & b \\ 0 & 0 & 0 & \dots & c & a \end{vmatrix}.$$

2.4 Linear Nonhomogeneous Equations: Method of Undetermined Coefficients

In the last two sections we developed the theory of linear homogeneous difference equations. Moreover, in the case of equations with constant coefficients we have shown how to construct their solutions. In this section we focus our attention on solving the kth-order linear nonhomogeneous equation

$$y(n+k) + p_1(n)y(n+k-1) + \cdots + p_k(n)y(n) = g(n), \tag{2.4.1}$$

where $p_k(n) \neq 0$ for all $n \geq n_0$. The sequence $g(n)$ is called the *forcing term*, the *external force*, the *control*, or the *input* of the system. As we will discuss later in Chapter 6, equation (2.4.1) represents a physical system in which $g(n)$ is the input and $y(n)$ is the output (Figure 2.2). Thus solving (2.4.1) amounts to determining the output $y(n)$ given the input $g(n)$. We may look at $g(n)$ as a control term that the designing engineer uses to force the system to behave in a specified way.

Before proceeding to present general results concerning (2.4.1) we would like to raise the following question: Do solutions of (2.4.1) form a vector space? In other words, is the sum of two solutions of (2.4.1) a solution of (2.4.1)? And is a multiple of a solution of (2.4.1) a solution of (2.4.1)? Let us answer these questions through the following example.

Example 2.28. Contemplate the equation

$$y(n+2) - y(n+1) - 6y(n) = 5(3^n).$$

(a) Show that $y_1(n) = n(3^{n-1})$ and $y_2(n) = (1+n)3^{n-1}$ are solutions of the equation.

(b) Show that $y(n) = y_2(n) - y_1(n)$ is not a solution of the equation.

(c) Show that $\varphi(n) = cn(3^{n-1})$ is not a solution of the equation, where c is a constant.

Solution

(a) The verification that y_1 and y_2 are solutions is left to the reader.

(b) $y(n) = y_2(n) - y_1(n) = 3^{n-1}$. Substituting this into the equation yields

$$3^{n+1} - 3^n - 63^{n-1} = 3^n[3 - 1 - 2] = 0 \neq 5(3^n).$$

(c) By substituting for $\varphi(n)$ into the equation we see easily that $\varphi(n)$ is not a solution.

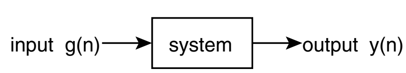

FIGURE 2.2. Input–output system.

Conclusion

(i) From the above example we conclude that in contrast to the situation for homogeneous equations, solutions of the nonhomogeneous equation (2.4.1) do not form a vector space. In particular, neither the sum (difference) of two solutions nor a multiple of a solution is a solution.

(ii) From part (b) in Example 2.28 we found that the difference of the solutions $y_2(n)$ and $y_1(n)$ of the nonhomogeneous equation is actually a solution of the associated homogeneous equation. This is indeed true for the general nth-order equation, as demonstrated by the following result.

Theorem 2.29. *If $y_1(n)$ and $y_2(n)$ are solutions of* (2.4.1)*, then $x(n) = y_1(n) - y_2(n)$ is a solution of the corresponding homogeneous equation*

$$x(n+k) + p_1(n)x(n+k-1) + \cdots + p_k(n)x(n) = 0. \tag{2.4.2}$$

Proof. The reader will undertake the justification of this theorem in Exercises 2.4, Problem 12. □

It is customary to refer to the general solution of the homogeneous equation (2.4.2) as the *complementary solution* of the nonhomogeneous equation (2.4.1), and it will be denoted by $y_c(n)$. A solution of the nonhomogeneous equation (2.4.1) is called a *particular solution* and will be denoted by $y_p(n)$. The next result gives us an algorithm to generate all solutions of the nonhomogeneous equation (2.4.1).

Theorem 2.30. *Any solution $y(n)$ of* (2.4.1) *may be written as*

$$\boxed{y(n) = y_p(n) + \sum_{i=1}^{k} a_i x_i(n),}$$

where $\{x_1(n), x_2(n), \ldots, x_k(n)\}$ is a fundamental set of solutions of the homogeneous equation (2.4.2)*.*

Proof. Observe that according to Theorem 2.29, $y(n) - y_p(n)$ is a solution of the homogeneous equation (2.4.2). Thus $y(n) - y_p(n) = \sum_{i=1}^{k} a_i x_i(n)$, for some constants a_i.

The preceding theorem leads to the definition of the *general solution* of the nonhomogeneous equation (2.4.1) as

$$y(n) = y_c(n) + y_p(n). \tag{2.4.3}$$

□

We now turn our attention to finding a particular solution y_p of nonhomogeneous equations with constant coefficients such as

$$y(n+k) + p_1 y(n+k-1) + \cdots + p_k y(n) = g(n). \tag{2.4.4}$$

Because of its simplicity, we use the method of *undetermined coefficients* to compute y_p.

Basically, the method consists in making an intelligent guess as to the form of the particular solution and then substituting this function into the difference equation. For a completely arbitrary nonhomogeneous term $g(n)$, this method is not effective. However, definite rules can be established for the determination of a particular solution by this method if $g(n)$ is a linear combination of terms, each having one of the forms

$$a^n, \quad \sin(bn), \quad \cos(bn), \quad \text{or } n^k, \tag{2.4.5}$$

or products of these forms such as

$$a^n \sin(bn), \qquad a^n n^k, \qquad a^n n^k \cos(bn), \ldots . \tag{2.4.6}$$

Definition 2.31. A polynomial operator $N(E)$, where E is the shift operator, is said to be an *annihilator* of $g(n)$ if

$$N(E)g(n) = 0. \tag{2.4.7}$$

In other words, $N(E)$ is an annihilator of $g(n)$ if $g(n)$ is a solution of (2.4.7). For example, an annihilator of $g(n) = 3^n$ is $N(E) = E - 3$, since $(E-3)y(n) = 0$ has a solution $y(n) = 3^n$. An annihilator of $g(n) = \cos\frac{n\pi}{2}$ is $N(E) = E^2 + 1$, since $(E^2+1)y(n) = 0$ has a solution $y(n) = \cos\frac{n\pi}{2}$. Let us now rewrite (2.4.4) using the shift operator E as

$$p(E)y(n) = g(n), \tag{2.4.8}$$

where $p(E) = E^k + p_1E^{k-1} + p_2E^{k-2} + \cdots + p_kI$.

Assume now that $N(E)$ is an annihilator of $g(n)$ in (2.4.8). Applying $N(E)$ on both sides of (2.4.8) yields

$$N(E)p(E)y(n) = 0. \tag{2.4.9}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the characteristic roots of the homogeneous equation

$$p(E)y(n) = 0, \tag{2.4.10}$$

and let $\mu_1, \mu_2, \ldots, \mu_k$ be the characteristic roots of

$$N(E)y(n) = 0. \tag{2.4.11}$$

We must consider two separate cases.

Case 1. None of the λ_i's equals any of the μ_i's. In this case, write $y_p(n)$ as the general solution of (2.4.11) with undetermined constants. Substituting back this "guesstimated" particular solution into (2.4.4), we find the values of the constants. Table 2.3 contains several types of functions $g(n)$ and their corresponding particular solutions.

Case 2. $\lambda_i = \mu_j$ for some i, j. In this case, the set of characteristic roots of (2.4.9) is equal to the union of the sets $\{\lambda_i\}$, $\{\mu_j\}$ and, consequently,

TABLE 2.3. Particular solutions $y_p(n)$.

$g(n)$	$y_p(n)$
a^n	$c_1 a^n$
n^k	$c_0 + c_1 n + \cdots + c_k n^k$
$n^k a^n$	$c_0 a^n + c_1 n a^n + \cdots + c_k n^k a^n$
$\sin bn, \cos bn$	$c_1 \sin bn + c_2 \cos bn$
$a^n \sin bn, a^n \cos bn$	$(c_1 \sin bn + c_2 \cos bn)a^n$
$a^n n^k \sin bn, a^n n^k \cos bn$	$(c_0 + c_1 n + \cdots + c_k n^k)a^n \sin(bn)$ $+ (d_0 + d_1 n + \cdots d_k n^k)a^n \cos(bn)$

contains roots of higher multiplicity than the two individual sets of characteristic roots. To determine a particular solution $y_p(n)$, we first find the general solution of (2.4.9) and then drop all the terms that appear in $y_c(n)$. Then proceed as in Case 1 to evaluate the constants.

Example 2.32. Solve the difference equation

$$y(n+2) + y(n+1) - 12y(n) = n2^n. \tag{2.4.12}$$

Solution The characteristic roots of the homogeneous equation are $\lambda_1 = 3$ and $\lambda_2 = -4$.

Hence,

$$y_c(n) = c_1 3^n + c_2(-4)^n.$$

Since the annihilator of $g(n) = n2^n$ is given by $N(E) = (E-2)^2$ (Why?), we know that $\mu_1 = \mu_2 = 2$. This equation falls in the realm of Case 1, since $\lambda_i \neq \mu_j$, for any i, j. So we let

$$y_p(n) = a_1 2^n + a_2 n 2^n.$$

Substituting this relation into equation (2.4.12) gives

$$a_1 2^{n+2} + a_2(n+2)2^{n+2} + a_1 2^{n+1} + a_2(n+1)2^{n+1} - 12a_1 2^n - 12a_2 n2^n = n2^n,$$
$$(10a_2 - 6a_1)2^n - 6a_2 n2^n = n2^n.$$

Hence

$$10a_2 - 6a_1 = 0 \quad \text{and} \quad -6a_2 = 1,$$

or

$$a_1 = \frac{-5}{18}, \qquad a_2 = \frac{-1}{6}.$$

The particular solution is

$$y_p(n) = \frac{-5}{18}2^n - \frac{1}{6}n2^n,$$

and the general solution is

$$y(n) = c_1 3^n + c_2(-4)^n - \frac{5}{18}2^n - \frac{1}{6}n2^n.$$

Example 2.33. Solve the difference equation

$$(E-3)(E+2)y(n) = 5(3^n). \tag{2.4.13}$$

Solution The annihilator of $5(3^n)$ is $N(E) = (E-3)$. Hence, $\mu_1 = 3$. The characteristic roots of the homogeneous equation are $\lambda_1 = 3$ and $\lambda_2 = -2$. Since $\lambda_1 = \mu_1$, we apply the procedure for Case 2.

