2. Categories for the Analysis of Culturally Shaped Conceptual Developments

What follows will outline the categories along which this study's analysis of culturally shaped processes of generalization and the conflicts fought over these processes primarily proceed.

The first category has already been described in the previous section: within the respective historical context, a concept will never appear isolated—rather, it is an element of a concept field that conveys its meaning to this element. Historical reconstruction thus must grasp how an element evolved together with its surrounding and supporting field.

The second category is that generalization, in many respects, turns out to have been a process of making *implied* meanings explicit. As Lakatos so nicely pointed out in his study on the polyhedra formula, a critical examination of conventional conceptual views leads time and again to uncovering "hidden lemmas" (Lakatos 1976, 43 ff.)—that is, to revealing assumptions implied. It is evident that such implied assumptions may prove to be inconsistent, or valid only for special cases, when we try to make them explicit or consciously reflect on them, thus calling for a conceptual frame modified by extension and generalization.

One of the consequences of this explicatory category is that concepts develop in a process of being differentiated ever further. Emerging concepts are in most cases still very general, in the sense that they can be used in wide fields of application. Bachelard pointed out that such original, unspecific generality, where experimentally testable features are hardly discernible, entail the danger of inducing gross mistakes, a fact that prompted him to assign overly general propositions as "premature" to the prescientific stage in his three-stages model of scientific development (Bachelard 1975, 55–72). In fact, while emerging concepts are being developed further, some of their fields of application become differentiated in subfields where the concept will then be given more specific meanings. Just think of the concept of function and its further differentiation into continuous functions, differentiable functions, etc.

An important further category of the process of explication is the concepts' sign aspect, their symbolism. There is a close connection between the sign and the signified, i.e., the concept. Upon encountering different sign representations, we as a rule cannot assume an identical concept meaning. This linguistic side of mathematical concepts has always had a special importance in periods of restructuring or modernizing the science.

Algebraization is closely linked to the category of sign representation. While the development of algebra has been based on elaborating suitable sign systems since Viète and Descartes, generalization was understood to be growing permeation of mathematics with algebra. Geometry, in contrast, was conceived of as confined to the limited frame of classical mathematics. Algebraization drives explication. This is why algebraization has been also reflected as "calculization"—as in Sybille Krämer's book (2001)—a process that not only makes mental entities graspable by the senses, but also signifies how they are produced.

Algebraization, as one of the central manifestations of generalization thus means a double process here:

- firstly, the process of transforming concepts originating from geometry into an algebraic form,
- and secondly, the process of internally changing algebraic propositions, which since Nesselmann's characterization of 1842 has generally been seen as taking place in three stages. The first stage is "rhetorical algebra," in which all propositions are presented exclusively in verbal form because of the total absence of signs. The second stage is the intermediate one of "syncopic algebra," which, while still operating verbally, introduces abbreviations for terms or operations frequently used. The third and last stage is "symbolic algebra," which relies on a sign language independent of words (Nesselmann 1842, 302).

Both the categories of the extent of sign use and of how far algebraization has progressed are subject to social variables. These scientification processes are dependent in their intensity on the ups and downs the estimation and valuation science as a whole encounters in the societies or nations concerned. These ups and downs find their classical expression in the polar opposition of a dominance either of the *analytic* or the *synthetic* method, and they will be documented as a pervasive category in the developments of concepts treated in this study.

At the same time, extent and modes of algebraization are also directly linked to cultural and epistemological traditions in the societies and professional communities concerned. These were conducive in the then leading countries of France, England, and Germany to variations of their own that remained stable over extended periods. Their differences will make the importance of culturally specific historical reconstructions particularly salient.

3. An unusual Pair: Negative Numbers and Infinitely Small Quantities

> The doctrine of opposite quantities and the calculus of the infinite have both experienced the same fate, inasmuch as one has been formerly happy with their correct application, thereby bringing pure mathematics to a high degree of perfection, while also enriching the mathematical sciences as such with the most important inventions by applying them, before the first concepts and principles on which they are based had been completely brought to light. [...] If one is convinced of a result's correctness beforehand, one will unconsciously ascribe more completeness and evidence to the concepts and principles from which it was derived in a shortcut than these actually have, and only he to whom both truth and its justification are new will feel the defects of the latter.
>
> —*Hecker* 1800 b, 4

The objects of this investigation into concept development shall be two concepts belonging to the foundations of mathematics: that of negative numbers and that of infinitely small quantities. Not only do the debates about clarifying the concepts of negative numbers and of "classical" infinitely small quantities both form typical cases of concept development for themselves, but the development of each of these concepts was by no means unrelated to that of the other.[1] Numerous mathematicians have made contributions to both problem areas. Their respective positions yield telling profiles on foundational issues of mathematics, but far from the claim that they show mechanist correspondences or directly interpretable "scales."

Both problem fields are typical indicators for the respective views on the relations between intuition, rigor, theoreticity (*Theoretizität*), and generality of mathematical concepts. At the same time, they are closely connected to contemporary epistemological views—both to general philosophical orientations

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1 The distinction between "classical" and "modern" infinitely small quantities, introduced here as one of the results of long-term analysis, will be explained later in Chapters V, VI, and VII.

and to specific ones for mathematics and the sciences. Therefore, the development of these concepts did not occur in an isolated way, but as part of more pervasive thought processes belonging to an epoch or a culture. This is why it was suggestive to pursue the very differences that are traceable for cultures existing side by side in Europe.

In traditional historical accounts, the development of the negative numbers appears to be just a small strand that can be presented without connecting it internally to other developments in the mathematical edifice. Actually, developing this very concept forms an essential element in the process of generalization in mathematics: it is pivotal not only for a more abstract understanding of mathematical operations, but also for the emergence of a concept of quantity that facilitates justifying the new, rigorous analysis. As Bos has shown, multiplication, for instance, had been executable only in a restricted manner as long as the concept of geometrical quantities prevailed, and this not only impeded an unrestricted application of the concept of equation, but also the introduction of foundational concepts of analysis in their then insufficiently general form (Bos 1974, 6 ff. and passim). As will be shown, one focus of the efforts at clarifying the concept of negative numbers was on making the operation of multiplication generally executable; in this process of development, the concept of geometrical quantities lost its status of fundamental concept for algebra and analysis.

In contrast to what historiography says, classical authors of function theory, and in particular of complex functions, at the apogee of establishing modern analysis were perfectly aware of the fact that progress in analysis was closely linked to extending the concept of number. Weierstraß, in his lectures on function theory, always presented the underlying concept of number, and was the first to develop a comprehensive and rigorous conceptual system for negative numbers (cf. Spalt 1991). Riemann, in his respective lectures on function theory, always emphasized in detail the introduction of negative numbers as a decisive step in the generalization process of mathematics:

> The original object of mathematics is the integer number; the field of study increases only gradually. This extension does not happen arbitrarily, however; it is always motivated by the fact that the initially restricted view leads toward a need for such an extension. Thus the task of subtraction requires us to seek such quantities, or to extend our concept of quantity in such a way that its execution is always possible, thus guiding us to the concept of the negative (Riemann 1861, 21).

It must be noted that Riemann, in particular contrast to received historiography, emphasized the systematic link between introducing negative numbers and the larger edifice of mathematical concepts: “This extension of the area of quantities incidentally entails that the meaning of the arithmetical operations is modified” (ibid.), quoting the operation of multiplication as an example.

Fontenelle, L. Carnot, and Duhamel shall be mentioned in this introduction as pioneers linking the foundational effort concerning the concepts of negative number and infinitely small quantity as mathematicians exemplary for this study's period of investigation at its beginning, middle, and end.

Carnot, who lived and worked in the middle of the transition from the Enlightenment to modern times, made a strong effort at anchoring mathematical concepts in intuition, and at establishing them rigorously at the same time. Negative numbers and limits, or infinitely small quantities, were his major points of approach toward rigor and generality. The contributions of Fontenelle's early attempts at founding theoretical concepts of mathematics are less familiar.

Bernard le Bovier de Fontenelle (1657–1757), permanent secretary of the Paris Academy of Sciences, is known as the author of *Eléments de la Géométrie de l'Infini* (1727), severely criticized by his contemporaries, in which he boldly developed a calculus for operating with the infinitely large and the infinitely small. At the same time, his volume also presented new contributions toward clarifying the concept of negative quantities: Chapters II to V of the first part dealing with *la grandeur infiniment petite*, and with incommensurable quantities are followed by chapter VI on *Des Grandeurs positives et negatives, réelles et imaginaires*. Based on the concept of quantity thus developed, Fontenelle investigates applications to geometry and to differential calculus. I shall treat in detail the pertinent sections of this in my own next chapter.

At the end of the entire period, there is J.M.C. Duhamel (1797–1872). While he himself was not a "Modern Man," he made an impressive effort to convey the general character of mathematical concepts in his teaching. Again, the two foundational concepts that form the bulk of his presentation of generalization in mathematics were negative numbers and infinitely small quantities.

Particularly instructive are his five volumes *Des Méthodes dans les sciences de raisonnement* (1865–1873), in which Duhamel summarized his own work extending over several decades of reflection and presenting the foundational concepts of the sciences. This work is noteworthy not only as a rare and late approach toward methodologically reflecting on mathematics, sciences, and even on parts of the social sciences, thus avoiding confinement to a narrow, technical specialization, but also because it again tackles the foundational issues, yet unresolved since the beginning of the nineteenth century, trying to solve them methodologically. As Duhamel says in his introduction, he had planned such a methodological work since his youth to assist in clarifying and removing the "obscurities" that he had observed already in the mathematics courses in school. He must thus have been first motivated around 1810, and obviously by doubts concerning the foundational aspects of algebra. These "obscurities" had not been removed at the time of his subsequent studies at the *École polytechnique* (1814 and later); instead, new ones had been added to the old (evidently concerning foundational concepts of analysis), and his fellow students had seen no reason for concern (Duhamel vol. I, 1875, 1). Duhamel's late work thus documents that there had been no decisive further development in the foundational issues in France from 1800/1810 until the century's last third.

At the same time, Duhamel’s work underlines how important it was for the methodology of science and epistemology to focus on foundational concepts, an approach typical of the way authors strove for rigor about the year 1800. In his section on the most frequent causes of error in thinking, Duhamel claims that the causes lay less in errors of deduction than in the inexactness of basic assumptions, the easiest way to err being in establishing general propositions (ibid., 21).

Although Duhamel’s views belong to the close of my study’s period of investigation, they may nevertheless be considered typical of how people strove for rigor across the entire period.

Eventually, the conceptual link between the developments of the concepts of negative numbers and of infinitely small quantities is also confirmed by a partisan opposing the view that algebra’s task was to generalize: by d’Alembert, who heatedly argued against admitting the two notions as mathematical concepts, thus indirectly contributing to their further clarification (cf. Chapter II. 2.8.).

Chapter II

Paths Toward Algebraization – Development to the Eighteenth Century. The Number Field

1. An Overview of the History of Key Fundamental Concepts

For the presumably first, albeit rather sketchy, historical study on textbooks about infinitesimal calculus, the conceptual basis had already been that analysis cannot follow up isolated concepts, but must rather pursue the connections within a concept field. The study listed, as elements of this field, "for differential calculus and integral calculus three concepts (are) basic: number, function, and limit" (Bohlmann 1899, 93).

Studying the two concept developments on which we intend to focus will be preceded by an introductory overview about how essential elements of this concept field were developed up to the eighteenth century. The intention is to sketch the conceptual frames, in line with the received literature, sufficiently to prepare for the subsequent in-depth analysis within this concept field. This introduction intends to present only those aspects that are relevant here for concept development. This is why I do not adopt Bohlmann's subdivision of the concept field into elements. Instead, I start from the general position that concepts are subject to continuous differentiation (*Ausdifferenzierung*, cf. Chapter I.). One original notion evolved, by continuous differentiation, into several separate and independent concepts. This kind of continuous differentiation can be established for the foundational concepts relevant here: the concepts of number and of function do not form entirely separate concepts, but have emerged by way of continuous differentiation from the holistic concept of *quantity*. If you take quantity as the original fundamental idea, you will see that yet another concept differentiated from this original makes up an element of this concept field: this is the concept of variable. Three foundational concepts thus must be considered to have been successively differentiated from that of *quantity*: the concepts of *number*, *variable*, and *function*. And what is called limit by Bohlmann constitutes but one element of the comprehensive field of limit processes in mathematics, which was eventually differentiated further into the concepts of *limit*, of *continuity*, and of *convergence*. The concept of the *integral* shall also be included in this overview, because it is of special importance for developing the concept of infinitely small quantities.

1.1. The Concept of Number

The concept of number is of interest here foremost in two of its aspects; firstly, in its capability for conceiving of limit processes, i.e., in particular for investigating intermediate values, completeness, etc. Secondly, the differentiation of the concept of number from that of quantity and from that of magnitude is relevant. The first aspect, which primarily is about how the concept of the real number is formed, has always been intensely investigated and presented in the literature (cf. Gericke 1970). The discovery of incommensurability and of the existence of irrational quantities in Greek mathematics, in particular, belongs to this aspect. The long-debated question whether the Greeks already had a notion of real numbers has meanwhile been decided with negative outcome. Likewise, there is agreement that there were, until the eighteenth century, no efforts by mathematicians to conceptually clarify the real numbers. The term real numbers had been in use since about 1700 to characterize rational and irrational numbers, as opposed to the complex or imaginary numbers. Due to the prevalence of the concept of geometrical quantities, the completeness of the domain of real numbers was implicitly assumed as given; this implicit assumption was also expressed in the terminology used. Quantities were divided into *discrete* and *continuous* quantities. Discrete quantities were understood to be both concrete and abstract, or pure numbers, while continuous quantities were understood to be real numbers given geometrically by segments of straight lines.[1]

How the concept of number became differentiated from the more general concepts of quantity and of magnitude has been less intensely investigated and presented. In the French usage, it is particularly clear that the character of *quantité* is more basic. The concept of *grandeur*, formerly often used synonymously with *quantité*, is now basically limited to meaning concrete numbers. For clarifying the concept of negative numbers, the separation between the concept of numbers and that of magnitudes or *quantités* will prove to have been decisive. In his account of the history of negative numbers, Sesiano chose the differentiation between quantity and number as the basic category for analyzing their development (cf. Sesiano 1985). It is all the more important to emphasize how drawn out and tedious the historical process of differentiation

[1] The characterization discrete-continuous for quantities was established already by Aristotle in his doctrine of categories (cf. Boscovich 1754a, V).

between quantities and numbers was, since scholars are apt to assume that a concept of pure numbers had surfaced already in ancient Greek mathematics, similar to other eminent achievements of the same. What had emerged, however, was a concept of number tied to geometry. Only the integers were understood as numbers (ἀριθμός) at all, while other number areas, in particular fractions, were understood to be quantities; and Euclid understood even the integers geometrically, as segments of straight lines. Arithmetic, at that time, formed an integral part of geometry.[2]

With regard to the possibility of infinitely small quantities being taken to mean non-Archimedean quantities in the eighteenth century, a more special concept development will be discussed at this point. In his Book V, on number theory, Euclid excluded non-Archimedean quantities. Felix Klein has pointed out that Euclid excluded non-Archimedean quantities from his own concept of number to enable himself to found the theory of proportions—and together with it an early concept of irrational numbers (Klein 1925, 221).

The only place where Euclid mentions the admissibility of non-Archimedean quantities is the angle concept. This is where he begins with admitting, besides rectilinear angles, angles whose boundary lines are formed by curves (Book I, Def. 8) as well. A special case of these so-called *hornlike* (or cornicular) angles is that of the mixtilinear angles, in which one of the two boundary lines is a straight line and the other a curve. Klein has shown in detail that the hornlike angles form a model of non-Archimedean quantities (ibid., 221–224).[3] Euclid used the hornlike angles, which are not commensurable with rectilinear angles, only once in his *Elements*, in Book III, 16. This proposition has subsequently provoked an extensive debate, however. It says:

> The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle (Heath, vol. II, 37).

Proclus, in his commentary on Euclid, rejected this assumed commensurability of mixtilinear or hornlike angles with rectilinear angles. Referring to the so-called Archimedean axiom, he declared that quantities having a ratio to one another, when multiplied, can exceed one another. Accordingly, therefore, the hornlike angle would be able to exceed the rectilinear one as well—this possibility, however, being excluded by it having been shown that the hornlike angle will always be smaller than any rectilinear angle (Proclus 1945, 251).

In modern times, the debate was resumed, due to Euclid's intensified reception prompted by the new printed editions: as debates whether the mixtilinear angles quoted in Euclid's above proposition were admissible and

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2 Cf. N. Rouche 1992, 169 ff.

3 For a nonstandard interpretation of angles of contingency cf. Laugwitz 1970.

feasible as angles between a circle and a tangent, now called *angulus contactus*, or angle of contingency. The confrontation between Jaques Peletier and Christopher Clavius, in particular, has become well known.[4] In his own edition of Euclid of 1557, Peletier had rejected using heterogeneous, incommensurable angles. The angle of contact, he claimed, was no genuine angle, and in general not even a quantity; rather, the straight line touching the circle coincided with the latter's circumference, the angle of contingency thus being zero, and the angle formed by a semicircle and its diameter being a right angle (M. Cantor, vol. 2, 1900, 559 f.). Against Peletier, Clavius, in the second edition of his edition of Euclid (1589), admitted the angle of contingency to be a genuine angle, that is a quantity infinitely divisible. He agreed with the Euclidean proposition that the angle of contingency will be smaller than any possible rectilinear angle and that at the same time, the angle of the radius will be smaller than right, but greater than any rectilinear acute angle. On the other hand, he conceived angle of contingency and rectilinear angle as heterogeneous quantities that are not comparable with each other (ibid., 560 f.).

The debate after Peletier and Clavius found a first conclusion with John Wallis who—in a treatise of 1656—*De angulo contactus et semi-circuli tractatus*, and in a later one of 1684—adopted Peletier's view according to which the angle of contingency was a "non-angulum" and a "non-quantity." Wallis was the first to introduce, at the same time, however, the term degree of curvature.[5] The curvature behavior of curves had indeed been the mathematical context for which the concept of the angle of contingency was intended. As the concept of curvature was specified and operationalized, in particular by developing differential calculus, the debate about the angle of contingency waned, the geometry textbooks of the eighteenth and the nineteenth centuries touching it only marginally, if at all, mentioning the concept of mixtilinear or that of curvilinear angle, and bare of any further discussion of its foundation or of its mathematical application.

While Proclus's use of the Archimedean axiom still shows an explicit reference to the concept of quantity, the notion of mixtilinear and cornicular angles was used in modern times neither for reflecting on the concept of quantity nor for extending or generalizing the concept of quantity or of number. Typical for this tendency is Tacquet's position concerning the controversy between Peletarius and Clavius, in his own geometry textbook of 1654, in which he completely excluded the angle concept from the concept of quantity proper. In a first hint at differentiating the concept of quantity and particularly at introducing angles as equivalence relations, Tacquet declared that angles were not quantities, but rather *modi* of quantities, comparable to one another only with regard to their congruence or noncongruence (cf. Klügel 1803, 290). Boscovich, one of the

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4 For relevant publications between 1550 and 1650 see Giusti 1994. On the debate between Clavius and Peletier cf. in particular Maierú 1990.

5 A new thorough analysis of both studies by Wallis and of their statements about the Peletier/Clavius debate is Maierú (1988).

most prolific authors regarding foundational issues of the eighteenth century, also examined—in his detailed study *De continuitatis lege* (1754)—various arguments as to whether infinitely small quantities objectively exist. He also discussed, as one of these possible cases, the angle of contingency as an angle that is infinitely small when compared to rectilinear angles. Boscovich adopted Tacquet's argument that angle was not a quantity—usually defined via inclination—but rather a *modus* of a quantity. Mixtilinear angles, he said, were in principle incomparable with rectilinear ones, hence had no ratio to a rectilinear angle at all, and for that reason could be neither infinitely large nor infinitely small (Boscovich 1754, XXXVII; cf. also Manara 1987, 179).

1.2. The Concept of Variable

While the question of how the concept of function was formed has always been an essential element of investigations and accounts within the history of analysis, less attention has been given to how the concept of variable emerged, although the concept of function presupposes the concept of variable, and although the latter expresses an equally fundamental change from Greek mathematics. In the latter's prevailing geometrical character, quantities had been understood to be *constant*. The history of the concept of variable has recently been discussed as a constitutive part of the development of the foundational concepts of analysis by E. Giusti (1984).

In modern times, acceptance of the concept of the *unknown* in algebra was the preliminary stage for establishing the concept of variable. In algebra, Descartes placed the unknown in a dualistic opposition to the known quantity (*quantité inconnue* versus *quantité connue*) and this dualism was transferred to the concept of variable. In all the definitions of variable since L'Hospital's textbook of 1696, variables are explained by their opposition to constants:

> One calls *variable* quantities those which increase or decrease continually, and by contrast *constant* quantities those which remain the same while the others change (L'Hospital 1696, 1).

Chr. Wolff's *Mathematisches Lexikon* of 1716 already contains the key terms *Quantitas constans* and *Quantitates variabiles*. The entry on constants says:

> *Quantitas constans*, an invariable quantity, is the name of a quantity which always maintains the same quantity, whereas others increase or decrease (Chr. Wolff in 1716, 1144).

Also, in d'Alembert's entry "variable" in the *Encyclopédie*, the opposition between constant and variable is decisive for the definition, but he relates the variation to "a certain law." As examples he names abscissae and ordinates of curves. Constant quantities "do not change," an example being the diameter of a circle (Encyclop., XVI, 840).

Besides the dichotomy of constant–variable, the frequent explicit demand that it change continuously is noteworthy in these definitions of a variable. Something that might be an additional quality, for ensuring the completeness of the domain of definition, was already integrated, by overgeneralizing, into the basic definition. Possible anomalies in case of limit processes could be thus excluded from the very outset. Some textbook authors even insisted on a dominant character of the demand for continuity in their own definitions, like the Oratorian priest Reyneau in his influential textbook *Analyse demontrée*, who required that the quantities increase or decrease *insensiblement* (Reyneau, vol. 2, 1738, 152).

In Euler, a substantial change in the concept of variable can be noted; he gave up the dichotomy between constant and variable, replacing it by a universality of the variable. For him, the constant presents a special case, since he understands the variable to be an indeterminate that is able to assume certain values:

> *A constant quantity is a determined quantity which always keeps the same value.[...] A variable quantity is one which is not determined or is universal, which can take on any value* (Euler 1988, 2).
>
> Since all determined values can be expressed as numbers, a variable quantity takes on the totality of all possible numbers.[...] Hence a variable quantity can be determined in infinitely many ways, since absolutely all numbers can be substituted for it (ibid., 2 f.)

We will again find the identical condition, based on geometrical ties, in the concept of function, where the entire domain of the real numbers is implicitly assumed in any case to be the domain of definition and of values.

1.3. The Concept of Function

Youschkevitch's voluminous contribution (1976) is valued as the classical account of the history of the concept of function. Dhombres published some supplements to it, in particular about Euler's concept of function (Dhombres 1986). A summary has been given by Medvedev (1991).

Functional relations were used not only in Greek mathematics—Ptolemy's *Almagest* being a well-known example—but by the Babylonians as well. The concept of "function," however, evolved from the general concept of quantity only in modern times, becoming an independent mathematical object, a concept proper. Issues taken up in physics, and above all the progress in kinematics, proved decisive for this process of ongoing differentiation. This context of application for a long time shaped what the concept of function contained, and how it developed. Another essential factor for this autonomization, however, was the progress in algebraic symbolism, which permitted the representation of even the most intricate equations and formulae by means of a limited number of signs. Because of this context, functions were understood to be *equations* between two variables. The attachment of the concept of function to the formula, i.e., to the "calculation's expression," was to continue determining this concept's form and content for a long time to come.

In Descartes's *Géométrie* (1637), we find an elaborate form of the concept of function in the shape of his formulation of a reciprocal dependency between two quantities given by an equation, both of which can assume an arbitrary number of values. According to this tradition, functions were at first restricted to algebraic functions. Newton and Leibniz, however, extended their investigations to transcendental functions. The problems of how to develop transcendental functions into series and the latter's impact on the meaning of the foundational concepts were to become the principal focus for research into analysis.

In his *Method of fluxions*, Newton introduced the distinction between independent and dependent (variable) quantities, as *quantitas correlata* and *quantitas relata*. In Leibniz's manuscripts, the term function is found for the first time in 1673, for the relation between ordinates and abscissae of a curve given by an equation. In his publications of 1684 and of 1686, he already divided functions into two classes: algebraic and transcendental. The first general definition of the new mathematical object of function was published by Johann Bernoulli in 1718:

> One calls function of a variable magnitude a quantity composed in a certain manner by that variable magnitude and by constants (Joh. Bernoulli *Opera Omnia*, tom. 2, 1968, 241).[6]

This definition gave no attention yet to the distinction between single-valued and multivalued functions.

Essential contributions to further elaborating the concept of function were made by Euler. His definition remained tied to the formula, to the calculation's expression, while he generalized it for analytic functions:

> *A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities* (Euler 1988, 4).

Euler understood as functions analytically expressible those that can be developed into infinite powers series. He admitted for this not only positive integer exponents, but also arbitrary ones. "Should anyone doubt," Euler argued, "his doubt will be eliminated by the very development of one or another function" into a power series (Youschkevitch 1976, 62). Euler's definition of function as an analytic expression whose most general form is a power series was to remain the predominantly recognized determination during the entire eighteenth century .

Euler developed the concept of function further in his own work mainly in two respects: firstly, in discussing the meaning of the continuity and the discontinuity of functions. For Euler, a function was continuous ("continua") if

6 "On appelle fonction d'une grandeur variable une quantité composée de quelque manière que ce soit de cette grandeur variable et de constantes."

its equation, its formula remained unchanged—hence, if the function was describable by just one calculational expression. Thus, the hyperbola with its two branches belonged to the continuous functions.

Conversely, he regarded a function as “discontinua” or “mixta” when it was composed of continuous parts, but subject to several equations—piecewise continuous functions, whose pieces could well be connected and whose graphs could be traced by a free stroke of the hand.[7] This distinction was later supplemented by Arbogast’s *fonctions discontigues*, whose various parts were conceived of as unconnected (see below, Section 1.5.). The second further development originated from discussion of the physical problem of the vibrating string: It made clear for the first time that functions are also representable as a superposition of trigonometric functions. The consequences for the concept of function and for analysis in general, however, were drawn only gradually. In his textbook on differential calculus, Euler already made allowance for these extensions by defining the concept of function more generally:

> If some quantities so depend on other quantities that if the latter are changed the former undergo change, then the former are called functions of the latter. This denomination is of broadest nature and comprises every method by means of which one quantity could be determined by others (Euler/Michelsen, Vol. 1, 1790, XLIX; Translation quoted from Youschkevitch 1976, 70).

1.4. The Concept of Limit

In analysis, the limit concept functions in a way analogous to that of the concept of quantity within mathematics as a whole. It forms the essential basic concept, while several other foundational concepts emerged from continuous differentiation, for purposes of studying limit processes, among them that of continuity for functions and that of convergence for sequences. We shall begin here by summarizing the history of the concept of limit.

Among the numerous historical accounts of how this concept was developed, Hankel’s contribution of 1871 is notable in that it distinguishes itself by considerable conceptual precision. The so-called method of exhaustion of Greek mathematics again and again provided the key source for all the conceptual developments in early modern times. Hankel pointed out that two different versions, between Euclid’s and Archimedes’s method of exhaustion, must be distinguished. Proposition 1 of Euclid’s Book X forms the common basis for the two methods:

> Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than

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7 See Youschkevitch 1976, 64 ff. It seems that Euler understood continuity in the sense of Aristotle as connectivity of adjacent parts; thus his formulation in the treatise *De usu functionum discontinuorum in analysi* (1763), cf. ibid., 67. Cf. Section 1.5.

its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out (quoted from Edwards 1979, 16).

This proposition finds its typical application in the inscribing polygons with an ever increasing number of sides into a curvilinear figure, say within a circle: a square inscribed within a circle is larger than half this circle's area. If we continue inscribing polygons having a number of sides 2^n, while applying the above proposition, the polygon's area will approximate the circle's area up to any quantity, however small.

Hankel noted that a general principle for limits of sequences implicitly underlies this method of exhaustion, formulating it as follows:

If the terms of two series of indefinitely increasing quantities:

$$a_1, a_2, a_3, \ldots \text{ and } b_1, b_2, b_3, \ldots$$

are in the ratios:

$$a_1 : b_1 = a_2 : b_2 = a_3 : b_3 = \ldots = \alpha : \beta$$

and if the a_i indefinitely approach a quantity A, the b_i a quantity B then A und B are in the same ratio:

$$A{:}B = \alpha{:}\beta \text{ (Hankel 1871, 186).}$$

Such a general principle is not found, however, in the mathematics of the ancient Greeks, although the frequent application of related conclusions might have us expect them to have become aware of a proposition about limits of series. In every demonstration, all the individual steps of proof are reiterated. Without such a proposition, the method of exhaustion, Hankel says, could not be characterized as scientific (ibid., 187).

The method of exhaustion used by Archimedes proceeds somewhat differently. To prove that a curvilinear figure C has the same area as a rectilinear figure D, he encloses C by inscribed and circumscribed polygon trains that ever more closely approach the value sought. One of the typical proof procedures is the following:

If one has two infinite series of quantities

$$E_1, E_2, E_3, \ldots, \quad U_1, U_2, U_3, \ldots,$$

and two fixed quantities C, D, and if one can prove that the former quantities E_1, E_2, $<D$ and $<C$ and that approximate C arbitrarily, and that moreover the quantities of the second series are $> D$ and $> C$ and that these also arbitrarily approximate C; then there must be $C = D$ (ibid., 188).

Archimedes, too, conducted proofs separately for every case while refraining from reducing them to a general theorem.

As Hankel stressed, these rigorous methods to determine limits were used by Euclid, Archimedes, and other Greek mathematicians, but these authors generally avoided infinite processes, never imagining the transition to the limit as something actually accomplished. Their avoidance of using and accepting the infinite thus already tended toward developing crucial elements for an algebra of inequalities.

In the Christian Middle Ages, however, scholastic philosophy, which dominated scientific debate, saw no epistemological obstacles in the transition to the infinite. In the mathematics of the early modern period in Western Europe, this philosophical acceptance led to establishing the method of the *indivisibles*: the method of exhausting curvilinear figures by an infinitely large number of rectilinear figures. The method's first known eminent representative was Kepler, who, starting from computing the volumes of wine casks, developed stereometrical principles according to which any continuously curved solid can be treated as a polyhedron having an infinite number of infinitely small sides. A typical example worth noting is computing the volume of the sphere. The sphere was understood as composed of infinitely many pyramids, their vertices lying in the sphere's center and their bases touching its surface from within. For F as base area and h as height, the pyramid's volume is

$$\frac{1}{3}Fh.$$

For arbitrarily small areas of the base, h can be regarded as identical with r (radius of the sphere). Hence, thanks to the already known area of the sphere $4\pi r^2$, the volume of the sphere results as (cf. Kepler 1615 and C.H. Edwards 1979, 102)

$$V = \frac{4\pi r^3}{3}.$$

It is typical for Kepler's method that:

- The indivisibles have the same dimension as the figure which has to be determined.
- The computation is done in each case for a particular geometrical figure, due to an ad hoc subdivision into indivisibles.

In his two volumes *Geometria indivisibilibus* (1635) and *Exercitationes geometricae sex* (1647), Cavalieri developed more general methods for determining volumes; he succeeded in making the indivisibles a widely accepted mathematical concept. In contrast to Kepler's ad hoc methods for the respective figure under scrutiny, Cavalieri established direct correspondences between the indivisible elements of *two* geometrical figures: the area or the volume of one of the two figures being known and the other figure being sought.

Moreover, Cavalieri assumed that the indivisibles of geometrical figures are quantities one dimension *smaller*. He understood, for instance, "all" indivisibles of an area to be the "aggregate" formed by an infinite number of parallel and equidistant segments of straight lines, interpreting the indivisibles of a solid accordingly as the aggregate of parallel and equidistant intersecting planes. On the basis of these two assumptions, Cavalieri's theorem can be understood:

> If two solids have equal altitudes, and if the sections made by planes, parallel to the bases and at equal distances from them are always in a given ratio, then the volumes of the solids are also in this ratio (Edwards 1979, 104).

An example of how to apply this theorem is to determine the volume of a circular cone of height h and base radius r. The cone is thus compared with a pyramid of identical height and with the unit square as base. The indivisibles as sections at the height x are to one another in the ratio πr^2. The result is

$$V(\mathrm{C}) = \pi r^2 \,.\, V(\mathrm{P}) = \pi r^2 \frac{h}{3}\,.$$

In Cavalieri's work, the method of the indivisibles reached its summit of explicit elaboration and application. Not only did it meet contemporary criticism because of the methodological and epistemological problems it raised, but at the same time it represented a developmental conclusion because of the emergence of a novel fundamental idea, which now began to prevail. Upon summing up the developmental stages of the limit concept hitherto accomplished, it is seen that all these attempts concerned "measuring" given geometrical solids, surfaces, or lines, by means of approximating curvilinear figures by rectilinear ones. The figures given were always fixed and invariable, the applied methods thus corresponded to static conceptions.

The emergence of the concept of function in the seventeenth century therefore marked a fundamental change for the meaning and scope of limit processes. Functions were at first understood primarily as *kinematic* objects—as quantities variable with time. They were functions of only *one* parameter, the name "fluents" being typical for this concept's reach (see Bourbaki 1974, 225 f.).

For Hankel, the "prehistory" of the limit concept ends in the seventeenth century. After summarily listing some mathematicians of the seventeenth century, he immediately switches to the modern concept of limit as established in the nineteenth century. He does not address Newton's essential suggestion toward developing the concept of limit further. This is why these achievements and the developments prompted by them in the eighteenth century will not be discussed in this overview, but instead in Chapter III.

1.5. Continuity

The received view about the concept of continuity in the historiography of mathematics is that Bolzano and Cauchy were the first to define the concept rigorously, and that while there had been some discussion about continuity during the eighteenth century, profound reflections on its meaning nevertheless did not begin before the close of the eighteenth century (cf. Grabiner 1981, 87 ff.). The memoir with which L.F.A. Arbogast (1791) won the prize offered by the St. Petersburg Academy in 1787 is considered to be the principal document of the emerging discussion about the concept. Edwards even considers this treatise to have been the first to clearly elaborate the intermediate value quality (Edwards 1979, 303).

Indubitably, the development of the concept of continuity is tied both to the formation of the concepts of variable and of function, and to advances in the study of sufficiently large classes of curves. Indeed, considerable progress was made only by the middle of the eighteenth century, due to the debates about the equation of the vibrating string, and in particular due to the Euler's contributions. On the other hand, the definitions of continuity, as given by Cauchy and Bolzano, were no sudden innovations without identifiable precursors. I will examine the actually provable contexts of origin in the Chapters III and V.2., in particular Cauchy's immediate context in the *École polytechnique*. The intention here is again to do no more than sketch the respective status of research in the historiography of mathematics.

As is seen from Euler's standard textbook *Introductio in Analysin infinitorum* of 1748, continuity was at first understood as a geometrical quality: as a quality of curves. Continuous curves were characterized by the fact that they could be expressed by an analytic expression. In contrast, discontinuous curves consisted of several segments that belonged to different functions and hence did not correspond to just one analytic expression, but to several. This explains why Euler called the non-continuous curves "discontinuous" or "mixed" curves:

> The idea of curved lines at once leads to their division into continuous and discontinuous or mixed ones. One calls a curved line continuous when its nature is determined by one specific function of *x*; however, it is called discontinuous or mixed and irregular when different parts of it, *BM*, *MD*, *DM* sc., are determined by different functions of *x* (Euler/Michelsen, vol. 2, 1788, 9).

The contemporary idea to call curves continuous when they were representable by a function, that is, by a single-valued analytic expression, is found in d'Alembert (cf. Bottazzini 1986, 23–24). Hence, in contrast to today's understanding, curves could be understood as discontinuous ones generated by an arbitrary movement of the hand, or representable by several analytic expressions (ibid., 25).

In his later treatise of 1763 *De usu functionum discontinuarum in analysi*, also on the problem of the integration of partial differential equations, Euler, specifying the concept of continuity, stressed that it is necessary for continuous curves to obey a single analytic law. A hyperbola's two branches thus form a continuous curve (Youschkevitch 1976, 7 f.).

The historical literature always refers to Arbogast's treatise of 1791 as to that which offered new conceptual proposals. This is said firstly because he explicitly formulated the intermediate value property for continuous functions (Edwards 1979, 303; Grabiner 1981, 92; Bottazzini 1986, 34) and secondly because he introduced a new term: "discontigue." While curves, according to Euler's specification, had been considered to be discontinuous as well if their various parts were attached to one another, provided that these were defined by different "laws," Arbogast now called curves *discontigue* if their various parts were unconnected (ibid.). In all these works, this continued conceptual differentiation is emphasized as an important achievement, because with it, and with the novel term, the discipline had come closer to the meaning of discontinuity as it is understood today.

It must be pointed out, however, that Arbogast's reflections on the meaning of *continue*, *discontinue* and *discontigue* still refer to curves, and that functions, for him, were only of secondary importance for representing particular parts of a curve. Arbogast assumed functions as basic concepts only when reflecting on intermediate values. It must be noted that with Arbogast, just as with many contemporary mathematicians, the concept "loi de continuité" occurs in a twofold meaning: both as the analytic (formulaic) expression of a curve or a part of it, and as the conceptual content of the function's property of continuity. Steps toward conceptually defining the continuity of functions could not be taken before the relationship between these two meanings had been clarified. One of the conditions for this was to abandon the hitherto prevailing epistemology of mathematics by adopting an algebraic–analytic view of mathematical objects, a move that conferred a more fundamental status on functions than on the curves they represent.

This kind of change took place in France after the Revolution of 1789, when the analytic method began to prevail (cf. Chapter IV.2). While the debate on the continuity of functions, which will be reconstructed in Chapter III.9. in more detail, now became more heated, it is not appropriate to consider Cauchy's works as the conclusion of clarifying the concept of continuity. Where not only a concept, but also its negation, has an independent meaning, this negation can be profited from to ascertain how far the original concept reached, and what it meant over a determinate period of time. In our case "discontinuity" may serve to determine the then intended reach of "continuity" more precisely. Cauchy's textbooks contain but few explicit reflections on discontinuity, while Ampère, who held lectures on analysis alternating with those of Cauchy at the *École polytechnique*, explicitly discussed "discontinuity" in these both before and after Cauchy's *Cours d'Analyse Algébrique* was published. Ampère explained discontinuity as "rupture de la continuité." Discontinuity, besides, was only a special form of continuity here: a piecewise continuity.[8] One can thus conclude that continuity, in the French context at the beginning of the nineteenth century, signified continuity in an interval, and not point-wise continuity—in contrast to Bolzano's view. This also implies that the further differentiation of the concept of continuity had not yet progressed far enough to grasp and examine the local behavior of functions in the full scale of its differentiatedness.