Thus,

$$(E-3)^2(E+2)y(n) = 0. \tag{2.4.14}$$

Now $y_c(n) = c_1 3^n + c_2(-2)^n$.

We now know that the general solution of (2.4.14) is given by

$$\tilde{y}(n) = (a_1 + a_2 n)3^n + a_3(-2)^n.$$

Omitting from $\tilde{y}(n)$ the terms 3^n and $(-2)^n$ that appeared in $y_c(n)$, we set $y_p(n) = a_2 n 3^n$. Substituting this $y_p(n)$ into (2.4.13) gives

$$a_2(n+2)3^{n+2} - a_2(n+1)3^{n+1} + 6a_2 n 3^n = 5.3^n,$$

or

$$a_2 = \frac{1}{3}.$$

Thus $y_p(n) = n3^{n-1}$, and the general solution of (2.4.13) is

$$y(n) = c_1 3^n + c_2(-2)^n + n3^{n-1}.$$

Example 2.34. Solve the difference equation

$$y(n+2) + 4y(n) = 8(2^n)\cos\left(\frac{n\pi}{2}\right). \tag{2.4.15}$$

Solution The characteristic equation of the homogeneous equation is

$$\lambda^2 + 4 = 0.$$

The characteristic roots are

$$\lambda_1 = 2i, \qquad \lambda_2 = -2i.$$

Thus $r = 2, \theta = \pi/2$, and

$$y_c(n) = 2^n\left(c_1\cos\left(\frac{n\pi}{2}\right) + c_2\sin\left(\frac{n\pi}{2}\right)\right).$$

Notice that $g(n) = 2^n\cos\left(\frac{n\pi}{2}\right)$ appears in $y_c(n)$. Using Table 2.3, we set

$$y_p(n) = 2^n\left(an\cos\left(\frac{n\pi}{2}\right) + bn\sin\left(\frac{n\pi}{2}\right)\right). \tag{2.4.16}$$

Substituting (2.4.16) into (2.4.15) gives

$$\begin{aligned}&2^{n+2}\left[a(n+2)\cos\left(\frac{n\pi}{2}+\pi\right) + b(n+2)\sin\left(\frac{n\pi}{2}+\pi\right)\right]\\ &\quad+\ (4)2^n\left[an\cos\left(\frac{n\pi}{2}\right) + bn\sin\left(\frac{n\pi}{2}\right)\right] = 8(2^n)\cos\left(\frac{n\pi}{2}\right).\end{aligned}$$

Replacing $\cos((n\pi)/2 + \pi)$ by $-\cos((n\pi)/2)$, and $\sin((n\pi)/2 + \pi)$ by $-\sin((n\pi)/2)$ and then comparing the coefficients of the cosine terms leads us to $a = -1$. Then by comparing the coefficients of the sine terms, we realize that $b = 0$.

By substituting these values back into (2.4.16), we know that

$$y_p(n) = -2^n n \cos\left(\frac{n\pi}{2}\right),$$

and the general solution of (2.4.15), arrived at by adding $y_c(n)$ and $y_p(n)$, is

$$y(n) = 2^n \left(c_1 \cos\frac{n\pi}{2} + c_2 \sin\left(\frac{n\pi}{2}\right) - n\cos\left(\frac{n\pi}{2}\right)\right).$$

Exercises 2.4.

For Problems 1 through 6, find a particular solution of the difference equation.

1. $y(n+2) - 5y(n+1) + 6y(n) = 1 + n.$
2. $y(n+2) + 8y(n+1) + 12y(n) = e^n.$
3. $y(n+2) - 5y(n+1) + 4y(n) = 4^n - n^2.$
4. $y(n+2) + 8y(n+1) + 7y(n) = ne^n.$
5. $y(n+2) - y(n) = n\cos\left(\frac{n\pi}{2}\right).$
6. $(E^2+9)^2 y(n) = \sin\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right).$

For Problems 7 through 9 find the solution of the difference equation.

7. $\Delta^2 y(n) = 16, \qquad y(0) = 2, \qquad y(1) = 3.$
8. $\Delta^2 y(n) + 7y(n) = 2\sin\left(\frac{n\pi}{4}\right), \qquad y(0) = 0, \qquad y(1) = 1.$
9. $(E-3)(E^2+1)y(n) = 3^n, \qquad y(0) = 0, \qquad y(1) = 1, \qquad y(2) = 3.$

For Problems 10 and 11 find the general solution of the difference equation.

10. $y(n+2) - y(n) = n2^n \sin\left(\frac{n\pi}{2}\right).$
11. $y(n+2) + 8y(n+1) + 7y(n) = n2^n.$
12. Prove Theorem 2.29.
13. Consider the difference equation $y(n+2)+p_1y(n+1)+p_2y(n) = g(n)$, where $p_1^2 < 4p_2$ and $0 < p_2 < 1$. Show that if y_1 and y_2 are two solutions of the equation, then $y_1(n) - y_2(n) \to 0$ as $n \to \infty$.
14. Determine the general solution of $y(n+2)+\lambda^2 y(n) = \sum_{m=1}^{N} a_m \sin(m\pi n)$, where $\lambda > 0$ and $\lambda \neq m\pi, m = 1, 2, \ldots, N$.

15. Solve the difference equation

$$y(n+2) + y(n) = \begin{cases} 1 & \text{if } 0 \le n \le 2, \\ -1 & \text{if } n > 2, \end{cases}$$

with $y(0) = 0$ and $y(1) = 1$.

2.4.1 The Method of Variation of Constants (Parameters)

Contemplate the second-order nonhomogeneous difference equation

$$y(n+2) + p_1(n)y(n+1) + p_2(n)y(n) = g(n) \tag{2.4.17}$$

and the corresponding homogeneous difference equation

$$y(n+2) + p_1(n)y(n+1) + p_2(n)y(n) = 0. \tag{2.4.18}$$

The method of variation of constants is commonly used to find a particular solution $y_p(n)$ of (2.4.17) when the coefficients $p_1(n)$ and $p_2(n)$ are not constants. The method assumes that a particular solution of (2.4.17) may be written in the form

$$y(n) = u_1(n)y_1(n) + u_2(n)y_2(n), \tag{2.4.19}$$

where $y_1(n)$ and $y_2(n)$ are two linearly independent solutions of the homogeneous equation (2.4.18), and $u_1(n), u_2(n)$ are sequences to be determined later.

16. (a) Show that

$$\begin{aligned} y(n+1) = {} & u_1(n)y_1(n+1) + u_2(n)y_2(n+1) \\ & + \Delta u_1(n)y_1(n+1) + \Delta u_2(n)y_2(n+1). \end{aligned} \tag{2.4.20}$$

(b) The method stipulates that

$$\Delta u_1(n)y_1(n+1) + \Delta u_2(n)y_2(n+1) = 0. \tag{2.4.21}$$

Use (2.4.20) and (2.4.21) to show that

$$\begin{aligned} y(n+2) = {} & u_1(n)y_1(n+2) + u_2(n)y_2(n+2) \\ & + \Delta u_1(n)y_1(n+2) + \Delta u_2(n)y_2(n+2). \end{aligned}$$

(c) By substituting the above expressions for $y(n)$, $y(n+1)$, and $y(n+2)$ into (2.4.17), show that

$$\Delta u_1(n)y_1(n+2) + \Delta u_2(n)y_2(n+2) = g(n). \tag{2.4.22}$$

(d) Using expressions (2.4.21) and (2.4.22), show that

$$\Delta u_1(n) = \frac{-g(n)y_2(n+1)}{W(n+1)}, \qquad u_1(n) = \sum_{r=0}^{n-1} \frac{-g(r)y_2(r+1)}{W(r+1)}, \tag{2.4.23}$$

$$\Delta u_2(n) = \frac{g(n)y_1(n+1)}{W(n+1)}, \qquad u_2(n) = \sum_{r=0}^{n-1} \frac{g(r)y_1(r+1)}{W(r+1)}, \tag{2.4.24}$$

where $W(n)$ is the Casoratian of $y_1(n)$ and $y_2(n)$.

17. Use formulas (2.4.23) and (2.4.24) to solve the equation

$$y(n+2) - 7y(n+1) + 6y(n) = n.$$

18. Use the variation of constants method to solve the initial value problem

$$y(n+2) - 5y(n+1) + 6y(n) = 2^n, \qquad y(1) = y(2) = 0.$$

19. Use Problem 16(d) to show that the unique solution of (2.4.17) with $y(0) = y(1) = 0$ is given by

$$y(n) = \sum_{r=0}^{n-1} \frac{y_1(r+1)y_2(n) - y_2(r+1)y_1(n)}{W(r+1)}.$$

20. Consider the equation

$$x(n+1) = ax(n) + f(n). \tag{2.4.25}$$

(a) Show that

$$x(n) = a^n \left[x(0) + \frac{f(0)}{a} + \frac{f(1)}{a^2} + \cdots + \frac{f(n-1)}{a^n} \right] \tag{2.4.26}$$

is a solution of (2.4.25).

(b) Show that if $|a| < 1$ and $\{f(n)\}$ is a bounded sequence, i.e., $|f(n)| \le M$, for some $M > 0$, $n \in \mathbb{Z}^+$, then all solutions of (2.4.25) are bounded.