1.6. Convergence

How the concept of convergence developed has been studied in detail by Grabiner (1981), in particular how novel Cauchy's works were compared to the mathematics of the eighteenth century. The study of infinite series, above all of

8 In his later work about "fonctions discontinues" (1849) as well, discontinuous functions are for Cauchy functions that experience an interruption of their continuity at isolated places. This is obviously connected with Cauchy's understanding functions also as equations, as given by terms, and not as a correspondence (Hawkins 1970, 11–12). On Ampère see below, Chapter VI.6.5.

power series, had become a major field of research in mathematics since Newton and Leibniz. On the other hand, the historiography of mathematics has kept wondering at the carefree manner or the lack of rigor in the summation of such series. An example of this phenomenon frequently quoted is the series $1-1+1-1+\ldots$. By 1700, there had already been debates whether the sum $\frac{1}{2}$ obtained from the formal development of $\frac{1}{1+x}$ for $x = 1$ is correct. Leibniz expressly confirmed this value in a letter of 1713 to Christian Wolff. His justification was to consider the two different series $1-1+1-1$ etc. and $1-1+1-1+1$ etc. and to halve the respective values 0 and 1. Although this argument might appear to be rather more metaphysical than mathematical, it was nevertheless justified, Leibniz said (Leibniz 1858, 382 ff.). Euler, too, expressly confirmed the value $\frac{1}{2}$ of the sum by considering the series formally as a development of $f(x) = \frac{1}{1+x}$, obtaining the sum $\frac{1}{2}$ as $f(1)$ (cf. Dieudonné 1985, 23 f.).

The understanding of convergence of series prevailing in the eighteenth century was that a series converges if its terms (seen absolutely) become ever smaller, approximating the value *0*. D'Alembert's convergence criterion must be understood in this sense, a criterion that examines the extent to which the respective ratios of successive terms generally become smaller (Grabiner 1981, 60 ff.). At the same time, there was no strict distinction yet between formal and numerical series.[9] Instead, there was the belief that summation was possible in the neighborhood of the point in question.

What eighteenth-century mathematicians understood the meaning of convergence to be was formulated in an exemplary fashion by Klügel in his mathematical dictionary of 1803: "A series is convergent if its terms become successively ever smaller." For Klügel, this explanation was already sufficient for convergence: "The sum of the terms then ever more approaches the value of the quantity which is the sum of the entire series when continued to infinity" (Klügel 1803, 555).

It is one of the standard propositions in the literature on the history of mathematics that clear distinctions between convergence and divergence, and rigorous research into convergence, were not published until Gauß's work of 1812/13 on the hypergeometric series, Bolzano's works of 1816/17 on the binomial theorem and on the intermediate value property, and Cauchy's 1821 *Cours d'Analyse Algébrique*. Grabiner has pointed out, however, that essential elements of Cauchy's innovations are already found in Lacroix, who treated both the definition of convergence and the exclusion of divergent series. The merit of Cauchy was the systematic development of the concept of convergence in analysis (Grabiner 1981, 101 ff.). Given the habit of mathematical

9 In Germany, a clear distinction had been established by the philosopher Fries, in particular within his philosophy of mathematics of 1822 (cf. Schubring 1990a, 154).

historiography of conferring priority of discovery to a concept's mere mention in passing, or to its rough definition, Grabiner's evaluation clearly intends to relativize Cauchy's pioneer achievements (and hence also those of Gauß). Since Lacroix's *Traité* presents basically a systematization of already available knowledge, it follows, therefore, that essential foundations for the concept of convergence have been elaborated as early as the eighteenth century.

As the first of these elaborations, the discussions about convergence and divergence in Euler's work of 1754/55 about divergent series need to be mentioned. Euler defined convergence just as we have already quoted from Klügel's dictionary, that is, by referring to ever diminishing and eventually disappearing terms. Divergence is determined not by the fact that the terms either do not decrease below a given limit or grow arbitrarily (cf. Bottazzini 1992, XLIX). This is also the same work in which Euler ascribed the value $\frac{1}{2}$ to Grandi's series $1-1+1-1+\ldots$.

By contrast, the literature tends to neglect Louis Antoine de Bougainville (le jeune, 1729–1811). In the latter's two volumes on integral calculus, which were intended to continue L'Hospital's textbook, he not only claimed convergent series to be the only summable ones, but had already developed a convergence criterion as well.

De Bougainville's first volume contains a Chapter "Théorie des Suites." While defining convergence by indefinitely diminishing terms just as his contemporaries did, he added the important remark that convergent series were the only true ones: "We will prove later on that these series are the only true ones" (Bougainville 1754, 302).

The fascinating thing about this chapter is that it contains a detailed discussion of the summability of series—indeed as basis for integrating developments of series—and this in combination with elaborating a criterion of convergence. Actually, Bougainville called a series "true" ("vraie") if its sum was identical with the expression from whose development it had emerged. And he indicated a ratio test as convergence criterion (ibid., 304). Applying his notions, Bougainville showed that Grandi's series was "faux," i.e., did not have the value $\frac{1}{1+1}=\frac{1}{2}$ (ibid., 311).

In their report of 1754 on Bougainville's book for the *Académie des Sciences*, Nicole and d'Alembert placed particular emphasis on his chapter about series and the convergence criterion: "He teaches the manner to form these series, the means to recognize their convergence or their divergence" (ibid., xxij[10]).

José Anastácio da Cunha (1744–1787) made another noteworthy contribution to the concept of convergence. It has repeatedly been discussed during recent years by historians of mathematics, in particular on the occasion of the

10 If the last digit in the roman pagination number of prefaces of French books was an "i" it was printed as a "j"; Thus an 8 in roman numerals was given as viij.

bicentenary of da Cunha's death. In his textbook *Principios Mathematicos*, published integrally in 1790, three years after his death, he gave a definition of convergence that basically agrees with the later so-called Cauchy criterion: for a series to be convergent, it is necessary and sufficient that the partial sums become arbitrarily small from a sufficiently high index on. Since the French translation of 1811 from the Portuguese errs in precisely this definition, I prefer quoting a modern translation into English:

> Mathematicians call convergent a series whose terms are similarly determined, each one by the number of preceding terms, so that the series can always be continued, and eventually it is indifferent to continue it or not, because one may disregard without notable error the sum of how many terms one would wish to add to those already written or indicated; and the latter are indicated by writing &c. after the first two, or three, or how many one wishes; it is however necessary that the written terms show how the series might be continued, or that this be known through some other way (Queiró 1988, 40).

As Giusti has criticized, da Cunha omitted to reflect on what the sum of an infinite series actually is (Giusti 1990, 105). Not only did da Cunha's concepts suffer from not extending them far enough, but the impact of his work was obviously much dampened as well, from the fact that it originated from Portugal, a nation at the periphery of the contemporary mathematical world. Although it saw a French translation in 1811 (in Bordeaux), no influence on mathematics in other countries has been noted as yet.

Nevertheless, this example of important innovations even at the periphery shows that a broader context of discussion existed for the concept of convergence as well.

The claim that there was no crucial difference between the positive knowledge about convergence in Lacroix's and in Cauchy's textbooks, as stressed by Grabiner, is not confirmed by Lacroix's voluminous *Traité*, to which Grabiner refers. The difference in the concept's status is much more evident upon comparing Lacroix's own two textbooks. While his voluminous edition, aiming at a learned audience, indeed discusses the concept of convergence, this discussion is absent from the concise version intended for a student audience, mainly at the *École polytechnique*. By contrast, Cauchy's later textbook of 1821, directed to the same student body, assigns a central function to the concept of convergence. Only at that time did the concept of convergence attain the status of a fundamental idea in analysis.

1.7. The Integral

The concept of integral differs from the foundational concepts discussed above in that it is not one of the "primary" fundamental ideas. It will receive mention here, however, in contrast to the analogous concept of derivative, because it later had an important role in effecting changes in the system of foundational concepts that will be studied in the chapters to come.

The concept of integral has seen a peculiar change as to its importance for analysis. While determining areas and volumes constituted the principal object for applying infinite processes since their very beginning, problems of this kind were transmuted into simple inversions of differentiation after the differential calculus had become established. The (indefinite) integral as an inverse of the differential was only of derived significance and had no role of its own in foundational studies throughout the eighteenth century. The concept of integral became independent again only subsequent to Cauchy's studies on the definite integral of 1814. Remarkably, this change was linked to a "resurrection" of the infinitely small.

The methods of exhaustion practiced by the Greeks had been taken up again by the methods of the indivisibles in modern times. Developed first mainly by Kepler for determining the volume of barrels, these methods climaxed with Cavalieri's indivisibles. Skillfully used, they permitted calculating volumes of solids, by comparing these to solids already known, and accordingly for areas. The conceptual basis of the methods was the atomistic assumption that any geometrical figure can be understood as composed of "indivisible," arbitrarily small quantities, thus forming a sum of elements (of "slices" having the lower dimension n–1).

Already in Newton's and Leibniz's first works on the new differential calculus, the integral calculus (then called the "inverse tangent problem") was conceived of as the former's inversion (cf. M. Cantor, vol. 3, 1901, 171). Medvedev has shown, however, that integral calculus, in Newton's early works, was not yet based on differential calculus. Rather, it was derived from the method of calculating areas by means of developing functions into infinite series (Medvedev 1974, 100 ff.). Medvedev also criticized the widespread view that credits Newton with the idea of the indefinite integral as a primitive function, and Leibniz with the idea of the definite integral as a limit of approximating sums (ibid., 117). Moreover, he showed that Newton had already introduced the concept of the definite integral as a limit of sums in 1686 (ibid., 120), the integration constant having first been used to solve a concrete problem in one of Leibniz's papers of 1694 (ibid., 121).

Wherever they treated integral calculus, the textbooks of the eighteenth century formally presented the integral as the inverse of differentiation with the task of determining the primitive function. Without discussing questions of existence, rules for determining (indefinite) integrals were examined. The most comprehensive study of how the concept of integral developed is given by Medvedev (1974). Characteristic for the eighteenth century's state of the concept is Euler's textbook on integral calculus (1768–1770). In its first volume, the extensive three-volume work contains a short general section giving definitions and explanations. The very first explanation introduces the integral as a problem of inversion: "The integral calculus is the method to find, from a given relation between differentials, the relation between the quantities themselves" (Euler 1828, 1).

And Euler adds a characterization of differential calculus and the integral calculus as analogously opposite operations, comparing them to the basic operations of arithmetic: “just as in analysis where always two operations are opposed to one another”—subtraction and addition, division and multiplication, extracting roots and exponentiation (ibid).

Medvedev also examined the developments that led to the rise of the definite integral in the second half of the eighteenth century. A major factor in favor of its increasing importance was the investigation of oscillations and their representation by trigonometric series. The research to determine the coefficients of these series showed that these could be most suitably determined by using definite integrals. A further impulse was provided by the problems raised by multidimensional integration; this is where presenting the integral as a sum becomes necessary. Potential theory necessitated the calculation of definite integrals. Lagrange used the concept of the definite integral throughout his *Méchanique analitique* as an important fundamental notion (Medvedev 1974, 154 and 159 ff.).[11]

Cauchy, however, was the first to raise the concept of the definite integral to the rank of a privileged fundamental notion, and the first to comprehensively make the concept and the existence of the integral a subject proper of mathematical research in his textbook of 1823. This is where the definite integral was introduced as a sum of infinitely small quantities.

2. The Development of Negative Numbers

2.1. Introduction

The history of the concept of the negative numbers has not been examined systematically. In M. Cantor’s seminal work one finds, in its second volume, for the period 1500 to 1668, several rather dispersed indications (M. Cantor 1900). The fourth volume of 1903 contains a topical account, by Cajori, on the development of algebra between 1750 and 1800; ten pages give an informed account on negative numbers for this period, which otherwise appears in usual historical reports as not having offered conceptual problems. Tropfke’s study, in its new version edited by K. Vogel, contains in its part on arithmetic and algebra a brief section describing the development from the Babylonians up to Peacock

11 Cf. also on the generalization of the concept of function as a premise for the development of the concept of integral: Hawkins 1970, Sections 1.1 and 1.2.

and Hankel (Tropfke 1980, 144–151). In accounts of the history of the concept of number (e.g., Gericke 1970, 51–57) what is held worthy of being reported on negative numbers in general ends with early modern times. The basic assumption implied in these traditional accounts is that the concept of negative numbers had been in essence clarified by the time of Stevin and Viète and no longer offered conceptual problems.[12]

A first approach to studying the gap of centuries—from the seventeenth to the nineteenth—came from an unexpected quarter: from French *didactique*, based on epistemological categories. It was a paper by Georges Glaeser (1981) on the history of negative numbers from the seventeenth to the nineteenth century, but mainly restricted to the rule of signs. The concept of his investigation turned out to be quite unsatisfactory. Glaeser constructed a frame of reference from an epistemological theory, that of "epistemological obstacles" developed by the French philosopher Gaston Bachelard (cf. Bachelard 1975). Bachelard's theory that the mental development of mankind occurs in three stages reveals decisive weaknesses in Glaeser's adaptation to the historical concept field of negative numbers. Glaeser registered six "obstacles" in the course of its evolution that allegedly prevented a full understanding of the concept of negative numbers; he marked with + or – leading mathematicians from the seventeenth to the nineteenth centuries according to whether they had overcome some of these "obstacles" (Glaeser 1981, 309). Earlier views judged by Glaeser as especially "backward" he commented with three exclamation marks (e.g., ibid., 320). Stagnating at the stage of concrete operations rather than progressing to the stage of formal operations is presented as a main obstacle. Even Euler is grouped there (ibid., 308 f.).

By such categorizations, Bachelard's teleological concept of scientific development is saliently revealed: as a progress necessary in the theoretical character of knowledge, as an eventual triumph of reason. In Bachelard, one recognizes the rationalistic vision of increasing mathematization as the necessary core within the process of developing the sciences. Bachelard distinguished three stages in this process: a concrete, prescientific stage in which the phenomena rule; a concrete–abstract stage in which the physical experience is complemented by some abstractions; and eventually the (present) abstract stage, which is determined by theoretical reasoning (cf. Bachelard 1975, 8). This gradation corresponds quite exactly to the three stages of mental development that Piaget ascribed to the individual: the succession of empirical, then of concrete, and eventually of formal operations.

Stagnation at the stage of concrete operations has appeared, since Glaeser's paper, in numerous mainly didactic publications as the "explanation" for a lack of understanding negative numbers in a modern sense. Historical development is again being understood to be linear, as converging to the modern status; moreover, differences—say between different cultures—are just as little

[12] An exception is Vredenduin's study of 1991 (in Dutch), which also considers the eighteenth and the nineteenth centuries.

considered as are embeddings of conceptual views into differing epistemological frames.

If one takes the historical texts seriously, however, and tries to understand their intentions, one will observe that presuming an *obstacle* where an earlier author did not "unify the number ray" (Glaeser 1981, 308) is rather unhistorical. Looking at concept development not under the limited aspect of the rule of signs, but under the more general one of the *existence* of negative numbers, one will observe that a foundational dimension underlies many of the controversies about their existence, which is not decidable by "true" or "false": this is the relation between *quantities* and *numbers*. The rather unspecific concept of quantity does not lend itself to conceptualizing the notion of "negative quantities," whereas the more specific concept of number proves to be more broadly applicable, and better adapted to developing the notion of "negative numbers"— that is, as more general. The decision for "quantity" as the fundamental idea, and for "number" as the derived concept, or for "number" as the fundamental idea, and for "quantities" as derived concept, depends on which architecture of mathematics is chosen or favored, and on the underlying epistemological concepts, and cannot be decided by truth values. The decision for one or the other side can just as little be qualified as a result of mental obstacles.

Moreover, such studies often assume anachronistic views of concept developments. For example, the alleged mental obstacle to unifying the number ray is induced by presenting this idea as having always been self-evident. As Bos has shown, however, geometrical quantities, like lengths or areas, were *not* scaled quantities as long as the requirement of dimensional homogeneity for these quantities was generally shared, so that the introduction of a unit length was unnecessary; hence, these quantities did not represent real numbers, and lines in algebraic geometry did not mean number rays (Bos 1974, 7 f.).

Another methodological problem is exemplified by the historian of mathematics Helena Pycior's new volume (Pycior 1997). In her Ph.D. thesis of 1976, she had examined the development of algebra in Great Britain from 1750 to 1850. In her new book, she studies—just as knowledgeably and carefully—the preceding period from about 1600. The development of the negative and the imaginary numbers constitutes her main focus. For evaluating this development, Pycior chooses a fixed *étalon*, namely "the expanding universe of algebra" (Pycior 1997, 27 and passim). Such a framework for comparison is not only too abstract and makes no allowance for cultural differences, it also assumes at the same time an inevitability in concept development that actually recognizes only cumulative advance. Her analysis therefore shows problems in conceptually grasping the ruptures present in her period of investigation. While she assures us that "British algebra did not develop in a fundamentally linear fashion," she flinches shortly thereafter from the consequences of that assertion, in line with her own model of universal progress, hastening to affirm, "This is not to say that there was no linear development" (ibid., 3). Undoubtedly, the explanatory pattern she applies to the British controversies about the status of negative and imaginary numbers is of particular relevance for this conceptual development. It is the contrast between propagators of the analytic method who are committed to algebraic procedures by means of "symbolical reasoning" and propagators of the synthetic method who accept merely geometrical foundational concepts.

Moreover, Pycior's book provides evidence of the difficulty in analyzing conceptual developments in their contemporary context and understanding. Her analysis refers to terms like *negative* ***numbers*** or *imaginary* ***numbers***. However, the majority of the authors examined by her use terms like *negative* or *imaginary* ***quantities*** or ***magnitudes***. Where she mentions the use of "quantity" or of "magnitude" in a quote from historical authors, she does not, however, systematically inquire into the intended meaning, but subsumes the respective position into her account as a contribution about "numbers" and their development. Not to include the difference between *quantity* (or *magnitude*) and *number* into the basic dimensions of the research design means to exclude from the analysis a majority of the most important contemporary problems in the development of the algebraic concept field, and thus to miss key historical insights, in particular concerning the extension of the multiplication operation.

Eventually, one notes a generally entirely complete account of concept development in the literature. Most authors report only the formulations of the definitions of negative numbers and justifications for the rule of signs. They omit, however, a discussion of how the respective conceptual view was applied in other mathematical concept fields—say, in analytic geometry. An exception is Tropfke's investigation of the history of quadratic equations, which he relates to the context of the history of negative numbers. He says, however, that there was only *one* normal form of quadratic equation since 1659, due to Hudde (Tropfke 1934, 102). As we will see, the question of the normal form was still controversial during the nineteenth century.

2.2 An Overview of the Early History of Negative Numbers

From Antiquity to the Middle Ages

A careful analysis as to when negative numbers were believed to exist was undertaken by Sesiano (1985). He gives a survey from antiquity to about 1500, intensely discussing what Italian mathematicians contributed during the Middle Ages. He bases his analysis on how the relation between quantities and numbers was viewed. For the overview intended here, I will thus follow his contribution for the era before modern times.

Generally, it can be stated—provided one deals exclusively with *quantities*—that controversies were about whether to admit negative solutions. A lesser issue concerned the problem whether negative quantities were admissible as intermediate quantities during the process of calculation. Where the problems treated became more abstract, intentions can be noted at "reinterpreting" negative solutions so as to transform them into positive ones, thus removing them from their doubtful status.

In *Old Babylonian* times (beginning of the second millennium BCE), a sign from cuneiform texts, translated by "lal," has been interpreted to indicate "being less"—but only in texts about economics, not in mathematical ones. It also occurs in subtractive writing of numbers, e.g. $20 - 1 = 19$ (Neugebauer 1934, 17). Quoting Neugebauer, Tropfke commented, "In serial texts, changing data carried out according to a certain pattern will occasionally lead to negative numbers, but we are ignorant of how that was understood and whether people calculated with such quantities" (Tropfke 1980, 144). Høyrup has taken up this aspect in his systematic re-analysis of the Babylonian texts to say, "The widespread legend that the Babylonians made use of negative numbers comes from misreading Neugebauer's treatment of the topic" (Høyrup 2002, 21). The only thing observable is the use of subtractive quantities/numbers, without, however, performing operations on them (ibid., 296).

It is well known that red rods were used for calculating with positive quantities and black rods for negative quantities in *Chinese mathematics*. Negative values were permissible as intermediate values during calculation, but not as solutions of systems of equations. The literature refers to a typical example, one of the tasks contained in the classical mathematics textbook *Chiu-chang suan shu* (mathematics in nine chapters, approx. 250 BCE). In modern terms, it can be written as the system of equations

$$2x + 5y - 13z = 1000$$

$$3x - 9y + 3z = 0$$

$$-5x + 6y + 8z = -600$$

having the solutions $x = 1200$, $y = 500$, $z = 300$ (Tropfke 1980, 145; Sesiano 1985, 107 f.).[13]

Diophantus's works (approx. 250 CE) form the apogee of algebra in *Greek mathematics*. Since compositions of quantities with additive and subtractive quantities in algebra were subjected to operations of multiplication and of division, Diophantus, in the introductory part of his *Arithmetica*, introduced the *rule of signs* (Diophantus/Czwalina 1952, 6). The rule of signs' function, however, was only to permit operating with so-called "complex" expressions like $(a+b)$ $(c-d)$, no reflection about or acknowledgment of the existence of

[13] Lay-Yong and Tian-Se strove to infer from the practice of such examples that the Chinese were the first to have had "the earliest negative numbers" and "the concept of negative numbers" (Lay-Yong, Tian-Se 1987, p. 222 and 236). They did not see that operating with subtractive quantities as intermediate entries means just one step in a long process of conceptualization.

negative quantities being implied. Rather, Diophantus rejected negative solutions, just as he did irrational or complex ones, avoiding roots of such kind by restricting the equations further, when necessary.

There is only one problem in which Diophantus obtained an equation with a negative solution. After formally operating with a system of nonlinear equations, he ended up with the linear equation

$$4 = 4x + 20,$$

commenting, however, that this this was "absurd" (ἄτοπον)), since "4 units[14] could not be smaller than 20 units." He therefore felt compelled to change his original hypotheses (Sesiano 1985, 106).

Hitherto, just a small part of the surviving texts of *Arabic mathematicians* have been evaluated. Nevertheless, the texts already studied give no indication that mathematicians within this cultural context considered negative solutions acceptable. The six classical types of algebraic equations of first and second degree were always conducive to positive solutions. Mathematicians who treated problems of indeterminate algebra adhered to Diophantus' model, selecting coefficients so as to obtain positive solutions (Sesiano 1985, 108).

In his careful analysis of Ibn al-Ha'im's (1352–1412 CE) commentary on algebra and related Arabic texts, Abdeljaouad has recently shown how their authors explicitly thought about repeated "negations" of quantities, and about operating with "subtracted" (*manfi*) and "confirmed" (*muthabbat*) numbers as intermediate entries (Abdeljaouad 2002).

Indian mathematicians developed mathematics predominantly in connection with astronomy. The textbooks known were mostly introductory parts to works on astronomy.

The view on negative numbers in *Indian mathematics* was not uniform. Brahmagupta (599–approx. 665) treated in an early textbook on astronomy how to calculate with negative quantities in all the elementary arithmetical operations, and in squaring and in extracting square roots, demonstrating the rule of signs.[15] He called positive quantities "property" or "fortune," and a negative quantity "debt" or "loss." Negative quantities were distinguished from positives by a superposed point. Thereby, he already established a general expression for equations of second degree.

Mahavira (around 850) used negative numbers and even discussed the taking of square roots for negative numbers, but declaring them to be impossible because negative numbers could not be squares (*Lexikon* 1990, 304).

Bhaskara II (1114–ca.1191) had a different view. His chapters on arithmetic and on algebra form the introductory part of a textbook on astronomy. In quadratic equations, he sometimes admitted only one solution, even if both

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14 I.e., the $4x$ in the above equation.

15 Cf. Colebrooke 1817, 339–343. Algebra was presented in the eighteenth chapter of Brahmagupta's textbook.

solutions would have been positive. For problems with concrete quantities, he rejected negative solutions, in the case of more abstract problems, however, he reinterpreted negative solutions so as to be able to admit them as positive ones.

An example of his former approach is a riddle about monkeys. The fifth part of a troop of monkeys less three, squared, had gone into a cave; only one monkey was still to be seen. What was the monkeys' number? Translated into modern style, the following equation arises:

$$\left(\frac{x}{5}-3\right)^2+1=x,$$

with the two solutions $x_1 = 50$ and $x_2 = 5$. Since one obtains for x_2 the value $\frac{x_2}{5} < 3$, Bhaskara stated that a double value arises here, but the second is not to be taken, because it is incongruous: "People do not approve a negative absolute number" (quoted from Sesiano 1985, 106).

While negative quantities were rejected here even as intermediate values in calculations, Bhaskara proceeded differently in the case of geometrical problems. While a negative number of monkeys did not appear to be reinterpretable in positive terms, he reinterpreted negative geometrical line segments as having the opposite direction. In a problem about determining the lengths of line segments on sides of triangles he obtained a negative result, to which instead of excluding it, he gave the following interpretation:

> This [i.e., 21] cannot be subtracted from the base [c = 9]. Wherefore the base is subtracted from it. Half the remainder is the segment, 6; and is negative: that is to say, is in the contrary direction.(quoted after ibid., 107).

Bhaskara's later works are thus considerably more reluctant to admit negative quantities than the earlier works by Brahmagupta. This would seem to make evident once again that there need not be continuity in scientific progress even within the very same cultural context.

Moreover, another interpretation seems possible to me. Bhaskara II's chapters on algebra contain a separate section on addition, subtraction, multiplication, and division of positive and negative quantities: "Logistics of Negative and Affirmative Quantities." Not only do the rules they present agree with those established by Brahmagupta, but they are formulated even more extensively and explicitly, and illustrated by examples (Colebrooke 1817, 131–135). The sum of –3 and –4, for instance, is given as –7 (more exactly: as the sum of $\dot{3}$ and $\dot{4}$ with the result $\dot{7}$ (ibid., 131)). Because of Bhaskara's rejection of isolated negative solutions, these rules were apparently intended for intermediate calculations, and hence did not basically differ from Diophantus's approach. Since Bhaskara's rules for calculations are even more explicit than those of Brahmagupta, it may be assumed that Brahmagupta wished only to establish similar rules for operating on subtractive quantities, and therefore also rejected isolated negative solutions. Brahmagupta indeed presents only one root in his own general solution of quadratic equations, giving only one solution, a positive one, for all the problems presented later as examples (ibid., 346 ff.).

While the Italian mathematicians began by adopting the traditions of Arabic mathematics, they gradually elaborated methods and approaches of their own. Possibilities of negative solutions arose in particular from the frequent problem type of having to calculate how to distribute goods among *n* persons under varying conditions. Sesiano examined the systems of linear equations occurring and their solution sets in detail (Sesiano 1985, 108 ff.).

The works on the solution of systems of equations of this epoch culminated in Leonardo Pisano's (Fibonacci) works, ca. 1170–ca. 1250, in particular in his book *Liber abaci* (1202/1228). Negative values appear in many of his problems, partly as solutions and partly as intermediate values. In each case, Leonardo discussed in detail how to treat negative values. While he generally rejected negative solutions *and* intermediate values as *insolubilis* (cf. ibid., 118), he resorted to reformulating the problem to permit a solution whenever a reinterpretation of such a value was possible. This is true for all problems concerning invested capital or monies. He interpreted negative values as debts, as borrowed money, or as capital invested by one participant in addition to monies jointly invested by several persons *n*. Where it was impossible to reinterpret negative quantities, for instance negative prices (ibid., 131) as positive ones, he rejected the negative solution as *inconveniens*.

A provençal manuscript of about 1430 in Occitan from the *Bibliothèque nationale* in Paris marks a quite revealing rupture with all prior traditions of coping with negative numbers. That this manuscript, *Compendi del art del algorisme* exists had been known for some time already, but exclusively in the history of literature, as one of the documents of Occitan culture. Its mathematical content was not examined until some years ago by Sesiano (Sesiano 1984). The text is the first to contain a negative solution accepted without restriction or reinterpretation.

The problem concerned was about buying a piece of cloth. From a system of five linear equations in six unknowns, and after choosing one indeterminate, the value $-10\frac{3}{4}$ was obtained for the unknown x_1 ("restan 10 et $\frac{3}{4}$ mens de non res," Sesiano 1984, 52). This negative value was accepted without any interpretation whatsoever. The only hint at its particularity is its exceptional verification by inserting the values in all the equations (Sesiano 1985, 133 f.).

This novel way to treat negative solutions did not remain an isolated case, but was continued. This is evidenced by Frances Pellos's *Compendion del abaco* written about 1460, and printed in Nice in 1492. Pellos copied the entire group of problems containing the negative solution from the earlier Provençal manuscript. He also refrained from comment. Pellos's volume is the first printed document to contain a negative solution (ibid., 134).

A step beyond this first instance of acknowledging negative numbers is Nicolas Chuquet's (ca. 1445–ca. 1488) manuscript *Triparty en la science des* *nombres,* authored 1484 in Lyon. Chuquet solved systems of linear equations in which the unknowns were pure numbers, and no longer quantities. In such systems, negative solutions appear as well. Chuquet accepted such negative values without attempting to reinterpret them, and only under the condition that these values satisfied the equations.

In a problem with five unknowns, Chuquet obtained the solutions: 30, 20, 10, 0, –10. Chuquet continues after this first appearance of zero and of negative numbers by explaining how to add and subtract zero and negative numbers. In an abbreviated form, this expressed the rule of signs. He also made remarkably explicit that the operations of adding and subtracting acquire a novel meaning from the novel negative numbers. First, he declared that adding or subtracting zero ("0") does not change the result of an addition or a subtraction. Then, he observed that when adding a negative number to another number or subtracting it, the addition will result in a decrease and subtraction in an increase: "Et qui adiouste ung moins avec ung aultre nombre, ou qui d'icellui le soustrayt, l'addicion se diminue e la soustraction croist" (ibid., 136).

Chuquet gave examples for these novel meanings; subtracting minus 4 from 10 gives 14 as difference. And he interpreted negative solutions: "minus 4" corresponds to a person who owns nothing but still owes 4. Chuquet's work contains a solution identical to that "negative" problem in the Provençal manuscript and in Pellos's book (ibid., 137).

In an appendix, Chuquet solved application problems for quantities. Here again, Chuquet went considerably further than his precursors. He even accepted negative amounts of commodities and negative prices; the only condition being here as well that the given equations be satisfied (ibid., 140 ff.).

Luca Pacioli's (ca. 1445–1517) approach is less radical, adhering somewhat more to tradition. He wrote an arithmetic in 1470, and his *Summa de arithmetica, geometria, proportioni et proportionalita* in 1494. Pacioli's general stance was to reject negative solutions; he even showed reserve where negative intermediate values were obtained. Nevertheless, he tolerated negative values in special cases, thus for pure numbers, but in one case for a price as well. In an abstract problem, he goes as far as calling a negative solution "un bellissimo chaso" (Sesiano 1985, 142 ff.).

2.3. The Onset of Early Modern Times. The First "Ruptures" in Cardano's Works

While differences of approach can already be noted in Europe at the beginning of early modern times, it is not yet possible to attribute these to differences between established cultural contexts.

Historiography ascribes an important modernizing role to Michael Stifel's textbook on arithmetic and algebra *Arithmetica Integra* (1544) Stifel (1486 or

1487–1567) lived first as a monk, became a pastor after the Reformation, and has been called a peregrine clergyman because of his frequent changes of residence (Cantor, vol. 2, 1900, 430). In the last period of his life, after 1559, he taught mathematics at the University of Jena (Chemnitius 1992, 8 f.). Noteworthy in his treatment of negative numbers is that he clearly states that these are less than zero. While Stifel termed positive numbers *numeri veri* and negative ones *numeri absurdi*, thus using only positive roots of equations (Cantor, vol. 2, 1900, 442), he characterized positive numbers as *supra 0* and negative numbers as *infra 0, id est infra nihil* (Stifel 1544, fol. 249v.). This is why Cantor emphasized, as Stifel's pioneering achievement, "the explanation of the negative number as being smaller than zero which thus entered into mathematics" (Cantor, vol. 2, 1900, 442).

To corroborate his evaluation, Cantor also noted that Stifel had declared the zero to be the common limit between positive and negative numbers. In fact, Stifel speaks of the zero as: "0, i.[d est] nihil (quod mediat inter numeros veros et numeros absurdos)" (ibid.; Stifel 1544, fol. 249v.). In the first historical account of the development of negative numbers, Karsten ascribed to Stifel that he had understood operating with negative numbers as the reversal of the usual ways of calculating, making that subtraction, for instance, effects an increase. Thus, Stifel is said to have obtained as result of subtracting –5 from –2 the positive result +3, hence a "numerum *supra* nihil, seu numerum verum" (Karsten 1786, 233). Actually, however, Stifel had worked with subtractive numbers in this case. Subtracting 0–5 from 0–2, he obtained 0+3 (Stifel 1544, fol. 249 v.). Stifel always used negative numbers in compositions, in "binomials"; there is no case in which he ended up with a purely negative result. While he explained the rules for operating with such compound expressions in his book's part on algebra, in particular also quite explicitly for multiplication and division (ibid., fol. 248 v–249), this treatment of numbers was accompanied by explicit epistemological reservations. Stifel not only opposed *numeri ueri* to *numeri ficti* (ibid., fol. 48) or "*absurdi*" (ibid., fol. 249v), but beyond that declared positive numbers to be real (*quae sunt*), while ascribing only an imagined existence to negative numbers (*quae finguntur esse*; ibid.). Stifel did not yet acknowledge fundamentally equal status for the two kinds of numbers.

Petrus Ramus's (or: Pierre de la Ramée, 1515–1572) algebra textbook was often reprinted in France after 1560. The systematic importance of this author for the development of generalization will be discussed in Section 2.7. In the two chapters on arithmetic preceding his algebra, Ramus introduced the four basic operations for integer numbers, for fractions, and for proportions. His algebra consists of two parts; the first expands the object—instead of (absolute) numbers there are figurate numbers, up to biquadratic—and operations on such with not exclusively positive quantities. The second treats equations, i.e., solving of problems.[16] For plus and for minus, he used the signs + and –, and instead of

16 The first edition was a separate publication of his algebra not preceded by arithmetic. It had merely 34 pages; while L. Schoner's thoroughly revised edition of 1586

a sign of equality Ramus used the term “aequat.” Ramus did not explicitly introduce or use negative numbers, nor did he use negative quantities, restricting himself basically to subtractive quantities. He used the opposition of signs, however, and elaborated the meaning of addition and subtraction as reversed: in case of opposite signs, addition effects subtraction, and the remainder receives the sign of the larger quantity:[17] “In contrariis signis additio est subductio et reliquus habet signum majoris” (Ramus 1586, 325).

As an example, he calculated: $(6q+8l)-(4q-4l)$ obtaining $10q+4l$.[18] Analogously, he explained the effect of subtraction for opposite signs; the result receiving the sign of the larger term: “S*ubductio in signis contrarijs est additio, cujus totus habet signum superioris*” (ibid., 327).

As an example he showed $(19q-8)-(14q+14)$ with the result $5q-22$. Because *6*q, *8*l, etc. signify figurate numbers, this implies that Ramus admitted absolutely negative values for some dimensions. Nevertheless, Ramus did not operate with isolated terms, but at least with a binomial expression. For multiplication and division, Ramus, without hesitation and without justification, formulated the rule of signs: “*Multiplicatio et divisio in signis iisdem faciunt plus, in diversis minus*” (ibid., 328).

Multiplying $9q-4l$ by *9*l he thus obtained $81c-36q$, and for $8q-9$ multiplied by itself $64\mathrm{b}q-144q+81$. Here, Ramus gave also the rule, explained by various examples: $(a+b)(a-b)=a^2-b^2$. He called opposite quantities like $+b$ and $-b$ “heterogeneous.” The second part of his algebra, about equations, contained essentially nothing but positive solutions. Only once, as solution of a quadratic equation, he gave a positive and a negative value—revealing, however, an elementary mistake in calculation (ibid., 349).

Ramus’s algebra shows that the foundations of operating on negative quantities belonged to the generally shared knowledge of his time.

Algebra saw a culminating point of its early-modern development in Girolamo Cardano’s (1501–1576) work. Beyond the already well-established solving of quadratic equations, his book *Ars Magna* (1545) made the solving of equations of third and fourth degree accessible to a larger public. While Cardano systematically developed operating on negative quantities further, he simultaneously was the first to reflect on the foundations of this new field of numbers, reflections that led him to the first ruptures in the development of this concept.

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also comprises only 43 pages, it is preceded by the arithmetic first published in 1569 but later revised as well. While the basic versions of 1560 and 1586 formulated in the second part correspond, the texts deviate substantially in the first part of the algebra, the meaning of the statements on subtraction, on opposite signs, and on the rule of signs remained unchanged.

17 Schoner added a reference to the respective passages in Diophantus in 1586.

18 The q-units are quadratic figurate numbers, and the l-units are linear numbers.

Cardano used the rules hitherto developed for operating on negative quantities. He admitted negative numbers as solutions of equations, calling them *radices fictae*—in contrast to *radices verae*, his term for positive solutions—and interpreted them in his *Practica Arithmeticae* of 1539 in the sense of debts, etc. (cf. M. Cantor 1900, 502). Cardano established the multiplicity of roots, e.g., that equations of the second degree have two solutions, and that, e.g. $x^2 = 9$ has both the positive root 3 and the negative root –,3, and accordingly for solutions for equations of third and fourth degree. Cardano stressed in his *Ars Magna* that one obtains the number 9 both by squaring 3 and by squaring –3: since "minus in minus ductum producit plus."[19] Furthermore, he explained, in an argument analogous to that of Chuquet, that to add something negative corresponds to subtracting something positive (*Ars Magna*, Cap. XVIII).

In his *Ars Magna*, however, Cardano took further fundamentally new steps. He began to operate on square roots of negative numbers. Thus he obtained as intermediate results $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$, and as their product the value 40 (since "minus –15" is identical with "+15" ; cf. M. Cantor 1900, 509; *Ars Magna* Chap. 37, problem 3). On the other hand, Cardano's thinking remained determined to a considerable degree by the tradition in algebra. As coefficients in equations, he admitted only positive numbers, treating a large number of cubic equations separately (Peiffer/Dahan 1994, 104), as the Arabic mathematicians had done.