(c) Suppose that $a > 1$ and $\{f(n)\}$ is bounded on $\mathbb{Z}^+$. Show that if we choose

$$x(0) = -\left(\frac{f(0)}{a} + \frac{f(1)}{a^2} + \cdots + \frac{f(n)}{a^{n+1}} + \cdots \right) = -\sum_{i=0}^{\infty} \frac{f(i)}{a^{i+1}}, \tag{2.4.27}$$

then the solution $x(n)$ given by (2.4.26) is bounded on $\mathbb{Z}^+$. Give an explicit expression for $x(n)$ in this case.

(d) Under the assumptions of part (c), show that for any choice of $x(0)$, excepting that value given by (2.4.27), the solution of (2.4.25) is unbounded.

2.5 Limiting Behavior of Solutions

To simplify our exposition we restrict our discussion to the second-order difference equation

$$y(n+2) + p_1 y(n+1) + p_2 y(n) = 0. \tag{2.5.1}$$

Suppose that λ_1 and λ_2 are the characteristic roots of the equation. Then we have the following three cases:

(a) λ_1 and λ_2 are distinct real roots. Then $y_1(n) = \lambda_1^n$ and $y_2(n) = \lambda_2^n$ are two linearly independent solutions of (2.5.1). If $|\lambda_1| > |\lambda_2|$, then we call $y_1(n)$ the *dominant solution*, and λ_1 the *dominant characteristic root*. Otherwise, $y_2(n)$ is the dominant solution, and λ_2 is the dominant characteristic root. We will now show that the limiting behavior of the general solution $y(n) = a_1\lambda_1^n + a_2\lambda_2^n$ is determined by the behavior of the dominant solution. So assume, without loss of generality, that $|\lambda_1| > |\lambda_2|$. Then

$$y(n) = \lambda_1^n \left[a_1 + a_2 \left(\frac{\lambda_2}{\lambda_1} \right)^n \right].$$

Since

$$\left| \frac{\lambda_2}{\lambda_1} \right| < 1,$$

it follows that

$$\left(\frac{\lambda_2}{\lambda_1} \right)^n \to 0 \quad \text{as } n \to \infty.$$

Consequently, $\lim_{n\to\infty} y(n) = \lim_{n\to\infty} a_1\lambda_1^n$. There are six different situations that may arise here depending on the value of λ_1 (see Figure 2.3).

1. $\lambda_1 > 1$: The sequence $\{a_1\lambda_1^n\}$ diverges to ∞ (unstable system).
2. $\lambda_1 = 1$: The sequence $\{a_1\lambda_1^n\}$ is a constant sequence.
3. $0 < \lambda_1 < 1$: The sequence $\{a_1\lambda_1^n\}$ is monotonically decreasing to zero (stable system).
4. $-1 < \lambda_1 < 0$: The sequence $\{a_1\lambda_1^n\}$ is oscillating around zero (i.e., alternating in sign) and converging to zero (stable system).
5. $\lambda_1 = -1$: The sequence $\{a_1\lambda_1^n\}$ is oscillating between two values a_1 and $-a_1$.
6. $\lambda_1 < -1$: The sequence $\{a_1\lambda_1^n\}$ is oscillating but increasing in magnitude (unstable system).

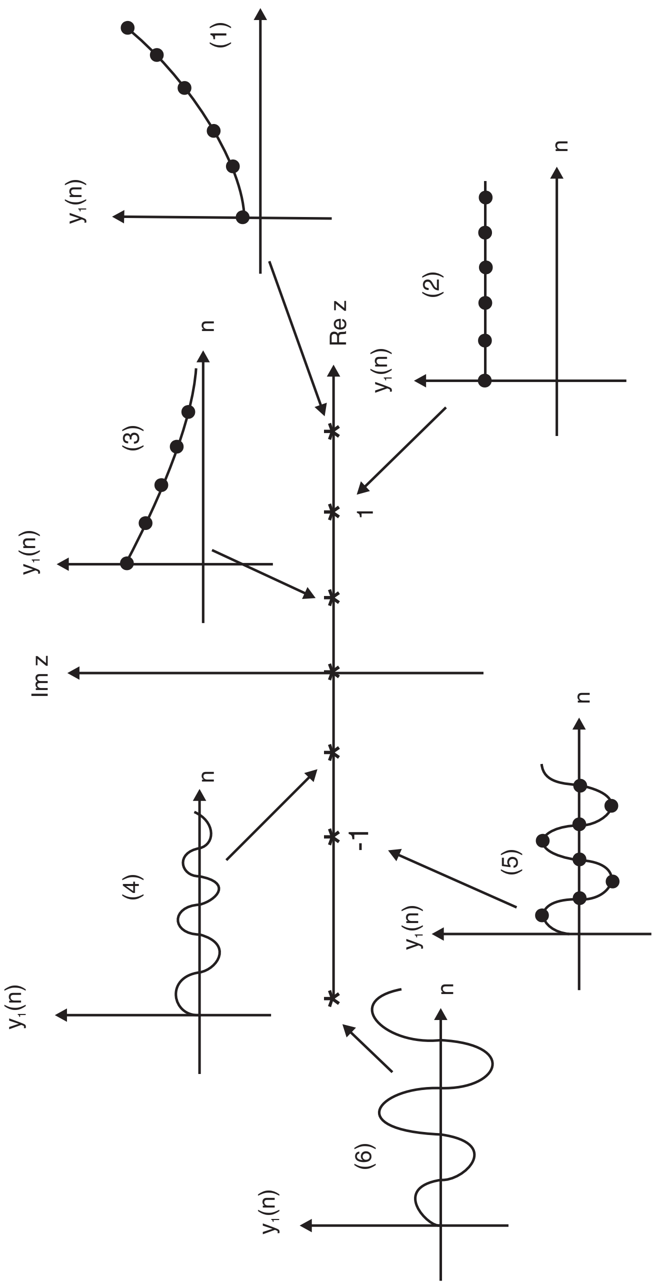

FIGURE 2.3. $(n, y(n))$ diagrams for real roots.

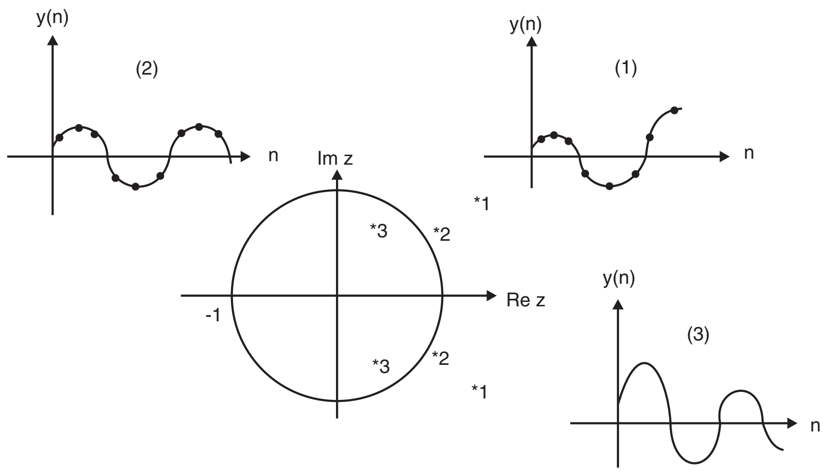

FIGURE 2.4. $(n, y(n))$ diagrams for comples roots.

(b) $\lambda_1 = \lambda_2 = \lambda$.
The general solution of (2.5.1) is given by $y(n) = (a_1 + a_2 n)\lambda^n$. Clearly, if $|\lambda| \geq 1$, the solution $y(n)$ diverges either monotonically if $\lambda \geq 1$ or by oscillating if $\lambda \leq -1$. However, if $|\lambda| < 1$, then the solution converges to zero, since $\lim_{n\to\infty} n\lambda^n = 0$ (Why?).

(c) Complex roots: $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, where $\beta \neq 0$.
As we have seen in Section 2.3, formula (2.3.12), the solution of (2.5.1) is given by $y(n) = ar^n \cos(n\theta - \omega)$, where

$$r = \sqrt{\alpha^2 + \beta^2}, \qquad \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right).$$

The solution $y(n)$ clearly oscillates, since the cosine function oscillates. However, $y(n)$ oscillates in three different ways depending on the location of the conjugate characteristic roots, as may be seen in Figure 2.4.

1. $r > 1$: Here λ_1 and $\lambda_2 = \overline{\lambda}_1$ are outside the unit circle. Hence $y(n)$ is oscillating but increasing in magnitude (unstable system).

2. $r = 1$: Here λ_1 and $\lambda_2 = \overline{\lambda}_1$ lie on the unit circle. In this case $y(n)$ is oscillating but constant in magnitude.

3. $r < 1$: Here λ_1 and $\lambda_2 = \overline{\lambda}_1$ lie inside the unit disk. The solution $y(n)$ oscillates but converges to zero as $n \to \infty$ (stable system).

Finally, we summarize the above discussion in the following theorem.

Theorem 2.35. *The following statements hold:*

(i) *All solutions of* (2.5.1) *oscillate (about zero) if and only if the characteristic equation has no positive real roots.*

(ii) *All solutions of* (2.5.1) *converge to zero (i.e., the zero solution is asymptotically stable) if and only if* $\max\{|\lambda_1|, |\lambda_2|\} < 1$.

Next we consider nonhomogeneous difference equations in which the input is constant, that is, equations of the form

$$y(n+2) + p_1 y(n+1) + p_2 y(n) = M, \tag{2.5.2}$$

where M is a nonzero constant input or forcing term. Unlike (2.5.1), the zero sequence $y(n) = 0$ for all $n \in \mathbb{Z}^+$ is not a solution of (2.5.2). Instead, we have the equilibrium point or solution $y(n) = y^*$. From (2.5.2) we have

$$y^* + p_1 y^* + p_2 y^* = M,$$

or

$$y^* = \frac{M}{1 + p_1 + p_2}. \tag{2.5.3}$$

Thus $y_p(n) = y^*$ is a particular solution of (2.5.2). Consequently, the general solution of (2.5.2) is given by

$$y(n) = y^* + y_c(n). \tag{2.5.4}$$

It is clear that $y(n) \to y^*$ if and only if $y_c(n) \to 0$ as $n \to \infty$. Furthermore, $y(n)$ oscillates[6] about y^* if and only if $y_c(n)$ oscillates about zero. These observations are summarized in the following theorem.

Theorem 2.36. *The following statements hold:*

(i) *All solutions of the nonhomogeneous equation* (2.5.2) *oscillate about the equilibrium solution* y^* *if and only if none of the characteristic roots of the homogeneous equation* (2.5.1) *is a positive real number.*

(ii) *All solutions of* (2.5.2) *converge to* y^* *as* $n \to \infty$ *if and only if* $\max\{|\lambda_1|, |\lambda_2|\} < 1$, *where* λ_1 *and* λ_2 *are the characteristic roots of the homogeneous equation* (2.5.1).

Theorems 2.35 and 2.36 give necessary and sufficient conditions under which a second-order difference equation is asymptotically stable. In many applications, however, one needs to have explicit criteria for stability based on the values of the coefficients p_1 and p_2 of (2.5.2) or (2.5.1). The following result provides us with such needed criteria.

---

[6]We say $y(n)$ oscillates about y^* if $y(n) - y^*$ alternates sign, i.e., if $y(n) > y^*$, then $y(n+1) < y^*$.

Theorem 2.37. *The conditions*

$$1 + p_1 + p_2 > 0, \qquad 1 - p_1 + p_2 > 0, \qquad 1 - p_2 > 0 \tag{2.5.5}$$

are necessary and sufficient for the equilibrium point (solution) of equations (2.5.1) *and* (2.5.2) *to be asymptotically stable (i.e., all solutions converge to* y^**).*

PROOF. Assume that the equilibrium point of (2.5.1) or (2.5.2) is asymptotically stable. In virtue of Theorems 2.35 and 2.36, the roots λ_1, λ_2 of the characteristic equation $\lambda^2 + p_1\lambda + p_2 = 0$ lie inside the unit disk, i.e., $|\lambda_1| < 1$ and $|\lambda_2| < 1$. By the quadratic formula, we have

$$\lambda_1 = \frac{-p_1 + \sqrt{p_1^2 - 4p_2}}{2} \quad \text{and} \quad \lambda_2 = \frac{-p_1 - \sqrt{p_1^2 - 4p_2}}{2}. \tag{2.5.6}$$

Then we have two cases to consider.

Case 1. λ_1, λ_2 are real roots, i.e., $p_1^2 - 4p_2 \geq 0$. From formula (2.5.6) we have

$$-2 < -p_1 + \sqrt{p_1^2 - 4p_2} < 2,$$

or

$$-2 + p_1 < \sqrt{p_1^2 - 4p_2} < 2 + p_1. \tag{2.5.7}$$

Similarly, one obtains

$$-2 + p_1 < -\sqrt{p_1^2 - 4p_2} < 2 + p_1. \tag{2.5.8}$$

Squaring the second inequality in expression (2.5.7) yields

$$1 + p_1 + p_2 > 0. \tag{2.5.9}$$

Similarly, if we square the first inequality in expression (2.5.8) we obtain

$$1 - p_1 + p_2 > 0. \tag{2.5.10}$$

Now from the second inequality of (2.5.7) and the first inequality of (2.5.8) we obtain

$$2 + p_1 > 0 \quad \text{and} \quad 2 - p_1 > 0 \quad \text{or} \quad |p_1| < 2$$

since $p_1^2 - 4p_2 \geq 0$, $p_2 \leq p_1^2/4 < 1$. This completes the proof of (2.5.5) in this case.

Case 2. λ_1 and λ_2 are complex conjugates, i.e., $p_1^2 - 4p_2 < 0$. In this case we have

$$\lambda_{1,2} = \frac{-p_1}{2} \pm \frac{i}{2}\sqrt{4p_2 - p_1^2}.$$

Moreover, since $p_1^2 < 4p_2$, it follows that $-2\sqrt{p_2} < p_1 < 2\sqrt{p_2}$. Now $|\lambda_1|^2 = \frac{p_1^2}{4} + \frac{4p_2}{4} - \frac{p_1^2}{4} = p_2$. Since $|\lambda_1| < 1$, it follows that $0 < p_2 < 1$.

Hence to show that the first two inequalities of (2.5.5) hold we need to show that the function $f(x) = 1 + x - 2\sqrt{x} > 0$ for $x \in (0, 1)$. Observe that $f(0) = 1$, and $f'(x) = 1 - \frac{1}{\sqrt{x}}$. Thus $x = 1$ is a local minimum as $f(x)$ decreases for $x \in (0, 1)$. Hence $f(x) > 0$ for all $x \in (0, 1)$.

This completes the proof of the necessary conditions. The converse is left to the reader as Exercises 2.5, Problem 8. □

Example 2.38. Find conditions under which the solutions of the equation

$$y(n+2) - \alpha(1+\beta)y(n+1) + \alpha\beta y(n) = 1, \qquad \alpha, \beta > 0,$$

(a) converge to the equilibrium point y^*, and

(b) oscillate about y^*.

Solution Let us first find the equilibrium point y^*. Be letting $y(n) = y^*$ in the equation, we obtain

$$y^* = \frac{1}{1-\alpha}, \qquad \alpha \neq 1.$$

(a) Applying condition (2.5.5) to our equation yields

$$\alpha < 1, \qquad 1 + \alpha + 2\alpha\beta > 0, \qquad \alpha\beta < 1.$$

Clearly, the second inequality $1+\alpha+2\alpha\beta > 0$ is always satisfied, since α, β are both positive numbers.

(b) The solutions are oscillatory about y^* if either λ_1, λ_2 are negative real numbers or complex conjugates. In the first case we have

$$\alpha^2(1+\beta)^2 > 4\alpha\beta, \quad \text{or} \quad \alpha > \frac{4\beta}{(1+\beta)^2},$$

and

$$\alpha(1+\beta) < 0,$$

which is impossible. Thus if $\alpha > 4\beta/(1+\beta)^2$, we have no oscillatory solutions.

Now, λ_1 and λ_2 are complex conjugates if

$$\alpha^2(1+\beta)^2 < 4\alpha\beta \quad \text{or} \quad \alpha < \frac{4\beta}{(1+\beta)^2}.$$

Hence all solutions are oscillatory if

$$\alpha < \frac{4\beta}{(1+\beta)^2}.$$

For the treatment of the general kth-order scalar difference equations, the reader is referred to Chapter 4, on stability, and Chapter 8, on oscillation.

Exercises 2.5.

In Problems 1 through 4:

(a) Determine the stability of the equilibrium point by using Theorem 2.35 or Theorem 2.36.

(b) Determine the oscillatory behavior of the solutions of the equation.

1. $y(n+2) - 2y(n+1) + 2y(n) = 0$.
2. $y(n+2) + \frac{1}{4}y(n) = \frac{5}{4}$.
3. $y(n+2) + y(n+1) + \dfrac{1}{2}y(n) = -5$.
4. $y(n+2) - 5y(n+1) + 6y(n) = 0$.
5. Determine the stability of the equilibrium point of the equations in Problems 1 through 4 by using Theorem 2.37.
6. Show that the stability conditions (2.5.5) for the equation $y(n+2) - \alpha y(n+1) + \beta y(n) = 0$, where α, β are constants, may be written as
$$-1 - \beta < \alpha < 1 + \beta, \qquad \beta < 1.$$
7. Contemplate the equation $y(n+2) - p_1 y(n+1) - p_2 y(n) = 0$. Show that if $|p_1| + |p_2| < 1$, then all solutions of the equation converge to zero.
8. Prove that conditions (2.5.5) imply that all solutions of (2.5.2) converge to the equilibrium point y^*.
9. Determine conditions under which all solutions of the difference equation in Problem 7 oscillate.
10. Determine conditions under which all solutions of the difference equation in Problem 6 oscillate.
11. Suppose that p is a real number. Prove that every solution of the difference equation $y(n+2) - y(n+1) + py(n) = 0$ oscillates if and only if $p > \frac{1}{4}$.

*12. Prove that a necessary and sufficient condition for the asymptotic stability of the zero solution of the equation
$$y(n+2) + p_1 y(n+1) + p_2 y(n) = 0$$
is
$$|p_1| < 1 + p_2 < 2.$$

13. Determine the limiting behavior of solutions of the equation

$$y(n+2) = \alpha c + \alpha\beta(y(n+1) - y(n))$$

if:

(i) $\alpha\beta = 1$,

(ii) $\alpha\beta = 2$,

(iii) $\alpha\beta = \frac{1}{2}$,

provided that α, β, and c are positive constants.

14. If $p_1 > 0$ and $p_2 > 0$, show that all solutions of the equation

$$y(n+2) + p_1 y(n+1) + p_2 y(n) = 0$$

are oscillatory.

15. Determine the limiting behavior of solutions of the equation

$$y(n+2) - \frac{\beta}{\alpha} y(n+1) + \frac{\beta}{\alpha} y(n) = 0,$$

where α and β are constants, if:

(i) $\beta > 4\alpha$,

(ii) $\beta < 4\alpha$.

2.6 Nonlinear Equations Transformable to Linear Equations

In general, most nonlinear difference equations cannot be solved explicitly. However, a few types of nonlinear equations can be solved, usually by transforming them into linear equations. In this section we discuss some tricks of the trade.