While Cardano thus adhered to the mathematical tradition in his algebra, he deviated—in his late work—from knowledge of algebra that had indisputably been approved at least since Diophantus: from the rule of signs. While admitting that it was indeed "opinio communis" that the multiplication of minus by minus gives plus, an opinion formerly also accepted by himself, he now stated that this was false. The result, he said, was not plus, but minus. This later assertion caused bewilderment in the historiography[20]—provided it was registered at all—and will hence be discussed here in more detail.

Cardano performed these conceptual reflections in his book *De Regula Aliza* (1570). By selecting *De operationibus plus et minus, secundum communem usum* as heading of the sixth chapter of this book, he indicated for the first time that he was taking his distance from the established rules of operation, while quoting the familiar rule of signs: "In the multiplication and in the division one obtains plus always from the same signs and minus from opposite signs."[21]

He did not provide an explicit critique of the tradition at this point, but gave it only later in his chapter 22, headed *De contemplatione plus et minus et quod*

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19 Cardano 1663, 222. "ducere in" means "multiply."

20 The *Mathematiker-Lexikon* speaks of an "error" (Lexikon 1990, 90). A recent Italian article speaks of "incomprehensible" justifications, and says that this error committed by Cardano does not diminish his eminence at all (Dell'Aquila/Ferrari (1994, 348 ff.).

21 "In multiplicatione et divisione plus fit semper ex similibus [et] minus ex contrariis" (Cardano 1663, 384).

minus in minus facit minus et de causis horum iuxta veritatem. This is where Cardano established a first alternative rule of signs: in the case of products and of divisions, in which at least one of the two signs is minus, the result also bears the *minus* sign. To justify this, he gave an example from Euclidean geometry (cf. Figure 1, which reconstructs geometrically Cardano's verbal argument).

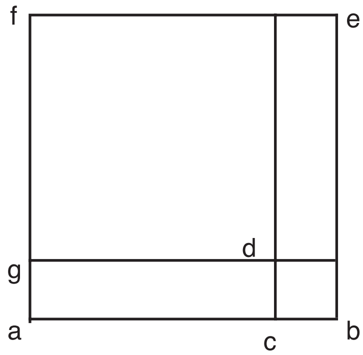

Figure 1, Cardano's refutation of the rule of signs

If a large square with side length *ab* = 10 is given, and the area of the square *df* is sought, with *cb* = 2, one must subtract the area of the two rectangles *gc* and *de*, i.e., 16 and 16, from 100. Since $100 - 2 \times 16 = 68$, one must subtract 4 in order to obtain 64. Thus, one gets −4 instead of the expected +4 (Cardano 1663, 399)! Cardano commented that for this reason minus multiplied by minus gives minus. The traditional error of those stating that minus multiplied by minus gives *plus* was evident, he said. Minus by minus would give no more plus than plus by plus gave minus (ibid.).

Actually, the equation $64 = 100 - 2 \times 16 - 4$ is not about multiplying an expression in parentheses: the latter would be written $(10 - 2)(10 - 2)$, and multiplying it would yield $100 - 2 \times 10 \times 2 + 4$.

While Cardano was well aware of this approach, he interpreted it, however, differently in line with his new view on foundations. Admitting to have hitherto argued along with the "opinio communis," he tried to explain why minus multiplied by minus only *seemed* to give plus and how this illusion might be comprehended (ibid.). He fell back on the same example: After taking away the two rectangles *gb* and *ce* from the large square, the small square *ab* had been subtracted twice instead of only once—it must therefore be added again, Cardano said. While it *seemed* as if minus by minus gave plus, this was false: "Sed non est verum" (ibid.). The +4, he said, was not the result of an operation of multiplication, but of applying Proposition VII of Euclid's second book (ibid., 400).[22] Essential for Cardano's reasoning was the following justification, for it

22 Cardano's argument shows that he was well aware that rectangle formation in Greek geometry was not to be understood as an equivalent to the algebraic operation of multiplication, as maintained since the end of the nineteenth century by the adherents of a "Greek algebra," in particular in Euclid's Book II.

shows that his decisive reason for rejecting the then already time-honored rule of signs was epistemological. For Cardano, the positives and the negatives constituted two strictly separate areas, as it were two different worlds. It was impossible to mingle them or simply to switch from one to the other, "because nothing can trespass its forces."[23] Upon multiplying plus by plus, he said, one would remain within the plus area. Likewise, multiplying minus by minus could not escape from the minus area—the result would remain minus. If plus was multiplied by something beyond its own area ("*extra ipsum*"), that is, by minus, the result would be minus (ibid.). In Cardano's late work, the overt rupture with the traditionally unproblematic application of the rule of signs was due to the conflict he was the first to realize between the extended rules for the operation of subtraction and their linkage to the operation of multiplication. Cardano conceived of this conflict as an epistemological one—as a separation between two different areas—and refrained from explicitly reflecting upon extending the concept of multiplication.

In *De Regula Aliza*, Cardano discussed no further application of his own alternative rule of signs; likewise, he did not discuss its effects on his earlier results obtained by applying the traditional rule. In a later text, the S*ermo de plus et minus*,[24] he underlined his new view that the multiplication of minus by minus would give *minus*. He offered a differentiated view, however, on the operation of division there. Minus divided by minus could in some cases give plus, and in other cases minus (Cardano 1663, 435), an assertion that did not altogether simplify his argument (cf. Marie 1883, 266).

2.4. Further Developments in Algebra: From Viète to Descartes

In the generation succeeding Cardano, excellent achievements were accomplished in algebra by François Viète (1540–1603) and Simon Stevin (1548–1620). Their work, however, did not advance the concept field of negative numbers along a straight line.

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23 Ibid., 400: "quia nihil potest ultra vives suas." Tanner analyzed Chapter 22 and the *Sermo de plus et minus* in detail, in particular to judge Cardano's influence on Harriot (cf. below, Section 8.2.); she noted the qualification of minus as "alienum" by Cardano, but did not consider its epistemological function. The separation of the areas of the positives and the negatives and the above justification has therefore been taken into consideration neither by Tanner (Tanner 1980b) nor by Pycior, who relies on her (cf. Pycior in 1997, 24 ff.).

24 This text has been published from Cardano's *Nachlass* (M. Cantor in 1900, 540). Since it refers to Bombelli's algebra of 1572, it was written between 1572 and 1576.

Viète, active as a lawyer in France, published a series of algebraic works after 1591. One of his contributions to the process of generalization was that he introduced the use of letters in algebra—not only for indeterminates, but for coefficients as well. His *In Artem Analyticem Isagoge* (1591) contains an explicit reflection about the conditions of the operation of subtraction. Subtraction, he says, is executable only if the subtrahend B is smaller than the minuend A. For this case, Viète uses as sign of operation "–," However, if it was not known at first which of the two quantities was larger, the sign "=" was to be used for this indeterminate direction of subtraction. Viète thus made clear that he considered possible only subtraction leading to a positive result (cf. J. Klein 1968, 331 ff.). While this was in line with how this operation was viewed traditionally, it had not previously come under debate. Viète's differentiation of signs suggested that there might also be another meaning of this operation.

The rule of signs in its traditional meaning was quite unproblematic for Viète (cf. ibid.). He did not accept negative roots of equations, however (cf. Cantor 1900, 636).

Stevin, active in the Netherlands mainly as an engineer, but also as an educator of Prince Moritz von Nassau, published numerous textbooks, his *L'Arithmétique* (1585) belonging among the more important. Herein, Stevin formulated the rule of signs as a theorem, "proving" it in the traditional way: for "complex" terms $(a–b)$ and $(c–d)$—what was taken away too much by multiplying the middle terms must be restored by the product of the two final terms.[25]

He operated with negative numbers, as had meanwhile become usual, but showed evident reserve in respect to admitting them as roots of equations. While Stevin declared that resolutions by minus ("*solutions par –*") did exist for some problems—e.g., $x^2 = 4x + 21$ is satisfied by $x = –3$—and gave all three roots for cubic equations, even if one of them was negative (cf. M. Cantor 1900, 628 and Stevin 1958, 667 f.), positive roots were his priority of interest, and he considered negative roots only inasmuch he would not risk disregarding any positive solution (cf. Stevin, Werke II B, 642 ff.).[26] Where solving quardratic equations is concerned he considered, however, only three of the four possible cases of combining the coefficients' positive and negative signs. He refrained completely from considering the fourth case $x^2 = -ax - b$, obviously because it gives negative solutions (ibid., 594).

Another generation later, Albert Girard (1595–1632), likewise an engineer in the Netherlands and the first editor of Stevin's works, had no reservations against negative solutions. He was one of the first to formulate the fundamental

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25 Stevin *Works*, Vol. II B, 560 f. The editors of these collected works canceled as allegedly insignificant the parts on subtraction in the preceding section introducing the arithmetic operations (ibid., 552).

26 Stevin commented on this example of a problem, noting that there were even more solutions than those shown by him. And although they "ne semblent que solutions songeés, toutesfois elles sont utiles, pour venir par les mesmes aux vraies solutions des problemes suivans par +" (ibid., 642).

theorem of algebra and thus granted equal status to all the roots of an equation: positive, negative, and imaginary.

In his *Invention nouvelle en l'algébre* of 1629, Girard expressly stated that negative roots of an equation must not be omitted, naming, e.g., +3 and –3 to be the square roots of +9. He was also the first in early modern times to interpret the negative numbers geometrically: the minus sign, he said, indicated an inverse movement (M. Cantor 1900, 787 f.). Incidentally, Girard belonged to the minority who adopted Viète's sign "=" to indicate the indeterminate difference.

René Descartes (1596–1650), the well-known rationalist philosopher and naturalist, active for some time in the military and otherwise as a private scholar, gave quite decisive impulses to the further generalization of mathematics by establishing algebraic geometry. In particular, the symbols he used, and his notation (he wrote equations with signs for plus and minus and the sign for equality, signs for powers by superscript exponents, used the radical sign, and introduced the practice of using the first letters of the alphabet for constant quantities and of the last letters for variable quantities) made mathematical texts easily readable.

In his famous *Discours de la Méthode* (1637), among whose appendices is *La Géométrie*, Descartes developed principles about how to clearly form concepts. There are no explicit reflections, however, about the foundations of arithmetic and algebra, and in particular none about negative numbers. Operating with negative numbers is nevertheless treated exhaustively in his texts. In the literature, his view on negative numbers, however, is assessed differently. Rouse Ball, for instance, says, "He realised the meaning of negative quantities and used them freely" (Rouse Ball 1908, 276).

A critical view is often deduced from the fact that Descartes called negative solutions "false" roots (cf. Tropfke 1980, 147; Dell'Aquila/Ferrari 1996, 322). Since Descartes at the same time called positive solutions "true roots," the term at first was used only to differentiate between positive and negative solutions (cf. Gericke 1970, 57). What is more revealing is that he characterized negative quantities as "less than nothing." Besides, he used the interesting name "defect of a quantity" for negative quantities:

> It often happens that some of the roots are false, or less than nothing. Thus if we suppose x to stand also for the defect of a quantity, 5 say, we have $x + 5 = 0$. (Descartes, *Œuvres* VI, Géométrie, 445).

On the other hand, Descartes's 1638 short account of his *Géométrie* contains a note that clearly shows the influence of Cardano's epistemological view, so that here as well we have an implicit allusion to the problems of extended multiplication:

> *Note*, that one must take care when multiplying by itself a sum which one knows to be less than zero, or where the greater terms have the sign –; for the product will be the same as if they had the sign +. Thus, $a^2 - 2ab + b^2$ is equally the square of $a - b$ as of $b - a$; so that, if one knows a to be less than b, one cannot multiply $a - b$ by itself, since it will produce a true sum in place of one that is less than nothing: which

will cause an error in the equation. (Descartes, *Œuvres* X, 662; cf. Dell'Aquila/Ferrari 1996, 323).

It has often been said that Descartes shirked considering negative roots; even Charles Adam and Paul Tannery, the editors of his *Collected Works*, commented on the section about solving quadratic equations as follows: “Descartes ne reconnait nullement les racines négatives des équations” (cf., e.g., Descartes *Œuvres* VI, 375). But only a couple of lines later in this text, Descartes gives both the positive and the negative solutions of a quadratic equation (ibid., 376).

Glaeser’s and Dell'Aquila/Ferrari’s summary criticism that Descartes suggested skillful tricks to eliminate negative solutions for equations is just as unjustified. While Descartes explained how to transform “fausses racines” into “vrayes,” without making the “true” at the same time “false” (ibid., 450), this refers to a method for transforming equations, and does not convey an eliminative attitude toward negative solutions. Elsewhere, he even showed how to reduce the number of positive solutions in favor of negative ones (ibid., 448).

There is a problem concerning the conceptual level in Descartes, however, actually in the relation between the concept of number and the concept of quantity, which has not been noted as yet in studies on the history of negative numbers. It appears in his treatment of quadratic equations. While Descartes speaks, upon introducing the algebraic treatment of geometrical problems, of *one* type of quadratic equation,

$$z^2 = -az + bb$$

(ibid., 373), this equation is not intended to be the normal form. Rather, there are different normal forms, depending on the signs with which the coefficients are provided. Actually, Descartes elsewhere presents a general form of the quadratic equation, which,because of the variability of the signs can maximally assume four different forms:

$$xx = + \text{ or } -ax + \text{ or } -bb$$

(ibid., 386). In fact, Descartes considered only three normal types in his account of solving quadratic equations,

$$z^2 = az + bb,$$
$$yy = -ay + bb,$$
$$z^2 = az - bb,$$

whereas the case in which only minus signs appear is not a subject of discussion (ibid., 374–376). Not only is it noteworthy that Descartes does not count the case of two negatives roots among the “normal” cases, but moreover, the implied concept of quantities is of fundamental importance. Contrary to the apparent generality of the domain of values, the coefficients a, b, etc. in the equations are not able to assume arbitrary values in the entire area of the defined numbers; rather, they are evidently understood to be absolute quantities that are able to assume only positive values. While Descartes had declared in the overview already mentioned that the first letters of the alphabet stand for known terms, that is, both for numbers and for quantities (“soit ligne, nombre, superficie, ou corps": Descartes *Œuvres* X, 672), because the coefficients in the equations represent geometrical objects, in particular lines, and hence substances. These quantities are understood as representing absolute values. A separation of the concept of number from the concept of quantity was not realized here, and hence no fusion of positive and negative numbers to the unified area of integers.

Although Descartes extensively operated with negative numbers, these were not completely employed on an equal basis with positive numbers, but with some restrictions due to epistemological reservations and to a specific concept of quantities. This ambivalence at the beginning of the intense modern development of algebra was to essentially affect the later history of the negative numbers, and not only their history. Thus it has already been noted often that Descartes, the founder of analytic geometry, always stayed within the first quadrant with his graphs of figures and curves (cf. M. Kline 1972, 311).

2.5. The Controversy Between Arnauld and Prestet

A New Type of Textbook

The texts mentioned above of authors from early modern times were for the most part research publications directed to a small audience of scholars; they were not objects for systematic teaching, say, at universities. A decisive change in this respect was initiated in France, by one of Antoine Arnauld's (1612–1694) textbooks.

The very title of his book expressed its bold program: *Nouveaux Élémens de Géométrie*—the title "elements of geometry" having hitherto been used in Europe exclusively for Euclid's textbook, "elements" and "Euclid" having become practically synonymous in mathematics. Not only did the title for the first time express the claim to present a textbook superior to Euclid's,[27] but its very concept contained a radical critique claiming to improve the realization of Euclid's intentions. In contrast to criticisms of Euclid aimed at improving the overall presentation or the wording of some propositions, Arnauld rejected Euclid's textbook, traditionally praised as the classical model of a rigorous and methodical presentation, for basic defects in methodical architecture. In doing so, Arnauld continued Petrus Ramus's (1515–1572) critique of Euclid, implementing it by restructuring.[28] In fact, Arnauld realized a new methodological structure. Instead of the pervasive switching between the more geometrical and more arithmetical/algebraic books in Euclid's work, Arnauld began with four books presenting the foundations of operating on quantities in

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27 After Arnauld, this title, "Nouveaux Élémens ..." has been used in France repeatedly by textbook authors (not, however, in Germany and in England).

28 For his critique of Euclid cf. Schubring 1978, 40 ff., and here Section 2.7.

general ("la quantité ou grandeur en général"), and only then developed the application of this theory of general quantities to geometry in the books that followed (Arnauld 1667, ii).

Beyond that, Arnauld's textbook had revolutionizing effects, since it adopted the new algebraic notation in equations elaborated by Descartes, addressing a larger public due to his algebraized, more readable style, and thus inaugurating the new textbook format. Subsequent to Arnauld's *Elements*, a large number of mathematics textbooks perfecting this new style were published in increasingly rapid sequence. While Arnauld's textbook had still been addressed to an unspecified general audience, the textbooks that followed were clearly motivated by purposes of university teaching. The publication of Arnauld's textbook may be considered as the onset of modern textbook production in general.[29]

ANTOINE ARNAULD

Antoine Arnauld was an eminent person who had an important impact on French intellectual life. Originally a theologian, he taught at the Sorbonne in Paris until expelled from there by the Jesuits; then he lived in the cloister Port-Royal near Paris; he was also a philosopher and a mathematician. He was one of Descartes's significant dialogue partners in questions concerning the method of cognition; and it was in particular by his own close contact with Arnauld that Leibniz, during his stay in Paris, became intensely acquainted with mathematical research (Bopp 1902, 203). Together with P. Nicole, Arnauld is the author of the famous Logic of Port-Royal: *La Logique ou l'Art de penser* of 1662, which, just as their grammar for linguistics, exerted profound effects on French philosophy. Together with other members of Port-Royal, Arnauld was one of the leading representatives of Jansenism in France. Jansenism, a reform-Catholic current in favor of a French national church, presented an acute challenge to the Jesuits, and they fought it violently. Eventually, the Jesuits succeeded in having the King of France destroy the Cloister of Port-Royal; because of its destruction, Arnauld left France and went into exile in the Netherlands and Flanders in 1679.

Arnauld's geometry textbook is particularly valuable for our context because it gave rise to the first known direct debate and controversy between two mathematicians over how to conceive of negative quantities. This controversy must be understood against the backdrop of a profound theological–philosophical conflict of long standing between Arnauld and Malebranche.

This quarrel is all the more unfortunate, actually, as Arnauld and Malebranche were convinced Cartesians, both actively promoting Cartesianism, in particular in mathematics. Their debate cannot be detailed here (cf. A. Robinet 1966), but must be mentioned briefly as the backdrop for the controversy about negative numbers. Nicole Malebranche (1638–1715), also a theologian and prominent ph

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29 In one respect, Arnauld differs from his successors: he obviously had another view of authorship. All his books appeared anonymously, both his logic and his geometry, the first edition of 1667 and its second of 1683 as well.

ilosopher, promoter of mathematics and the sciences, started a theological career, was ordained a priest, and became a member of the Oratorian Order. In 1699, he was elected to the Paris *Académie des Sciences*. He was the principal representative of a Christian Cartesianism in France. By means of occasionalism, he attempted to solve the soul–body problem caused by the Cartesian dualist doctrine of the two substances. Arnauld replied with his 1683 *Des Vrayes et des Fausses Idées* to Malebranche's two major works, *De la Recherche de la Vérité* (1674/75) and *Traité de la nature et de la grâce* (1680). Besides the theological concept of grace, an epistemological question was at stake as well. Arnauld criticized Malebranche's thesis that the knowledge of objects required ideas that exist independently of the human mind, and were distinct from it and its perception (W. Doney 1972, Mittelstraß 1984, Piclin 1993). The related question as to what extent theoretical concepts have to be representable by empirical knowledge constituted the core of the controversy between Arnauld and Malebranche's disciple Prestet.

In his geometry textbook, Arnauld's introductory chapters on the doctrine of general quantities are on general conditions, or "suppositions." The second of these conditions was the "knowledge" that the multiplication of two numbers is commutative (Arnauld 1667, III). Explanations of general terms like axiom, definition, theorem, followed, and then explanations of signs used, like +, –, = , and the proportion sign :: ; he continued with *principes generaux*, on the relation of part and whole, and with axioms on equality and inequality. Based on this, Arnauld introduced the four basic operations. Subtraction was explained quite generally as the remainder $b–c$ of two quantities (ibid., 6). And by introducing letters, Arnauld stressed extensively that upon designating quantities by letters one no longer needed to pay attention to the letters' meaning, since the task now was only to investigate identities and relations. Besides, he added, it was one of his book's particular advantages that it trained the mind to understand things in a "spiritual manner," without the help of any "sensible" images (ibid., 4).

After that, Arnauld began treating "complex," i.e., compound, quantities. Their introduction contains a hint at the concept of opposite quantities. The plus and the minus of the same quantity cancel one another, resulting in zero: "Le plus et le moins d'une même grandeur ou terme sont égaux à rien, ou valent zero. Car l'un ostant ce que l'autre a mis, il ne demeure rien" (ibid., 9).

Using the multiplication of complex quantities $(a–b)$ and $(c–d)$, Arnauld explained and "proved" the rule of signs. The last of its four cases reveals a noteworthy influence of Cardano's last works.[30] Only reluctantly, Arnauld

30 The Leyden edition of Cardano's works appeared in 1662, so that it might have influenced Arnauld quite directly. On the other hand, Cardano and Arnauld seem isolated with their critique of the rule of signs. However, this can also be an error of our historical perspective: according to Rider's bibliography, between 1570 and 1667, at least 82 algebra textbooks were published, which have hitherto not yet been evaluated with a focus on this question (Rider in 1982, 26–49)!

quoted the traditional rule here that minus by minus makes plus, adding that their product properly gives, due to its inherent nature, *minus*:

> MINUS by *minus* gives *plus*: that is to say that the multiplication of two terms, both of which have the sign *minus*, gives a product which must have the sign plus. […] This appears rather strange, and in fact it cannot be imagined that this could happen other than by accident. For of themselves, *minus* multiplied by *minus* can only give *minus* (ibid., 13).[31]

Arnauld neither explained the inherent reasons for which the product should yield a minus sign, nor why the plus result should appear only accidentally. The justification for the plus sign's occurrence given by Arnauld following the quotation above corresponded to the traditional justification for compound expressions: because of the terms having a minus sign, one subtracted too much, and this excess had to be compensated by adding the final term. No randomness of the "plus" result was evident from this reasoning, either. Obviously, Arnauld was prompted by profound epistemological concerns, similar to those of Cardano, that operations within the negative area would remain there with their result as well, making mathematical justifications proper marginal.

The statement that a proposition concerning the foundations of mathematical operations was of random validity had to cause a stir. Moreover, it was strange that Arnauld had nevertheless presented this proposition, which was allegedly true only by accident, as a valid rule.[32] Within the very foundations, there existed, hence, a striking contradiction to the otherwise rigorous and methodical structure of his book.

JEAN PRESTET

One of the contemporary reactions to this contradiction has come down to us: that of Jean Prestet (1648–1691).[33] Prestet, of poor origins, was ordained as a priest only late in his life; as adolescent, he had been a servant to the *Oratoire* in Paris. He soon became one of Malebranche's scribes, then his disciple in mathematics; eventually, Prestet, guided by Malebranche, began writing a textbook in Arnauld's new style in 1670. After publication in 1675, the *Oratoire* delegated him to positions for teaching mathematics, the most prominent of these being the new mathematical chair of Angers in 1681. He died soon after having published the second edition of his textbook (Robinet 1960).

The title of the textbook was even more ambitious than Arnauld's: *Elémens des Mathématiques*. As is shown by how the title goes on, Prestet's intention was to extend the usually prefatory general part about quantities to a separate

31 "Car de soy-même *moins* multiplié par *moins* ne peut donner que *moins*."

32 In the other parts of the volume, Arnauld did not return to the validity of the rule of signs, firstly since its main part treated elementary geometry, and secondly because he was entitled to apply the traditional rule due to its attested contrariness.

33 P. Schrecker was the first to give access to this debate in 1935 (Schrecker 1935).

book on arithmetic and algebra. It continues, *ou Principes généraux de toutes les Sciences qui ont les Grandeurs pour Objet*. Prestet's volume contained no geometry,.In spite of its apparent claim at totality, the book rather intended a program of algebraization for mathematics, as emphasized in the preface, and as demonstrated in its second edition with challenging innovativeness (see below).

Prestet's textbook indeed contains the first account of the concept of negative numbers in which the negative numbers are presented as having the same status as positive numbers, and in which the rule of signs is "proved" not geometrically, but algebraically.[34] At the very beginning of his presentation of the concept of quantities, Prestet explains that the quantities are a composite of a positive area and of a negative area, each alone being infinitely large. The starting point for this conceptualization was the introduction of the zero, which did not have any absolute character or precarious exceptional status, but was understood to be a relative quantity between the positive area and the negative area that enables a comparison of the relations between quantities from the two areas: "Le rien ou le zero nous sert de milieu[35] pour faire les comparaisons des grandeurs, et pour juger de leurs rapports" (Prestet 1675, 3).

Prestet's concept of quantity, however, is not exempt from contradictions. While he stresses that he is not interested in the nature of quantities and in what they are *in themselves* ("dans elles mêmes"), saying that only the *relations* between them are essential, he characterizes the positive and the negative quantities, despite this relational view, in terms of *existence*, as "being" and "non-being" ["non estre"] respectively, and as bearing an increased or a reduced degree of reality:

> Magnitudes have more or less reality as their being takes them further from zero, and they have less reality when their non-being takes them further from this same zero. It became customary to call positive or true every magnitude which adds to zero, and negative or false every magnitude which takes away from this same zero. (ibid.).

The positive quantities extend to where nothing can be added any more that the quantity does not already have, this being the infinitely true magnitude: "grandeur infiniment vraie." And the negative quantities extend to where nothing can be subtracted any more that has not already been subtracted: this being the infinitely false magnitude, "grandeur infiniment fausse." In defining subtraction, Prestet restricted the quantities to be subtracted to positive quantities, in order to avoid confusing the algebraic sign with the sign of operation, an error committed by many later authors: "The addition of true magnitudes are marked by the sign +, which signifies plus, and the taking away or the subtraction of these same magnitudes are marked by the sign –, which

34 Prestet thus continued—after the interim ruptures in the understanding of the concept since Cardano—Chuquet's approach; but what had remained mostly implicit and had become clear only in the application, was developed here to an explicit foundational part of its own.

35 "Milieu," probably to be understood as the "middle;" in the second edition substituted by "terme ou un point fixe" (Prestet 1689, Vol. I, 3).

signifies minus" (ibid.). Prestet was the first to present addition and subtraction very lucidly as the operations inverse to one another:

> The + and the – of equal magnitudes are each mutual takings away, the + takes away from the –, and the – takes away from the +. The position or the possession of a thousand écus takes away the negation or the privation of a thousand écus, and the negation or the privation of a thousand écus takes away the position or the possession of a thousand écus, […]. Or what is the same thing, +1000 écus –1000 écus are equal to zero (ibid., 10).

Although Prestet did not speak specifically of "opposite quantities" here, his lucid confrontation of *positing* ("*position*") and of *canceling* ("*négation*") refers to this concept as a possible next step of development. As a "clear" conclusion, but without formal proof, Prestet added the rule of signs; we quote here the rule for the equal signs: "D'où il est clair, 1°. que plus plus ou + + est égal à moins moins ou – –, et que – – = + +" (ibid.).

In his discussion of the four basic operations on integers,[36] Prestet took up the rule of signs once more. Thus, he explained it for multiplication by interpreting multiplication as a repeated addition. The product $(-2) \times (-4)$ was to be understood as a twofold negative addition of –4; since a singular negative addition of –4 gave +4, a twofold had to give +8 (ibid., 20).

Of particular interest under the aspect of foundations of arithmetic is that Prestet did not restrict the definition of subtraction to the case of a positive difference. Rather, this definition is entirely general, and this is why Prestet also specifically notes that the difference can be negative: "When the number to be taken away is greater than the number from which one takes it, the difference or the remainder is negative" (ibid., 17).

He exemplified several such negative *remainders*. Moreover, in a comment, Prestet explained for those who might at first be surprised at the result $4 - (-3) = 7$ that such a subtraction must be understood as "descending" from the positive into the negative area: Firstly as descending ("*descendre*") by 4 units from +4 to 0, and then as descending from 0 to –3 by three more units, thus altogether by 7 units (ibid., 18).

Thanks to its extensive, coherent, and consistent presentation, Prestet's textbook of 1675 was the hitherto most advanced presentation of the negative numbers. It represents a significant summit in the history of the negative numbers.

THE CONTROVERSY

Within a period of only eight years, two textbooks had thus been published that transcended by far the traditional level of explicity for the concept of negative numbers while assuming divergent approaches. This alone would have

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36 "grandeurs entières," later in France with a restricted meaning: positive integers, i.e., natural numbers.

suggested a controversy between the two French authors.[37] Moreover the heated theological–philosophical dispute between Arnauld and Prestet's teacher and master Malebranche may have fueled this debate in mathematics. Prestet gave no dates for his correspondences with Arnauld, but wrote in 1689 that the exchange of letters had taken place "more than ten years ago" (Prestet 1689, Vol. 2, 366), that is, before 1679, the year when the conflict between Arnauld and Malebranche broke out. The period of 1681/82 is more probable, however.

Prestet published his debate with Arnauld in his own textbook's second edition. It had begun with a letter that he directed to Arnauld, the content of which he reported only briefly. Arnauld's reply is rendered in more extensive excerpts, and Prestet's answer probably almost completely. Prestet gave as his motive for publishing the correspondence that he had observed what difficulties even intelligent and enlightened people experienced when trying to obtain an unambiguous concept of the nature of roots and of negative quantities for themselves, and all the more so one of imaginary quantities. His answer to the difficulties expressed by Arnauld might be helpful, Prestet argued.

As he explained in an annex to his second edition, he had held Arnauld's geometry textbook in great esteem for its order and method. Because of his wish that the latter's geometry book should contain only completely certain knowledge, Prestet said, he had suggested to Arnauld to modify his paragraph on the rule of signs and had presented him his reasons why, to his mind, minus by minus could not give minus (Prestet 1689, Vol. 2, 366).

Arnauld had answered that he had already made up his mind to change this part in a new edition he was preparing. He said to have himself reflected the matter some time ago and become convinced that one could in fact say that minus by minus makes plus (ibid). In what followed, Arnauld, however, had given four reasons why he himself had conceptual reservations with regard to isolated negative quantities (*ces moins sans rapport à aucun plus*). These four reasons were the following (ibid., 366–367):

1. It was clear that one can subtract two *toises* (fathoms) from five *toises*. But it was impossible to subtract seven *toises* from five *toises*. It was thus incomprehensible for him ("inconceivable"), how one could say "five *toises* minus seven *toises*." Arnauld assigned a meaning to arithmetic operations as far as they were executable with concrete quantities.
2. Arnauld declared it to be likewise incomprehensible why the square of –5 could be the same as the square of +5. Obviously, Arnauld was assuming like Cardano that results of operations in the negative area had to remain within it.
3. Moreover, Arnauld had been the first to introduce the so-called proportion argument, which was to develop a high power of persuasion for the future. If (–5) (–5) = (+5) (+5), the proportion

[37] The first edition of Prestet's volume was published anonymously, just as Arnauld's. While it was not difficult to detect Arnauld as the author, many thought Malebranche to be the author of the other volume.

$$1 : -4 :: -5 : 20$$

had to be valid as well. This proportion, Arnauld had said, arose from the "fondement de la multiplication," according to which unity is to a factor as another factor is to the product; this was true for integers and for fractions. In the above proportion, however, the rule ever applicable for all other proportions did not hold: if the proportion's first term exceeded the second, the third must also exceed the fourth.

Arnauld had said he himself had already considered whether the problem could be solved by disregarding the signs + and – in the proportion, that is, by considering absolute numbers (*en faisant abstraction des signes*). He had, however, feared that this might result in undesirable consequences, since negative quantities could actually exist; they should accordingly have to be made allowance for in the proportions.

4. Finally, Arnauld had explained that he rejected isolated negative quantities. A negative quantity like –10.000 écus could be a real thing, such as a man's debts, which could cause him to be thrown into debtor's prison to compel him to find the money to somehow pay his creditors. One could say that a man had –10.000 écus only if one could attribute to him some power (*puissance*) however fictitious to obtain 10.000 écus to satisfy his creditors. Negative quantities, Arnauld had said, were to be credited as possible only in relation to some plus, but without such a relation they were but a fiction, *une chimère*, *une montagne sans vallée*.

In his response, Prestet had reduced the four problems to two: to whether the subtraction was executable and to the argument of proportion.

With regard to the first set of problems, Prestet referred first to an epistemological issue: whether abstract concepts were admissible. People possessed a notion of plus and minus; this did not comprise quantities that, whether disparate, or not, were certain ways to understand quantities either as added or as subtracted ones. While applying this to the results of operations, he, Prestet, continued to rely, however, on the substantialist concept of "existence," for his own reflections were based not on numbers, but on quantities existing in real life. Only positive quantities existed, he said, they were characterized by the positive idea of plus (Prestet 1689, vol. 2, 368). Conversely, negative quantities did not exist and had better be designated as "rien" or "zero." For to be a negative quantity and to be no quantity at all appeared to be the same thing.

Prestet tried to escape this contradiction by means of an *operational* approach: while –2 *toises* did not exist, one might actually use this expression without relating it to any expression in plus—as an *indication* that 2 *toises* are lacking to return up to zero again. Such an operative meaning indicating the absence of 2 *toises,* Prestet declared, was no chimera or fiction.

Regarding the proportion argument, Prestet did not address Arnauld's problem of contradiction to the definition directly. Instead, he argued by quoting those parts of the rule of signs that had been accepted by Arnauld, i.e., for mixed terms. As Arnauld had pointed out in his own textbook, minus by plus gave minus, just as plus by minus gave minus. Therefore, +1 divided by –1 gave –1, just as –1 divided by +1. Since the results were identical, the ratios were identical, too; compounded, this yielded the following proportion:

$$-1 : +1 :: +1 : -1.$$

Just as admissible and valid, hence, for him was the proportion

$$1 : -4 :: -5 : 20.$$

Prestet did not recognize the principal importance of the problem of how to define an enlarged multiplication. In the same way, he dealt with Arnauld's question concerning the equality of $(-5)^2$ and $(+5)^2$: equality resulted from the two proportions

$$1 : -5 :: -5 : 25 \text{ and } 1 : +5 :: +5 : 25.$$

As a complement, Prestet demonstrated the equality of the two squares, by means of how he had interpreted the definition of multiplication already in his own textbook: –5 times –5 signified subtracting –5 five times; subtracting once gave +5, subtracting five times hence gave 25 (ibid., 370).

THE DEBATE'S EFFECTS ON THEIR TEXTBOOK REEDITIONS

The two opponents each had a second edition of their mathematics textbooks printed—Arnauld in 1683, and Prestet in 1689. Which effects of their debate can be observed in the new editions? Let us begin with Prestet. He changed the title and made it analogous to Arnauld's by preceding it by a *Nouveaux*. Actually, the textbook had been profoundly revised. The one-volume work having 428 pages had become two-volumed with more than 1100 pages; its concept had been changed as well. While the preface of the first edition had favored the analytic method, the second edition's considerably enlarged preface pleaded with fervor for the analytic method and for algebra's superiority over the synthetic method and over geometry (cf. below, Section 2.7).

While Prestet had left the concept of negative quantities virtually unchanged, he now abstained from the qualifications "existence" and "non-existence," and from attributing greater or lesser degrees of reality. The comparison with zero continued to be pivotal:

> We call *positive* or *real* or *true magnitude*, every magnitude which adds to zero, or which is worth more than nothing; and *negative* or *defective* or *false magnitude*, every magnitude which takes away from zero, or which is worth less than nothing. (Prestet 1689, vol. I, 59).

One change, however, was caused by the new structure. While the first edition had been entirely based on the concept of *quantity*, the second began with *number* as the fundamental idea. After an introductory part on methods, a part titled "science générale des nombres" followed; it virtually constituted an independent arithmetic based on *"nombres entiers* ou *naturels."* The definition of subtraction in this arithmetic was restricted to cases in which the subtrahend was smaller than the minuend (ibid., 23). Algebra as the calculus with letters constituted the work's third part, and it was based on the concept of quantity: *La* *science générale des grandeurs.* Here, subtraction was again defined without restriction concerning the relations of quantities.

Prestet had entangled himself in considerable technical difficulties, however. To keep the two meanings of the minus sign as algebraic sign and as sign of operation separate, he had implicitly restricted the use of letters to positive quantities alone. He was thus able to characterize positive quantities as those endowed with the sign +, or without sign, and negative quantities as those endowed with a minus sign (ibid., 59). Subtraction, as in the first edition, was thus restricted to positive quantities in both terms: "take away incomplex numbers from each other when they are both true" (ibid., 60). This prompted Prestet to prescribe use of the minus sign for the addition of negative quantities (ibid., 61), and use of the plus sign for their subtraction (ibid., 62). This kind of conceptual differentiation opposing time-honored use proved untenable in practice.

His presentation of the rule of signs had remained basically unchanged (ibid., 62 ff.). Since multiplication was understood as repeated addition, and subtraction of negative quantities as some kind of addition, the rule of signs of multiplication resulted already from the rules for subtracting these quantities (ibid., 62).

In *Arnauld's* second edition, there was a series of changes.[38] It is important to note that he did not use the term "negative numbers" or quantities, just as he had not done in the first edition, but only discussed the use of the minus sign. A relevant change was in how he defined subtraction: it was now confined to cases giving *positive* results: "Soustraire, ou *soustraction*, c'est retrancher une moindre grandeur d'une plus grande" (Arnauld 1683, 7).

Arnauld thus continued to reject isolated negative solutions. Against that, he had changed his mind regarding the rule of signs. The propositions saying that the product of twice minus gives again minus, or that plus arises only accidentally, no longer appeared. He accepted the rule of signs without reserve, undertaking to prove it completely. For this purpose, he elaborated the concept of multiplication, as Prestet had used it, even more systematically.

For the product, Arnauld introduced the distinction between multiplier (*multiplicant*) and multiplicand (*multiplié*). For the proof of the rule of signs, he considered the four cases:

$$\begin{array}{llll} 1. & + \times + & 3. & - \times + \\ 2. & + \times - & 4. & - \times - \end{array}$$

In the two first cases having a positive multiplier, in essence the usual meaning of the multiplication is valid: as a reiterated execution—in the first case of addition, in the second case already by an extension of the concept, namely of

38 Robinet has told without giving evidence that this edition contains a polemic against Prestet (Robinet 1961, 210). Actually, this is not true. The publication of their letters by Prestet six years later certainly occurred to support Malebranche in his controversy with Arnauld and to show the "narrownesses" of Arnauld.

subtraction; hence, the sign of the multiplicand remains the same. In the two last cases, however, a new meaning of the operation of multiplication is revealed, an extended meaning: as a *subtraction* of the multiplicand—it must be subtracted as many times as the multiplier indicates. Hence, in both last cases the sign of the multiplicand must be changed:

> In the 4th case where the multiplicand has a *minus*, the product must have a *plus*; […] to multiply –3 by –5 is to take away 5 times –3. Now, to take away one times –3, is to set down +3, as has been said on the subject of subtraction; thus to take it away 5 times, is to set down +15; which was what had to be proved (ibid., 18).