Type I. Equations of Riccati type:

$$x(n+1)x(n) + p(n)x(n+1) + q(n)x(n) = 0. \tag{2.6.1}$$

To solve the Riccati equation, we let

$$z(n) = \frac{1}{x(n)}$$

in (2.6.1) to give us

$$q(n)z(n+1) + p(n)z(n) + 1 = 0. \tag{2.6.2}$$

The nonhomogeneous equation requires a different transformation

$$y(n+1)y(n) + p(n)y(n+1) + q(n)y(n) = g(n). \tag{2.6.3}$$

If we let $y(n) = (z(n+1)/z(n)) - p(n)$ in (2.6.3) we obtain

$$z(n+2) + (q(n) - p(n+1))z(n+1) - (g(n) + p(n)q(n))z(n) = 0.$$

Example 2.39. The Pielou Logistic Equation

The most popular continuous model of the growth of a population is the well-known Verhulst–Pearl equation given by

$$x'(t) = x(t)[a - bx(t)], \qquad a, b > 0, \tag{2.6.4}$$

where $x(t)$ is the size of the population at time $t; a$ is the rate of the growth of the population if the resources were unlimited and the individuals did not affect one another, and $-bx^2(t)$ represents the negative effect on the growth due to crowdedness and limited resources. The solution of (2.6.4) is given by

$$x(t) = \frac{a/b}{1 + (e^{-at}/cb)}.$$

Now,

$$\begin{aligned} x(t+1) &= \frac{a/b}{1 + \left(e^{-a(t+1)}/cb\right)} \\ &= \frac{e^a(a/b)}{1 + (e^{-at}/cb) + (e^a - 1)}. \end{aligned}$$

Dividing by $[1 + (e^{-at}/cb)]$, we obtain

$$x(t+1) = \frac{e^a x(t)}{\left[1 + \frac{b}{a}(e^a - 1)x(t)\right]},$$

or

$$x(n+1) = \frac{\alpha x(n)}{[1 + \beta x(n)]}, \tag{2.6.5}$$

where $\alpha = e^a$ and $\beta = \frac{b}{a}(e^a - 1)$.

This equation is titled the *Pielou logistic equation.*

Equation (2.6.5) is of Riccati type and may be solved by letting $x(n) = 1/z(n)$. This gives us the equation

$$z(n+1) = \frac{1}{\alpha} z(n) + \frac{\beta}{\alpha},$$

whose solution is given by

$$z(n) = \begin{cases} \left[c - \dfrac{\beta}{\alpha - 1}\right] \alpha^{-n} + (\beta/(\alpha - 1)) & \text{if } \alpha \neq 1, \\ c + \beta n & \text{if } \alpha = 1. \end{cases}$$

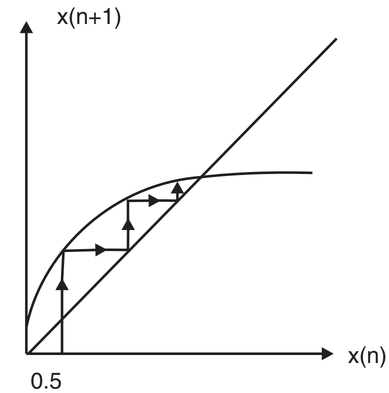

FIGURE 2.5. Asymptotically stable equilibrium points.

Thus

$$x(n) = \begin{cases} \alpha^n(\alpha-1)/[\beta\alpha^n + c(\alpha-1) - \beta] & \text{if } \alpha \neq 1, \\ \dfrac{1}{c+\beta n} & \text{if } \alpha = 1. \end{cases}$$

Hence

$$\lim_{n\to\infty} x(n) = \begin{cases} (\alpha-1)/\beta & \text{if } \alpha \neq 1, \\ 0 & \text{if } \alpha = 1. \end{cases}$$

This conclusion shows that the equilibrium point $(\alpha - 1)/\beta$ is globally asymptotically stable if $\alpha \neq 1$. Figure 2.5 illustrates this for $\alpha = 3$, $\beta = 1$, and $x(0) = 0.5$.

Type II. Equations of general Riccati type:

$$x(n+1) = \frac{a(n)x(n) + b(n)}{c(n)x(n) + d(n)} \tag{2.6.6}$$

such that $c(n) \neq 0, a(n)d(n) - b(n)c(n) \neq 0$ for all $n \geq 0$.

To solve this equation we let

$$c(n)x(n) + d(n) = \frac{y(n+1)}{y(n)}. \tag{2.6.7}$$

Then by substituting

$$x(n) = \frac{y(n+1)}{c(n)y(n)} - \frac{d(n)}{c(n)}$$

into (2.6.6) we obtain

$$\frac{y(n+2)}{c(n+1)y(n+1)} - \frac{d(n+1)}{c(n+1)} = \frac{a(n)\left[\dfrac{y(n+1)}{c(n)y(n)} - \dfrac{d(n)}{c(n)}\right] + b(n)}{\dfrac{y(n+1)}{y(n)}}.$$

This equation simplifies to

$$\begin{aligned} y(n+2) + p_1(n)y(n+1) + p_2(n)y(n) &= 0, \\ y(0) = 1, \quad y(1) &= c(0)x(0) + d(0), \end{aligned} \tag{2.6.8}$$

where

$$\begin{aligned} p_1(n) &= -\frac{c(n)d(n+1) + a(n)c(n+1)}{c(n)}, \\ p_2(n) &= (a(n)d(n) - b(n)c(n))\frac{c(n+1)}{c(n)}. \end{aligned}$$

Example 2.40. Solve the difference equation

$$x(n+1) = \frac{2x(n)+3}{3x(n)+2}.$$

Solution Here $a = 2, b = 3, c = 3$, and $d = 2$. Hence $ad - bc \neq 0$. Using the transformation

$$3x(n) + 2 = \frac{y(n+1)}{y(n)}, \tag{2.6.9}$$

we obtain, as in (2.6.8),

$$y(n+2) - 4y(n+1) - 5y(n) = 0, \qquad y(0) = 1, \qquad y(1) = 3x(0) + 2,$$

with characteristic roots $\lambda_1 = 5, \lambda_2 = -1$.

Hence

$$y(n) = c_1 5^n + c_2(-1)^n. \tag{2.6.10}$$

From formula (2.6.9) we have

$$\begin{aligned} x(n) &= \frac{1}{3}\,\frac{y(n+1)}{y(n)} - \frac{2}{3} = \frac{1}{3}\,\frac{c_1 5^{n+1} + c_2(-1)^{n+1}}{c_1 5^n + c_2(-1)^n} - \frac{2}{3} \\ &= \frac{(c_1 5^n - c_2(-1)^n)}{(c_1 5^n + c_2(-1)^n)} = \frac{5^n - c(-1)^n}{5^n + c(-1)^n}, \end{aligned}$$

where

$$c = \frac{c_1}{c_2}.$$

Type III. Homogeneous difference equations of the type

$$f\left(\frac{x(n+1)}{x(n)}, n\right) = 0.$$

Use the transformation $z(n) = \frac{x(n+1)}{x(n)}$ to convert such an equation to a linear equation in $z(n)$, thus allowing it to be solved.

Example 2.41. Solve the difference equation

$$x^2(n+1) - 3x(n+1)x(n) + 2x^2(n) = 0. \tag{2.6.11}$$

Solution Dividing by $x^2(n)$, equation (2.6.11) becomes

$$\left[\frac{x(n+1)}{x(n)}\right]^2 - 3\left[\frac{x(n+1)}{x(n)}\right] + 2 = 0, \tag{2.6.12}$$

which is of Type III.

Letting $z(n) = \frac{x(n+1)}{x(n)}$ in (2.6.12) creates

$$z^2(n) - 3z(n) + 2 = 0.$$

We can factor this down to

$$[z(n) - 2][z(n) - 1] = 0,$$

and thus either $z(n) = 2$ or $z(n) = 1$.

This leads to

$$x(n+1) = 2x(n) \quad \text{or} \quad x(n+1) = x(n).$$

Starting with $x(0) = x_0$, there are infinitely many solutions $x(n)$ of (2.6.11) of the form

$$x_0, \ldots, x_0; 2x_0, \ldots, 2x_0; 2^2x_0, \ldots, 2^2x_0; \ldots .[7]$$

Type IV. Consider the difference equation of the form

$$(y(n+k))^{r_1}\,(y(n+k-1))^{r_2} \cdots (y(n))^{r_{k+1}} = g(n). \tag{2.6.13}$$

Let $z(n) = \ln y(n)$, and rearrange to obtain

$$r_1 z(n+k) + r_2 z(n+k-1) + \cdots + r_{k+1} z(n) = \ln g(n). \tag{2.6.14}$$

Example 2.42. Solve the difference equation

$$x(n+2) = \frac{x^2(n+1)}{x^2(n)}. \tag{2.6.15}$$

Solution Let $z(n) = \ln x(n)$ in (2.6.15). Then as in (2.6.12) we obtain

$$z(n+2) - 2z(n+1) + 2z(n) = 0.$$

The characteristic roots are $\lambda_1 = 1 + i, \lambda_2 = 1 - i$.

Thus,

$$z(n) = (2)^{n/2}\left[c_1 \cos\left(\frac{n\pi}{4}\right) + c_2 \sin\left(\frac{n\pi}{4}\right)\right].$$

---

[7] This solution was given by Sebastian Pancratz of the Technical University of Munich.