Although this approach was more reflected and more explicit than those of previous authors and was even suitable to make the rule of signs plausible, it was as yet no real proof. The last justification already assumed by its – –3 = +3 what had to be proved. Moreover, Arnauld did not discuss the consequences of the distinction between multiplier and multiplicand for the commutativity of multiplication which he had assumed in his textbook's introduction (ibid., 2).[39]

In spite of this "solution" of his problem with the rule of signs, the proportion argument continued to present a stumbling block for Arnauld. He devoted a note to its discussion in his second edition, which covers more than a page in small type, where he mentioned for the first time that due to the new multiplication concept he had been able to solve "la plus grande difficulté" that had led him earlier to assert the accidental character of the plus result. He then presented the proportion argument, first presented in his answer to Prestet as one still bothering him. Arnauld gave the argument with so many doubts of his own that the reader was unable to decide whether he himself considered it refuted or not. Arnauld had added an observation that shows that he was the first to have become aware of the fact that operating on negative quantities required a new, extended concept of multiplication and that the "usual concept of multiplication" did not suffice for this purpose:

> I see no other answer to this [concerning the proportion argument] than to say that the multiplication of minus by minus is carried out by means of subtraction, whereas all the others are carried out by addition: it is not strange that the notion of ordinary multiplications does not conform to this sort of multiplication, which is of a different kind from the others (ibid., 19).

Later mathematicians, however, for a long time did not take this hint at enlarging the concept of multiplication, but preferred to focus on the proportion argument Arnauld had presented here for the first time, attempting either to refute or to apply it. Thus, the argument began to have an intensive aftereffect: $1:-5 = -3:15$ could not be valid, since the second term was smaller than the

39 He had presented it indeed as unimportant which of the two factors was chosen for which function (Arnauld 1683, 16). But he had obviously operated on numbers only. The general case is, however, that only the multiplier needs to be a pure number, a scalar, while the multiplicand can be a quantity.

first, requiring that the fourth also ought to be smaller than the third, which actually was larger (ibid).[40]

Summing up the controversy between Arnauld and Prestet, one will observe that Prestet had left his concept of negative quantities practically unchanged. Although he still considers the linkage between the number concept and empiric substances as given, the decisive aspect for him was a criterion of intra-mathematical consistency as to whether quantities are able to satisfy the given equations. The only change in Prestet's concept was that he excluded negative quantities from arithmetic, referring them to the next stage of generality, to algebra. This kind of hierarchization of mathematical knowledge was later to become typical of how the French saw the relationship between arithmetic and algebra.

Two points are noteworthy with regard to Arnauld's positions. Firstly, he changed his mind in the course of the controversy from postulating an only accidental validity of the rule of signs to affirming its full validity. The artful distinction between *multipliant* and *multiplié* he seems to have first introduced illustrates how poorly reflected and explained were the foundational aspects of the mathematical operations hitherto intensely used. It became explicit that the implied premise was that mathematical statements should be always valid over the entire field of numbers and quantities known; no explicit differentiation between ranges of argument and ranges of values had as yet been established. In his closing reflection on the proportion argument, Arnauld for the first time voiced the idea that validity—at this point the validity of the definition of proportions—might be restricted to certain subranges of numbers, i.e., might be invalid for negative numbers.

The second remarkable point appears in the changed definition of subtraction: in spite of its general formulation in the first edition, Arnauld apparently had not intended it to be unrestrictedly executable. His correspondence with Prestet shows his steadfast refusal of isolated negative quantities; the debate had made Arnauld aware of the consequences of his, according to him, own definition, which he himself considered as too generally formulated. In the second edition, he thus explicitly excluded isolated negative numbers.

The reasons advanced by Prestet and Arnauld in their debate "spanned" the largest part of the space from which the arguments on the status of the negative numbers were obtained in the times that were to follow.

The concept of multiplication is a privileged instance illustrating how different views of the same concept could "coexist" within the self–same person and thus provoke specific conceptual ruptures. The algebraic view of multiplication, as an iterated addition and containing the differentiation of

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40 Arnauld had here himself mentioned an extended notion of multiplication, namely the product concept as introduced by Descartes (Descartes 1974, 370): as the fourth proportional to the unity, to the multiplier, and to the multiplicand (Arnauld 1683, 19); however, because of the proportion argument, he saw no possibility to apply this extended notion to negative quantities.

multiplier and multiplicand, a view that incidentally agreed exactly with the view of multiplication in Indian mathematics, had not been assumed to be exclusive by Arnauld. In the second edition, he complemented it by the Cartesian notion of product for geometrical lines: "the most natural notion of multiplication in general […] is that the unit […] must be to the multiplier as the multiplicand is to the product" (Arnauld 1683, 19).

Prestet, who had used the algebraic view of the rule of signs for his "proof" of the rule of signs, however, used the geometrical concept as well. A productive use of this geometrical version is his discussion of imaginary quantities. To my knowledge, the first indication of their geometrical interpretation appears in Prestet. He qualified negative quantities to be "linear" and imaginary quantities to be "plane" quantities, in the form of mean proportionals:

> imaginary quantities, originating from the second degree, imply planes and they are complex [compliquées], as when one wishes to take a mean proportional $\sqrt{-ab}$ between a positive magnitude $+a$ and the negative $-a$ (Prestet 1689, II, 371).

2.6. An Insertion: Brief Comparison of the Institutions for Mathematical Teaching in France, Germany, and England

In even another regard, Arnauld's and Prestet's textbooks present a change which was to become constitutive for the future. While the authors discussed before had published their treatises mainly in Latin, thus addressing an international (learned) audience, these new textbooks were written in French and, hence, oriented with priority toward the French education system and the culturally interested French public. Teaching mathematics now became a part of the emerging national education systems. This integration into different systems began to shape mathematics itself in a "style" specific for each case. Hence, it is necessary to compare the educational systems of the three countries with the mathematical cultures best established in a brief overview:[41] France, Germany and England. This comparison should serve as a matrix for a structural analysis of how the concept of negative numbers, and likewise later that of infinitely small quantities, underwent distinctive developments in these communities.

UNIVERSITIES AND FACULTIES OF ARTS AND PHILOSOPHY

A major basis of comparison is the status mathematics had within the universities. How things began in the Middle Ages is relatively uniform for Western Europe. Mathematics, as the quadrivium, then consisting of arithmetic, geometry, astronomy, and music, formed a stable, although marginal,

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41 A more detailed analysis of this institutional development in Europe is given in (Schubring 2002).

component of the propaedeutic teaching provided by the faculty of arts. Teachers were in general not especially qualified for this subject and changed very often, as courses were given by young lecturers who strove to rise to the better and more renowned faculties. The relative uniformity of this structure throughout Europe dissolved into differentiation at the beginning of early modern times. The emergence of national or territorial states and the schism of the Christian faith were conducive to marked differences among the universities. These began to be redefined into components of individual states' sovereignty. The fates suffered by the universities' faculties of arts is an indication of this progressive development.

The position of the traditional faculties of arts was in jeopardy from two sides, firstly from their precarious situation with regard to the three other "superior" or professional faculties. Wherever they were able to attain some degree of independence and a status beyond their propaedeutic functions, the subjects taught there had better opportunities of development toward disciplines of their own. Secondly, they were endangered by their relation to the system of secondary schools, which had expanded since Humanism, since both types of institution competed as to their propaedeutic tasks. Beyond the case in which secondary school and faculty of arts functioned consecutively, there were the two extreme forms under which either the faculty of arts was "soaked up" by the secondary schools or the secondary school became an integral part of the faculty of arts as its preliminary stage.

The decisive factor in the emergence of the various functions and structures was the conflict between Protestantism and Catholicism, which had been ongoing since the Reformation. While the faculty of arts was able to attain a relatively independent position documented in its rise to "faculty of philosophy" in Protestant territories, in particular Lutheran ones, it not only remained confined to subordinate functions in territories of Catholic faith, but was even in essential parts replaced by colleges having the status of secondary schools.

This radical change of function in Catholic territories was basically a consequence of the work of the Jesuit order. It had been intent on establishing a Catholic education system since about 1550, during the first century of the Counter Reformation. The almost total disappearance of the faculty of arts under the Jesuit system was directly connected with the success of a new structural element in the education system, the establishment of secondary schools providing an instruction systematically organized and serving to prepare students for university studies. In their adaptation of this model originating from Humanism, the Jesuits succeeded in transferring practically the entire teaching program of the faculty of arts to their own colleges, named *Kolleg* in Germany and *collège* in France. Their curriculum became restricted to the study of Latin as a transformation of the *trivium*, and to philosophy. The faculty of arts was therefore to all intents and purposes reduced to holding the *examinations* necessary for passing to the superior faculties.

In contrast to the professors at the Protestant faculties of philosophy, the teachers at the Jesuit colleges were ordained priests. Possible personal scientific orientations had to submit to rigid reglementation by the order's superiors and to the predominant task of spiritual education. Starting from the seventeenth century, other orders began to rival the Jesuits in Catholic states with *collèges* of their own, such as the Benedictines and the Oratorians in France, but these remained within the structures established by the Jesuits and did not attempt institutional change.

In comparison, the situation can be summarized as follows: In Germany, conflicting structures existed side by side in Catholic and in Protestant countries. In states with Lutheran denomination, the faculties of philosophy were able to hold their own against the *Gymnasien*, the classical secondary schools, and to develop into nuclei of subsequent growth of academic disciplines. A somewhat fragile division of labor was established between scholastic instruction in the preparatory subjects and a higher, general scientific education. In the Catholic territories of Germany, the Jesuits succeeded in gaining control over the totality of the faculties of arts, and to substitute them by their own *Kollegs* (cf. Hengst 1980, Meuthen 1988, J. Steiner 1989). The German classes possessing *Bildung* were almost exclusively oriented toward the universities. Eminent scholars outside the university system, such as Leibniz, were exceptional.

The French structures were very different from the German. In France, the universities were uniformly Catholic.[42] The teaching tasks of the faculty of arts had been pervasively transferred to *collèges,* reducing these faculties to holding the examinations necessary for the admission to the three professional faculties (Julia/Verger 1986, 141– 152). One of the effects of this development was that in France, these faculties did not foster the emergence of scientific disciplines. Scientific culture in France, on the other hand, was not confined to the universities and their context. In the aristocracy in particular, considerable groups existed as support for scientific activity. The *Académie des Sciences* in Paris, founded in 1666, soon became a crystallizing core for mathematical and scientific research. Beyond that, the universities were confronted with increasingly acute competition, in particular from the first half of the eighteenth century, by the expanding military schools, which became institutions offering comprehensive instruction in mathematics and in the sciences (cf. Taton 1986: Ve Partie). They were also challenged by the *Collège royal*, founded in 1530, which was unique in that it provided lectures and courses in modern science without leading to any examination or degree.

The structures in England were different again, the "Collegiate University" prevailing (McConica 1986a). By the middle of the sixteenth century, the

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42 By the occupation of Alsace in 1681, the Protestant university of Strasbourg became French but remained a special case in France, with an independent faculty of philosophy.

transition occurred to the university exclusively organized into *colleges*.[43] Now not only did the students have to live together in *colleges*, supervised by tutors assigned to each, but teaching was basically done within these *colleges* as well. In contrast to the Catholic model established by the Jesuits, teaching in these English colleges was not restricted to the faculty of arts's subjects proper, but included the subjects offered by all the faculties (cf. McConica 1986b). In Oxford and Cambridge, there was thus a double structure of *colleges* and of faculties. While exams were left to the faculties (Fletcher 1986, 185 ff.; Leader 1988, 102 ff.), there were also salaried professors giving "public" lectures as a part of the activities of faculties side by side with the colleges' *lecturers*. Teaching within the faculty of arts was not only the most extensive part, but the students had to devote most of their studies to its subjects. The dynamism unfolded in the humanist reform phase during the first half of the 16th century had to submit to orthodoxy due to the so-called Elizabethan reform of 1570.

The Status of Mathematics in Various Systems of National Education

With regard to mathematics as well, grave differences between European states and their ruling Christian denominations developed at the beginning of modern times. Schöner is the first author to have described the rather marginal function of mathematics at the end of the Middle Ages—a situation rather uniform for the universities in Europe—in his profound study (Schöner 1994).

Philipp Melanchthon, Luther's advisor in all questions of teaching and education, was a fervent advocate of a strong position of mathematical teaching in schools and at universities. In Lutheran territories, it was thus possible for mathematics to undergo a continuous development at the universities. Mathematics was among the first subjects for which tenures for specialized professors were established as part of the process of reorganizing the faculties of arts into faculties of philosophy.

The situation in the Catholic states was considerably different. In the founding document for Jesuit college instruction, the "Ratio atque institutio studiorum Societatis Jesu" of 1599, mathematics was a marginal subject having no relevance for exams. Two things became decisive for mathematics within the Catholic–Jesuit system of education. Mathematics was no longer a part of the teaching of the faculty of arts; instead, lessons in the colleges were organized strictly according to a hierarchy of classes. Teachers were not in general professional specialists, as in the Protestant system, but more or less generalists, their teaching resembling the medieval system of courses held by ever varying lectors. The second structural pattern was that teaching of mathematics was intended only for the end of the curriculum, in the closing two-year philosophy

43 The transition to the "Collegiate Society" is detailed for Oxford in the third volume of the new university history (McConica 1986a), and for Cambridge in the first volume of the university history (D.R. Leader 1988).

course, namely as part of the physics course held in the second year of the study of philosophy. Accordingly, the subject matter of mathematics was reduced to astronomy and to some basic notions of mathematics required for that.

In France, where the Jesuits' model of instruction had been adopted also by their subsequent competitors, the Benedictines and the Oratorians, teaching mathematics at universities remained in general restricted to that part of the physics class in the final course of the *collèges* attended by only a minority of students. The absence of an academic structure, however, made for creating a number of complementary institutions. The *Collège de France* in Paris had at first two professorships for mathematics, astronomy, and physics in the seventeenth and eighteenth centuries and then five beginning in 1768/69 (cf. Collège de France 1930, 15 f.). During the seventeenth century, mathematical professorships were established at some universities by a number of endowments offered by the king, or by urban foundations, as in Angers. The audience was composed not only of the regular college students, but also of a more general public, for instance of young gentlemen preparing for a military career (Belhoste 1993). Eventually, this professional formation became institutionalized separately from the universities in military schools providing a strong component of mathematics and the sciences. This explains the substantial production of mathematical writings in France beyond the academic context.

Mathematics experienced a quite peculiar development in England. The literature always mentions the extraordinary position of mathematics in Cambridge University: it was the almost exclusive exam subject for the *undergraduate* students. Gascoigne emphasizes the often overlooked development that a basic curriculum reform had been effected in Cambridge about 1700, by accepting Newtonism (Gascoigne 1989, 7). The ruptures caused by Humanism and religious schisms are thus essential for understanding the position of mathematics at the English universities, just as they are for Germany and France. Developments in England after the period of Humanism confined mathematics to marginality for a long time in a way analogous to that in the Catholic states, but also to Calvinist territories like the Netherlands, where mathematics went into decline after the first strong impetus in favor of it during the seventeenth century.

The eminent role that mathematics eventually attained in Cambridge after the mid-eighteenth century due to the *Senate House Examination* was not really conducive to a progressive development of mathematics.[44] While the subject was indeed intended only for that minority of the students who strove for an "honors" degree, it determined the style of the entire university studies. Obviously, mathematics served as a substitute for logic, a subject prescribed since 1570 but meanwhile considered outdated. In line with this one-sided function, the study of mathematics was primarily restricted to geometry, in its Euclidean version (Gascoigne 1989, 270 ff.). This context favored neither an

44 In the nineteenth century, the name changed to *Mathematical Tripos*.

advancement of mathematics nor a transition toward generalization by means of algebraic methods.

NEW APPROACHES IN THE EIGHTEENTH CENTURY

In most European states, qualitative reforms of the educational systems occurred in the eighteenth century, predominantly during its second half. Among the aspects of these reforms were attempts at specialization within the universities; they were initiated at Northern German universities, and constituted an essential contribution to the institutional development of mathematics.

As is well known, this structural change is related to two universities that were outstanding among the large number of minor universities as new establishments, and were the first to practice a new style: Halle (1694), in the Kingdom of Prussia, and Göttingen (1734/7), in the Dukedom of Hanover. Halle, a center of Pietism and of Enlightenment in Northern Germany, had a great force of attraction for students. The utilitarian orientation of these two ideological currents produced a modernization of teaching. Göttingen had already been established in the wake of the state-promoted concept of mercantilism. To attract a large number of students, professors were obliged for the first time to publish research results in the hope that they would become known and attractive. This is the first form of the "research imperative" to come. Personalities like Christian Wolff, Johann Andreas Segner, Christian Hausen, Abraham Gotthelf Kästner, and Johann Karsten are representatives of this new approach.

For France, it was a new type of school established by the Crown that rivaled the functions of the universities, and eventually made them obsolete: the military schools. For the young noblemen who prepared for a career as a military officer, some mathematics teachers had already been attached to regiments in the seventeenth century. Professional training was first institutionalized for future artillery officers. In 1720, the Crown founded *écoles régimentaires d'artillerie* for five garrisons of artillery regiments. Each of these schools employed a mathematics teacher. Among these were also known mathematicians like Bernard Forest de Bélidor (1693–1761), Sylvestre François Lacroix (1765–1843), and Louis François Antoine Arbogast (1759–1803). Mathematics formed the central part of theoretical training and shortly afterward as well of the entrance examination the aristocratic candidates had to pass after 1755. For this exam, a new function was created, that of the *examinateur permanent*, held by members of the *Académie*. Other military schools for young noblemen, but at a lower level, i.e., for officers' careers in infantry and cavalry, requiring less mathematical knowledge, were likewise founded in this period.

For future naval officers, there were, besides the royal *chaires* in Jesuit *collèges*, naval schools proper, employing teachers for training in mathematics and engineering. The formalization of these officers' formation led to establishing private preparatory schools, in particular because of the entrance examinations with their increasing focus on mathematics. All this made for an ever stronger presence of mathematics in the general culture.

The teaching of mathematics undoubtedly attained its highest and most innovative level in this period at the *École du Génie* in Mézières, the school founded in 1748 for training military engineers, in particular in fortification technology, the military formation valued as particularly *savant*. Both the entrance and the final examinations at this school were assigned to the same *examinateurs permanents*: to a mathematician and member of the Paris academy.

The Catholic states had become a privileged region for realizing particularly profound reforms. On the one hand, the time lag to be caught up in order to satisfy the requirements of mathematics and the sciences occasioned by mercantile policy, on the other hand, the dissolution of the Jesuit Order required the Crown to take urgent substitutional action.

The most radical structural reform for mathematics was realized in Catholic Portugal. Among these comprehensive reform measures was the complete reorganization of its only university, in Coimbra, in 1772. The university was now structured into six faculties of essentially equal status. Besides the traditional faculties for theology, for civil law and for canonical law, and for medicine, two new faculties were created, one for philosophy, and one for mathematics. The philosophical faculty was actually a science faculty. The mathematics faculty, like the philosophy faculty, was for the first time entitled to organize its own courses, with graduation rights of its own; in particular, teaching posts were reserved for its graduates. With the establishment of this first mathematical faculty, a specialized scientific education in mathematics was institutionalized for the first time.

2.7. First Foundational Reflections on Generalization

As shown in Section 2.5., Arnauld's and Prestet's textbooks form the starting point for the development of the modern textbook. Their work is a landmark in the process of generalizing mathematics, since it permitted mathematical productions to reach a larger audience. At the same time, however, they oritented the process of generalization in an even more fundamental sense. Their writings heralded the subsequent triumph of the analytic method. With their contributions to the familiar *querelle des anciens et des modernes*, the authors were pioneers in claiming priority for the modern age in mathematics as well.

In mathematics, the debate between tradition and modernity until the nineteenth century was to a large extent a conflict about whether the synthetic or the analytic method should prevail. This pair of opposites had been traditionally known from Pappus's (around 300 CE) *Collectio Mathematica*[45] and was treated as complementarity between the method of "composition" and the method of "resolution" for solving problems without attributing particular priority to one of the two poles. The unproblematic opposition turned, however, into a debated one due to Ramus's sharp critique of Euclid's Elements, and of their underlying synthetic method.

Petrus Ramus may be considered one of the most eminent humanists. A representative of modernization, he was a sharp critic of Aristotelianism and in particular of the Jesuits. Similarly to Melanchthon in Germany, he advocated for an enhanced role for mathematics in schools and at universities. Stinging rhetoric repeatedly made him a victim of persecution who had to flee the country. His conversion to Calvinism in 1562 was logical, but led to new persecution. In 1572, he was murdered in the Saint Bartholomew's Day Massacre.[46]

Ramus, who had himself endowed the *Collège royal* with a chair for mathematics, was intent on anchoring mathematics in society. His critique of Euclid was thus motivated didactically. His own volume *Scholarum Mathematicarum* (1569), the first methodological reflection on mathematics in print, examined why mathematics was held in so little esteem by scholars, and by the public at large. Mathematics, he said, was basically accused of two things (reproaches which were to be voiced in later epochs as well), which constituted the *pestifera duplex opinio*: of *inutilitas* and of *obscuritas* of mathematics (Ramus 1569 [quotations after the edition of 1599], 39). Ramus discussed the two charges in two extensive chapters. Convinced that the utility of mathematics would be admitted by many, he considered *obscuritas* the more serious problem by far. Even those who recognized mathematics' utility, he said, were still persuaded that it was incomprehensible (ibid., 39 and 72). As the cause of this general view, Ramus identified the classical text for mathematics instruction, precisely the [then] 15 books of a text surpassing in obscurity anything else ever written by a human hand: Euclid's *Elements* (ibid., 72). Euclid's work, he said, had nevertheless been estimated to be above criticism, and hallowed all over the world, for almost 2000 years (ibid., 74). Remarkably, Ramus interpreted the familiar anecdote where one of the Ptolemies asked Euclid for a simpler approach to mathematics, getting the answer that there was no "royal road" to mathematics, as proving the unintelligibility of Euclid's *Elements*. That there was no royal road was the guideline for Ramus's critique of the *Elements*.

In contrast to later critiques of Euclid, Ramus did not criticize individual mathematical or methodological problems. He did not hold any mathematical errors against Euclid for the simple reason that the latter, as Ramus repeatedly confirmed, was not known for any mathematical discoveries, but only for giving

45 In the preface to Book VII (Pappus/Jones 1986, 82–83).

46 An extensive literature analyzing Ramus's works has been published. With regard to mathematics one should mention M. Cantor 1857, Hooykaas 1958, Verdonk 1966.

proofs and for generally presenting things anew (e.g., ibid., 82). Rather, the reason for Euclid's overall *obscuritas* arose from the *Elements*' matter and form. The first part of Ramus's critique is a methodological inquiry into the subject matter. Thus, he said, some parts were absent that should necessarily be present in any Elements. In particular, Ramus criticized that there was no introduction of the four basic operations for integers, and for fractions. How was some learner, yet ignorant of addition, subtraction, multiplication, and division, to learn mathematics from Euclid's *Elements* without these parts (ibid., 82–83)?

The next aspect of *obscuritas* in Euclid concerned the relationship between logic and grammar to mathematics, which for Ramus was still central. In this relationship, Ramus held "redundantia" to be a major fault: firstly, by the introduction of superfluous terminological differentiations, secondly by unnecessary repetitions (ibid., 84 ff.).

Ramus saw the main cause for *obscuritas*, however, in a methodological and epistemological fault of Euclid's textbook that Ramus was the first to discuss, i.e., the *Elements*' very structure, which went against the most elementary rules of a methodological architecture. For the ideal he strove for, Ramus coined the keyword of "architectura methodica" (ibid ., 94).

The central issue of Ramus's critique was that Euclid's *Elements*—notwithstanding the high esteem in which the work was held, was no model at all for methodological order and logical consistency. For our purposes, it is important to emphasize that Ramus's major criticism stressed that Euclid gave no thought at all to the process of generalizing mathematical knowledge. Ramus understood arithmetic to be the more fundamental ("prius"), more general and simpler discipline of mathematics, whereas he saw geometry "by its nature" as a particular discipline based on arithmetic, while Euclid, by contrast, had begun his *Elements* with geometry (ibid., 97). Even in geometry, Euclid had not adhered to the basic rule of developing the general before developing the particular (ibid., 98–99).

Ramus's critique of method was continued and even intensified by Arnauld and Prestet. It is typical for these two authors, who were to become so eminent for the further development of mathematics in France, to have been the first to take up this issue as well. And it is just as typical for these two advocates of rationalism to have stressed different aspects again. More precisely, it was Prestet again who was more radical in this field. While he had to a great extent already opted for an autonomy of theoretical concepts in mathematics in his textbook, he was also the first in the field of methodology to go beyond criticizing the "Ancients" (i.e., Greek mathematicians) by energetically proclaiming the superiority of the "moderns" and of their analytic method. For all his biting critique of Euclid, Arnauld seems to have been more moderate in generally criticizing "les géomètres" (quoting Euclid merely as a case in point) while refraining from making a distinction between "ancients" and "moderns."

Arnauld's position shall be presented first. He demonstrated his own concept of method in detail in the fourth section of the famous *Logique* of Port-Royal.

He authored this volume jointly with Pierre Nicole, the fourth section being ascribed to Arnauld alone (Brekle et al. 1993, 513). This is where Arnauld introduced the significant innovation of distinguishing between method of research (*méthode de résolution*) and method of instruction or presentation (*méthode de doctrine*) instead of merely reflecting on a single method, as had been traditional, in particular in Descartes and Pascal:

> Thus there are two sorts of methods: The one to discover the truth, which is called *analysis*, or the method of *resolution*, and which can also be called the *method of invention*: and the other to make understood to others what one has found, which is called *synthesis*, or the *method of composition*, and which can also be called the *method of doctrine* (Arnauld, Nicole 1662, 368).

For this method of presentation, Arnauld established specific rules as well, as a part of his eight principles for “la méthode des sciences.” According to the seventh rule, the method of presentation has to follow the natural, true order of things—and that means to start from the more general and the simple and to progress to the special (ibid., 408).

Arnauld’s critique of Euclid’s *Elements* is based on these methodological rules. Not only did he criticize partial aspects of Euclid’s textbook by saying that the frequent use of proofs “par l’impossible,” which, while persuading the mind, did nothing to elucidate, or saying that propositions were proved that did not need any proof, or that proofs were too far-fetched. What he criticized above all was the pervasive major fault of Euclid’s neglecting the “genuine order of nature.” The geometers’ most important error, Arnauld said, was to confuse everything instead of adhering to natural order:

> It is herein that lies the greatest fault of the Geometers. They are of the opinion that almost no order would be preserved unless first propositions were able to be used to prove subsequent ones. And thus, without taking the trouble to employ the rules of the true method, which is always to start with things that are the most simple and the most general, in order to pass on later to more compound objects and to particular cases, they muddle up everything, and treat pell-mell lines and surfaces, and triangles and squares: proving properties of simple lines from figures, and they make an infinite number of other inversions which disfigure this beautiful science (ibid., 402)..

Arnauld claimed that this fault permeated Euclid’s *Elements*; Euclid had begun by treating extension in the first four books, then generally switched to proportions for all types of magnitudes in the fifth—only to return, in his sixth book, to extension, treating numbers from the seventh to the ninth books, speaking of extension again in the tenth. To quote all the examples of this chaos (*désordre)*, one would have to transcribe Euclid in its entirety (ibid., pp.402–403). Arnauld made this critique the basis of his own *new* elements, as he emphasized in the preface:

> [...] because the Elements of Euclid were so muddled and confused, that far from being able to provide the mind with the idea and sense of true order, they could only on the contrary accustom the mind to disorder and confusion (Arnauld 1667, X).

And Arnauld stressed that his own new arrangement not only facilitated understanding geometry by finding principles more fruitful and proofs better than those ordinarily used, but at the same time contained proofs self-evident from the principles established, and comprising a large number of new propositions (ibid., xii).

On the basis of this reflection on methodological procedure and structure, Prestet stridently advocated four new aspects:

- priority for the *generality of method;*
- greatest generality attainable only by *algebra* (which he called *analysis*);
- algebra underlying all the other mathematical disciplines;
- *analysis*, as developed by the moderns, thus being infinitely superior to the "geometry of the ancients" (Prestet 1789, v. I, [8]).

Already in the first edition of his textbook of 1675, Prestet had claimed priority of arithmetic and algebra over geometry. His reasoning is not only more extensive in the second edition of 1689—26 pages of the preface now as compared to the former 9—but also more programmatical, carried along by such a rhetorical burst of reformatory optimism that one is led to conclude that Prestet was sure of being backed by an entire group with conceptions of their own. This group may be assumed to have been that which centered around Malebranche, a group that had a key role in modernizing mathematics in France (cf. Robinet 1961 and 1967).

Prestet's leitmotiv was to develop the learner's intellectual abilities such as to enable him to invent new knowledge on his own: "inventer par luy-même." The necessary basis for this, he said, was to establish the general method, and to avoid getting mired in the thousand volumes of possible particular discoveries:

> The general method is what one ought principally to establish, without uselessly taking pains over all the truths one could discover. For in the end, if all the particular discoveries that could be made were to continue to be piled up in a thousand volumes, one would never finish and no lives of men would suffice to read them, since an infinity of them could be found. (Prestet 1689, I, [8]).

As faults of the "geometry of the ancients," he listed their lack of method, their accumulation of detail, and their attachment to sensual intuition. In their disregard of method, the "ancients" had even occasionally placed things upside down, since they sought less the natural sequence of things than how to use some of them to prove others, up to deducing the simplest things from the most complicated. Prestet proudly pointed out that the improved knowledge of the moderns had overcome the almost superstitious admiration of the works of antiquity:

> Besides the fact that their rather difficult and very limited method is not to the taste of a century where one is better taught […] no longer affecting, as formerly, an almost superstitious admiration to all their works (ibid.).

Prestet's starting point is the "analyse des modernes" (ibid., [5]). For him, all truths are relationships, "rapports." The only science admitting "verités exactes," hence the perfect science, was mathematics. Only in this science were all *rapports* exactly determinable (ibid., [11–12]).[47]

As all relations could be expressed by *numbers*, he said, the discipline teaching the necessary operations on numbers constituted the basis of all the other sciences. In particular, it is the condition of all application to *quantities*:

> All exactly known relations being therefore expressible by numbers, it is evident that numbers contain all magnitudes in an intelligble way, and thus the science which teaches how to make all the necessary comparisons in numbers, so as to know the relations, is a general science or the principle of all the exact sciences. For it is only necessary to apply to types of magnitudes what one has discovered in general for numbers, to have a knowledge of almost all particular sciences (ibid., [12]).

Arithmetic thus presented itself as a universal science, basic for numerous other sciences: "l'Arithmétique ou la science des nombres [est] une science universelle dont tant d'autres dépendent" (ibid.). There was another science, however, that was even more general and of greater extent, the *analyse* as developed by the moderns:

> This science which I call *Analysis*, and which is normally called *Algebra*, serves marvellously to enlighten, extend and perfect Arithmetic itself and Geometry, and all other parts which mathematics contains (ibid., [13]).

Due to its general character, this *analysis* at the same time already represents, for Prestet, the general method sought. It permitted the ease that allows one to discover concealed truths as well. And due to its short and simple expressions, it was an excellent guide for investigations (ibid).

In a mode without precedent, full of disrespect and reformatory zeal, he argued that the prevalence traditionally claimed for geometry was quite unsubstantiated, and that geometry was unambiguously second to *analysis*. *Analysis* went far beyond what could be inferred from viewing or from simple "imagination"; it was based on methodologically guided operations of the *esprit* (ibid., [11]). Its proofs were more general and simpler than the geometrical ones, and hence the more natural as well (ibid., [14]).

While *analysis* as a fundamental science need not borrow from other areas, geometry was *imparfait*; it could not do without the means and results of arithmetic and *analysis*. It is understandable from that why Prestet was able to call his textbook *Élemens des Mathématiques*, although it treated only arithmetic and *analysis*:

> It is evident that these Elements comprise the general science or the principle and the foundation of all of mathematics, and not geometry, which depends in several places on the knowledge of these Elements and which would undoubtedly be highly imperfect and very limited if it did not borrow support from the sciences we explained (ibid., [15]).

To give an example, Prestet referred to proportions: without them, almost nothing could be discovered in geometry. And to demonstrate the nature and the

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47 The preface is not paginated, the numbers in square brackets give the corresponding numbers.

properties of these proportions, the geometers were obliged to take recourse to the multiples, equimultiples, and aliquota of magnitudes that one compared and could not determine without numbers. Without numbers, Prestet added, the geometers were unable to compare their lines and figures, the commensurables or incommensurables, except very imperfectly.

Arithmetic and analysis were by contrast entirely different. They could be extended to the infinite without any support by lines, figures, or anything. They were above all the other sciences, which had to rely on them for treatment by method, or for perfection. Prestet proclaimed that *analyse* was infinitely more fertile for discovering truths than geometry, and without its support an infinite number of geometrical problems were unsolvable:

> Analysis is infinitely more productive than figures for discovering truth, and it is quite impossible, if one does not engage its help, to solve an infinity of problems in Geometry (ibid., [15–16]).

The methodical and methodological reflection beginning with Ramus led to a first apogee with Arnauld and Prestet, who were the first to formulate the program of generalizing mathematical knowledge as a challenge postulating the preeminence of the analytic method. This is where the subsequent revolution of conceptual rigor was essentially prepared. The object of foundational reflection had to be the very mathematical fields that had already been sufficiently developed by contemporaries. While Ramus's reflections had still been essentially confined to elementary geometry, Arnauld and Prestet were able to include recent progress in algebra.[48]

2.8. Extension of the Concept Field to 1730/40

2.8.1. France

The conceptual clarifications achieved by Arnauld and Prestet and their presentation of negative numbers gradually spread in France. This dissemination was essentially effected by the work of professors belonging to the order of the Oratorians.[49] The Oratorians, strictly opposed to the Jesuits, can be generally considered to have been the promoters of the processes of algebraization and

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48 Robinet's critique that Prestet did not include the new differential and integral calculus (Robinet 1960, 98 f., 104; Robinet 1961, 232 ff.), is anachronistic: the first research publication of this new calculus had occurred only in 1684, by Leibniz. Prestet died in 1691, while the first textbook appeared in 1696. Prestet's achievements concerning the proof of the uniqueness of the prime factor decomposition is examined by C. Goldstein (Goldstein 1992).

49 The role of the Oratorians for French educational history is studied by Lallemand 1888, and their contributions to mathematics by Belhoste 1993. The case of Angers, one of the few universities held by the Oratorians, is studied by Maillard (1975).

generalization within mathematics in France. It was in particular the group of the *Malebranchistes* at the end of the seventeenth and at the beginning of the eighteenth century, Malebranche's adherents who exercised a decisive influence on the Paris *Académie des Sciences*, and together with it on the development of science in France as well (cf. Robinet 1967).

Two members of the Oratorians, in particular, have the merit of having extended Prestet's pioneering program and secured broader acceptance for it: Bernard Lamy (1640–1715) and Charles-René Reyneau (1656–1728). Their highly successful textbooks were used until the second half of the eighteenth century.

Bernard Lamy, a priest of the order of the Oratorians and one of Malebranche's friends, was not only a disputatious theologian, but also an author successful both with grammar and mathematics textbooks. Between 1661 and 1668, he taught poetics and rhetoric. After 1673, he was active at the University of Angers as a philosophy professor. He was soon suspended, however, by intervention of the Jesuits and the Crown for his Cartesian views. Since he was also accused of antimonarchist teaching, he was no longer permitted to teach at a *collège* or at universities, but taught only intermittently in clerical seminaries (cf. Brekle 1993, 808 f.). This did not affect his productiveness as a textbook author. In mathematics, he published two textbooks, which, while applying Arnauld's and Prestet's concepts, were clearly more successful than the two.

The first of these volumes concerned algebra, which was understood as the basis of all mathematics: *Traité de la grandeur en général qui comprend l'arithmétique, l'algèbre, l'analyse et les principes de toutes les sciences qui ont la grandeur pour objet* (1680). The second was devoted to geometry: *Les Elémens de géométrie ou de la mésure du corps* (1685). The success of both textbooks is evident not only from the fact that each of them saw four editions during the author's lifetime (with changes and extensions, and in titles as well), but from the fact that there were further, even revised, posthumous editions for another half century.[50] The two textbooks could be also used independently of each other, because Lamy, in the later editions of the geometry book, had inserted a chapter on arithmetic covering the four basic arithmetical operations and their application to geometrical quantities—in particular to prepare the theory of proportions.

Lamy's textbooks are revealing documents for the process of reflecting the foundations of mathematics and its operations, and for the intention to create the ability to generalize. The enormous impact of Descartes's work on this process

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50 To mention only the major editions: for algebra: 1680^{1}, 1689^{2}, 1704^{3}, 1706^{4}, 1731^{5}, 1738^{6},1741^{7},1741^{8}, 1765^{9}, and for geometry: 1685^{1}, 1692^{2}, 1695^{3},1701^{4}, 1731^{5}, 1734^{6}, 1758^{7}. Moreover, there were numerous parallel editions, by other printers, in particular in Amsterdam (s. Brekle 1993, 806 f.). A study of Lamy's geometric concepts is Barbin 1994.

of generalization is very evident. Each of the two textbooks has a voluminous closing chapter *De la Méthode*. The textbooks' sections are for the most constructed as a sequence of rules; noteworthy is the respective major section *Les Principales Regles de la Méthode* which appear, in line with Pascal and Arnauld, as eight rules. These chapters show less fundamental intention than Arnauld's and Prestet's reflections on method; rather, they mark the transition to reflections pertaining more to intramathematical issues. Not only are the methodological parts more concretized on arithmetical/algebraic and on geometrical subjects and rules; at the same time, Lamy had repeatedly revised the respective parts on method in the different editions, substituting, in particular, the basic tenor of the discussion on *analyse* and *synthèse* as general methods present in the first editions by reasoning concerning more concrete mathematical subject matter.