Therefore,

$$x(n) = \exp\left[(2)^{n/2}\left\{c_1 \cos\left(\frac{n\pi}{4}\right) + c_2 \sin\left(\frac{n\pi}{4}\right)\right\}\right].$$

Exercises 2.6

1. Find the general solution of the difference equation
$$y^2(n+1) - 2y(n+1)y(n) - 3y^2(n) = 0.$$
2. Solve the difference equation
$$y^2(n+1) - (2+n)y(n+1)y(n) + 2ny^2(n) = 0.$$
3. Solve $y(n+1)y(n) - y(n+1) + y(n) = 0$.
4. Solve $y(n+1)y(n) - \frac{2}{3}y(n+1) + \frac{1}{6}y(n) = \frac{5}{18}$.
5. Solve $y(n+1) = 5 - \dfrac{6}{y(n)}$.
6. Solve $x(n+1) = \dfrac{x(n)+a}{x(n)+1}, 1 \neq a > 0$.
7. Solve $x(n+1) = x^2(n)$.
8. Solve the logistic difference equation
$$x(n+1) = 2x(n)(1 - x(n)).$$
9. Solve the logistic equation
$$x(n+1) = 4x(n)[1 - x(n)].$$
10. Solve $x(n+1) = \dfrac{1}{2}\left(x(n) - \dfrac{a}{x(n)}\right), a > 0$.
11. Solve $y(n+2) = y^3(n+1)/y^2(n)$.
12. Solve $x(n+1) = \dfrac{2x(n)+4}{x(n)-1}$.
13. Solve $y(n+1) = \dfrac{2 - y^2(n)}{2(1 - y(n))}$.
14. Solve $x(n+1) = \dfrac{2x(n)}{x(n)+3}$.
15. Solve $y(n+1) = 2y(n)\sqrt{1 - y^2(n)}$.
16. The "regular falsi" method for finding the roots of $f(x) = 0$ is given by
$$x(n+1) = \frac{x(n-1)f(x(n)) - x(n)f(x(n-1))}{f(x(n)) - f(x(n-1))}.$$

(a) Show that for $f(x) = x^2$, this difference equation becomes

$$x(n+1) = \frac{x(n-1)x(n)}{x(n-1)+x(n)}.$$

(b) Let $x(1) = 1, x(2) = 1$ for the equation in part (a). Show that the solution of the equation is $x(n) = 1/F(n)$, where $F(n)$ is the nth Fibonacci number.

2.7 Applications

2.7.1 Propagation of Annual Plants

The material of this section comes from Edelstein–Keshet [37] of plant propagation. Our objective here is to develop a mathematical model that describes the number of plants in any desired generation. It is known that plants produce seeds at the end of their growth season (say August), after which they die. Furthermore, only a fraction of these seeds survive the winter, and those that survive germinate at the beginning of the season (say May), giving rise to a new generation of plants.

Let

$$\begin{aligned}
\gamma &= \text{number of seeds produced per plant in August,}\\
\alpha &= \text{fraction of one-year-old seeds that germinate in May,}\\
\beta &= \text{fraction of two-year-old seeds that germinate in May,}\\
\sigma &= \text{fraction of seeds that survive a given winter.}
\end{aligned}$$

If $p(n)$ denotes the number of plants in generation n, then

$$\begin{aligned}
p(n) &= \begin{pmatrix}\text{plants from}\\ \text{one-year-old seeds}\end{pmatrix} + \begin{pmatrix}\text{plants from}\\ \text{two-year-old seeds}\end{pmatrix},\\
p(n) &= \alpha s_1(n) + \beta s_2(n),
\end{aligned} \tag{2.7.1}$$

where $s_1(n)$ (respectively, $s_2(n)$) is the number of one-year-old (two-year-old) seeds in April (before germination). Observe that the number of seeds left after germination may be written as

$$\text{seeds left} = \begin{pmatrix}\text{fraction}\\ \text{not germinated}\end{pmatrix} \times \begin{pmatrix}\text{original number}\\ \text{of seeds in April}\end{pmatrix}.$$

This gives rise to two equations:

$$\tilde{s}_1(n) = (1-\alpha)s_1(n), \tag{2.7.2}$$

$$\tilde{s}_2(n) = (1-\beta)s_2(n), \tag{2.7.3}$$

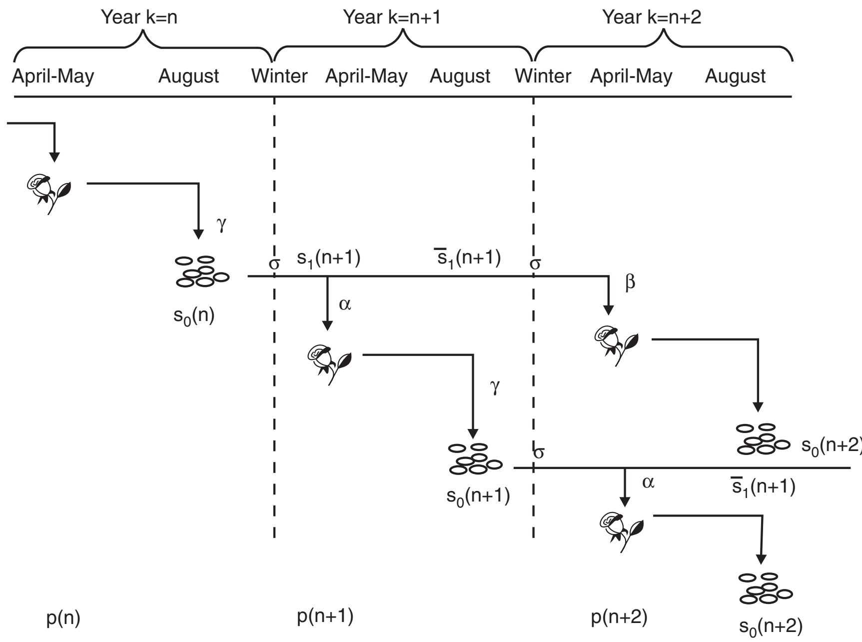

FIGURE 2.6. Propogation of annual plants.

where $\tilde{s}_1(n)$ (respectively, $\tilde{s}_2(n)$) is the number of one-year (two-year-old) seeds left in May after some have germinated. New seeds $s_0(n)$ (0-year-old) are produced in August (Figure 2.6) at the rate of γ per plant,

$$s_0(n) = \gamma p(n). \tag{2.7.4}$$

After winter, seeds $s_0(n)$ that were new in generation n will be one year old in the next generation $n+1$, and a fraction $\sigma s_0(n)$ of them will survive. Hence

$$s_1(n+1) = \sigma s_0(n),$$

or, by using formula (2.7.4), we have

$$s_1(n+1) = \sigma \gamma p(n). \tag{2.7.5}$$

Similarly,

$$s_2(n+1) = \sigma \tilde{s}_1(n),$$

which yields, by formula (2.7.2),

$$\begin{aligned} s_2(n+1) &= \sigma(1-\alpha)s_1(n), \\ s_2(n+1) &= \sigma^2\gamma(1-\alpha)p(n-1). \end{aligned} \tag{2.7.6}$$

Substituting for $s_1(n+1), s_2(n+1)$ in expressions (2.7.5) and (2.7.6) into formula (2.7.1) gives

$$p(n+1) = \alpha\gamma\sigma p(n) + \beta\gamma\sigma^2(1-\alpha)p(n-1),$$

or

$$p(n+2) = \alpha\gamma\sigma p(n+1) + \beta\gamma\sigma^2(1-\alpha)p(n). \tag{2.7.7}$$

The characteristic equation (2.7.7) is given by

$$\lambda^2 - \alpha\gamma\sigma\lambda - \beta\gamma\sigma^2(1-\alpha) = 0$$

with characteristic roots

$$\lambda_1 = \frac{\alpha\gamma\sigma}{2}\left[1 + \sqrt{1 + \frac{4\beta}{\gamma\alpha^2}(1-\alpha)}\,\right],$$
$$\lambda_2 = \frac{\alpha\gamma\sigma}{2}\left[1 - \sqrt{1 + \frac{4\beta}{\gamma\alpha^2}(1-\alpha)}\,\right].$$

Observe that λ_1 and λ_2 are real roots, since $1-\alpha > 0$. Furthermore, $\lambda_1 > 0$ and $\lambda_2 < 0$. To ensure propagation (i.e., $p(n)$ increases indefinitely as $n \to \infty$) we need to have $\lambda_1 > 1$. We are not going to do the same with λ_2, since it is negative and leads to undesired fluctuation (oscillation) in the size of the plant population. Hence

$$\frac{\alpha\gamma\sigma}{2}\left[1 + \sqrt{1 + \frac{4\beta}{\gamma\alpha^2}(1-\alpha)}\,\right] > 1,$$

or

$$\frac{\alpha\gamma\sigma}{2}\sqrt{1 + \frac{4\beta(1-\alpha)}{\gamma\alpha^2}} > 1 - \frac{\alpha\gamma\sigma}{2}.$$

Squaring both sides and simplifying yields

$$\gamma > \frac{1}{\alpha\sigma + \beta\sigma^2(1-\alpha)}. \tag{2.7.8}$$

If $\beta = 0$, that is, if no two-year-old seeds germinate in May, then condition (2.7.8) becomes

$$\gamma > \frac{1}{\alpha\sigma}. \tag{2.7.9}$$

Condition (2.7.9) says that plant propagation occurs if the product of the fraction of seeds produced per plant in August, the fraction of one-year-old seeds that germinate in May, and the fraction of seeds that survive a given winter exceeds 1.