Lamy shared Arnauld's and in particular Prestet's critique of the *anciens*: their works, he said, not only lacked *netteté* and *clarté*, they were also too long and too complicated—and above all they were deficient in methodological order (Lamy 1692, *Préface*, [12]). This is why he had tried to transform their demonstrations into "general" ones to make them prove several truths at once (ibid., [14 f.]). Since he was treating the *grandeur en général* in his algebra textbook, this book provided the foundation for mathematics as a whole and hence the true "Elements" of mathematics. Euclid had considered, Lamy said, only one "particular species" of quantities, i.e., the geometrical. This was didactically particularly dangerous, because such a textbook supported those who were forever in need of pictures and figures for their demonstrations. Imaging, however, was always a considerable cause for errors. His own textbook, against that, did not require "de se representer des corps," i.e. ,no figurative images (ibid., [16 f.]),

In his presentation of negative numbers, Lamy continued the previous stage of operative understanding; in particular, developments in the sense of opposite quantities can be noted; on the other hand, Lamy's concept was at first still characterized by an epistemological reserve similar to that of Descartes and Arnauld, which no longer appeared, however, in later editions.

In algebra, which Lamy understood as *Arithmétique plus parfaite* (ibid., 57), he introduced subtraction without restrictions concerning the relative sizes of subtrahend and minuend (ibid., 61 f.). This was intentional, for Lamy was at ease operating with negative results. He gave no explicit definition of positive and negative results; rather, he used a concept of opposite quantities mutually canceling one another. Thus he posed the problem, "when the magnitudes which must be added are the same and have opposite signs," hence, e.g., $2d$ and $-2d$ give 0, which he explained at length (ibid., 66).

Lamy justified the rule of signs in the section on multiplication of "grandeurs complexes ou composées," but unlike Arnauld, who did this by means of the conceptual differentiation between multiplier and multiplicand, in the traditional way according to which the excess subtracted must be restored (ibid., 72). Lamy had applied this rule already, when subtracting complex quantities like $c+f$ and $b - d$. One did not want to subtract from $c+d$ the entire b, thus $c+f - b$, but somewhat less. One thus had to change the algebraic sign of d from – into +, so as to perform the operation $c+f - b+d$ (ibid., 68).

Nevertheless, Lamy still expressed epistemological reserve concerning this rule of signs, which shows that Descartes's and Arnauld's arguments were effective around 1700, a situation evident from Lamy's closing remark:

> It is not necesssary to search for any mystery here: it is not that *minus* is able to produce a *plus* as the rule appears to say, but that it is natural that, when too much has been taken away, one puts back the too much that has been taken away (ibid., 72 f.).

What is salient here is the contradictory situation of an operative practice of handling negative numbers coupled with an epistemological conviction of a strict separation between the "minus" area and the "plus" area, no transition from one to the other being possible. It is thus symptomatic that Lamy gave no definition of negative or opposite quantities, and restricted himself to introducing them only in their operative execution.

In the sixth edition of his geometry textbook (1734), he omitted the epistemological caution, in the part containing the concise version of arithmetic and algebra for the application in geometry. After establishing the rule of signs, he said briefly, without restricting conditions:

> Minus times minus therefore makes plus, that is, that at the end of the product there is the sign + , because having taken away too much, the too much that was taken away has to be put back (Lamy 1734, 132).

Evidently, the concept of negative numbers propagated by Prestet had meanwhile met with broader acceptance, the epistemological cautions and contradictions becoming marginal.

Lamy's textbook on *Grandeur en général* contains no account of how to treat equations of the second or higher degree. This is why Lamy's operative use of his concept of negative numbers cannot be investigated in this context.

While Lamy's textbooks can be considered to consolidate the already established practice of operating with negative numbers, Reyneau was able to advance the theoretical understanding of the concept of number significantly. Reyneau continued Prestet's work in several regards. Not only was he Prestet's immediate successor as mathematics professor at the University of Angers (keeping this chair from 1683 to 1705), but he also handled Prestet's scientific *Nachlass*. Moreover, Reyneau, an Oratorian like Prestet and Lamy, was one of Malebranche's close cooperators and friends. After giving up his Angers chair in 1705 because of deafness, he went to Paris, where he was elected an *associé libre* of the *Académie* in 1716.

During his scholarly leisure in Paris, Reyneau made his Angers lectures accessible to a basically universitarian audience in two influential textbooks:

- *Analyse demontrée, ou la Méthode de résoudre les problêmes mathématiques, et d'apprendre facilement ces sciences* (vol. 1: 1708, 2nd edition 1736; volu. 2: 1708, 2nd edition 1738); a textbook on algebra and on differential and integral calculus,

- *La Science du Calcul des Grandeurs en Général, ou les Élémens des Mathématiques* (vol. 1: 1714, 2nd edition 1739; vol. 2 posthumously 1736).

The *Science du Calcul* is a textbook on arithmetic and elementary algebra completely relying on Prestet's concept, as evidenced by its very title. In the preface, Reyneau stresses indeed that arithmetic and algebra form the "general science" of mathematics, and that one learning mathematics must begin with them (Reyneau 1714, xviij).

In his presentation of negative numbers, Reyneau closely adhered to Prestet as well, developing the concept of opposite quantities considerably further in several respects. For Reyneau, positive and negative numbers are not absolutely established as to their meaning, but are rather relative quantities that mutually refer to each other. The positive was no longer privileged from the outset. He motivated positive and negative quantities not only by applications to assets and debts, but illustrated them also by straight line segments of opposite direction ("sens") in geometry. In a figure (Figure 2), Reyneau not only showed horizontal segments as examples of such opposite quantities, but also vertical ones:

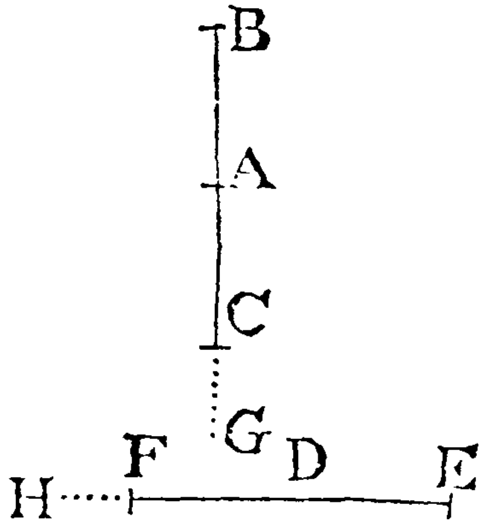

Figure 2, relative position of opposite segments (Reyneau 1714, 14)

> Magnitudes can be distinguished as *positive* and *negative*. In commerce, for example, the fortune of a Merchant is a positive magnitude; his debts are negative magnitudes. With lines, and all magnitudes that can be represented by lines, in order to distinguish between the way a line like *CAB* should be understood, in going from bottom to top, and the way of understanding the same line *BAC* taken in the opposite direction, returning from top to bottom, the one taken in the one direction is called

> *positive* and the one taken in the opposite direction is called *negative*. Thus if we suppose the line *CAB*, going from *C* to *B* to be understood as positive, it will be *negative* taken in the opposite direction descending from *B* to *C*. In the same way, if *FDE* is taken as *positive*, in going from left to right, it will be *negative*, in going from right to left from *E* to *F* (Reyneau 1714, 14).

Reyneau has explicated Prestet's concept of opposite quantities further by a detailed discussion of the internal connection between positive and negative quantities, understood as "retranchements mutuels," as a mutual taking away, where quantities equal in size but of opposite directions canceled another out:

> From which it can be seen that these two sorts of magnitudes, positive and negative, are mutual subtractions of each other: for example, the positive magnitude *CAB*, going from *C* to *B*, being drawn, if the negative smaller [length] *BA*, going back from *B* toward *C*, is superposed, then *BA* is taken away from the positive quantity *BAC*, and all that is left of the positive [length] is *CA*; and if also the negative *AC*, which joined to *BA* is equal to the positive *CB*, is added, it will take away completely the positive *CA,* and zero will be left. If a negative magnitude *BG*, greater than *CB*, is superposed onto the positive *CB*, then the negative *CG* will be left over (ibid.).

With regard to the epistemological objections against connecting the positive area with the negative area, Reyneau's way of taking up and continuing Prestet's de-ontologizing zero must be considered a pioneering achievement. Instead of talking of a metaphysical nothing, Prestet had introduced the zero as an intermediate "term" between the positive and the negative quantities; Reyneau went beyond that in emphasizing that it was by convention that one or the other was called positive or negative:

> It is evident that zero, or nothing, is the term between the positive and negative magnitudes that separates them one from the other. The positives are magnitudes added to zero; the negatives are, as it were, below zero or nothing; or to put it a better way, zero or nothing lies between the positive and negative magnitudes; and it is as the term between the positive and negative magnitudes, where they both begin. For example, with the lines, the point *C* on which lie the negatives *CG*, is the term which separates them, where they begin, and from which they depart toward opposite regions. We call this term the *origin* of the positive and negative magnitudes; and at this term there are neither positive nor negative magnitudes; thus there is zero or nothing. In the same way, *F* is the origin of the positive magnitudes *FD*, *FE* which go to the right, and the negative magnitudes like *FH* which go to the left, and at the point *F* there are neither positive nor negative magnitudes; hence there is zero. It should be noted that, from the opposite directions of the positive and negative magnitudes, it is arbitrary in which way the positive is chosen, the negatives being taken in the other direction; but when one of these two directions has been chosen in a Problem, this should be maintained throughout the Problem (Reyneau 1714, 15).

Finally, it was a quite conceptual clarification as compared to his predecessors that Reyneau was to my knowledge the first to note the two possible functions of the signs + and – , and to reflect on their relation. While he introduced the signs + and – as algebraic signs of numbers and of quantities, however only after introducing the positive and negative quantities (ibid., 16), he explained afterwards, in his next section, that these "also" served as signs of operation, to indicate the operations of addition and subtraction. Reyneau then explained how to proceed if two signs concurred in their different functions of sign of operation and of algebraic sign (ibid.). For this purpose, he explained the conception of opposite quantities still further: indeed, the "–" as sign of operation actually required transition to the opposite quantity:

> From which it can be seen that the sign – before a magnitude simply indicates an opposite. If that magnitude, in front of which is the sign –, is positive or negative, the sign – indicates that one takes the opposite magnitude. Thus $- +a = -a$ and $- -a = +a$ (ibid., 17).

This last determination, however, implied a contradictory definition of negative numbers. While Reyneau had at first given a purely relational definition of positive and negative numbers, he now introduced—in order to explain the signs + and – as algebraic—a requirement already implied in Prestet's conception that presupposed using absolute numbers: One had to put the sign "+" before positive *quantities* and the sign "–" before negative *quantities*. He even added that one had to put the sign "–" before negative quantities in any case (ibid., 16).

Provided that this requirement did not resort presupposing the exclusive existence of absolute numbers, as in traditional mathematics, Reyneau's second definition for negative numbers signified a lack of reflection on the new generality of calculating with letters, as opposed to calculating with numbers. For in the examples that followed, Reyneau used only numbers to explain his definition, e.g., –2 as an example for the algebraic sign of a negative quantity, while in fact general quantities like a, b, can be equally positive or negative, depending of their respective admissible range of values.

Reyneau was virtually the only Frenchman after Arnauld to discuss problems of the concept of multiplication, again prompted by the rule of signs. To prove this rule, he deviated from his previous presentation in introducing the Cartesian concept of proportion for the product, explaining this with the unproblematic example of two positive quantities. He did not attain any conceptual clarification, however, since he relied only on Prestet's conception to derive the extended multiplication from subtraction, although he used Arnauld's differentiation between multiplier and multiplicand. Thus he tried to solve the really difficult problem of establishing a relation between the (positive) unity and a negative multiplier by simply positing that this negative number originated from the unity by subtractions:

> The positive unit +1 can appear, as it were, through subtraction in the multiplier, or rather it may be taken away when the multiplier $-a$ is negative. Therefore if the multiplicand $-b$ is also negative, it must be taken away from the product as many times as the positive unit +1 is taken away from the multiplier $-a$ (ibid., 70).

Reyneau extended operating with negative numbers to exponents as well, forming series with ordered negative exponents (ibid., 128 ff., 135). In the section "Extracting roots from literal numbers," he showed that a^2 has two roots:

$+a$ and $-a$, and that this multiplicity could be expressed by the sign $\pm a$ (ibid., 205).

Reyneau did not treat solving equations in his work *La Science du Calcul*, but did so in the parallel textbook *Analyse demontrée*. This is where he confirmed that he understood quantities at first as absolute, and thus as having positive values: "[…] thus in $x = a$, the root a is positive; but when the value of the unknown is negative, as in $x = -b$, we say the root is negative" (Reyneau 1736, 57).

The volume confirms that Reyneau indeed always understood coefficients as permitting only positive values. When treating equations of second degree, he hence distinguished positive and negative coefficients by different combinations of their algebraic signs; he devised thus six different equation types of second degree: four mixed types (where he accepted negative roots, also admitting the type $x^2 + px + q = 0$), as well as two pure types: $x^2 - p = 0$ and $x^2 + p = 0$ (Reyneau 1736, 58). For equations of third degree, he pioneered a simplification in combining (in his own sense) positive and negative coefficients with a ± sign into one equation. In spite of this he still obtained four types of equation, since he conceived of cases in which the terms of first and second degree disappear as of separate types (ibid., 59). Reyneau had no problems with operating on imaginary and complex numbers (cf. ibid., 197).

Thanks to a document from Reyneau's *Nachlass* published in Johann Bernoulli's correspondence, we have the fortune rare for this early period of being able to get a glimpse of Reyneau's "workshop." We can thus not only observe his learning process in a relevant field of problems, but also get an insight into the complexity of this field concerning negative numbers, which shows that the new developments of analysis substantially contributed to deepening the understanding of negative numbers. The notes taken by Reyneau document his newly acquired mathematical knowledge during a sojourn in Paris in 1700: "Mémoire de ce que j'ay appris dans mon voyage de Paris en juillet–aoust de 1700." The published note relates the new things in mathematics Reyneau had learned from Pierre Varignon, a leading member of the Malebranche group. It begins with the problem how to invalidate the proportions argument against negative quantities, which must have been relevant still for the group. The "solution" consisted in declaring the minus signs in the proportion to be inessential for the proportion itself, and in separating its qualitative meaning—like indicating an opposite direction—from the concept of ratio:

> On 14 July 1700, I learnt from Mr. Varignon that the ratios of positive and negative magnitudes of the same type are equal to the ratios of the same magnitudes all being taken as positive, the plus and the minus being only signs for calculating the magnitudes, that is to say, for adding or subtracting, and that the magnitudes, supposing they are lines, are on different sides with respect to the point they start from, that is, with respect to the origin. He proves it by the proposition +2:–4::–4:+8, which, according to all mathematicians, is true, the product of the extremes being equal to that of the means. Therefore if $\frac{+2}{-4}$ is that by which +2 exceeds –4, it must

> be that in the equal ratio $\frac{-4}{+8}$, –4 also exceeds +8, which cannot be the case. Thus these ratios are the same as those between positive magnitudes, thus $\frac{+a}{-2} = \frac{-a}{+2}$, always half of a. (Joh. Bernoulli 1988, 349).[51]

In his textbook of 1708, Reyneau did not use this argumentation, but rather the more general conclusion that negative solutions indicate a value on the opposite side. This is what Reyneau called an inference not only for the common *calcul,* but also for differential and integral calculus. Indeed, the additional value of this "workshop report" consists in the fact that Reyneau testified how differential calculus and integral calculus had become the new area for applying the concept field of negative numbers and how results of this new *calcul* called for a clarification of hitherto controversial questions about understanding negative quantities. The note continues namely thus:

> Thus when working, either ordinarily or by the integral and differential calculus, a negative solution is found, it only indicates that the magnitude that gives the solution is on the other side of the origin, opposite to the one that was taken to indicate positive magnitudes.
>
> Thus in the quadrature of hyperbolas, or the area between the hyperbola and one of its asymptotes, if one finds that the sum of the parallel lines that fill that region is negative, this does not mean that the area is infinite, but only that it lies on the other side of the origin, and that it is the quadrature of the region that lies between the other asymptote and the hyperbola. For there are hyperbolas of certain degrees such that the hyperbola approaches one of its asymptotes more closely than it does the other, and that this area is less, with respect to one of the asymptotes, and greater with respect to the other (ibid.).

This interaction between negative numbers and the first results of integral calculus thus boosted the improvement of conceptions of analytic geometry and the representation of curves in the various quadrants.

Reyneau, with his comprehensive operative understanding of negative numbers, was indeed the first to develop a virtually complete analytic geometry in his second textbook. In his preface to the second volume of the *Analyse demontrée*, he had already emphasized "the perfect correspondence of the *analyse* with geometry and even with nature itself," demonstrating how the "lines and figures of geometry are better represented by letters of the alphabet" and how relations between lines and figures are transformable into a *calcul* by means of these letters (Reyneau 1738, iv f.).

This new understanding was based on conceiving the plane as describable by a coordinate system—after having chosen a straight line in the plane as the starting segment for the abscissae *(coupées*), and of its beginning as the *origine* of the coordinate system. The infinite set of parallel lines intersecting the basic line segment, the *ordonnées,* together with the *coupées*, form the coordinates of

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51 Such a general acceptance of this proportion had not taken place, as already shown and as will be discussed even more extensively!

the curve to be examined; at the same time, Reyneau adapted the new concept of variable in analysis for analytic geometry, calling it *changeante* (ibid., vj).

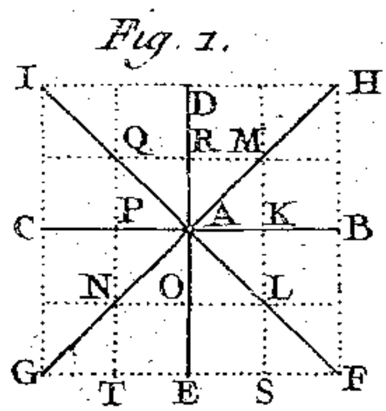

Figure 3, the four quadrants (Reyneau 1736, Planche I)

In a separate section “Sur l’usage des signes + et – par raport à la Geometrie,” Reyneau was probably the first to explicitly introduce the four quadrants in the coordinate system of the plane, a novelty that appears self-evident to us today. Using Figure 1 (our Figure 3), he explained, if the straight lines *DAE* and *CAB* intersect at right angles in point *A*, and if one is held to distinguish between parallels to *AB* going right or going left by the problem given, as well as to distinguish between parallels to *DAE* “descending” or “rising,” one should call those going right positive, giving them the sign of plus, and call those going left negative, giving them the sign of minus. In contrast to modern convention, he called the descending lines positive, and the rising lines negative (ibid., 9). By means of this, he introduced four quadrants, always with referring to his figure 1. Due to the difference of convention, what is now our fourth quadrant formed the first quadrant for him:

> Calling the angle *EAB* the first, *DAB* the second, *CAE* the third, and *DAC* the fourth, the lines of the first will all be positive; of the lines of the second, those which go to the right are positive and those which rise are negative; in the third, those which go to the left are negative and those which descend are positive; and in the fourth, both are negative (ibid., 10).

As an example of how Reyneau applied this method to representing curves in the four quadrants, we reproduce his figure 22 (our Figure 4) which serves to examine a hyperbola’s properties. The axes themselves show no designations, but it may be deduced from the designations of points on the curves that points lying symmetrically to an axis in the positive or negative area are indicated by lower and upper cases of the same letter, to show their correspondence.

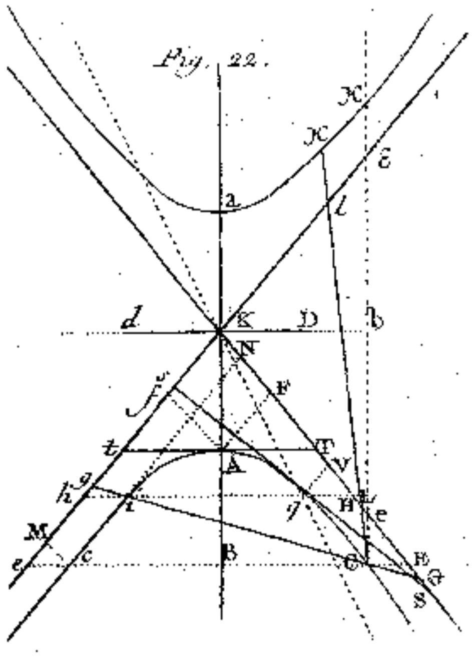

Figure 4, the hyperbola in the quadrants (Reyneau 1736, Planche II)

This introductory part for analytic geometry, however, also hinted at a new conceptual problem. In his "workshop report" of 1700, Reyneau had still interpreted negative areas to express a change of position. In his textbook, however, he now interpreted the product of a line oriented positively and one oriented negatively to be a negative area:

> The rectangle *AH* made by *–AD* and *+AB* will be negative. The rectangle *AG* made by *+AE* and *–AC* will be negative. But the rectangle *AI* made by *–AD* and *–AC* will be positive; and on the opposite side of the negative rectangle *AH* made by *–DA* and *+AB*. […] From which it can be seen that the areas which are on the opposite sides of the line which has been taken as the division between the positive and negative magnitudes are on the one hand positive, and on the other negative (ibid., 10 f.).

Reyneau did not reflect how his inferring a negative area relates to the fact that a positive result is obtained if one of the lines is substituted by its oppositely oriented counterpart. Nor did he argue conceptually or operatively with regard to the meaning of a negative area.[52]

Although Pierre Varignon (1654–1722), essentially a self-educated mathematician and physicist, belonged among the scientifically most productive members of the Malebranche group, his textbook *Élemens de Mathématique*,

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52 In a *mémoire* for the St. Petersburg Academy of 1753, Heinrich Kühn tried to conceptualize negative values of areas as a real substrate for legitimating the construction of imaginary quantities in geometry (Kühn 1753).

published posthumously in 1734, contained no in-depth reflection of the concepts. Varignon was not only active as a scholar in the Paris Academy, but was an enthusiastic teacher as well, after 1688 as a professor of mathematics at the *Collège Mazarin*, one of the *collèges* of the Paris University, and after 1694 in addition to that as professor of the *Collège Royal*. His textbook, before its main part on geometry, contained a short introduction to arithmetic and algebra. Positive and negative quantities were not introduced explicitly; subtraction in the domain of algebra was not unambiguously formulated. At first, he limited subtraction to subtracting a smaller from a larger quantity (Varignon 1734, 12). In his formulation of calculations with letters, however, this restriction was not repeated (ibid., 14). His justification of the rule of signs is remarkable, however. The logical–linguistic argument that a double negation signifies an affirmation appears here for the first time: "The negation of a negation is an affirmation" (ibid., 21).

On the other hand, Varignon did not accept zero as a number; upon introducing the place value system, he explained that the digit 0 signified nothing in itself: "ce chiffre 0 qu'on appelle zero ne signifie rien seul; mais seulement lorsqu'il est mis après les autres dont il augmente la valeur" (ibid., 8).

The next textbook was authored by Dominique-François Rivard (1697–1778). Rivard belonged to no order and was also not a priest; he had just as little connection to the Academy in Paris. Rather, he was—for almost forty years—professor of philosophy and mathematics at one of the *collèges* belonging to the corporation of the Paris University, the *Collège de Beauvais*. Rivard represents the new normality of a university mathematics lecturer; his textbook was destined for his audience. In the preface to his *Elémens des Mathématiques* (1732) he thanked the university for having introduced mathematics into the philosophy course of its *collèges* some years ago.

The textbook, in its fourth edition already in 1744, consists of a first part on arithmetic and algebra, and a second on geometry. Rivard, as the most explicit among the authors hitherto discussed, introduced here negative numbers as quantities opposite to the positives. At the same time, he introduced negative quantities as equivalent to the positives, and as equally legitimate mathematical objects, in the until then most challenging mode:

> It should be remarked that negative quantities are magnitudes opposite to positive quantities. […] With this notion of positive and negative quantities, it follows that both are equally real and that, consequently, negatives are not the negation or absence of positives; but they are certain magnitudes opposite to those which are regarded as positive (Rivard 1744, 66).

And he introduced subtraction within algebra regardless of small/large relations (ibid., 69), while it was defined for arithmetic in the restricted manner (ibid., 17). Just like Reyneau, however, he made parallel use of the definitions of positive and negative quantities by algebraic signs for quantities understood exclusively as absolute numbers: "Non-complex quantities [that is, not composed quantities] which are preceded by the sign + are positive; and those preceded by the sign – are negative" (ibid., 66).

At the level of the general calculus with letters, Rivard used this parallel definition coherently without, however, differentiating between algebraic sign and sign of operation, as Reyneau had done. In particular, he used this definition to explain that a subtraction may have the form of an addition in the domain of algebra (ibid., 69). After having explained the rule of signs by numerical examples, he explained it without any epistemological reservations for isolated negative quantities:

> Suppose it is a question of taking away a negative quantity on its own; it is still clear that the sign must be changed from minus to plus: for example, if one wishes to subtract $-c$ from a, one must write $a + c$. For, to take away a negative quantity is to add a positive one (ibid.).

Although connected with the implied concept of exclusively absolute letter quantities, these reflections became productive for future conceptual clarifications, since they made one aware of the different nature of algebraic quantities as compared to the traditional concepts of arithmetic. In fact, just these reflections were later taken up by Condillac in his *Langue des calculs* to demonstrate the new, theoretical level of the operations in algebra (cf. below, Chapter IV.1.2.).

In Rivard's exceptional methodologically well elaborated and well structured work—thus he discussed for the basic operations in each case in a separate section how to perform the test of operation—there is another noteworthy innovation. Rivard is to my knowledge the first author who explicitly stressed that a^2 arises as a square both from $+a$ and from $-a$. (Rivard 1744, 93). In his third part, about solving equations, Rivard treated negative solutions. For the general method, Rivard also pointed out that there are two roots. From $2ax - xx = b$ one obtains not only the root

$$x = a + \sqrt{a^2 - b}\text{, but also } x = a - \sqrt{a^2 - b}\,.$$

There is "également" a "racine positive" and a "racine négative" (ibid., 251 f.).

Rivard's textbook can be considered the presentation of negative numbers most vigorously elaborated among all the texts hitherto discussed in their theoretical and operational respects.

One must be aware that the textbooks analyzed here up to now are not representative, with regard to the degree of generalization they attained in forming the concepts of negative numbers, of the totality of mathematical production in France up to the first third of the eighteenth century. These textbooks, first written by "dissenters" like the Jansenist Arnauld, by the Oratorians, in particular by the Malebranche group, and finally, by the emerging group of mathematics professors at *collèges* of the universities, are more representative of a reform-minded group in France interested in promoting the sciences. Beside these, there were more authors of textbooks among amateurs, private teachers, and lecturers in other educational facilities who, while adopting the innovations, amalgamated them with traditional conceptions without more profound reflection.

Three examples of this broader circle of authors shall be mentioned here: Jacques Ozanam (1640–1717) was educated as a priest, but was predominantly active as a private tutor of mathematics—at first in Lyons, later in Paris. The author of "Dictionnaire Mathématique" (1690), he published a five-volume *Cours de mathématiques* in 1693. The first volume contained an *Abrégé d'Algèbre* (1702) developed more in detail in a separate volume *Nouveaux Elemens d'Algèbre*).

The lack of differentiation between algebraic sign and sign of operation caused Ozanam to generally understand any quantity to be a negative if provided with a minus sign—even for the subtraction of monomials: the second term to be subtracted, for him, was a negative quantity (Ozanam 1697, Vol. 1, 13). Ozanam claimed the subtraction of a larger from a smaller quantity to be absolutely impossible. He explained that only the inverse operation was performable in this case, and that one had to provide the result with the minus sign to indicate that it concerned a quantity that is smaller than nothing, hence a "grandeur fausse" (ibid.).

In contrast to his substantialist rejection of negative quantities, Ozanam justified the rule of signs by the new differentiation between multiplier and multiplicand, adhering in this to Arnauld (Ozanam 1702, 19; according to Dell'Aquila/Ferrari 1996, 330).

The textbooks by the Abbé Deidier (1696–1746) present another example. Trained in a *collège* run by Oratorians, he studied theology with the Jesuits. He became first a priest, then assumed a philosophy professorship in a seminary, teaching primarily mathematics. After that, he accepted the position of a private tutor for the children of a marquis, until he changed to the famous military school of La Fère to teach artillery (Hoefer 1855, Vol. 13, 31). He became the author of numerous mathematics textbooks for military engineers. His textbook on differential and integral calculus will be considered in Chapter III.7. His two-volume textbook *L'Arithmétique des Géométres* (1739) is relevant for this chapter.

In his first volume, Deidier presented the usual arithmetic as well as the commercial arithmetic; in his second volume he covered algebra, *analyse*—i.e., solving equations—and progressions, etc. He himself claimed to have treated everything with the greatest possible "ordre et clarté" (Deidier 1739, Avertissement). Within algebra, for "grandeurs littérales," Deidier devoted a short section to negative and positive quantities. Deidier used the traditional terminology of Descartes's time, calling positive quantities "réelles," and negatives "fausses" or the expression of a "défaut." While introducing both as opposite quantities canceling one another, "s'entredétruisent," he also designated them by the algebraic sign + for positive and by the algebraic sign – for negative quantities (ibid., 8). Deidier's comments reveal that negative quantities were strongly opposed in his time, particularly among the lay public. This opposition prompted Deidier to assert that nothing was simpler than these concepts when one applies his method of motivating them through analogy with debts, or the like:

> Many persons rise up against these negative magnitudes, as if they were objects difficult to conceive, yet there is nothing at the same time more simple nor more natural (ibid.).

He justified the rule of signs, as he said himself, by "la nature même de la multiplication," meaning Prestet's method of pretending to reduce all multiplications to iterated additions (ibid., 12 f.).

In his further algebra and *analyse,* Deidier reverted to pre-Arnauld textbook style as compared to the textbooks analyzed above. His major text was on presenting and solving particular problems, treating solving procedures for isolated cases without explicit theoretical or systematic structure. These parts do not require any negative solutions, and Deidier actually gave rules of thumb on how coefficients have to be chosen to make sure of obtaining only positive solutions (cf. ibid., 115). As coefficients could be only positive for Deidier as well, he presented four types of equations of second degree (ibid., 174). Somewhat later, he systematized the solutions, obtaining eight "general" solutions (ibid., 191).

Deidier's textbook is not only a revealing example for the possibility of different epistemological concepts coexisting in a "superposition" of various historical "layers." It serves at the same time as a first proof that developments of concepts and style need not occur in a "one-directional" and cumulative way, but that ruptures and reversions to forms prevailing at an earlier time may very well occur.

A third example is Bernard Forest de Bélidor (1697–1761), a double-career engineer who lectured at the famous military school *École de la Fère* and attained the rank of field marshal. In 1756, he was elected to the Paris Academy. Bélidor authored numerous textbooks for civil and military engineers, among them of his *Cours de Mathématiques* of 1725, which extensively treated the application of mathematics to the most diverse fields of practice in the manner of an encyclopaedia containing a "crash course" on the algebraic operations in the introductory part on geometry. Positive and negative quantities are introduced rather summarily, again as we already know, as quantities provided with a plus or a minus sign (Bélidor 1757, 11). Bélidor justified the rule of signs by differentiating between multiplier and multiplicand in a mode analogous to Arnauld's, albeit in an abbreviated form (ibid., 14). Similar to Rivard, Bélidor attributed to negative quantities a status equal to that of positive ones: "negative quantities are not less real than positive quantities, but are simply opposite to them: they can therefore by multiplied just like the others" (ibid., 18).

Bélidor applied this view in his discussion of equations of second degree to present an application of negative solutions free of problematizing. Noteworthy and rare among his contemporaries is his explicitness in explaining the existence of two square roots: one with a plus and another with a minus sign (ibid., 159).

In general, he said, the problem presented decided which solution to accept. Bélidor made a point of warning against neglecting or even suppressing negative solutions. Convinced of the analytic method, he pointed out how knowledge profited by these additional solutions, which would never have been found but for the analytic method (ibid., 159).

At the same time, his treatment of equations of second degree is a case in point for the then contradictory generality of mathematical treatment. While Bélidor always presented the general solution with a ± sign preceding the root, he gave like Reyneau six "general" forms for the equation of second degree, because the coefficients are always intended to be only positive numbers.

The development in France up to the first third of the eighteenth century can be summed up as follows: Negative numbers were acknowledged as legitimate mathematical objects; they were more or less explicitly understood to be quantities opposite to positive quantities. Practicing an operative calculus had progressed far; zero was predominantly no longer understood as a metaphysical limit. Conversely, negative quantities were acknowledged exclusively in algebra, while arithmetic, separate from algebra, was understood to be the domain of operating with absolute numbers. While algebra was defined, in contrast to arithmetic, as the domain of general operating on quantities, contemporary mathematicians understood the general letter quantities a, b, c, etc. to be confined to the domain of absolute numbers.

Eventually, the contradiction between the intended generality of algebra and the restriction of its understanding to absolute quantities became an element of the mathematical crisis as to whether negative numbers were mathematically admissible, which became acute in the 1750s, having been triggered, remarkably, by members of the Paris Academy.

Before discussing this critical development for France, developments in England and in Germany shall be examined for resemblances or differences.

2.8.2. Developments in England and Scotland

The first English attempts to develop algebra can be dated with two textbooks published simultaneously in 1631: William Oughtred's *Clavis mathematicae* and Thomas Harriot's posthumous *Artis analyticae praxis*, both authors having been strongly influenced by Viète. The two authors are notable for their enthusiastic use of signs; their textbooks contain remarkably little "prose," inventing and applying a great number of signs. This is why Pycior judges them to have been the beginning of symbolic algebra in England.

Oughtred (1574–1660), educated at Cambridge University, was a parson doing mathematics as a private hobby. His textbook, which gave a summary of arithmetic and algebra in a mere 88 pages, had been written for private teaching (Cajori 1916, 17). Indeed, Oughtred distinguished between arithmetic and *specious*, i.e., algebraic, operations, introducing negative quantities in his *specious* subtraction. For practical purposes, however, he confines their use to polynomial expressions, as subtractive quantities. He used only positive quantities for solving quadratic equations. In contrast to that, Oughtred introduced negative exponents, marking them by overlined positive numbers, designating –1, for example, by $\overline{1}$ (Pycior 1997. 49 ff.).

Harriot (ca. 1560–1621), initially occupied as a surveyor after his studies in Oxford, later worked undery the patronage of the Earl of Northumberland, where he was in a position to carry out his mathematical, astronomical, and physical research. His textbook shows no special innovations with regard to the concept of negative numbers; negative equation roots do not appear. He even proves that only positive roots of equations are possible (Cantor II, 1900, 792, Pycior 1997, 58 f.). By contrast, Tanner's analysis of manuscripts from the *Nachlass* yielded that Harriot was the first mathematician to systematically experiment with the rule of signs. His *Nachlass* contains numerous pages with notes in which Harriot tried to establish a consistent connection between the alternative rule of multiplication—*minus* by *minus* gives *minus*—and the other basic operations (Tanner 1980a).

Harriot's approach to that was the "mixed" multiplication of plus by minus, respectively minus by plus. Because first attempts to calculate the result as plus were not satisfactory, Harriot introduced his own sign for mixed multiplication to facilitate his experiments: first a $\vdash$, and later a $\top$. Even then, however, Harriot was unable to attain consistent results: "no interpretation of the intermediary signs gives the desired result in association with the unorthodox 'minus into minus'" (Tanner 1980b, 134).

While Tanner considers this alternative experimentation only as "an entirely unique footnote to the history of mathematical notation in its conceptual aspect" (ibid., 128), the influence exerted by Cardano's *Aliza*, and its having been quoted by Commandino, an influence admitted by Harriot, shows that discussing an alternative concept of multiplication was by no means an isolated instance, but rather regular until about the year 1800.

Two other algebra textbooks that gradually enhanced the acceptance of negative equation solutions in England were also authored by nonprofessional mathematicians. There was the textbook *Teutsche Algebra* published in 1659 by Johann Rahn in Zurich, translated into English and published in 1668 as *An Introduction to Algebra* by John Pell a scholar who had already strongly influenced Rahn's German original, as well as the two-volume textbook *The Elements of That Mathematical Art Commonly Called Algebra* published in 1673–74 by John Kersey (1616–1701) (Pycior 1997, 88 ff.).

A clear break with these hesitant and tentative approaches was made with the ideas conceived by John Wallis (1616–1703), geometry professor in Oxford, one of the most eminent English mathematicians in the mid seventeenth century. While his contribution to the concept of negative number is frequently mentioned in the literature, this is mostly done with little understanding. A view is ascribed to him according to which the negative numbers are simultaneously smaller than nothing, and larger than infinite. Since Cantor (1900, 12), this proposition has virtually been alleged again and again to be the core of his conception (for the more recent literature, cf. Kline 1980, 116, and S. Haegel 1992, 12). What is disregarded here is the specific context of Wallis's considerations, which is that of the transition from $\frac{a}{0}$ to $\frac{a}{-1}$. The more essential thing in this is the context's object proper of integral calculus. In Wallis's time it was already well known that the integral of x^m is given by $\frac{1}{m+1}x^{m+1}$, and the area of ax^m between 0 and 1 by $\frac{a}{m+1}$. Wallis's concern in this part of his own *Arithmetica Infinitorum* of 1655, which has been so frequently reviewed in the literature, was to generalize this solution for negative exponents *m.* This attempt proves Wallis's ability to make advanced operative use of negative numbers.

To reconstruct Wallis's concept of negative numbers accurately, one must look where he presents the basic concepts of arithmetic and algebraic operations. This he does in his textbook *Mathesis Universalis* of 1656, where Wallis introduces subtraction (*subductio*) separately for arithmetic and for algebraic operations. While he declares it "impossible" (Wallis 1972, vol. 1, 70) to subtract a larger from a smaller number in the field of arithmetic ("subductia numerosa," Wallis 1972, vol. 1, 70), i.e., in the field of operating with positive numbers, he extends the area of numbers in his chapter on "Subductio Algebrica." While he initially introduces positive as well as negative numbers as the *quantitates* beset with plus, respectively minus, he continues by declaring them to be opposite quantities (ibid., 70 f.).

Subsequently, Wallis unfolds an extensive operative application of negative quantities and numbers, in particular also of "isolated" negative solutions, which is free of any epistemological reserve. He justified the rule of signs by an abbreviated version of an argument analogous to that of Arnauld concerning the functioning of multiplier and multiplicand (ibid., 104). Wallis was most probably also the earliest English textbook author to explicitly state that there are *two* solutions for every square number, thus for $r^2 = 27d^2$, equally $r = 3d\sqrt{3}$ as $r = -3d\sqrt{3}$. In solving equations, he thus also extensively used the signs $\pm$ and $\mp$ (cf. ibid., 233). Wallis did not present solving equations of second or higher degree in these instructional texts.

He presented the solving of equations in a later textbook on algebra, in his *Treatise on Algebra* of 1685, and in its extended Latin reedition of 1693.[53] In a

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53 In this historically structured work, Wallis credits his compatriots Oughtred and Harriot with all the essential recent progress, to the detriment of the French mathematicians, in particular of Descartes. Wallis thus seems to have been the first case of nationalist behavior in the history of mathematics (cf. Scott 1981, 133 ff., and Pycior 1997, 128).
One of Wallis's major achievements in this second textbook is that he was the first to have given a geometric interpretation of imaginary quantities (Wallis 1693, 286–287; cf. Scott 1981, 162). In particular, he gave a description of the four quadrants of analytic geometry by means of combining the plus and minus signs of multiplication (Wallis 1693, 287 f.).

mode analogous to his earlier textbook, Wallis introduced negative quantities into the operations of *Arithmetica Speciosa*, explaining the rule of signs just as analogously as something following from "the nature of multiplication" (Wallis 1693, 78). In the solving of equations of second and higher degrees, discussing all the root values is the self-evident basis for him. He has no problem with treating, say, in case of quadratic equations, the case with two negative roots together with the other cases on the same basis (ibid., 141 f.). As he discusses different types of equations according to the respective signs of their coefficients, it is a total of four cases for the quadratic equations (ibid., 143).

Wallis argued often intensely in favor of the superiority of algebra over geometry, considering algebra as the representative of modernity (cf. Pycior 1997, 118 ff.). Quite in contrast to the "modernists" in France, however, Wallis was ferociously attacked for this in his own time; not only by the philosopher Thomas Hobbes (1588–1679), in whose empiricist philosophy geometry represented both the origin of mathematical cognition from the senses and their certainty, but also by his famous mathematical colleague Isaac Barrow(1630–1677), who saw geometry as developed by the "ancients," the Greeks, as the secure fundament of all of mathematics. In the controversies of Hobbes and Barrow with Wallis the negative numbers, however, had no role; they were recognized by both of his opponents (Pycior 1997, 143 and 160).[54]

The next influential English author is Isaac Newton (1643–1727). During his time as professor of mathematics in Oxford (1669–1701), Newton also held lectures on *Arithmetica Universalis*, in which he presented the very foundations with remarkable clarity and extensiveness. *Arithmetica Universalis* was first published as a textbook in 1707.

Before presenting the basic operations, Newton first introduced the various kinds of numbers and the signs. From the very beginning, he introduced positive and negative quantities (*quantitates affirmativae/negativae*), as larger, respectively smaller, than zero (*majores nihilo/nihilo minores*). Upon introducing the signs + and of – , however, he added that common use was to designate the negative quantities by a minus sign placed before them, and the positive ones by a plus sign placed before them, that is, speaking again generally of quantities, and not of numbers. The remarkable thing here is that Newton introduces the indeterminate sign of ± and its counterpart of ∓ (Newton 1964, 3 f.).

In a part separate from the signs, Newton then introduced the operative signs of plus and minus, considering operations like –5 + 3= –2 to be quite self-evident (ibid., 4). He listed the zero without problem among the integers ("integrorum numerorum"). Before, he had stressed that he conceived of numbers not as a plurality of units, but rather more abstractedly as relationships (ibid., 2).

54 Cf. Cajori 1929, Scott 1981, Chap. 10; Pycior 1987, 135 ff.; Maierú 1994.

The basic operations subsequently represented are immediately treated across all the kinds of numbers (positive, negative, fractions, powers) introduced beforehand, actually both for numbers and algebraic quantities in the same respective section. For the operation of addition, Newton stresses at once that negative quantities are added just like positive ones; –2 + –3 thus giving –5. Algebraic subtraction is defined as unrestricted; only the signs were to be changed. The rule of signs is implicitly used, but not justified.

In the part that follows, *De extractione radicum,* Newton does not treat negative roots. This is done only later, in the parts on solving equations. In its introductory section *De forma aequationis*, Newton first explains the designations used, and for the first time introduces a designation for coefficients (without naming them as such): he was using the letters p, q, r, s, etc., he said, for "arbitrarily other quantities from which the x [sought] is also determined, if theses quantities are determined and known" (ibid., 54).

It would seem that Newton, with these coefficients, really meant numbers from all the number areas he had introduced, for after this follow normal forms for the equations from first to fourth degrees. For each degree, he is the first to give but a single equation. The general equation of second degree, for instance, is given as

$$xx - px - q = 0 \text{ (ibid.).}$$

In particular, he explains that the general solution for $xx - ax + bb = 0$ is

$$x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa - bb} \text{ (ibid., 58 f.).}$$

The parts following also show that Newton, going beyond Wallis, and in international and intercultural comparison as well, is the author to operate most comprehensively with negative quantities, giving them a legitimate function as mathematical objects.[55]

Another typical textbook is *The Elements of Algebra* (published posthumously in 1740) by the famous blind Cambridge mathematics professor Nicholas Saunderson (1682–1739). He adheres so closely to Newton in his own conception of negative quantities that details are unnecessary here. What is remarkable is the didactical ethic that prompts him to openly discuss all the possible objections against this conception. Saunderson introduces negative quantities as "less than nothing," explaining why this is neither a "very great paradox" nor a "downright absurdity." To substantiate this, Saunderson points to the difference between quantities and numbers; in particular, he shows how "narrow minds" are perplexed by not taking into account that there exist numerous quantities—such as bodies or substances—for which there is no opposite quantity (Saunderson 1740, 50 f.).

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55 Although Pycior devotes an entire chapter to Newton's *Universal Arithmetick*, she did not pursue these concrete forms of Newon's operating in the concept field of negative numbers (cf. Pycior 1997, 192 ff.).

Remarkable is his justification of the rule of signs, which he presents quite extensively, fearing it might be difficult to digest, particularly by people with a weak constitution (ibid., 54 ff.). For this, he draws on arithmetical series and on how to preserve their properties in case of multiplication by a constant factor. In order to show that –3 multiplied by +4 gives –12, one was to consider the part +3, 0, –3 of an arithmetic series, first obtaining +12 and 0 in multiplication, and hence the last term had to give –12, etc. (ibid., 56 ff.).

The clear conceptual penetration of the concept field shows in particular in Saunderson's presentation of the theory of equations. For the quadratic equations, he shows that all equations can be reduced to only one normal form: to $Axx = Bx + C$. Saunderson was the first to explicitly assign both positive and negative values to the coefficients A, B, C (ibid., 172).

The next important work is the *Treatise of Algebra* by Colin MacLaurin (1698–1746), mathematics professor in Aberdeen and later in Edinburgh, the textbook being published posthumously in 1748 from MacLaurin's *Nachlass*. In an excellently explicit way, his textbook develops an operative conception of opposite quantities. He introduces the two basic operations of addition and subtraction as operations having contrary effects. Quantities are either additive, or subtractive:

> Hence it is, that any quantity may be supposed to enter into algebraic computations by two different ways which have contrary effects, either as an *increment* or as a *decrement*; that is as a quantity to be added or to be subtracted (MacLaurin 1748, 4).

And the operations are unrestrictedly executable. If a is smaller than b, "then $a - b$ is itself a decrement" (ibid.). Just as addition and subtraction are opposite, there is an analogous opposition between the quantities' "affections," which are considered in mathematics. MacLaurin calls additive quantities positive, and subtractive quantities negative: "they are equally real, but opposite to each other" (ibid, 6). Negative quantities are thus defined by the operation, not by a sign of an absolute number.[56]

MacLaurin is very precise in describing the significance of the mutual cancellation:

> When two quantities equal in respect of magnitude, but of those opposite kinds, are joined together, and conceived to take place in the same subject, they destroy each other's effect, and their amount is *nothing* (ibid., 5).

MacLaurin also emphasizes the difference between numbers and quantities. While abstract quantities can be both negative and positive, he says, concrete

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56 Glaeser, who wrongly and with a wholly inadequate approach believed himself justified in attributing to MacLaurin an inability to master two "obstacles" in the concept of negative numbers (Glaeser 1981, 317), drew on MacLaurin's French translation of 1753, which indeed shows remarkable deviations from the original in its re-interpretation to suit the French conception prevailing at the time. The translator, Le Cozic, teacher at military schools and successor to Bélidor and Deidier in La Fère, took liberties with the text, abbreviating some parts and reformulating others to change their meaning, as in inserting a definition of the positive and negative quantities via the signs of absolute quantities (ibid., 316 f.).

quantities are not always capable of assuming the quality of being opposite to each other (ibid., 6). Just as explicitly, MacLaurin elaborated the conceptual distinction between a quantitative aspect and the necessarily associated qualitative aspect of opposite quantities: equality in algebra, he said, required not only equality of quantity, but also of quality (ibid.).

He makes a point of characterizing negative quantities as just as legitimate as positive quantities: "But a negative is to be considered no less a real quantity than the positive" (MacLaurin 1748, 7).

It is also notable that MacLaurin gives a definition of *coefficients*, actually as that number with which a letter quantity appears multiplied. This number states how often the quantity represented by the letter should be taken.

The introductory parts on quantity types are followed by chapters on the basic operations. In treating addition, MacLaurin already presents cases where addition quite self-evidently gives negative results, such as adding a larger negative term, or as an addition of exclusively negative terms. Quite novel, too, is his presentation of the rule of signs. He does not claim to prove the rule of signs, but only to "illustrate" it by arguments. The argument is based on distributivity: it followed from $+a - a = 0$ that $+na - na = 0$. And it resulted for $(+a - a)(-n)$ that the first factor was $-na$, and the second factor need hence be $+na$ (ibid., 12). To my knowledge, MacLaurin also was the first to elaborate the difference between multiplier and multiplicand in the rule of signs: the multiplier, he said, was always regarded only as a number (hence also the letter n in the rule of signs). For quantities of any kind could be multiplied only with numbers, but generally not with quantities (ibid., 13 f.).

For his chapter on powers and roots, MacLaurin clearly took the multiplicity of roots—their having positive, negative, and imaginary values—as his basis. He insists on permitting the possibility of all the roots for an equation being negative (ibid., 144). For equations from first to fifth degree, he explicitly gives the normal form—in each case with one equation only; that is, the coefficients are understood to have a general range of values. MacLaurin even used analytic geometry for applying his concept of negative numbers. In the closing third part of his textbook, about how to apply algebra and geometry to one another, he uses in his figures all four quadrants, while inserting coordinates, even if he does not name the axes. Line segments of equal length, e.g., in the first and third quadrants, are designated by y, respectively by $-y$ (cf. ibid., 318).

MacLaurin's textbook is an excellent example of conceptual clarity and broad operative use of negative numbers and quantities.

At the same time, however, MacLaurin is another example of the fact that a mathematician's work need not be coherent as a whole. In his textbook *A Treatise in Fluxions* published as early as 1742, we find a section about general characters in algebra in which he discusses whether it is possible and whether it makes sense to use generalizing signs in algebra. This discussion was necessary for MacLaurin, since he had attempted to prove the validity of the Newtonian methods of calculus with purely geometrical arguments (MacLaurin 1742, 575) in the book's main part. He began by distancing himself from approaches not specified by himself that had used a complication of symbols to conceal abstruse doctrines that were unable to stand the harsh light of geometrical form.

As a first example, he discussed "the use of the negative sign in algebra." He explained that it was necessary to differentiate between the absolute value of a quantity ("real value of the quantity") and its quality of having the potential to be opposite to another. He thus stressed the achievement of generalization this permitted because it had become possible to group together several cases, and to use their analogy (ibid.). He immediately went on from that to discuss the proportion argument not taken up by English authors as yet. He attempted to declare the problem of negative terms in proportions to be nonexistent by stating that proportions of lines depended only on their absolute magnitude, but not on their quality, i.e., not on their direction. This was true not only for lines, he said, but generally for all quantities. Hence, the proportion of the quantities $-b$ and a was the same as for the quantities b and a. While MacLaurin added a justification of plausibility for the rule of signs, he also made clear that a separate justification for it was not really called for, since the proportion $1 : -n :: -b : nb$ must agree with the proportion $1: n :: b: nb$ because of the fact that the absolute values were the same (ibid., 576 f.). It is very probable that this was where d'Alembert, who had most diligently studied MacLaurin's *Treatise on Fluxions*, found the reasoning most central for his own writings (see below, Section 2.9.3).

It is typical for the close connection between the concept of negative quantities and epistemological foundations that this first sign of a break with traditional views emerges in the context of a foundation of mathematics as purely geometric as possible—and not in the context of an algebraic one.

The four textbooks on which this section focused show over a period of one hundred years in England (and Scotland) a remarkably intense development of the concept of negative numbers in which the legitimacy of this mathematical concept was quite naturally assumed. Since all four authors were influential university professors, a wide dissemination of their views may be taken for granted. Their radical rejection beginning in England about 1750 is thus all the more surprising.

2.8.3. The Beginnings in Germany

The first textbook author in Germany to adapt the modern style developed by Arnauld and Prestet was Christian Wolff (1679–1754). After his studies in Jena and his *Habilitation* in Leipzig, he became professor for mathematics and natural theory at the University of Halle in 1707. In 1723, he was driven away from there because of theological conflicts, finding sanctuary at the University of Marburg. In 1740, he was reinstated in Halle. His four-volume textbook *Anfangs-Gründe aller mathematischen Wissenschaften* of 1710 became, just as its Latin version *Elementa Matheseas Universae* of 1713, the most successful

German textbook of the entire eighteenth century, in ever new editions. The *Encyclopédie* highly praised this textbook for its quality (*Encyclopédie*, tome V, 1755, 497).

Indeed, the fourth volume, on algebra, and on differential and integral calculus, also shows an effort at conceptual clarity. Thus, Wolff reflected on the relationship between quantities and number, designating quantities as "indeterminate numbers" (Wolff 1750, 1551). Conversely, his book does not contain any independent reflection on the nature of negative quantities, and nothing going beyond the average level of contemporary literature in France. His remarks on that topic are even very brief. In the various reprints edited by him, Wolff did not change these parts.

After introducing the signs of plus and minus, Wolff introduces the negative quantities with the unrestrictedly executable subtraction, without, however, giving them a name of their own. He designates them in a substantialist mode as an expression of "defect," while the positive quantities state an existence:

> All quantities marked with the sign of – are defective, and against that those having the sign of + exist. If I am thus called to add of both kinds, the latter will level out the defect, although the addition has to be converted into a subtraction (ibid., 1557).

Wolff added as a footnote:

> The quantities marked with the sign of – have to be regarded as nothing else but debts, and by contrast the others bearing the sign of + as ready money. And therefore the former are called less than nothing, because one must first give away enough to settle one's debt before having nothing (ibid.).

In the practice of calculation, Wolff had no problem with the admissibility of these quantities. He continues by explaining how subtractions must be executed if "the larger must be subtracted from the smaller" (ibid., 1588).

In multiplication he justified the rule of signs, again very summarily, saying that one obtained the sign of plus in multiplication within a complex in order to level out again the too large "defect" (ibid., 1560). While he did not mention the plurality of roots in his treatment of extracting roots, he pointed out in the solving of equations the existence of two roots, also inserting them in further representation (ibid., 1588).

The Latin version of his algebra—he called it *Arithmetica Speciosa*—is, like other parts of the textbook, more concept-oriented. Here, Wolff uses the terms positive and negative quantities. And the definition he gives is familiar from France: quantities provided with a sign of plus are called *positive*, or *affirmativa*, respectively *nihil major*, and those preceded by a sign of minus are called *privativa*, or *negativa*, or *nihilo minor* (Wolff 1742, 299). Wolff was the first to connect this definition with an explicit mention of absolute quantities. In a corollary to his definition, he said how positive and negative quantities developed from absolute quantities (called *vera*, but not thematized before):

> Quantitas positive prodit, si vera aliqua additur, e.gr. $0 + 3 = +3$, $0 + a = +a$; privativa relinquitur, si quantitas aliqua vera ex nihili subtrahitur, e.gr. $0 - 3 = -3$, $0 - a = -a$ (ibid.).

Wolff preferred the designation *privativa* to that of *negativa.* Indeed, he assigned other qualities of being to the positive quantities than to the negative ones, which expressed a lack (*defectus*). Because of the different qualities, positive and negative quantities were heterogeneous to each other. Positive quantities were able to be in ratio to one another, just as negative quantities could be to another—but there could be no ratio between positive and negative quantities (no. 24, ibid., 300). Wolff justified this view, on the one hand, with his own definition of homogeneous and heterogeneous already presented in his part on arithmetic: homogeneous were only those quantities that, after being multiplied (*aliquoties sumta*), could exceed one another (ibid., 26). Since taking a defect several times, however, made the defect even larger, and could never exceed the positive quantity, positive and privative quantities were heterogeneous (no. 23., ibid., 300). Wolff also tried to justify with this that, e.g., $-3a$ did not relate to $-5a$ as $+3$ related to $+5$, and that 1 did not relate to -1 in the same manner as -1 related to 1 (ibid.). Wolff's intention in this reasoning was to invalidate Arnauld's argument of proportions (also adopted by Leibniz): the proportion $1 : -1 = -1 : 1$, he said, was formed of quantities heterogeneous to each other and hence inadmissible (ibid.). The correct starting point of his ad hoc argumentation was that proportions had originally been formed exclusively for positive quantities, and that nobody had ever tested whether they were applicable to the new number area.

On the other hand, Wolff linked positive and negative quantities in common operations, in contrast to his own view that they were heterogeneous to each other. Thus, he also gave a hint at the concept of opposite quantities: $+a$ and $-a$ mutually canceled out one another ("se mutuo destruunt," ibid.). In his Latin edition, he justified the rule of signs more extensively than in the German version, but only geometrically, by means of a comparison of areas within a parallelogram (ibid., 304). Although Wolff immediately adopted Newton's *Arithmetica universalis*, referring to it in the very first German and Latin editions of his own work, he was still strongly determined by Descartes's concepts. This is particularly evident in his part on solving equations. The plurality of roots is treated consistently, but the positive roots are called *vera*, and the negative roots *falsa*. Coefficients are always understood to be positive so that in quadratic equations, for instance, the cases

$$x^2 + ax = b^2,\ \ x^2 - ax = b^2,\ \text{ and } x^2 - ax = -b^2$$

are solved separately. The fourth case, of two negative solutions, is not mentioned at all (ibid., 342 f.).

A deeper reflection of concepts is contained in the textbook *Elementa Matheseos*, by Christian August Hausen (1693-1743), a mathematics professor at the University of Leipzig. In the historiography of mathematics, Hausen is practically unknown today, but during the second half of the eighteenth century, the conceptual achievements of his textbook were pointed out again and again. We shall indeed get to know them not only when treating the concept of continuity, since his presentation concerning the negative numbers had an innovative effect as well. Thus, Wenceslaus J.G. Karsten (1732–1787), himself the author of an important textbook, in 1786 ascribed the prevalence of the modern conception to Hausen's "most excellent book," in which the elementary concepts of positive and negative quantities are from the very beginning presented in the most perfect light" (Karsten 1786, 243). And J.A.C. Michelsen (1749–1797), translator of Euler's textbooks, and intensely involved in clarifying the basic concepts, also underlined Hausen's role in 1789 (Michelsen 1789, 15). Hube, one of Kästner's disciples, even listed Hausen among the top mathematicians in a publication of 1759:

> What else could have moved the greatest measuring artists [i.e., mathematicians] like Newton, de la Hire, de l'Hôpital, Simson, Euler, Hausen, and many others, to place this theory [of the conics] in a better light? (Hube 1759, [v]).

Hausen introduced the subtraction $A - B$ unrestrictedly: for numbers just as for letter quantities. For the case $A = B$, the difference was 0, and for the case $A > B$ he stated with ease that the difference was negative. He represented negative differences as an expression for opposite quantities: opposite to another quantity that, although being of the same kind, was regarded as positive. Hausen tried to grasp the concept of opposition more precisely. It was a case of two contrary determinations, one of which involved the other's absence: "Determinationes oppositas hic quaslibet duas, quarum una involvit absentiam alterius, ut contradictorium sit utramque adesse" (Hausen 1734, 13).

As examples Hausen quoted the sun, which could not at the same time be a certain quantity over the horizon and below it, and the fact that a quantity could not at the same time increase and decrease within the same relation. Opposite equal quantities cancel one another:

> Quoties determinatiom ejusmodi concursus fit, effectibus oppositis se mutuo destruunt, et in statu ejus, quod afficerent solitariae, nihil mutant conjuctae. (ibid., 14).

Hausen supplemented this by an extensive discussion of opposition and mutual cancellation. He also used the term of "absolute quantities" here: positive quantities could also be regarded as "absolute" ones. Subsequent to the discussion of principles of the concept of opposition, Hausen explained the operations of addition and subtraction for opposite quantities.

Also very remarkable is Hausen's extended discussion of the concept of proportions. He used it to define multiplication as a proportion—the product as the fourth proportional to the unit and to the two given factors—in order to arrive thus without further justification at an extension of the admissibility of multiplication for opposite quantities (ibid., 3). In contrast to Wolff, but without mentioning the latter, he declared positive and negative quantities to be of the same genus ("ejusdem est generis," ibid., 14). And without taking recourse to the originally geometric meaning of proportions, he conceived of them exclusively arithmetically, as composed of arithmetic ratios, which he again conceived of, while applying his own concept of opposite quantities, as "rationes oppositae," which led to "numeri negativi" (ibid., 19–21). Hausen concludes that the expression $+1 : -1 = -1 : +1$ is an admissible proportion, and, more generally,

$$A : -B = -C : +D$$

as well, provided that A, B, C, and D are proportional. Hausen then applies this concept of proportion to justify the rule of signs for multiplication and division.

2.9. The Onset of an Epistemological Rupture

2.9.1. Fontenelle: Separation of Quantity from Quality

The above description of how the concept of negative numbers developed in Europe did not offer any clue of an impending crisis, or rupture, in the direction this development was taking. Precisely such a rupture, however, occurred about 1750; it began in France and then spread to England, while there was no influence on Germany for a long time. The crisis originated with scholars belonging exclusively to the Paris *Académie des Sciences*. Their other common feature was that they did not do any teaching. The first theoretical impulse emanated from Bernard le Bouvier de Fontenelle (1657–1757), a further, educationally motivated contribution was made by Alexis-Claude Clairaut (1713–1765), and the decisive formulation, which had real impact, was provided by d'Alembert (1717–1783). Since previous developments give no hint at an imminent crisis, it is suggestive to search for a novel element that may have had a strong effect. This novel element will be found in the concept of *logarithms*. Discovered at the outset of the seventeenth century as an auxiliary computing tool, they had increasingly evolved into an object of mathematical theory in the course of the seventeenth century, having become indispensable for clarifying and making coherent the concept within the field of algebra's and analysis's foundational concepts.

Fontenelle, the Paris Academy's secretary of long standing, published a book in 1727 that was a provocation for many: *Éléments de la géométrie de l'infini*. The most provoking parts were those in which Fontenelle developed an algebraic calculus relying on infinitely large quantities. In spite of its title, the volume was not a textbook for purposes of teaching, but a scholarly contribution reflecting foundational mathematical concepts. That the author was perfectly aware that a real need for clarifying the concept of number had arisen can be seen from his discussing not only the concept of the infinite, but also devoting a separate chapter to the concept of number, and in particular to that of negative quantities.

Fontenelle was the first to examine not only negative numbers, but also the connection to imaginary numbers: *Des Grandeurs Positives et Negatives, Réelles et Imaginaires* is the heading of his book's *Section VI*. At first glance, this seems to be where Fontenelle intends to deepen the concept of opposite quantities. He argues against understanding "the negative" only as a subtracting, as a "retranchement" (Fontenelle 1727, 169). He is the first author to elaborate the differentiation between a *quantitative* aspect and a *qualitative* one, that of oppositeness:

> From this it follows that the idea of positive or negative is added to those magnitudes which are *contrary* in some way. […] All *contrariness* or opposition suffices for the idea of positive or negative. […] Thus every positive or negative magnitude does not have just its *numerical* being, by which it is a certain number, a certain quantity, but has in addition its *specific* being, by which it is a certain *Thing* opposite to another. I say *opposite to another*, because it is only by this opposition that it attains a specific being (Fontenelle 1727, 170).

Fontenelle explains oppositeness by the fact that one term cancels the other, negating it, and hence is negative:

> When two magnitudes are opposite, the one excludes or repudiates the other, and consequently is negative with regard to the other, which is positive (ibid., 171).

Upon closer analysis, however, one will note that things positive and negative do not assume symmetrical functions for Fontenelle. Basically, only the negative quantities are endowed just with specific quality, while the only positive quantities have the "privilege" of being endowed with numerical quantity. As is evident from Fontenelle's introductory reflections on the concept of number, there can be no sequential number series

$$..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...$$

for him, since the zero is not a quantity for him, as he most emphatically proclaims. Before, he had defined as "quantity" what lends itself to an increasing or a decreasing. Fontenelle obviously identifies the zero with the metaphysical nothing; the zero, he says, is no quantity, because it is not capable of being increased or decreased. "A nothing cannot be a greater or a lesser nothing" (ibid., 2).

The zero does not even have a numerical character (ibid., 171). For Fontenelle, the smallest number thus is the one. The one represents the element from which all the other numbers can be generated.

Fontenelle indeed refrains from constructing a general concept of positive and negative quantities as a new common concept of number. An essential condition remaining implicit for this is that he does not distinguish conceptually between numbers and quantities. Rather, he just states that there are no opposites for some quantities (that velocity, for instance, had no *opposé*), and that for quantities having such an *opposé*, the positive part may exist for itself alone—as a numerical quantity. The negative part, against that, necessarily contained the negative quality of oppositeness:

> The negative magnitude, when taken on its own, necessarily includes its specific being in its idea; but the positive, taken in the same way, does not include it necessarily, and so is only positive *improperly*, because it is considered or posited, but not *properly* and with respect to an opposite magnitude (ibid., 171).

Fontenelle's refusal to conceive of negative quantities as of things produced from *retranchements*, i.e., from subtractions, is on the one hand due to his having studied as well whether fractions with integers and powers with roots are able to form opposite quantities with regard to multiplication. In doing so, Fontenelle was the first to implicitly inquire whether there was a second kind of oppositeness—a multiplicative besides the additive one. Since he had neglected to closely inspect the "neutral" element required for this, his result was bound to be negative: neither fractions nor powers could belong to the *grandeurs opposées* (ibid., 172). Although the exponents n and $-n$ were opposed, he said, their powers were due to *retranchements*.

The second reason why Fontenelle assigned to the negative quantities a quality of their own separate from that of positive quantities lay, however, in a conceptual field linked with negative numbers: that of imaginary numbers. Fontenelle tried to grasp the conceptual difference between real and imaginary numbers more exactly. For this purpose, he analyzed the concept of multiplication, or more precisely of multiplication between numbers and quantities, while taking note of the difference between numbers and quantities, multiplication involving, in his own terms, *quantités revêtues d'une idée specifique* (ibid., 176).

His point of departure was to reflect on the concept of multiplication. The only product he declares to be admissible is that of a scalar—"un pur nombre"—multiplied by a quantity. Conversely, a product of a quantity a with another quantity b made no sense; Fontenelle nevertheless arbitrarily posited that such a product ab could only be a scalar: one would have "necessairement" to take away the *idée spécifique* of quantity; the *idée spécifique* disappeared completely in the product ab, which was thus *purement numérique* (ibid., 175 f.). In case of a negative product $-ab$, such as one consisting of a debt $-a$ and a number b, against that, the specific idea subsisted: "But in the product $-ab$, the idea of specific being remains, and in effect, this idea is properly attached to negative magnitudes" (ibid., 176).

Fontenelle relied on such reasoning to show that a product $-aa$ could not be conceived of as a square: It was not a purely numerical square, for if a is a pure number, $-a$ is not a pure number. At the same time $-aa$ neither could be a square with an *idée spécifique*, because $-a$, understood, say, as debts, could not be detached from its specific idea because of Fontenelle's own willful positing, and hence could not be inserted in a product (ibid., 177).

This chain of reasoning eventually helps Fontenelle to attain his goal. If one takes $-a^2$ as a square, one is taking it for something it cannot be. If one intended, however, to take the square root $\sqrt[2]{-a^2}$, one would have to assume $-a^2$ to be a square. Hence, $\sqrt[2]{-a^2}$ was a completely *imaginary* quantity, which could not be real in any sense.

These concept determinations represent first tentative efforts, and as their author was not called on to make them explicitly coherent in his teaching, he had the privilege of being satisfied with what he himself had posited ad hoc. For later developments, however, it turned out to be crucial that he had determined negativity as a specific quality of oppositeness while neglecting, respectively reducing, the operative side of negative quantities. “Moins que rien,” he declared, was neither a mathematical nor a physical concept, but rather “only a moral one” (ibid.).

2. Clairaut: Reinterpreting the Negative as Positive

The next work in which we similarly find arguments against integrating the negative numbers into an overall conception of real numbers was, in contrast to Fontenelle’s a textbook, explicitly addressing laymen. It was likewise authored by a scholar and member of the Paris *Académie des Sciences* who was not involved in any teaching, by Alexis-Claude Clairaut (1713–1765), a scientist mostly known for his research into mathematical physics. Clairaut wrote two elementary textbooks, both intended for a marchioness, and more generally for an elegant public intent on dabbling in leisurely mathematics without having to shoulder any real effort (cf. Glaeser 1983). The two textbooks on geometry (1741) and algebra (1746) attempt to realize a methodological approach according to which the respective mathematical concept field evolves in a seemingly “natural way” from simple inquiries or from useful problems.

While Clairaut himself did not have any problems with operating with negative numbers and quantities, and also quite clearly exposed the plurality of values for roots, such as the two values in equations of second degree (Clairaut 1757, 163), his major concern was to avoid scaring off beginners (*commençants*), and a particular stumbling block (“écueil,” Clairaut 1797, 3) in his eyes was multiplication. Since he did not consider isolated negative solutions acceptable for *commençants*, he adopted part of Fontenelle’s arguments in favor of separating positive from negative quantities.

For Clairaut, negative numbers did not represent a mathematical problem, but rather a didactical one. Of the fifteen pages of his preface, he devoted more than three and a half to describing his own approach of guiding beginners gradually toward an understanding of the necessity of operating with negative numbers, and of the appropriate rules—in particular the rule of signs. This path led to a discussion of the multiplication of isolated negative quantities: “des quantités purement négatives” (Clairaut 1797, 5).

Operations with negative quantities are indeed extensively presented in an educationally well-considered way. Thus, Clairaut explains the difference between “ajouter” and “augmenter” for addition, and between “soustraire” and “diminuer” for subtraction (ibid., 58 ff.), using more abstract terms that are less dependent on intuition with the intention of leading away from the inability to grasp the fact that adding a negative term will give a smaller result, and subtracting the same will give a larger. On the other hand, Clairaut assumed that the occurrence alone of operations with negative numbers—e.g., divisions $\frac{300}{-10}$

or $\frac{-400}{-10}$—would involve the learners with metaphysical problems: they might fear executing "mauvais argumens métaphysiques" (ibid., 95). He thus showed other paths to attaining the result without such operations. Clairaut therefore was all the more convinced that his audience intuitively shunned negative solutions. He thus developed a method of interpreting negative solutions away. Already in his introduction, Clairaut had explained that he was liberating operating with negative quantities from everything "shocking," permitting the reader to recognize the nature of negative problem solutions. One should assume the unknown to be of opposite direction:

> When it happens that the unknown in a solution is found to be negative, it must be taken in a sense opposite to that which had been used in expressing the Problem (ibid., 6).

The textbook part did the interpretation by changing direction, but not at all developing this gradually, or "naturally," but rather as an abrupt positing. Clairaut had begun by developing the rule of signs in solving a system of equations concerning the following problem: two sources of different strength fill two different ponds within definite periods. For x and y, the volumes of water provided by the two sources, the results were –30, respectively +40 units (ibid., 94 ff.). Subsequent to the equation's "abstract" solution, Clairaut then inquired how the *autre espèce d'embarras* could be solved: the meaning of the negative value of x. For this, he explained, one had to go back to the problem's conditions, and that meant going back to the initial equations.

The remarkable thing, in particular as compared to later developments among French authors, is that Clairaut does not change the equations in order to attain positive values, but instead discusses extensively how the values obtained must be understood to have them satisfy the given equations. He advances as an interpretation posited by him without further explanation that the first source did not pour water into the pond, but instead drew water from it. Clairaut did not understand this interpretation of assuming the unknowns in an opposite sense as a rejection of the method of *analyse*, but rather as the confirmation of the generality of this method, which provided more results than originally intended:

> One sees on this occasion an example of the generality of analysis, which allows cases to be found in a question which one had not first of all anticipated as being able to be included (ibid., 99).

Clairaut thus admitted genuine negative values of unknowns, a fact relieving him from changing the equations, as subsequent authors found themselves compelled to do. He even extended this conception by permitting negative values as well for the coefficients—in contrast to his having defined them in his general definition of negative quantities as quantities preceded by a minus sign, as was the then current practice in France (ibid., 56). He did not substitute according to that, however, as later authors did, say, the coefficient b by $-b$, but inserted for b the negative value –3 (ibid., 100), after having generally explained before that it was permissible to take not only the unknowns, but also the "connues" in the inverted sense (ibid., 99).

It is not only the mathematical recognition of negative quantities that prevails in Clairaut; it is also of prime importance for him that the values satisfy the given equations. He was the first to directly tackle, in a textbook of modern times, the question of how to interpret negative solutions in equations with concrete quantities. Whereas his predecessors had stressed the admissibility of negative solutions, they had not progressed as far as this level of application.

3. D'ALEMBERT: THE GENERALITY OF ALGEBRA: AN INCONVÉNIENT

While the writings of Fontenelle and Clairaut contained new approaches in their thought on negative numbers and quantities, they did not become elements of crisis before d'Alembert's publications. These effected an acute and incisive rupture with everything developed up to the time. D'Alembert wrote articles in the *Encyclopédie* he coedited radically criticizing the then current conception of negative numbers for their false metaphysics, thus attaining a much greater impact than every mathematical author before him. He recognized virtually nothing but positive numbers as admissible mathematical objects. Just as radically, he rejected the generality given by algebra in the solving of equations, labeling it a "disadvantage." Jean le Rond d'Alembert (1717–1783), one of the leading representatives of Enlightenment in France, and at the same time among the most eminent philosophers, mathematicians, and physicists of his time, was a member of the Paris Academy of Sciences. He never accepted any other post, and in particular did not teach. His works on the foundations of negative numbers contradicted other aspects of his work in which he had advocated algebraization and generalization. This raises the question of what the mathematical and/or epistemological reasons led d'Alembert to reject what had been developed up to his time.

D'Alembert did not author a textbook on algebra, or any other integral presentation of his own view of algebra. There are, however, a number of treatises and articles that lend themselves to scrutiny. D'Alembert had no problem with the hitherto established rules for operating with negative quantities and numbers. Thus, he explicitly stressed that these rules were generally recognized to be exact: "the rules of algebraic operations on *negative* quantities are accepted by everybody and generally received as exact," independent of the meaning ascribed to these quantities (d'Alembert, *Négatif*, 1765, 73 left column). He saw the problem exclusively in finding the appropriate "metaphysics" of this concept. If one considered the precision and simplicity of algebraic operations with these quantities, he said, one was tempted to believe that the precise idea one had to ascribe to the negative quantities needs to be a simple idea, and must not be derived from a sophisticated metaphysics (ibid., 72 right column).

D'Alembert's book *Essai sur les Éléments de Philosophie* may serve as an introduction to the metaphysics of negative quantities, a book he wrote in 1758, during a time of crisis when his *Encyclopédie* project was in jeopardy of being wrecked by political opponents. He intended this book to comprehensively present his own views regarding the various fields of knowledge. In his chapter on *Algèbre,* he said that algebra was the leading science in mathematics, but noted at the same time that is was in some aspects not yet free of obscurities—at least in the current textbooks. In a footnote, he quoted as an example that he did not know any textbook in which the theory of negative quantities was *parfaitement éclairci* (d'Alembert 1805/1965, 291). In a subsequent *Eclaircissement sur les élémens d'algèbre*, d'Alembert made this critique more explicit. Again, he highlighted the negative quantities as an example of the lack of exactitude in the textbooks on algebra. He criticized not only the view of negative quantities as smaller than "nothing," and their interpretation as debts as too narrow, but also the view of understanding them as opposite to positive quantities, since geometry also provided examples showing that negative quantities should be taken in the same sense as positive ones:

> Some regard these quantities as *below nothing*, an absurd notion in itself: others, as expressing *debts*, a very restricted notion and for that reason alone hardly exact: others still, as quantities that must be taken in an opposite sense to quantities which are supposed to be positive; an idea for which geometry easily provides examples, but which is subject to frequent exceptions (ibid., 301).

His own answer to the question of what negative quantities really are is more extensively presented in several articles in the *Encyclopédie.* The best known among these is his article *Négatif* of 1765 in the eleventh volume, where he defines these quantities in the very introduction in the ways common in France, that is, as quantities preceded by a minus sign, but at once rejects the view advocated by "several mathematicians" that they are smaller than zero (*plus petites que zéro*)[57]: such an idea, he said, was incorrect (*pas juste*), as would be instantly seen (d'Alembert, *Négatif*, 1765, 72 right col.).

Actually, d'Alembert did not give a straight justification in this article, but tried to refute an argument given by an author not named who had said that 1 was incomparable to –1, and that 1 was in another ratio to –1 than –1 was to 1. The arguments quoted by d'Alembert correspond exactly to those given by Chr. Wolff (see Section 2.8.3), whose primary concern actually had not been to show that the negatives are smaller than zero, but rather that Arnauld's argument of proportions did not make sense because proportions are not defined for negative quantities. D'Alembert's refutation thus did not apply to Wolff's reasoning: While 1 was constantly being divided by –1 in algebraic operations, the proportions argument was about the geometric theory of proportions.[58]

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57 This time thus not below "nothing," but smaller than "zero."

58 D'Alembert gave a direct justification in his treatise on the logarithms of negative numbers: while he recognized the proportion $1 : -1 :: -1 : 1$ as correct, on the basis of algebraic calculus, he had no qualms using the traditional geometric proportions argument: if the negative numbers were smaller than zero, there would have to be simultaneous validity of $1>-1$ and $-1 > 1$ because of the sequence of terms (the text

D'Alembert derived his own determination of negative quantities from geometry. In geometry, negative quantities were often represented by real quantities, which differed "only" by their position with regard to a point they have in common on a line, as he had shown in the article *Courbe*. Consequently, one could infer that the negative quantities ocurring in the *calcul* were indeed real quantities. One had to ascribe to them, however, another idea than that originally assumed: negative quantities *indicated* **positive** quantities—the minus sign simply pointing out an error made in posing the problem and thus a false position requiring correction:

> Thus *negative* quantities really indicate in the calculation a false position. The sign – found in front of a quantity serves to redress and correct an error made in the hypothesis (ibid., 72 left col.).