2.7.2 Gambler's Ruin

A gambler plays a sequence of games against an adversary in which the probability that the gambler wins \$1.00 in any given game is a known value q, and the probability of his losing \$1.00 is $1-q$, where $0 \leq q \leq 1$. He quits gambling if he either loses all his money or reaches his goal of acquiring N dollars. If the gambler runs out of money first, we say that the gambler has been ruined. Let $p(n)$ denote the probability that the gambler will be ruined if he possesses n dollars. He may be ruined in two ways. First, winning the next game; the probability of this event is q; then his fortune will be $n+1$, and the probability of being ruined will become $p(n+1)$. Second, losing the next game; the probability of this event is $1-q$, and the probability of being ruined is $p(n-1)$. Hence applying the theorem of total probabilities, we have

$$p(n) = qp(n+1) + (1-q)p(n-1).$$

Replacing n by $n+1$, we get

$$p(n+2) - \frac{1}{q}p(n+1) + \frac{(1-q)}{q}p(n) = 0, \quad n = 0, 1, \ldots, N, \tag{2.7.10}$$

with $p(0) = 1$ and $p(N) = 0$. The characteristic equation is given by

$$\lambda^2 - \frac{1}{q}\lambda + \frac{1-q}{q} = 0,$$

and the characteristic roots are given by

$$\begin{aligned}
\lambda_1 &= \frac{1}{2q} + \frac{1-2q}{2q} = \frac{1-q}{q}, \\
\lambda_2 &= \frac{1}{2q} - \frac{1-2q}{2q} = 1.
\end{aligned}$$

Hence the general solution may be written as

$$p(n) = c_1 + c_2\left(\frac{1-q}{q}\right)^n, \qquad \text{if } q \neq \frac{1}{2}.$$

Now using the initial conditions $p(0) = 1, P(N) = 0$ we obtain

$$c_1 + c_2 = 1, \qquad c_1 + c_2\left(\frac{1-q}{q}\right)^N = 0,$$

which gives

$$c_1 = \frac{-\left(\frac{1-q}{q}\right)^N}{1-\left(\frac{1-q}{q}\right)^N}, \qquad c_2 = \frac{1}{1-\left(\frac{1-q}{q}\right)^N}.$$

Thus

$$p(n) = \frac{\left(\dfrac{1-q}{q}\right)^n - \left(\dfrac{1-q}{q}\right)^N}{1 - \left(\dfrac{1-q}{q}\right)^N}. \tag{2.7.11}$$

The special case $q = \frac{1}{2}$ must be treated separately, since in this case we have repeated roots $\lambda_1 = \lambda_2 = 1$. This is certainly the case when we have a fair game. The general solution in this case may be given by

$$p(n) = a_1 + a_2 n,$$

which with the initial conditions yields

$$p(n) = 1 - \frac{n}{N} = \frac{N-n}{N}. \tag{2.7.12}$$

For example, suppose you start with \$4, the probability that you win a dollar is 0.3, and you will quit if you run out of money or have a total of \$10. Then $n = 4$, $q = 0.3$, and $N = 10$, and the probability of being ruined is given by

$$p(4) = \frac{\left(\dfrac{7}{3}\right)^4 - \left(\dfrac{7}{3}\right)^{10}}{1 - \left(\dfrac{7}{3}\right)^{10}} = 0.994.$$

On the other hand, if $q = 0.5$, $N = \$100.00$, and $n = 20$, then from formula (2.7.12) we have

$$p(20) = 1 - \frac{20}{100} = 0.8.$$

Observe that if $q \leq 0.5$ and $N \to \infty$, $p(n)$ tends to 1 in both formulas (2.7.11) and (2.7.12), and the gambler's ruin is certain.

The probability that the gambler wins is given by

$$\tilde{p}(n) = 1 - p(n) = \begin{cases} \dfrac{1 - \left(\dfrac{1-q}{q}\right)^n}{1 - \left(\dfrac{1-q}{q}\right)^N}, & \text{if } q \neq 0.5, \\[2ex] \dfrac{n}{N}, & \text{if } q = 0.5. \end{cases} \tag{2.7.13}$$

2.7.3 National Income

In a capitalist country the national income $Y(n)$ in a given period n may be written as

$$Y(n) = C(n) + I(n) + G(n), \tag{2.7.14}$$

where

$C(n)$ = consumer expenditure for purchase of consumer goods,

$I(n)$ = induced private investment for buying capital equipment, and

$G(n)$ = government expenditure,

where n is usually measured in years.

We now make some assumptions that are widely accepted by economists (see, for example, Samuelson [129]).

(a) Consumer expenditure $C(n)$ is proportional to the national income $Y(n-1)$ in the preceding year $n-1$, that is,

$$C(n) = \alpha Y(n-1), \tag{2.7.15}$$

where $\alpha > 0$ is commonly called the *marginal propensity to consume*.

(b) Induced private investment $I(n)$ is proportional to the increase in consumption $C(n) - C(n-1)$, that is,

$$I(n) = \beta[C(n) - C(n-1)], \tag{2.7.16}$$

where $\beta > 0$ is called the *relation*.

(c) Finally, the government expenditure $G(n)$ is constant over the years, and we may choose our units such that

$$G(n) = 1. \tag{2.7.17}$$

Employing formulas (2.7.15), (2.7.16), and (2.7.17) in formula (2.7.14) produces the second-order difference equation

$$Y(n+2) - \alpha(1+\beta)Y(n+1) + \alpha\beta Y(n) = 1, \qquad n \in \mathbb{Z}^+. \tag{2.7.18}$$

Observe that this is the same equation we have already studied, in detail, in Example 2.38. As we have seen there, the equilibrium state of the national income $Y^* = 1/(1-\alpha)$ is asymptotically stable (or just stable in the theory of economics) if and only if the following conditions hold:

$$\alpha < 1, \qquad 1 + \alpha + 2\alpha\beta > 0, \qquad \alpha\beta < 1. \tag{2.7.19}$$

Furthermore, the national income $Y(n)$ fluctuates (oscillates) around the equilibrium state Y^* if and only if

$$\alpha < \frac{4\beta}{(1+\beta)^2}. \tag{2.7.20}$$

Now consider a concrete example where $\alpha = \frac{1}{2}, \beta = 1$. Then $Y^* = 2$, i.e., Y^* = twice the government expenditure. Then clearly, conditions (2.7.19) and (2.7.12) are satisfied. Hence the national income $Y(n)$ always converges in an oscillatory fashion to $Y^* = 2$, regardless of what the initial national income $Y(0)$ and $Y(1)$ are. (See Figure 2.7.)

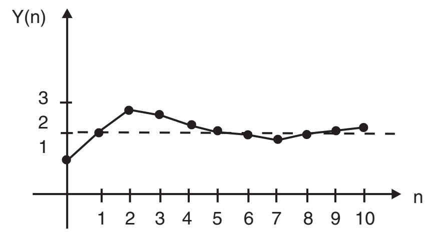

FIGURE 2.7. Solution of $Y(n+2) - Y(n+1) + Y(n) = 1, Y(0) = 1, Y(1) = 2$.

The actual solution may be given by

$$Y(n) = A\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{n\pi}{4} - \omega\right) + 2.$$

Figure 2.7 depicts the solution $Y(n)$ if $Y(0) = 1$ and $Y(1) = 2$. Here we find that $A = -\sqrt{2}$ and $\omega = \pi/4$ and, consequently, the solution is

$$Y(n) = -\left(\frac{1}{\sqrt{2}}\right)^{n-1} \cos\left[\frac{(n+1)}{4}\pi\right] + 2.$$

Finally, Figure 2.8 depicts the parameter diagram $(\beta - \alpha)$, which shows regions of stability and regions of instability.

2.7.4 The Transmission of Information

Suppose that a signaling system has two signals s_1 and s_2 such as dots and dashes in telegraphy. Messages are transmitted by first encoding them into a string, or sequence, of these two signals. Suppose that s_1 requires exactly n_1 units of time, and s_2 exactly n_2 units of time, to be transmitted. Let $M(n)$ be the number of possible message sequences of duration n. Now, a signal of duration time n either ends with an s_1 signal or with an s_2 signal.

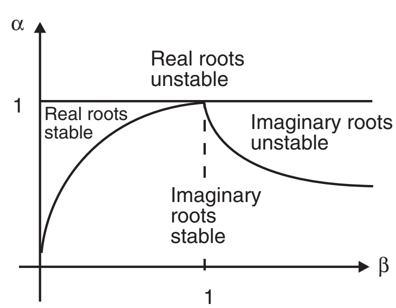

FIGURE 2.8. Parametric diagram $(\beta - \alpha)$.

FIGURE 2.9. Two signals, one ends with s_1 and the other with s_2.

If the message ends with s_1, the last signal must start at $n - n_1$ (since s_1 takes n_1 units of time). Hence there are $M(n-n_1)$ possible messages to which the last s_1 may be appended. Hence there are $M(n - n_1)$ messages of duration n that end with s_1. By a similar argument, one may conclude that there are $M(n - n_2)$ messages of duration n that end with s_2. (See Figure 2.9.) Consequently, the total number of messages $x(n)$ of duration n may be given by

$$M(n) = M(n - n_1) + M(n - n_2).$$

If $n_1 \geq n_2$, then the above equation may be written in the familiar form of an n_1th-order equation

$$M(n + n_1) - M(n + n_1 - n_2) - M(n) = 0. \tag{2.7.21}$$

On the other hand, if $n_1 \leq n_2$, then we obtain the n_2th-order equation

$$M(n + n_2) - M(n + n_2 - n_1) - M(n) = 0. \tag{2.7.22}$$

An interesting special case is that in which $n_1 = 1$ and $n_2 = 2$. In this case we have

$$M(n + 2) - M(n + 1) - M(n) = 0,$$

or

$$M(n + 2) = M(n + 1) + M(n),$$

which is nothing but our Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$, which we encountered in Example 2.27. The general solution (see formula (2.3.14)) is given by

$$M(n) = a_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + a_2 \left(\frac{1-\sqrt{5}}{2}\right)^n, \qquad n = 0, 1, 2, \ldots. \tag{2.7.23}$$

To find a_1 and a_2 we need to specify $M(0)$ and $M(1)$. Here a sensible assumption is to let $M(0) = 0$ and $M(1) = 1$. Using these initial data in (2.7.23) yields

$$a_1 = \frac{1}{\sqrt{5}}, \qquad a_2 = -\frac{1}{\sqrt{5}},$$

and the solution of our problem now becomes

$$M(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{2.7.24}$$

In information theory, the capacity C of the channel is defined as

$$C = \lim_{n\to\infty} \frac{\log_2 M(n)}{n}, \tag{2.7.25}$$

where $\log_2$ denotes the logarithm base 2.