As an example recurring in variations, d'Alembert quotes the problem of finding a number x that gives 50 when added (*ajouté*) to 100. The rules of algebra yield $x = -50$. This, however, showed that x really was equal to 50, the problem actually having to be formulated as follows: Seek a quantity x and subtract it from 100, a remainder of 50 being left (ibid.). D'Alembert summed this up to say that there are no isolated negative quantities as mathematical objects:

> Thus there is in reality and absolutely no isolated *negative* quantity: –3 taken abstractly offers no idea at all to the mind; but if I say that a man has given another –3 écus, that means, in an intelligible language, that he has taken 3 écus from him (ibid.).

D'Alembert did not reflect on how to maintain his view in the case of quantities for which there is no natural element of opposition as in giving and taking.

He was so radical in his view that he went so far as to declare the long-discussed problem of founding the rule of signs to be obsolete: to multiply $-a$ by $-b$, if one removed the error from the problem and formulated it correctly, was nothing but multiplying $+a$ by $+b$ (ibid.). Whereas later authors adopted many of d'Alemberts approaches and arguments, this argument (of MacLaurin) was never taken up again.

A further typical aspect is contained in the article *Équation* which appeared in the fifth volume in 1755. Its first third has been adapted from the corresponding article of the English encyclopedia, as d'Alembert notes himself, and the latter originated in its essentials from Newton's *Arithmetica Universalis*; it is on methods of solving equations.

After proving the fundamental theorem of algebra, he commented on the significance of the two roots for equations of second degree. While two positive roots solve the equation in the same sense, how should one assess mixed roots? For an example, d'Alembert selects the following problem: Sought is a number

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shows, due to a printing error, $-1 < 1$); hence, no proportion existed (d'Alembert 1761, 201).

x, x being smaller than 1, for which $(1-x)^2 = \frac{1}{4}$ holds. Solving first gives $1-x = \pm\frac{1}{2}$ (this is one of the few times where the sign ± appears in d'Alembert's writings); hence $x = \frac{1}{2}$ and $x = \frac{3}{2}$. Among these, only $\frac{1}{2}$ solves the roblem, because of the condition $x < 1$. D'Alembert, however, continued the inquiry: why does one get another real, positive root? It was the solution, he said, to the problem, sought is a number x, x larger than 1, for which holds $(x-1)^2 = \frac{1}{4}$ (d'Alembert, *Équation*, 1755, 849 f.).

The algebraic translation of the former task, he said, was from its nature more general than this former task itself; it contained at the same time the second task as well. D'Alembert now adds a polemic commentary, obviously directed against his fellow academy member Clairaut:

> Many algebraists consider this generality a richness of algebra, which, they say, answers not only what is asked of it, but even more what has not been asked of it, and what one has not thought of asking of it (ibid., 850).

This indeed corresponds to Clairaut's evaluation of the solutions found additionally (see above, Section 2.9.2.). D'Alembert, however, strongly rejected this program of generalization, which had been held in high esteem since Prestet. He was forced to admit, he said, that this richness and this generality presented an inconvenience for him: "For my part, I cannot avoid asserting that this alleged richness appears to me to be an inconvenience" (ibid.).

He considered particularly disruptive the case in which the solutions do not all have the same sign, but in which mixed positive and negative solutions appear. In these cases, he said, interventions into these equations were particularly difficult to carry out: "Negative roots, I repeat, are an inconvenience, above all those mixed up with positive ones" (ibid.).

Let us sum up what has hitherto been obtained with regard to d'Alembert's breaking with the traditional development of the concept of negative numbers:

- For him, *quantities* are the basic and initial concept of mathematics; *numbers* have only a derived status, and not an independent one.
- Since the classical quantities are only positive-valued, the mathematical zero and the metaphysical nothing signify the same; there are no quantities smaller than zero/nothing.
- Where isolated negative quantities appear in a solution, they indicate an error in formulating the problem.[59] In contrast to Clairaut, who was satisfied with a subsequent reinterpretation of the problem conditions, such an error, for d'Alembert, requires one to intervene into the problem formulation and to change the signs of the corresponding equation.

More generally, it can be noted that d'Alembert adopted and radicalized Fontenelle's conception of the two different *êtres* of negative quantities: they

[59] Occasionally, d'Alembert emphasizes, without going into detail, that there is a second meaning of negative quantities still: they could also indicate solutions of the same problem under an aspect slightly different from that assumed in the problem, but nevertheless analogous to the latter's sense (e.g., d'Alembert 1805/1965, 302).

have a numerical quantity and agree in this with the positive quantities; at the same time, they possess quality, an *idée*. This quality, however, has no connection to the first *être*. It designates only a potential difference—one he did not consider relevant—of position with regard to the coordinate axis, as well as a command to remove this as if it were a normal quality by speedily shifting to positive solutions.

Let us return to the inquiry into the causes or motives for this epistemological rupture. D'Alembert himself has extensively described them in his treatise of 1761. The rupture was precipitated by a problem in applying imaginary quantities, a problem in conceiving of the logarithm function concerning negative numbers. D'Alembert wished to demonstrate the real-value character of these logarithms at any cost, and this compelled him to overturn the traditional conception of negative quantities. This reasoning of d'Alembert's developed into a controversy with Euler, and it can be reconstructed from their correspondence.

A controversy about the nature of the logarithms of negative numbers had already been conducted in 1712/13 between Leibniz and Johann I. Bernoulli. Leibniz had declared them to be imaginary-valued, whereas Bernoulli considered them to be real. No agreement was reached, but the public did not learn anything about their controversy. Only the publication of their correspondence in 1745 by G. Cramer made it accessible. D'Alembert's claim in a memoir of 1761 that he had got to know this correspondence only shortly before[60] is difficult to accept, since Cramer's edition certainly must have become rapidly known at the Paris Academy. Soon after this edition appeared at the beginning of December 1746, d'Alembert sent a *mémoire* to the Berlin Academy with results on integral calculus, the first part of which contained a proof of the fundamental theorem of algebra, and at the same time a text in which he explained his own view that the logarithms of negative numbers are real.[61] After Euler had presented his own conception in his reply of the end of December saying that the logarithm of –1 was not real, but imaginary, and even had many imaginary values, a controversy extending over two years ensued, in which no agreement was reached. D'Alembert's arguments in this controvery were virtually never taken up. Youschkevitch and Taton, who comment the Euler–d'Alembert correspondence in detail, note quite generally:

> We should silently pass over the latest arguments of d'Alembert, a certain mixture of reasons which he himself describes as metaphysical, drawn from an improper extension of the properties of ordinary logarithms, and of pseudo-geometric

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60 Actually, d'Alembert wrote this treatise, published in 1761 in the *Opuscules*, already in 1752 (cf. Youschkevitch/Taton, in Euler 1980, 18).

61 The pricese wording of this text is not known, since Euler refrained from publishing it at d'Alembert's request, and the original manuscript no longer exists (information from the Archive of the Berlin-Brandenburgische Akademie der Wissenschaften). In his reply of December 29, 1746, however, Euler gave a brief rendering of the argument already used by Bernoulli (Euler *Opera* IV A, vol. V, 252).

considerations, etc. He lacks above all a good definition of logarithm (Youschkevitch, Taton , in Euler 1980, 19).

A similar position was taken by Verley, who briefly remarks in his description of the Euler-d'Alembert controversy that d'Alembert never attained an understanding of multi-valuedness (Verley 1981, 126). The connection to a conception of negative quantities had not been seen, or not been evaluated.

D'Alembert's reasoning in the correspondence extending over two years keeps turning around a point that seems to have been critical for him: whether Log(–1) is real and whether in particular Log(–1) = Log(1) = 0, and after Euler's many exhausting mathematical argumentations he withdrew to asking whether one could not at least *assume* Log(–1) to be real? It is indeed remarkable, on the one hand, how d'Alembert resorts to ad hoc constructions (such as admitting a negative unit for logarithms, as quoted in Euler *Opera* IV A, vol. V, 261) to maintain his own position, or challenges Euler's concrete and extensive reasoning only with some general, unsubstantiated doubts, such as questioning Euler's explicit description of the indefinitely many imaginary values of Log(–1) as to whether this formula was actually complete and whether a real value did not nevertheless exist after all (ibid., 268 and 290).

On the other hand, Euler, too, had to concede points, and to revoke some things. Thus he was compelled to withdraw (ibid., 270) his own argument saying that e could not be negative for reasons of its expansion into a series alone (ibid., 264) after d'Alembert's objection that expansions into a series will sometimes yield false values, and not the totality of values (ibid., 267). Again and again, reasons went back and forth in their debate whether e^x was a unique or twofold function. First, Euler admitted that the function had two values for $x = \frac{1}{2}$ (ibid.) then d'Alembert conceded that e^0 did not have two, but only one value, adding, however, that one might indeed suppose it to have two (ibid., 273).

Eventually, Euler modified his standpoint with regard to the question so essential for d'Alembert as to whether e^x was multi-valued so extensively that he did not permit his own *mémoire* "sur les logarithmes des nombres négatifs et imaginaires," presented to the Berlin Academy on September 7, 1747, to be printed, but published a different version later in which this point in particular had been changed. As Euler wrote after some pause to d'Alembert on February 2, 1748, he had reflected further on e^x, having come to the conclusion that there were twofold values not only for $x = \frac{1}{2}$, but also generally for $x = \frac{n}{2}$, n being an odd number: there were thus any number of conjugated negative values, these existing, however, as isolated points, and no longer a continuous curve. Such a curve, he said, was present only above the axis (ibid., 280). D'Alembert concluded from that for his own purposes, however, it could therefore be stated that negative quantities could indeed have a real logarithm (ibid., 286).

In August 1747, Euler had referred d'Alembert to his *mémoire* presented to the Berlin Academy, which would soon be printed (ibid., 270). D'Alembert declared himself most willing to submit to these arguments, but after further objections from d'Alembert, Euler wrote back that his forthcoming "piece" would probably not remove all of d'Alembert's doubts (ibid., 275). Euler indeed withdrew this *mémoire* and revised it completely, while he wrote to d'Alembert, obviously at the same time, that he himself was not familiar enough with the issue of the imaginary logarithms any more to give well-founded answers to d'Alembert's new remarks; he thus had to refer him to taking up the matter again some time later (letter dated September 28, 1748; ibid., 293). Euler no longer hoped to be able to convince d'Alembert, and thus ended the controversy. Euler's revised version was published in the Berlin Academy's *Abhandlungen* of 1749 in 1751. In response, d'Alembert sent a *mémoire* to the Berlin Academy in 1752; since it remained unprinted, he published it himself in 1761, in the first volume of his *Opuscules*.

Euler's first version of his *mémoire*, the version he had given in the 1747 lecture in Berlin, already represented an excellent didactical elaboration, which made the then novel matter of distinguishing between areas of definition and infinite complex values comprehensible.[62] It contained, however, a number of allusions to contemporary representatives of Bernoulli's views (Euler *Opera* I, vol. 23, 421 and 425) that would have amounted to Euler making his controvery with d'Alembert public. Euler wished to avoid this, and in his revised version he was indeed in a position, in his review of Bernoulli's arguments, to discuss all of d'Alembert's objections, including the new ones, without naming him. The revised version of 1749/1751 (Nr. 168 on Eneström's list) represents a didactical masterpiece of a technical and unemotional reflection on all the difficulties having arisen as yet in the concept of the logarithmic function, and in its dissolution within the new concept of an infinitely valued complex function. He began with four justifications reflecting J. Bernoulli's position that the logarithms of negative numbers were real and equal to those of positive numbers. Against this, Euler raised six objections. Then he presentend, while quoting three justifications, Leibniz's view that the logarithms of negative numbers were imaginary and differed from those of positive numbers by an imaginary constant. Against Leibniz's views as well, Euler enumerated three objections. Since the respective objections could not be completely refuted, Euler's lucidly conducted demonstration showed that neither side was correct, and that the seemingly clear concept of logarithm must thus be contradictory!

By a subsequent theorem and by three propositions, Euler proved that the contradiction could be removed only by changing the concept of logarithm. To every number now corresponded an arbitrary number of function values, and only in the case of positive numbers was one of these values real (Euler *Opera* I, vol. XVII, 1915). Euler did not explicitly discuss his own concept of negative numbers in either of the two versions of his *mémoire*. He presented them later, in his textbook on algebra (see below, Chapter IV.1.1.).

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62 This *mémoire*, Nr. 807 of Eneström's list, was published in 1862 by Fuss in the *Opera postuma*.

D'Alembert, in his answer of 1752 published in 1761, did not discuss Euler's new conception of logarithms in detail. Rather, he tried to insist by some general remarks that the logarithms of negative numbers *could also* be real, proposing indeterminate definitions of logarithm ("une suite de nombres en progression Arithmétique *quelconque*, répondans à une suite de nombres en progression Géométrique *quelconque*") without developing them operatively, declaring it to be unproblematic to assume Log(nx) = Log(x). D'Alembert concluded that the logarithms of negative numbers could just as well be assumed either to be real or imaginary, the issue being dependent on the choice of system alone (d'Alembert 1761, 181–198).[63]

It is particularly revealing how d'Alembert made the connection between his argumentations on logarithm and his conception of negative quantities explicit here. D'Alembert again confirmed that negative quantities, for him, were essentially identical with the positive, they differed only, he said, by their "opposite" position; a property that he refrained from describing in more exact detail. He inserted it operatively to attain the goal he aspired to with the logarithm: Log($-x$) = Log(x). This at the same time yielded the real value of the logarithm of negative quantities: "lx and $l.{-x}$ give the logarithmic [curve] two branches, which are equal and similarly placed with regard to the axes" (d'Alembert 1761, 195).

Also important for d'Alembert was Fontenelle's distinction between a quantity's numerical being and its specific, qualitative being, d'Alembert strictly separating the two types of being. He stressed that the negative quantities were "just as real" as the positive (ibid., 202), admitting that they form another progression 0, –1, –2, –3, etc., *qui revient* [...] *en sens contraire* to the progression of positive numbers to which it formed the "complément" (ibid., 187). The minus sign, he said, had no influence on the nature of these quantities:

> The sign – which the algebraic expression carries […], only indicates its position, and has no effect on its quantity. […] Negative quantities have no other difference from positive quantities than that they are taken on the opposite side (ibid., 202).

The oppositeness of position was of no consequence for d'Alembert, since he saw relations of quantities determined only via the first, numerical, being: "These magnitudes have no other ratios to each other than that of their quantities" (ibid., 202).

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63 It has been neglected in the literature on the history of mathematics that all of d'Alembert's arguments against Euler by which he intended to prove his own proposition of Log(–1) = 0—even those put forward in obscure texts—have been discussed and refuted in detail, in a treatise by a mathematical outsider of 1801 written quite according to the conception professing dominance of the analytic method. Its author, A. Suremain-Missery, a former artillery officier, most excellently continued Euler's arguments of the study of the complex multiplicity of the logarithm values (in particular A. Suremain-Missery, 1801, 23 ff.). Although Grattan-Guinness mentioned the author's correspondence with Lacroix and with the *Institut* named this treatise, its significance escaped him (Grattan-Guinness 1990, 257).

D'Alembert made quite explict his own epistemological point of view recognizing only the absolute values of numbers as numbers:

> There can be no ratio between –*a* and *b* other than if one compares the magnitude of *a* with that of *b*; the sign – is only a denomination. […] In a word, every quantity on its own has the sign + (ibid., 203).

For this view as well, an epistemological conception can be found to be implicitly determining in d'Alembert. As he had made explict in his *Encyclopédie* article *Quantité* (*Philosophie*), he assumed the concept of quantity to be bound ontologically. In this article, he had criticized the view of quantity as a substance, since this substance then must also participate in a quantity's changes. D'Alembert was intent on excluding such changeability. He thus underlined that any quantity must be firmly bound to an object, without such a tie, a quantity was a pure abstraction:

> We imagine *quantity*, as an abstract notion, as like a substance, and increases and reductions as modifications, but there is nothing real in this notion. Quantity is not an attribute susceptible of different ways of determining it, some constant, others variable, which characterize substances. *Quantity* requires an attribute in which it resides, and outside of which it is nothing but pure abstraction (d'Alembert, *Quantité* 1765, 653).

In retrospect, we may state that Euler's and d'Alembert's efforts at attaining generality and rigor both had their price. An increase in rigor implied a loss in generality. Euler had to abandon his principle that (real) functions have to be understood as being generally defined, without stating a restricted field of definition. This point of view was a consequence of his own epistemological position of seeing generality represented by algebra. Since algebra assured general validity, deviations were considered to be exceptional values that did not detract from general validity. Fraser has succinctly characterized the identification of algebraization with generality as follows:

> In Euler's or Lagrange's presentation of a theorem of the calculus, no attention is paid to considerations of domain. The idea behind the proof is always algebraic. It is invariably understood that the theorem in question is generally correct, true everywhere except possibly at isolated exceptional values. The failure of the theorem at such values is not considered significant (Fraser 1989, 329).

For the loss of generality, Euler obtained in compensation a rigorous understanding of the logarithm function. In his seminal textbook *Introductio in analysin Infinitorum* (1748), he defined the logarithm only for positive numbers (Euler 1748/1983, §102, 76). D'Alembert intended to maintain the general definition of the logarithm at any cost, but had to accept a restriction of the area of real number in exchange.[64] It is remarkable that a later resistance in Germany against Euler's restriction of the field of definition went along with support for d'Alembert's conception of absolute numbers (cf. below, Chapter II.2.10.4.).

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64 He was unaware of the consequence that he again restricted the area of the logarithm in doing so.

At the same time, d'Alembert is a case in point showing that a mathematician's epistemological views need not comprehend and "direct" his entire field of mathematical activity, but can also affect part of them without entering into conflict with the other parts and their possibly underlying epistemological conceptions. We noted at the beginning of this section that d'Alembert had acknowledged the entirety of rules of operation for negative quantities developed before. An even more far-reaching example is that d'Alembert developed algebraic geometry much more explicitly toward opposite quantities than Descartes did. While Descartes applied his own new method of coordinates virtually only within the first (positive) quadrant, d'Alembert very clearly demonstrated that complete gaphs of curves can be obtained only by taking all four quadrants into consideration. D'Alembert presented this further development of algebraic geometry consistently, in particular in his article *Courbe* in the fourth volume of his *Encyclopédie*, an article he later referred to with great pride as the state of affairs he had achieved. This is where he extensively justifies the necessity of including all positive and all negative axes in determining a curve (d'Alembert, *Courbe* 1754, 379). He stressed that the totality of positive and negative values form the necessary basic unit (ibid.).

D'Alembert's conception of algebraic geometry implicitly and unproblematically assumes that negative values are smaller than positive values, respectively that they lie "below" zero, and below the positive values. The figures illustrating the article *Courbe* in the accompanying *Planches* volumes of the *Encyclopédie* show the entire coordinate surface appropriately situated for curves. The coordinate axes are not separately designated; while points lying symmetrically to the axes are not designated as opposite ones by designations like $M/-M$, their oppositeness is hinted at by designations like M/m.

Generally, this section shows as it were quite incidentally how inappropriate the common notion of mathematics in the eighteenth century is: it is seen as a naïve advance unconcerned with foundations. Frequently, this attitude is ascribed precisely to d'Alembert, whose advice to a beginner, *Allez en avant, et la foi Vous viendra*, is interpreted as unconcern with secure foundations. By contrast, his controversy with Euler shows how intensely he worked on foundational issues. He must thus be taken seriously when he asked Euler not to consider him stubborn when he demanded to be enlightened in order to be convinced:

> You find me, perhaps, very importunate and opinionated to return again to the same things. But the truth can only be found by a great deal of patience and superstition, and I am only seeking to be enlightened so as to give up (Letter of 7.9.1748; Euler *Opera*, IVA, vol. V, 291).

2.10. Aspects of the Crisis to 1800

D'Alembert's break with the conceptual development of negative quantities up to his time had not the same impact all over, but one that varied depending on the respective institutional and cultural contexts of crisis. Notable in France, in particular, is the different effects in the two competing institutional contexts of the universities and the military and engineering schools.

2.10.1. Stagnant Waters in the French University Context

The textbooks for the *collèges* of the French universities do not show any essential impact of d'Alembert's new conception. They continued to use the traditional concepts, albeit without reflection of their own, and even in a stagnant and bowdlerizing fashion, as compared to the earlier authors. That d'Alembert would encounter resistance here had become clear from the very *Encyclopédie* itself. It contained an article parallel to d'Alembert's own that held a contrary position: *Quantité, en termes d'Algèbre*. It was signed "E," which meant that this article had been written by Abbé Joannes B. de la Chapelle (1710–1792), at the same time author of an influential mathematics textbook for the university context: *Institutions de Géométrie* (1746). De la Chapelle was not himself a teacher of mathematics, but a *censeur royal*—a post salaried by the Crown for the examination of manuscripts submitted to determine whether they were to be authorized for printing with royal *approbation*, or *privilège.* The parallel *Encyclopédie* article classifies the *quantités algébriques* into positive and negative ones, determining their concept as follows: positive quantities shall be called those larger than zero ("au-dessus de zéro"), and negative those that *are* not smaller than nothing (as the positive *are* larger than zero), but *are regarded to be* less than zero—a formulation that most probably was a concession to the Encyclopedia's coeditor d'Alembert (*qui sont regardées comme moindres que rien*: de la Chapelle *Quantité* 1765, 655). The text then identifies positive quantities with additive, and negative quantities with subtractive quantities:

> All that is needed to produce a positive *quantity* is to add a real quantity to nothing; for example $0 + 3 = +3$; and $0 + a = +a$. In the same way, to produce a negative *quantity* all that is needed is to take a real *quantity* away from 0; for example $0 – 3 = –3$; and $0 – a = –a$ (ibid.).

De la Chapelle continued by discussing the view held by "some" authors unnamed according to whom negative quantities were "deficits" (*les défauts*) in positive quantities. Actually, this was but one author, Christian Wolff, whom de la Chapelle discussed just like d'Alembert, quoting him even more extensively than the latter to say that negative quantities were homogeneous among one another, but heterogeneous to the positive ones; one could thus establish, between $-3a$ and $-5a$, the relation 3 : 5, but the relation of 1 : –1 was different from the relation of –1 : 1. De la Chapelle countered this Wolffian argument by saying that the proportion 1 : –1 :: –1 : 1 was valid, since the product of the outer components was equal to the product of the inner components. De la Chapelle thus added the remark that the concept of negative quantities was not *parfaitement exacte* in (some) authors (ibid.). To close, de la Chapelle listed the most important rules for operating with negative quantities.

In his own mathematics textbook,[65] he said much more, in the part on algebra preceding that on geometry. He did not present a unified conception, however, but merely juxtaposed fragments of different origin. His first determination of concept did not conceive of negative quantities as of subtractive ones, but as a general expression of any kind of subtraction, a conception that disregarded the differentiation between algebraic sign and sign of operation that had been clarified long before:

> Algebraic quantities preceded by the sign + are called *positive*, and those preceded by the sign – are called *negative*. The quantity $a + b$ shows that $+b$ is a positive, and in $p - m$ one can see that $-m$ is a negative (Chapelle 1765, 156).

Somewhat later, Chapelle gave a second determination of concept, which came from another conceptual tradition, defining algebraic quantities as opposite quantities. Here, Chapelle emphasized that positive and negative quantities were similar and hence capable of forming a common concept field:

> Positive quantities are directly opposite to negative quantities to which they are similar, and so these quantities wipe themselves out reciprocally (ibid., 159 f.).

And according to this conceptual tradition, Chapelle stressed that the two kinds of quantities were equally legitimate: “By consequence negative quantities are just as real as positive ones” (ibid., 160). As “proof” for the rule of signs, he used Arnauld’s reasoning concerning the function of multiplier and multiplicand (ibid., 165 f.).

Chapelle did not develop anything further on his own. In his part on equations, he described the multiplicity of roots (ibid., 232 ff.). In another of his textbooks, in his *Traité des Sections Coniques* (1750), everything including the figures shows that he used the full system of coordinates and all four quadrants to examine and represent the curves, but points lying symmetrically to the axes are not characterized as correlating points, neither by opposite signs nor by analogous designations (like *M*/*m* in d’Alembert).

An even more trivializing conception is found in Abbé Sauri’s (1741–1785) textbook *Institutions mathématiques*. This textbook, first published in 1770, saw five reprints—until 1834!—and was widely disseminated in the context of universities as well. Sauri was *professeur de philosophie* at the University of Montpellier, offering in this capacity the mathematics courses within the philosophy class of the *collège*. Sauri’s texts no longer contain any hint at the

65 The first edition appeared in 1746, the last in 1765.

conception of opposite quantities, and no reflection on concepts either. Only one definition is given at the beginning of the part on algebra:

> Algebraic quantities preceded by the sign + are called *positive*, and those preceded by the sign – are called *negative*. The quantity $a + b$ shows that $+b$ is a positive, and in $p - m$ one can see that $-m$ is a negative (Sauri 1777, 36).

This definition, briefly illustrated, is followed by a naked presentation of the rules for operating with positive and negative quantities without any specifications or clarifications by the author himself. Thus, Sauri notes that it was inconvenient to confuse *augmenter* with *ajouter*: to add to b the term $-d$ gave $b - d$, that is, a diminuition (ibid., 39). He did justify the rule of signs separately for isolated quantities: by means of the differentiation between multiplier and multiplicand in Arnauld's sense (see ibid., 40 f.).

The deficit in determining positive and negative quantities by way of the algebraic signs of numbers implicitly assumed to be absolute has already been mentioned. It is thus no wonder that the wide dissemination of unsatisfactory conceptions eventually led to some kind of fundamentalist reactions (see below, Chapter IV.1.4.).

A first approach at conceptual clarification is finally found in a textbook first published in 1781 for the university context, in the *Éléments de Mathématiques* by Roger Martin. Martin (1741–1811), who obviously did not pursue extended studies, became a professor for philosophy at the *Collège Royal* in Toulouse at the age of less than 20 years. In 1782, he took the college's newly created post for experimental physics. As a freemason, he became at once politically active with the beginning of the Revolution, assuming many political and administrative positions in Toulouse. From 1795 to 1799, he was a member of the first chamber of the French Parliament *(Conseil des Cinq-Cents*), getting involved mainly in questions of education. After that, he returned to Toulouse, became professor for physics at the *École centrale*, and when courses in natural sciences were no longer being held at the new *Lycées* after the École's dissolution in 1803, the city founded an *École spéciale des Sciences et des Arts*, at which Martin again taught physics. In 1808, when the *Université Impériale* was founded, this institution was integrated as *Faculté des Sciences*, Martin finally becoming a professor of this faculty.[66]

Martin's textbook was reprinted in a somewhat revised version in the year X (1802).[67] Martin was the first author in France to use the term "opposite num-*bers*" (*nombres opposés*) and to make them the basis of his own conception.

Half of the 84-page *Discours Préliminaire* is devoted to discussing the concept field of opposite numbers. Martin was the first of the university textbook authors after d'Alembert to again undertake a comprehensive reflection on foundations. The innovation in his conceptions was first that he made the concept of number the basis of his arithmetic and algebra, the concept of

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66 An exhaustive biography was published by Gros in 1919.

67 I was able to study this edition thanks to Colette Laborde (Grénoble), who has this edition in her possession.

quantité no longer being the exclusive basis for the first time. Martin started from the concept of unit (*unité*), establishing that any quantity can be conceived of as a number, with reference to the underlying unit: *Une quantité quelconque peut être conçue comme un nombre* (Martin 1781, viij).

At the same time, Martin's approach shows how multidimensional and complex the transition from the general concept of quantity to the concept of number was. Martin went on to divide numbers into "abstract" and "concrete" according to whether their unit was abstract or concrete, and he declared it to be his position of principle that certain qualities could be connected only with the *concrete* unit—resulting in the fact that only the concrete numbers could form such numbers, like heterogeneous or opposite numbers, because of their unit's different "attributes" (ibid.). As we shall see, this position of principle involved Martin in some conceptual difficulties, but it permits us at the same time to analyze how effective the epistemological beliefs were that were shared by Martin and many French authors.

Since Martin based both his arithmetic and his algebra on the concept of number, he gave in his part on arithmetic a general definition of subtraction not confined to positive remainders. Subtraction, he said, provided the parts the addition of which gave the sum (Martin 1781, 6 ff.). And Martin also was the first to give an exclusively operational definition of opposite numbers—as numbers whose addition signified a subtraction, and whose subtraction gave an addition: *on peut dire en géneral que leur addition doit se changer en soustraction, et leur soustraction en addition* (ibid., xiij).

Martin stressed that he had for the first time used this property to define opposite numbers, and that it represented an essential concept in number theory despite its simplicity (ibid.). Although formulated quite verbally, it already contains structural analogies to the algebraic definition simultaneously developed in Germany (see below, Chapter VII).

Martin also was the first to criticize the French university tradition for hitherto defining negative numbers virtually exclusively by their being "plus petits que zéro" (ibid., xiv). This, however, was not a critique of the implicit conception of understanding quantities only as absolute, respectively positive, ones; thus a definition of negative quantities via their algebraic sign was considered to be sufficient. For he made a point of stating that one could not imagine any quantity smaller than that which had already attained the term of zero, since zero was the extreme limit of diminishing or decreasing a quantity: *puisque zéro étant la limite de tous les décroissemens possibles d'une quantité* (ibid.).

Martin admitted the notion of "smaller than zero" only as a relative way of speaking: a negative number was relatively farther distant from a positive number than the zero (ibid.).

Martin actually did not adhere consistently to his own conception of taking the number concept as basis. While Martin introduced negative numbers, in contrast to all French authors before him, in the first part of his textbook on arithmetic,

> Two numbers are said to be *opposite kinds*, or simply *opposites*, when the addition of the one reduces the other, and the subtraction of the one increases the other. To distinguish between them, those of the same kind are called *positive* and preceded by the sign +; and those of the opposite kind are called *negative*, and preceded by the sign – (ibid., 10).

Stressing that the two conditions have been chosen at will, he repeated the definition for the representing terms in the second part of his textbook on algebra, which he himself considers to be that science, which investigates the general properties of all kinds of numbers by means of the symbols representing them. Now, however, he again tacitly assumed that the terms merely represented the positive numbers:

> Since there are positive and negative numbers, algebraic terms, which represent them must be distinguished by the opposite signs + and –. Here therefore, as in Arithmetic, terms preceded by the sign + are called *postive*, and those preceded by the sign – are called *negative* (ibid., 44).

Martin later inserted mostly only positive values for coefficients and unknowns, but gave one normal form of $x^2 + px + q = 0$ in the case of quadratic equations, inserting then, after deriving the general solution, a positive value for p, and a negative value for q (ibid., 162). It was undoubtedly one of Martin's decisive steps to choose numbers for his basic concept, and to desist from operating with the concept of quantity in algebra, taking representative terms instead, but the transition to the general concept of the representing sign was not yet complete, because of still prevailing epistemological notions pertaining to substantialist views. While Martin's idea to conceive of oppositeness and negativity as of additional qualities of numbers corresponds to Fontenelle's thoughts (see Section 2.8.1.), the operational novelty of interpreting the additional quality as an "attribute," and hence as constitutive element of a "concrete number," is his own. It guided Martin to reflecting on the concept of multiplication, a reflection that shed sudden light on the fundamental and long unsolved problems of multiplying quantities, and on the immediate connection between extending the area of numbers and defining all the basic operations; at the same time, it led the author into a situation without exit from which he could escape only by willful positing. Martin used the familiar concept of multiplication, according to which the multiplier had to be an "abstract" number, and only the multiplicand was permitted to be a "concrete" number or quantity. Since he himself had postulated, however, that opposite numbers were "concrete" numbers, he saw no possibility of admitting a product from such numbers; products had to agree, accordingly, with the multiplicand in their "dimension." Just as inadmissible for Martin was a product of a quantity of space and one of time, and neither a product of two linear quantities, nor a product of quantity of motion and a quantity of mass (ibid., xv f.).

Martin made his escape from this contradiction by declaring—in a way somewhat comparable to that of Fontenelle and d'Alembert—a potential minus sign preceding the multiplier to be negligible, “it is [done] by abstracting the sign from the multiplier and considering this factor as a pure abstract number” (ibid., xvj).

The concrete method was to be the following: If the task, for instance, was to multiply –6 by –3, the first step was to multiply –6 by 3 as *nombre abstrait*, and after that the result of –18 was to be subtracted from a real term, or an assumed term A. Martin justified this “trick” by explaining that there were sometimes propositions in mathematics that, taken literally, seemed absurd while expressing, if understood correctly and in the “sense of their inventors,” precisely a method that could be easily memorized (ibid., xvij).[68]

Such cloaking of inconsistencies could not serve the clarity of concept proclaimed, but was an expression of strong underlying epistemological views. Since these were too strong, it probably was not feasible for Martin to look for an alternative justification of opposite numbers: justifying them not as concrete but as abstract numbers.

The second conceptual problem Martin became entangled in lies in his further basic assumption that positive and negative quantities were *heterogeneous*. By explicitly quoting Christian Wolff (“Wolf”), and by implicitly rejecting Wolff’s critique by d’Alembert and de la Chapelle, Martin even extended Wolff’s conception. While Wolff had established in a few sentences that positive and negative quantities were heterogeneous, and that no proportion 1 : –1 :: –1 : 1 could therefore exist (see above, Section 2.8.3.), Martin extended these propositions into a larger system by comprehensively reflecting the concept of ratio (*raison*). His basic assumption here was: a ratio could only exist between two quantities one of which was contained as a part within the other—and which, however, are homogeneous to each other (ibid., xxxiij ff.). There could thus be no ratio between heterogeneous quantities, respectively numbers. In addition to that, he required, for measuring ratios, to use not the arithmetic means, the fractions, but rather to maintain the original concept of measure (ibid., xxvij ff.); since he had postulated positive and negative numbers as heterogeneous before, this made clear that there could be no ratio between these either, and that Arnauld’s proportion argument was thus without foundation as well (ibid., xxx ff.).

Here again, Martin had introduced an ad hoc construction in order to avoid another consequence: that of restricting the concept of ratio to positive quantities. Instead of reflecting this concept’s original restrictedness in the field of geometric quantities, Martin tried also to maintain the basic epistemological trait of a generality as comprehensive as possible for the concepts. He did not become aware of the fact that he had confined the field of application by his own definition of heterogeneity virtually in the same way.

Martin was also subjected to criticism by one of his teacher colleagues in Toulouse. In his contribution to Cantor’s handbook, Cajori mentions a treatise of 1784 by Gratien Olléac, mathematics teacher at the Collège national de

68 Martin advanced analogous arguments in favor of division (ibid, xx–xxiv).

Toulouse: *Sur des théories nouvelles des nombres opposés, des imaginaires et des équations du troisieme degré*. Cajori recorded only the explicit critique of Christian Wolff:

> One should give up as absurd Wolff's idea of the heterogeneity of numbers according to which they cannot have any relations with one another, and adopt Descartes's idea of the reality of both negative and positive numbers (Cajori 1908, 86).[69]

Besides openly criticizing Wolff, Olléac actually challenges Martin, without naming him, by quoting the latter's definition of *nombres opposés* (Olléac 1794, 8). Olléac is correct in criticizing Wolff's and Martin's notion of heterogeneity for being inconsistent in merely excluding the existence of ratios between heterogeneous quantities while nevertheless executing arithmetic operations between them (ibid.).

Olléac's own justification of why positive and negative numbers were homogeneous, however, was not suited to solve the conceptual problems. Olléac based his own approach on rejecting Martin's assigning *espèces opposés* to positive, respectively negative, numbers, and virtually assuming conceptual identity for them in implicit adherence to d'Alembert. Quoting the geometric example of two radii lying in opposite directions in a circle, he claimed them to be essentially identical, differing merely in one "aspect," by means of a mental operation that, however, did not change their nature:

> The numbers which are expressions [of the two radii] are called, one positive and the other negative, or opposite numbers of a common number; but these quantities were of the same kind before being envisaged from this point of view, since each one was a radius; thus they must be so after, since the way in which a thing is envisaged is only a purely mental operation which cannot change its nature; […] so the numbers which express the value of these radii, no matter that they are opposite, are homogeneous (ibid., 11 f.).

After that, Olléac had no difficulty in justifying the rule of signs, since multiplying a negative multiplicand meant nothing but multiplying a positive one, which was being considered only *sous un point de vue contraire*. Otherwise, Olléac adhered to his colleague Martin in understanding a negative product as a subtractive term (ibid., 18 ff.).

Martin remained unimpressed by Olléac's objections to the heterogeneity claimed, and did not revise the second printing of his own textbook of 1802 in this respect.

It can be generally said that Martin's conflict between number concept and proportion concept illustratively demonstrates the sharp contrast between the traditional geometric founding of mathematics and the dimensions of restructuring mathematics under the program of algebraization.

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69 I was unable to unearth this work in any library in France, in particular not in the libraries of Toulouse, and neither in Toulouse's municipal archives. One exemplar, however, is present in the New York Public Library.

It was unfortunately impossible to find further bibliographical data on Olléac. It is only clear that he taught mathematics at least after 1793 at the same institutions where Martin taught physics. I am very grateful to Madame Jocelyne Deschaux, Bibliothèque Municipale de Toulouse, for her extensive research.

2.10.2. THE MILITARY SCHOOLS AS MULTIPLIERS

Shortly after the onset of establishing the military school system in France began the systematic introduction of textbooks, in particular for mathematics. The fact that textbook authors and examiners (*examinateurs permanents*) were in most cases one and the same caused these textbooks to be widely disseminated and applied. Among the three major fields of education—navy, artillery, and engineers—the military engineers (*corps du génie*) were the first to respond. The examiner of the *École de Mézières*, Charles Etienne Camus (1699–1768), was the first to have been officially charged with elaborating a mathematics textbook: the four-volume *Cours de mathématiques* (1749–1752 in first edition). Because artillery was for some time lumped together with the *corps du génie* in 1755, Camus became examiner for artillery as well, which resulted in his textbook being used there as well. After the two *corps* had been separated in 1758, Camus's textbook was criticized by the artillery as too elementary, and too little oriented to practice (Hahn 1986, 529 ff.), yet it remained in use until Camus retired in 1768. With four printings in less than twenty years, it was a quite successful textbook.