From (2.7.24) we have

$$C = \lim_{n\to\infty} \frac{\log_2 \frac{1}{\sqrt{5}}}{n} + \lim_{n\to\infty} \frac{1}{n} \log_2 \left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n \right]. \tag{2.7.26}$$

Since $\left(\frac{1-\sqrt{5}}{2}\right) \approx 0.6 < 1$, it follows that $\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0$ as $n \to \infty$.

Observe also that the first term on the right-hand side of (2.7.26) goes to zero as $n \to \infty$.

Thus

$$C = \lim_{n\to\infty} \frac{1}{n} \log_2 \left(\frac{1+\sqrt{5}}{2}\right)^n,$$

$$C = \log_2 \left(\frac{1+\sqrt{5}}{2}\right) \approx 0.7. \tag{2.7.27}$$

Exercises 2.7

1. The model for annual plants was given by (2.7.7) in terms of the plant population $p(n)$.
   (a) Write the model in terms of $s_1(n)$.
   (b) Let $\alpha = \beta = 0.01$ and $\sigma = 1$. How big should γ be to ensure that the plant population increases in size?
2. An alternative formulation for the annual plant model is that in which we define the beginning of a generation as the time when seeds are produced. Figure 2.10 shows the new method.

   Write the difference equation in $p(n)$ that represents this model. Then find conditions on γ under which plant propagation occurs.
3. A planted seed produces a flower with one seed at the end of the first year and a flower with two seeds at the end of two years and each year thereafter. Suppose that each seed is planted as soon as it is produced.
   (a) Write the difference equation that describes the number of flowers $F(n)$ at the end of the nth year.
   (b) Compute the number of flowers at the end of 3, 4, and 5 years.
4. Suppose that the probability of winning any particular bet is 0.49. If you start with \$50 and will quit when you have \$100, what is the probability of ruin (i.e., losing all your money):

(i) if you make \$1 bets?

(ii) if you make \$10 bets?

(iii) if you make \$50 bets?

5. John has m chips and Robert has $(N - m)$ chips. Suppose that John has a probability p of winning each game, where one chip is bet on in each play. If $G(m)$ is the expected value of the number of games that will be played before either John or Robert is ruined:

   (a) Show that $G(m)$ satisfies the second-order equation

   $$G(m+2) + pG(m+1) + (1-p)G(m) = 0. \tag{2.7.28}$$

   (b) What are the values of $G(0)$ and $G(N)$?

   (c) Solve the difference equation (2.7.28) with the boundary conditions in part (b).

6. Suppose that in a game we have the following situation: On each play, the probability that you will win \$2 is 0.1, the probability that you will win \$1 is 0.3, and the probability that you will lose \$1 is 0.6. Suppose you quit when either you are broke or when you have at least N dollars. Write a third-order difference equation that describes the probability $p(n)$ of eventually going broke if you have n dollars. Then find the solution of the equation.

7. Suppose that Becky plays a roulette wheel that has 37 divisions: 18 are red, 18 are black, and one is green. Becky can bet on either the red or black, and she wins a sum equal to her bet if the outcome is a division of that color; otherwise, she loses the bet. If the bank has one

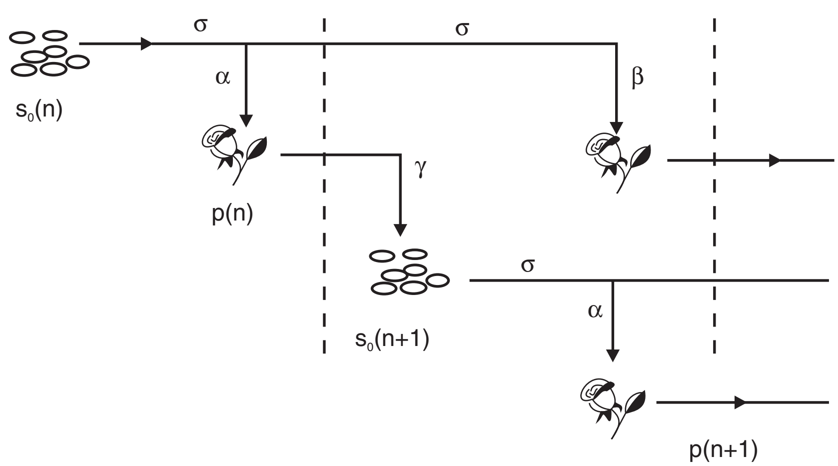

FIGURE 2.10. Annual plant model.

million dollars and she has \$5000, what is the probability that Becky can break the bank, assuming that she bets \$100 on either red or black for each spin of the wheel?

8. In the national income model (2.7.14), assume that the government expenditure $G(n)$ is proportional to the national income $Y(n-2)$ two periods past, i.e., $G(n) = \gamma Y(n-2)$, $0 < \gamma < 1$. Derive the difference equation for the national income $Y(n)$. Find the conditions for stability and oscillations of solutions.

9. Determine the behavior (stability, oscillations) of solutions of (2.7.18) for the cases:

   (a) $\alpha = \dfrac{4\beta}{(1+\beta)^2}$.

   (b) $\alpha > \dfrac{4\beta}{(1+\beta)^2}$.

10. Modify the national income model such that instead of the government having fixed expenditures, it increases its expenditures by 5% each time period, that is, $G(n) = (1.05)^n$.

    (a) Write down the second-order difference equation that describes this model.

    (b) Find the equilibrium value.

    (c) If $\alpha = 0.5, \beta = 1$, find the general solution of the equation.

11. Suppose that in the national income we make the following assumptions:

    (i) $Y(n) = C(n) + I(n)$, i.e., there is no government expenditure.

    (ii) $C(n) = a_1 Y(n-1) + a_2 Y(n-2) + K$, i.e., consumption in any period is a linear combination of the incomes of the two preceding periods, where a_1, a_2, and K are constants.

    (iii) $I(n+1) = I(n) + h$, i.e., investment increases by a fixed amount $h > 0$ each period.

       (a) Write down a third-order difference equation that models the national income $Y(n)$.

       (b) Find the general solution if $a_1 = \dfrac{1}{2}, a_2 = \dfrac{1}{4}$.

       (c) Show that $Y(n)$ is asymptotic to the equilibrium $Y^* = \alpha + \beta n$.

12. (Inventory Analysis). Let $S(n)$ be the number of units of consumer goods produced for sale in period n, and let $T(n)$ be the number of units of consumer goods produced for inventories in period n. Assume that there is a constant noninduced net investment V_0 in each period.

Then the total income $Y(n)$ produced in time n is given by $Y(n) = T(n) + S(n) + V_0$.

(a) Develop a difference equation that models the total income $Y(n)$, under the assumptions:

(i) $S(n) = \beta Y(n-1)$,

(ii) $T(n) = \beta Y(n-1) - \beta Y(n-2)$.

(b) Obtain conditions under which:

(i) solutions converge to the equilibrium,

(ii) solutions are oscillatory.

(c) Interpret your results in part (b).

13. Let $I(n)$ denote the level of inventories at the close of period n.

(a) Show that $I(n) = I(n-1) + S(n) + T(n) - \beta Y(n)$ where $S(n), T(n), Y(n)$ are as in Problem 12.

(b) Assuming that $S(n) = 0$ (passive inventory adjustment), show that

$$I(n) - I(n-1) = (1-\beta)Y(n) - V_0$$

where V_0 is as in Problem 12.

(c) Suppose as in part (b) that $s(n) = 0$. Show that

$$I(n+2) - (\beta+1)I(n+1) + \beta I(n) = 0.$$

(d) With $\beta \neq 1$, show that

$$I(n) = \left(I(0) - \frac{c}{1-\beta}\right)\beta^n + \frac{c}{1-\beta},$$

where $(E-\beta)I(n) = c$.

14. Consider (2.7.21) with $n_1 = n_2 = 2$ (i.e., both signals s_1 and s_2 take two units of time for transmission).

(a) Solve the obtained difference equation with the initial conditions $M(2) = M(3) = 2$.

(b) Find the channel capacity c.

15. Consider (2.7.21) with $n_1 = n_2 = 1$ (i.e., both signals take one unit of time for transmission).

(a) Solve the obtained difference equation.

(b) Find the channel capacity c.

16. (Euler's method for solving a second-order differential equation.) Recall from Section 1.4.1 that one may approximate $x'(t)$ by $(x(n+1) - x(n))/h$, where h is the step size of the approximation and $x(n) = x(t_0 + nh)$.

    (a) Show that $x''(t)$ may be approximated by
    $$\frac{x(n+2) - 2x(n+1) + x(n)}{h^2}.$$

    (b) Write down the corresponding difference equation of the differential equation
    $$x''(t) = f(x(t), x'(t)).$$

17. Use Euler's method described in Problem 16 to write the corresponding difference equation of
    $$x''(t) - 4x(t) = 0, \quad x(0) = 0, \quad x'(0) = 1.$$
    Solve both differential and difference equations and compare the results.

18. (The Midpoint Method). The midpoint method stipulates that one may approximate $x'(t)$ by $(x(n+1) - x(n-1))/h$, where h is the step size of the approximation and $t = t_0 + nh$.

    (a) Use the method to write the corresponding difference equation of the differential equation $x'(t) = g(t, x(t))$.

    (b) Use the method to write the corresponding difference equation of $x'(t) = 0.7x^2 + 0.7, x(0) = 1, t \in [0, 1]$. Then solve the obtained difference equation.

    (c) Compare your findings in part (b) with the results in Section 1.4.1. Determine which of the two methods, Euler or midpoint, is more accurate.

Springer

http://www.springer.com/978-0-387-23059-7

An Introduction to Difference Equations
Elaydi, S.
2005, XXII, 540 p., Hardcover
ISBN: 978-0-387-23059-7