The textbook's elementary character is less due to Camus himself, but rather a first typical expression of the institutional "framing" of the subject matter and methods of textbooks. Camus authored his textbook indeed on commission: it was intended for the training of military engineers at the engineering school founded in 1748 by Count d'Argenson, the later famous *École de Mézières*. M.A.R. Paulmy Voyer, Count resp. Marquess d'Argenson (1722–1787) had in 1748, as *Sécrétaire d'État* and war minister, transformed the hitherto informal modes of training for these engineers into a formal, institutionalized education. To ensure the quality of this education, he had at the same time commissioned Camus with authoring a textbook, establishing for this purpose both type and extent of the knowledge required, as well as the method of teaching. It is markedly typical for the sectoring between university and military education that d'Argenson decreed the method for the education of engineers to be the "synthetic" one. The linkage between method of teaching and educational objective had been strictly intentional, as is evident from Camus's introduction to his first volume, on arithmetic:

> M. le Comte d'Argenson […] has decided on the degree of competence that must be demanded of Candidates: he has even been kind enough to provide all the details of their instruction; […] he has ordered me to bring together in a single Work, treated synthetically, all the Theory which an Engineer needs to have (Camus 1749, i).

What was understood by the "synthetic" method for engineers can be directly seen from Camus's textbook. It consists of volumes on arithmetic, geometry, statics, and hydraulics, but contains no part on algebra. Camus himself explains this absence of algebra, which might astonish some, as execution of the synthetic method: he had not presented algebra, which he calls *Calcul littéral*, together with the *Calcul numérique*, as other authors had done, because it did not belong among the knowledge mandatory for engineers:

> But having treated the principal parts in which an Engineer needs to be instructed by Synthesis alone, I thought I must reserve literal Calculus for Analysis (ibid., iv).

Camus was so rigorous in his method as to exclude algebra altogether from his textbook for engineers:

> I advise, then, that here I shall not talk about the literal Calculus and Analysis until after having kept my promise and given the Treatises which I have just announced making use only of Synthesis (ibid.).

Camus did not author a textbook of this kind for a more general public. At the same time, it becomes evident that grossly operationalizing the synthetic method in concentrating exclusively on geometry corresponded with a basic conviction widespread among users of mathematics as well.

On the occasion of a navy reform in 1763, Étienne Bézout (1730–1783) not only became examiner for the navy, but was also commissioned with authoring a textbook. His *Cours de mathématiques à l'usage des Gardes du Pavillon et de la Marine* appeared in six volumes starting in 1764. Together with the edition for artillery, where Bézout had assumed the function of examiner in 1768, too, the work led the market;[70] his navy edition saw eleven reprints until 1791, and the artillery version two (1770/72 and 1781) (see Lamandé 1987, 373).

There was one competitor, however, to Bézout's textbook. At the *École de Mézières*, Abbé Charles Bossut (1730–1814), until then mathematics professor of the school of engineers, succeeded Camus as the school's examiner in 1768. In his function of professor, he had striven to raise the level above that of Camus's elementary textbooks. As examiner, he published a multi-volume *Cours de mathématiques à l'usage des éleves du corps royal du Génie*, starting in 1771:

- *Traité élémentaire d'arithmétique* (1772),
- *Traité élémentaire d'algèbre* (1773),
- *Traité- Traité élémentaire de géométrie et d'application de l'algèbre à la géométrie* (1775),
- *Traité élémentaire de mécanique: statique* (1772),
- *Traité élémentaire d'hydrodynamique* (2 volumes, 1771).

Beyond that, Bossut also published revisions of the textbook, such as:

- *Cours de mathématiques à l'usage des écoles royales militaires*, in two volumes (1782),

a volume intended for the formation of infantry and cavalry officers. In his textbooks, Bossut adhered to the algebraization tradition. Thus, he emphasized algebra's high degree of generality, which permitted easy and manifold applications:

> It is evident […] that by one and the same algebraic calculus one resolves all problems of the same kind, proposed in all the generality of which they are

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70 "Gardes du pavillon" (guardians of the flag) is an ancient name for French navy officers.

> susceptible; and that the applications of this calculus to all particular cases are no more than subsequent operations, reduced to their greatest degree of simplicity (Bossut 1781, 286).

Bossut introduced negative quantities via the concept of oppositeness:

> [quantities] can be […] opposite to each other, as to the way they exist; and to mark this oppositeness, these quantities are distinguished in general as *positive* and *negative* quantities (Bossut 1773, 8).

Subsequently, Bossut also quoted the traditional definition by preceding signs according to which a quantity provided with a minus sign is a negative quantity (ibid.).

As in the university tradition, Bossut also emphasized that *soustraire* did not always mean *diminuer*, but rather was an increase in case of negative quantities (ibid., 14). In line with the level of the textbook's audience, no further conceptual reflections followed. Nor was the rule of signs justified in detail.

In the section on powers and roots, Bossut explained the multiplicity of roots (ibid., 74 ff.). He also used the ± sign, for example for square roots. In representing equations of second degree, Bossut not only explained that there was only one normal form, $x^2 + ax = b$, but also gave as general solution $x = -\frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}}$ (ibid., 184 ff.), introducing, in contrast to his own definition of the signs of positive and negative quantities, the coefficients a and b pointedly as *quantités réelles, positives ou négatives* (ibid., 186). In his examples, however, he inserted only positive values for the coefficients.

Bossut was a quite successful textbook author. The abbreviated version of his textbook of 1782 saw its third printing already in 1788. The version for engineers attained four printings by 1789; a revised edition appeared in 1800.

Bossut had no hope, however, of catching up, in the dissemination of his textbooks, with Bézout, whose textbooks saw a general dissemination and acceptance that no other mathematician before him had attained in France. A comparable success after him was achieved only by Lacroix. Étienne Bézout (1730–1783), member of the Paris Academy since 1758, became mathematics teacher and examiner for navy officers in 1763. About 1800, together with the new educational system's consolidation, we find *four* authors simultaneously publishing reeditions of Bézout: Peyrard, Reynaud, Lacroix, and Garnier! In manifold reeditions, these textbooks appeared until 1868 (cf. Lamandé 1987, 375). In the survey of the ministry of education of the year VII (1799) among the mathematics teachers of the *Écoles Centrales*, 50 of 69 reported that they were relying entirely, or in parts, on Bézout's textbook. L.F.A. Arbogast, who regretted in his answer that he was compelled to use this textbook, of which he was critical, explained its use with the fact that it had already been in the hands of the majority of students (Lamandé 1990, 32).

Bézout based the algebra part of his textbook on d'Alembert's conception of negative quantities, elaborating it so far as to enable it to achieve its true dissemination in practice; in some parts, he adopted d'Alembert's formulations word for word.

For an analysis of Bézout's conception, it is sufficient to examine the volume on algebra in his principal work, his textbook for the navy.[71] This is where Bézout steps forth as the first author who voiced a clear programmatic challenge to the self-understanding then current in French mathematics: to the belief that the analytic method is the best for taking the path toward increasing generalization. This is evident from the fact that in the six-volume work, algebra, as the general science of the quantities, no longer precedes geometry, but forms only the third part, *after* arithmetic and geometry. Beyond that, Bézout explicitly weighed the advantages of the analytic and the synthetic methods for his part on algebra. In contrast to the tradition since Ramus and Arnauld of ascribing better success in learning to the analytic method, Bézout was the first to introduce the distinction between *génie* and the public at large, declaring the analytic method to be suitable only for the "inventeurs," while the synthetic method was more appropriate for the majority: whereas the synthetic method gave the rules to be applied directly, the analytic method led via a complex sequence of operations and *raisonnements* toward the general rules (Bézout 1781, v). While this might raise beginners' curiosity, it entailed the danger that they might feel humiliated if the *raisonnements* did not appear by themselves and from "the depth of their own mind," as desired:

> This last method may seem preferable to the first, in that it appears it must flatter the self-esteem of beginners and stimulate their curiosity. But on reflection, while attention is necessarily divided between three objects, namely the statement of the question, the reasoning needed to express it algebraically, and the operations which need to be carried out with the aid of signs whose significance escapes one, the more easily to the extent that one is less experienced in representing these ideas in an abstract manner, it appears to me that it is doubtful that this method is the best to begin with, for the greater number of readers. Will it not produce, on the contrary, an effect totally opposed to that which some claim for it? The reasonings it demands, however simple to start with, where, doubtless, one treats only simple questions, these reasonings, I say, where the one who uses them is faced with having to draw on his very own resources, will they not humiliate him when they are not apparent to him? The method of invention presupposes always a certain finesse; it is the method inventors ought to follow, and consequently the method of men of genius; now they are certainly not the greater number (ibid., vj).

At the same time, to Bézout is due the merit of having been the first to transform the premise for algebra hitherto implicitly practiced in France into an explicitly formulated requirement. The basic concepts of algebra, the letter

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71 The analogous volume in his later, parallel textbook for artillery is the second of the four volumes. As to their conception, the two versions of Bézout's algebra are identical; they are even widely identical in text. The artillery manual is in parts of somewhat better quality of print, and more extensive in some presentations. It is notable that text problems are differently "cloaked" despite containing the same numeric values: Thus, a problem presented in the navy issue with pure numbers (Bézout 1781, 67) was reformulated for the artillery textbook with a vocational reference: with the corresponding number of "rounds for muskets" (Bézout 1779, 52).

quantities, are presented, and assumed, only as absolute quantities: "Letters represent only the absolute value of the quantity" (ibid., 79).[72]

The fascinating question is always how far an author keeps such a view throughout his entire textbook, or whether he nevertheless inserts negative values. This can be easily tested by how the solution of equations is treated. For Bézout, the test is difficult, since he does not discuss, or represent, any normal forms for equations of various degrees, but only individual concrete cases. And in contrast to all the textbooks analyzed here, he does not give any *formula* for the solutions of mixed equations of second degree, but a verbal formulation consisting of *nine* lines (ibid., 129 f.). From the other examples, however, it can be seen that Bézout remained consistently true to his own conception of absolute quantities. The zero, however, he had already admitted without problem in his arithmetic when introducing digits.

If there are essentially only absolute quantities, what can negative quantities then be? Bézout devoted an entire paragraph of eight pages to this question: *Réfléxions sur les quantités positives et les quantités négative*. These reflections are far more important than those of, say, de La Chapelle or of Sauri. Bézout saw the conceptual solution in the fact that quantities dispose of an *inner* property: they can be regarded under two opposite aspects, that is, as having the capacity of increasing another quantity, or as having the capacity of decreasing another quantity. As long as the quantity is represented only by a letter or a number, one could not make out its capacity aspect. To indicate its effect on other quantities besides the numerical quantity, one made use of appropriate signs:

> The same quantity can be considered from two opposite points of view, as being able to increase [another] quantity or being able to diminish it. While the quantity is represented only by a letter or by a number, nothing shows which of these two aspects is being considered. […] The most natural way for the difference to be revealed is to show it by a sign which indicates the effect which the [quantities] can have on each other (ibid., 79).

As such signs one used the signs of plus and minus, which Bézout clearly distinguishes from their function as signs of operation. On the basis of this symmetric ascription of function, Bézout was led to establish that negative quantities are just as real as positive quantities: "Negative quantities therefore have an existence just as real as that of positive quantities" (ibid., 80).

He thus had no problem with representing the multiplicity of roots, and with showing, in particular, that both $+a$ and $-a$ are roots of a^2 (ibid., 125 f.). Bézout also acknowledged the rules for operating with negative quantities. He justified the rule of signs only by the application of the distributive rule upon multiplying the two composite terms ($a–b$) and ($c–d$) (ibid., 18 f.). On the other hand, Bézout continues in the same quotation as follows: "and they are no different except that they have an opposite sense in calculations."

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72 It is remarkable that the designation "valeur absolue" is used here as something evidently familiar. The history of the concept of absolute value has seen little research as yet. Cf. Duroux 1983, Gagatsis 1995.

There was thus no longer oppositeness of "position," as with other authors of his kind, but oppositeness of "meaning"—a term that was at first indeterminate. Bézout now explained the difference of "meaning" in a way not to be expected by someone familiar with the presentation used up to then. Compared to the hitherto symmetric functions of the positive and the negative, it really represents a rupture that Bézout does not admit negative solutions, designating them as "impossible," or more precisely that he understands them as a demand to "reverse" the problem formulation. In this concrete mode of dealing with negative solutions, one may immediately recognize an elaboration and didactical implementation of d'Alembert's views concerning the epistemological exclusion of negative numbers:

> 70. If then, after having resolved a question, it happens that the value of the unknown found by these methods is negative; for example, if one arrives at a result like this, $x = -3$, it must be concluded that the quantity one has designated by x does not have the properties one had assumed in carrying out the calcuation, but quite contrary properties. For example, if one proposes this question. Find a number which added to 15 gives 10; this question is evidently impossible; if one represents the desired number by x one will have the equation $x + 15 = 10$ and, consequently, by virtue of the rules given above, $x = 10 - 15$ or $x = -5$. This last conclusion makes me see that x which I had considered before to be added to 15 to make 10, by right ought on the contrary to be subtracted. Thus all negative solutions indicate something false in the proposition of the question; but at the same time indicating the correction, in that it shows that the desired quantity ought to be taken in a sense completely opposite to the one that had been chosen.
>
> 71. Let us conclude then from this that, if having resolved a question in which some of the quantities were taken in a certain sense; if, I say, one wishes to resolve this same question in taking these same quantities in an entirely opposite sense, it suffices to change the signs the quantities carry at the moment (ibid., 81 f.).

It was this very epistemologically justified rejection of negative solutions, which, in their form of reformulating the equations with the objective of securing their positive character, was to have the most profound effects in France.

2.10.3. Violent Reaction in England and Scotland

As described in II.8.2, foundational controversies about the respective role of algebra and geometry had become public during the second half of the seventeenth century, conflicts, in particular debates, about how to justify concepts. Geometrical and empiricist justifications confronted algebraic generalizing systems of signs. While negative quantities had not been affected by the debates of that period, they now became the core of the conflict.

Robert Simson (1687–1768) can be identified as a pioneer opponent of algebraization and negative numbers in the first half of the eighteenth century. Simson, a self-educated mathematician (he had studied Euclid's *Elements*) was a mathematics professor at the Scottish University of Glasgow from 1711 on. An admirer of ancient mathematics, he was active as an editor of the works of Greek geometers. In this capacity, he sharply criticized modern algebraic development, condemning in particular Newton's use of algebraic methods in geometry. Simson did not publish these views, however, discussing them only with his disciples. He voiced his rejection of negative solutions of equations in 1764 in one of his letters (Pycior 1976, 49 ff.).

The first publication of this kind of approach is due to Thomas Simpson. Simpson (1710-1761) was a mathematics teacher in an area of applications and thus quite distinct from the universities, at the English Military Academy of Woolwich. To support his courses, he published his *Treatise of algebra* in 1745, the long-term influence of which is attested by a total of ten reeditions.

Simpson professed in his preface the belief that was to become the leitmotiv for a revisionism stressing the advantages of the synthetic method of the ancients over modern analysis with regard to rigor:

> We will see the Advantages which the ancient synthetic Method of Reasoning has, in many Cases, over the modern Analysis, especially in what regards Neatness and Perspicuity (Simpson 1745, vii).

At first, no divergence is apparent in his textbook as compared to the established view in England; he introduced negative quantities and operating on them in the familiar way, and the rule of signs as well. The difference is revealed only upon closer inspection. The operations and the rule of signs are defined for compound terms ("compound quantities"), in a way analogous to that of Diophantus. In discussing their applicability to isolated negative quantities, however, he made clear that he considered quantities smaller than zero to be absurd: they were just as impossible as imaginary quantities:

> For it ought to be considered that both $-b$ and $-c$, as they stand alone, are, in some Sense, as much impossible Quantities as $\sqrt{-b}$ and $\sqrt{-c}$; since the Sign –, according to the established Rules of Notation, shews the Quantity, to which it is prefix'd, is to be subtracted, but to subtract something from nothing is impossible, and the Notion or Supposition of a Quantity actually less than Nothing, absurd and shocking to the Imagination (ibid., 24).

It was thus ridiculous, he said, to pretend one might be able to prove by some kind of reasoning what the product of $-b$ times $-c$ should be, "when we can have no Idea of the Value of the Quantities or Expressions to be multiply'd." Only operations on real, positive quantities were admissible; numbers as algebraic expressions were legitimate only as measures of geometrical quantities:

> All our Reasoning regards real, affirmative Quantities, so the algebraic Expressions whereby the Measures of those Quantities are exhibited, must be likewise real and affirmative (ibid., 24 f.).

This fundamental belief, however, did not keep Simpson from applying virtually unchanged procedures in his concrete operations, such as using both a positive and a negative square root (ibid., 99).

The first radically consistent rejection of algebraization and generalization was published by Francis Maseres (1731–1824) in his treatise *Dissertation on the Use of the Negative Sign in Algebra* in 1758. Maseres had studied in Cambridge and had concluded his studies of mathematics with the good qualification of fourth “Wrangler.” In 1755, he became a Fellow of Clare College in Cambridge and competed in 1760 for the Lucasian mathematics professorship. Failing to obtain it, Maseres switched to the profession of lawyer. It is not known who prompted his ideas on mathematics, or who had influenced him; influences from France have not been investigated as yet.

Basically, Maseres confined the meaning of negative numbers to subtractive quantities, defining subtraction exclusively for positive remainders (Maseres 1758, 1). He thus did not admit isolated negative quantities; for him, an expression like $(-5) \times (-5)$ made sense only in the form 5×5, without making allowance for the minus sign (ibid., 2). Maseres thus also denied as a matter of principle that there were two square root values.

His concept of numbers that admitted only absolute numbers becomes particularly clear from how he treated the solving of quadratic and cubic equations. For equations of the second degree, he declared that all such equations are reducible to the following three forms:[73]

$$xx + px = r,$$
$$xx - px = r,$$
$$px - xx = r.$$

p and r being assumed here as strictly positive. The possible negative form of the coefficients must be expressed by the signs of operation. Furthermore, it is typical that the fourth form of combining the signs + and –, $xx + px = r$, does not appear, just as it did not in Descartes (cf. above, Section 2.4.) its solutions becoming negative (ibid., 20). Analogously, for equations of third degree, Maseres treated the various cases of allowed combinations of the coefficients’ plus and minus signs as separate problems and cases.

In a programmatic manner, Maseres emphasized that the two equations $xx + px = r$ and $xx - px = r$, for instance, by no means expressed the same problem, but rather formed different equations for different problems:

> This method of uniting together two different equations may perhaps have its uses; but I must confess, I cannot see them: on the contrary, it should seem that perspicuity and accuracy require, that two equations, or propositions, that are in their nature different from each other, and are the results of different conditions and suppositions, should be carefully distinguished from each other, and treated of separately, each by itself, as it comes under consideration (ibid., 29).

The main part of Maseres’s 300-page treatise is devoted to studying equations of third degree. The author succeeded in distinguishing sixty different cases of respective constellations of coefficients and their algebraic signs, a result inducing him to claim that there were just as many “normal forms” (ibid., 200 ff.).

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73 In mathematical historiography, the underlying number concept has been given little attention. The lack of care in relating Maseres’s ideas provides some examples; Pycior reports it thus: “He considered two possible forms of a quadratic equation” (Pycior 1976, 57), while Arcavi/Bruckheimer speak *of four* forms (Arcavi, Bruckheimer 1983, 11).

William Frend (1757–1841) published his anti-algebraic views even more extensively and radically in his voluminous algebra textbook *Principles of algebra* of 1796. Frend had also studied in Cambridge and had passed his exams in 1780 as second "Wrangler." He began working as a tutor at the famous Jesus College in Cambridge, but had to give up that post later because of confessional conflicts. Frend reviewed MacLaurin's algebra textbook incisively for his admission of negative quantities. The legitimate concern of his critique was that MacLaurin introduced his notions not conceptually, but by examples of applications:

> Now, when a person cannot explain the principles of a science without reference to metaphor, the probability is, that he has never thought accurately upon the subject (Frend 1796, x).

Frend made clear his own epistemological principles, saying that only positive numbers were permissible, and that algebraic operations must not change this "nature" in any way. Concerning positive numbers, he proclaimed:

> No art whatever can change their nature. You may put a mark before one, which it will obey: it submits to be taken away from another number greater than itself, but to attempt to take it away from a number less than itself is ridiculous. Yet this is attempted by algebraists, who talk of a number less than nothing, of multiplying a negative number into a negative number and thus producing a positive number, of a number being imaginary. Hence they talk of two roots to every equation of the second order, and the learner is to try which will succeed in a given equation (ibid.).

Frend's algebra textbook is based on rejecting negative numbers, and in particular multiple roots. Since it is not immediately evident in case of algebraic differences which of the two terms constitutes the larger, Frend introduced a sign for general subtraction, in the form of a horizontal S, virtually in the function of the absolute value:

> $\sim$ is the mark of difference: $a \sim b$ means the difference of the two numbers a and b, which will be either $a - b$ or $b - a$, according as a is greater or less than b; and we read $a \sim b$ thus, the difference of a and b (ibid., 4).[74]

Like classical Greek authors, Frend presented the rules of signs in the context of multiplying compound terms. He did this without justification, however, in particular for multiplying two terms having a minus sign (ibid., 19).

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74 Klügel reported in a treatise of 1795 that this sign occurred for the first time in volume 51 of the *Philosophical Transactions* of the Royal Society. That article was pubished in 1761, being the review of an astronomical treatise. The same sign is used there on page 928, but rather marginally and without explanation. Hence, one must suppose that the sign was well known among the contemporaries, at least in England. The sign fulfills the same function as Viète's sign "=" (cf. above Section 2.4.). The author was Henry Pemberton (1694–1771), member of the Royal Society since 1720, physician and professor of medicine at Gresham College in London. He had also published on mechanics and astronomy. Indeed, this sign was known in England, in any case, by Oughtred. It appears first in this function in 1652 in an appendix to his algebra textbook (Pycior 1997, 48).

After having presented the basic operations, Frend explained extensively in an elementary way suited to the addressed level of students how to solve equations of first degree by algebraic manipulation, for natural numbers. He then went on to simple problems, which were solved with linear equations. Only after that, he introduced fractions, ending with powers and roots. The basic method for treating roots was that there were only positive root values in any case (ibid., 89 ff.).

In the subsequent section on equations of second degree, Frend also posited as fundamental premise that only positive solutions were possible (ibid., 104 ff.). Here, he extensively examined four cases as independent problems: the pure quadratic equation $x^2 = b$, and the three forms of mixed quadratic equations Maseres had already used. Frend went so far as to exclude negative terms *within* the process of solving the equation as well. Thus, he performed estimations during this process to determine the positive roots, thereby obtaining only the positive value for mixed roots (ibid., 106). The type of equation that may yield two positive roots raised more difficulties. Calculating an example, Frend surprised the reader with the result that *two* (positive) values satisfied the equation (ibid., 111), without explaining how to handle the two solutions and why there were two solutions in this case instead of only one, as had been shown before. In an analogous mode, Frend exhaustively examined various coefficient constellations for the equations of third degree as separate isolated cases ("forms").

One of the numerous texts in the appendix to Frend's textbook is noteworthy: "A remark on an error in the reasoning of the late learned French mathematician, Monsieur Clairaut, in that part of his *Elements of algebra* in which he endeavours to prove the rules of multiplication laid down by writers on algebra concerning negative quantities" (ibid., 514–518). Frend here indeed revealed an error in the argumentation of many authors who wanted to give a "proof" for the rule of signs. Clairaut had started from the polynomials $(a - b)$ and $(c - d)$, multiplying them as $ac - bc - ad + bd$; claiming on this basis that a and c might be assumed to be zero, thus obtaining $-b \times -d = +bd$. Frend, however, showed that Clairaut had at first assumed $(a - b)$ and $(c - d)$ to be positive, and that it was neither justified nor legitimate to extend the admissible operations at that point to purely negative terms. Frend additionally argued that Clairaut had defined negative quantities only as subtractive quantities and thus had not been entitled to use these as isolated quantities:

> The author on this occasion seems to have forgot his own definition of the sign –, by which he made it to be a mark of the subtraction of the quantity to which it is prefixed from the quantity that goes before it; from which definition it is plain that the said sign always supposes the existence of two different quantities, of which the one is to be subtracted from the other, and consequently that it can have no meaning when applied to a single quantity, as *b* or *d*, independently of some other and greater quantity, as *a* or *c*, from which it is to be subtracted. There cannot therefore exist any such quantities as $-b$, or $-d$; and consequently no propositions concerning them can be either true or false. Consequently the proposition which the author there

> endeavours to demonstrate, to wit, "that $-b\times-d$ is equal to $+bd$; when $-b$ and $-d$ are not preceeded by two greater quantities a and c from which they are substracted, but are considered as single and independent quantities," is so far from being *true* that it is not even *intelligible* according to the only idea of the sign – which the author has given us in all the preceeding parts of the book (ibid., 517).

While it was incorrect of Frend to claim that Clairaut had had only subtractive quantities in mind, Frend's critique clearly showed the deficiencies of defining negative quantities only by their algebraic signs. Frend's critique concerns section 60 in Clairaut's algebra textbook (pp. 96–97 in Lacroix's edition of 1797).[75]

In the same appendix, Frend sharply criticized Euler's algebra textbook for including a multitude of errors, caused by Euler's use of the "perplexing" and absurd doctrine of negative quantities for which Frend attributed an unusual predilection to Euler (ibid., 518). Frend did not detail the alleged errors, however.

An often quoted article by John Playfair (ca.1748–1819) also proves general reservations against the methods of algebra. Playfair had first been active as a priest, then as a private tutor, and eventually as mathematics professor in Edinburgh. Although his treatise read before the *Royal Society* in 1778 was not directly about negative numbers, but rather about imaginary numbers, it reflects the paradoxical situation that operating with meaningless algebraic signs may nevertheless lead to a rigorous result—meaning, for Playfair, a geometrical result (Playfair 1778). It was this very reflection on "artificial symbols" (ibid., 319) and on the possibility of such symbols mutually "compensating" one another with the result of obtaining a "real quantity" that turned out to be so tempting for Carnot (see below, Chapter V.I.3.).

Cajori, who analyzed English publications of the time, asserts a great impact of Maseres's and Frend's concepts in England, for the first half of the nineteenth century as well (Cajori, in: Cantor, Vol. IV, 1908, 85 ff.). While Pycior mentions some voices criticizing that mainstream, too, she has to admit that the concepts presented above dominated debates in England until about 1840 (Pycior 1976, 61 ff.). Pycior attempted to identify an anti-Newtonian attitude as the common feature of these views. This would utterly contradict, however, the traditional interpretation according to which the small progress of mathematics achieved in England is explained by the exclusive partisanship and admiration for Newton's infinitesimal calculus. The authors oppose Newton only insofar as the latter had taught newer algebra as well. A more conclusive approach is to see the motives for Simson, Maseres, and Frend in the absolute model function of Euclid's *Elements*, which still waxed in England in the course of the eighteenth century. Within the frame of this Euclidean epistemology, mathematics basically consisted of investigating isolated cases. A generalizing algebra did not fit

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75 This section followed the deduction of the rule of signs by means of an example: the two springs problem (see above Section 2.9.2).

within this epistemology. Confining oneself to isolated cases without generalizations permitted one to circumvent negative solutions, though at some cost. The model function of Euclid maintained by the British educational system until after 1900 raised this epistemology to the status of a fundamental cultural attitude.

2.10.4. THE CONCEPT OF OPPOSITENESS IN GERMANY

Developments in Germany during the eighteenth century remained to a great extent exempt from being influenced by the radical changes in England and France. Wolff's views found no influential followers, while the now important textbooks were more or less strongly based, in critical dispute with Wolff, on Hausen's views. The quintessential point of the conceptual development was formed by the concept of 'oppositeness,' which was deepened philosophically and historically. In Germany, its linkage to the concept field of the foundations of mathematics was reflected much more explicitly and comprehensively; a challenge producing new solutions repeatedly was constituted by the problem of finding an adequate frame for the operation of multiplication. In this, the tendency toward algebraization came into conflict with the resort to justification by geometrical objects and applications.

Hausen's work, published in Latin, saw only one edition and served primarily mathematics professors themselves. A decidedly more successful author was Johann Andreas Segner (1704–1777), who continued Hausen's views. Segner became the first mathematics professor of the Göttingen reform university in 1735; in 1755 he moved to the university of Halle. In his textbook *Rechenkunst und Geometrie* of 1747, he distinguished, analogously to Hausen, between quantities and numbers, assuming numbers to be the more basic concept. His work was the first to clarify that the signs of the *numbers* represent the criterion for positiveness or negativeness:

> Upon using these signs + and – , the quantities whose numbers are designated by + are regarded as really positing something and are thus also called *positive.* By contrast, those quantities whose numbers have the sign – are regarded, with respect to the former, as *negative* or *privative* because they always negate or destroy from the former considered to be positive just as much as they themselves amount to (Segner 1767, 27).

Segner emphasized that no "internal constitution" made the quantities positive or negative, but rather that the respective attribution was willful positing. Segner had begun his introduction of positive and negative quantities by saying that there were quantities that were diminished by addition, or increased by subtraction (ibid., 25). He also elaborated on the underlying conception of oppositeness (ibid., 27).

Segner discussed the rule of signs in this textbook only in a brief algebraic section—in some kind of elementary algebraic geometry (*Anfangsgründe der Berechnung ausgedehnter Größen*). Although Segner had developed the conception of oppositeness within the frame of arithmetic; for numbers, he fell back for the rule of signs, that is, for the operation of multiplication, on geometrically justified notions, that is on the theory of proportions. For this purpose, he used Hausen's version of the product as the fourth proportional, striving scrupulously toward inferring lucid conclusions, from the ratios between the first two proportional terms to the last two, by the four possible combinations of signs in the multiplication (ibid., 641–645). As he had to admit himself, he fell back for this on "examples," since "otherwise, we might be at a loss for words" (ibid., 643). Since Segner assumed that the algebraic signs of the third and fourth proportional are determined by those of the first and second proportionals, the exemplary basic question for him was to show the ratio $1{:}-a$ to be possible. For this, he had to clarify how $-a$ can originate from the unit, thus from 1. Segner resorted for this purpose to the visualization of having a person walk a certain distance "toward evening," up to $+a$. Afterwards, the person was to walk back "toward morning," first until the starting point, thus making the unity vanish, and then to continue walking the distance a "toward morning" to make the distance $-a$ originate:

> $-a$ will thus always originate from 1, while the unity is destroyed step by step, and by having the quantity which has made vanish the unit, grow still farther thereafter in the same manner (ibid., 643).

Further illustrations referred to the debt/asset example.

Abraham Gotthelf Kästner (1719–1800) followed Segner as mathematics professor in Göttingen in 1756. By his well-elaborated lectures and by the numerous editions of his textbooks, he had an enormous impact, becoming, in this respect, the genuine successor to Chr. Wolff in Germany. This was caused in particular by the fact that the university of Göttingen became the center of mathematical studies precisely during the time he was active there. Kästner had studied in Leipzig with Hausen, and emphasized later over and over again Hausen's great influence on his own mathematical views (cf. the preface in Kästner 1792, vi). Kästner also exerted an important influence on the development of mathematics in Germany by not only working on foundational questions himself, but by also stimulating his disciples to like research.

Kästner's major work, "*Die mathematischen Anfangsgründe*," appeared in 1758 as a first edition in four volumes, later extended to eight. He developed his basic conception of negative numbers in his first volume, on arithmetic, adopting, just as Hausen and Segner had done, the number concept as basis and developed from that the notion of oppositeness for numbers and for quantities. This concept was to generally understand *quantities* as *numbers*, so that the laws of arithmetic were applicable not only to numbers, but also to quantities:

> When quantities of one kind, i.e., quantities where one can be a part of the other, can be compared in such a way that one examines how many times the one is completely contained in the other, or that one measures both against a part they have in common; then one can consider them once and for all as numbers, and this subjects all quantities which can be measured in such a way, to arithmetic, permitting one to apply the doctrines of numbers to all such quantities (Kästner 1792, 71).

As examples, he quoted concrete numbers and opposite quantities. Kästner based his definition of these quantities on a general concept of oppositeness he posited, but did not justify in detail:

> Quantities of the same kind which are considered under conditions that one diminishes the other shall be called opposite quantities. E.g., assets and debts, walking forward and walking backward. One of these quantities, as one likes, shall be called positive or affirmative, and its opposite negative or denying (ibid.).

Kästner's principle of oppositeness implied from the very outset that positive and negative quantities are "of the same kind," hence homogeneous. Implicitly, he thus criticized Christian Wolff's notion.[76] By quoting further examples, Kästner underlined how general quantities, if capable of oppositeness, can be transformed into one another: "Debts are denied assets, and assets can be regarded as denied debts" (ibid., 72).

Kästner discussed the relation between algebraic sign and sign of operation without, however, reflecting on it extensively (ibid.). Kästner stressed, also in implicit criticism of Wolff, that negative and positive quantities were of equal status: "This negative which remains is a real quantity, only opposite to that considered to be positive" (ibid.).

A new element as compared to the previous discussion in Germany was Kästner's reflection on the "nothing," and his differentiation between an absolute and a relative "nothing." He explained, "in themselves [...] all denied quantities are more than nothing because they are real quantities." But adding a negative quantity to its opposite affirmative quantity yielded zero, or nothing. From this, it followed, on the one hand, that one could call negative quantities, as compared to positive qunatities, to be "*less* than nothing." On the other hand, from this also followed the relative significance of Nothing:

> This expression "less than nothing" assumes a meaning of the term "nothing," which, in a certain way of considering it, relates to the "something" (*nihilum relativum*) and which can be distinguished from an unrelated "nothing" without a relation (*nihilum absolutum*) (ibid., 73).

After these explanations of concept, Kästner introduced the arithmetic operations on opposite quantities. Addition and subtraction were, in the already established way, reversals of the respective operations for natural numbers. He derived the rule of signs for multiplying quantities from Hausen's concept of product, as an arithmetic calculation with ratios, refraining from justifying it in detail, but giving a rather summary explanation, adding, "This is accepted by everybody without proof" (ibid., 79).

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76 In the preface, Kästner explicitly defended Wolff against precipitate critics: His textbook is pioneering the new trend of reflecting on the foundations of mathematics instead of only accumulating scholarship. Hence, the defects of his book are pardonable (Kästner 1792, [ii–iii]). Kästner permitted himself, however, some criticism: thus that it was "a big mistake against method" to found "the doctrine of fractions on that of ratios" (ibid., [iv]).

Kästner treated solving equations in the third volume of his *Anfangsgründe*, the *Analysis endlicher Größen*. The multiplicity of roots was self-evident to him (Kästner 1794, 52). Moreover, he noted with a clarity and explicitness rarely attained before that coefficients are not limited to the positive area, but are capable of values both in the positive and in the negative area (ibid., 55). For equations of second degree, he gave as solution the single general formula bearing the +/–sign.

The elements of analytic geometry contained in the same volume, and presenting the properties of parabola, ellipse, and hyperbola, show quite clearly that Kästner had no problem with using all four quadrants for curves, and operated with both positive and negative coordinates (ibid., 204 ff.). The coordinate axes are not scaled and not presented as number lines.[77]

Analytic geometry can indeed serve as a particularly illustrative indicator for the degree of acknowledgment and operativity of negative numbers. This is particularly evident from a textbook authored by M. Hube, one of Kästner's disciples, and advised by the latter in this endeavor; it was probably the first elementary textbook of analytic geometry ever published (Hube 1759). For the figures, the coordinate axes also are not designated or provided with numbers; the text clearly introduces the coordinate axes as spanning the entire plane: each axis contains both all positive and all negative numerical values, the axes not necessarily being right-angled, but also being skewed-angular (Hube 1759, 3). Kästner extensively stressed in his preface to Hube's textbook the advantages of this first representation of the conic sections by the analytic method, by means of equations, as compared to the traditional synthetic approach (Kästner 1759).

A conception different from that of Segner and Kästner was developed in the third textbook series that became influential during the second half of the eighteenth century in northern Germany: the eight-volume *Lehrbegriff der gesammten Mathematik* (1767–1777) by Karsten. Wenceslaus Johann Gustav Karsten (1732–1787), raised in the small duchy of Mecklenburg Schwerin, began as a lecturer at the duchy's universities of Rostock and Bützow, and became in 1778 Segner's successor as mathematics professor at the prestigious university of Halle.[78]

In contrast to Hausen, Segner, and Kästner, Karsten introduced oppositeness not for numbers, within the frame of arithmetic, but in the second part of his algebra textbook under the heading "general art of arithmetic." For Karsten, the basis of this algebra was newly conceived basic operations, namely, the "four general kinds of operation," generalized from the basic arithmetical operations

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77 In his preface to Hube's textbook on analytic geometry, Kästner, by contrast, sharply crititized Wolff for his concept of negative quantities: The expression "nothing" had induced him, by a wordplay, "to expel the denied quantities [...] from the realm of true quantities." After such examples, one must not be surprised, "when other geometers arrive at deductions, which seem unbelievable to themselves," and in particular did not distinguish between signs and objects (Kästner, in Hube 1759, [xxxi]).

78 For Karsten's biography, see below Chapter III.9.

by including opposite quantities. Karsten did not refer the concept of oppositeness to numbers, but rather to quantities, which he obviously considered to be more general. For this purpose, however, he neither defined the concept of "quantity" nor discussed how it related to numbers.[79] For Karsten, the concept of quantity essentially implied the meaning of variable, because he simultaneously ascribed to it the meaning of a continuous movement:

> If two quantities are in such a relation to each other that the one decreases just as much as the other one increases, and vice versa, then they are called *opposite quantities* (Karsten 1768, 64).

With this definition referring to quantities, Karsten implicitly disproved Wolff's view that positive and negative quantities were heterogeneous. Atlthough the positive and the negative part can be considered independently and may thus seem to be of different kind, they can be subsumed in a common "superior" quantity and will then form a homogeneous quantity:

> Such opposite quantities, considered for themselves, are quantities of a different kind, or are to be regarded as having different denominations. However, they are always situated under a common principal concept, and can in so far be considered as quantities of the same kind (ibid., 65).

Karsten introduced the designation of a positive or negative quantity as an abbreviation in order to avoid naming the subconcepts and the principal concepts separately:

> If one assumes that one speaks of the principal concept which the two opposite quantities have in common, one can indicate each of them by negating its opposite (ibid., 66).

Karsten was very cautious in linguistic denomination; instead of *negative*, or *denied* quantity it was more appropriate to speak of a *denyingly expressed* quantity, and accordingly of a *positively expressed* quantity. Karsten has called the new meaning of this basic operation, which arose from adding opposite quantities, a "general addition" (ibid., 68), and their subtraction analogously a "general subtraction."

For the multiplication (and division) of opposite quantities as well, Karsten introduced a more general notion. Due to his preference for the traditional concept of quantity, he also relied on a traditional fundamental concept, on that of ratio, which he understood, however, in a way analogous to Hausen and Segner, largely as a concept to be thus arithmetized. The product results as the fourth proportional to unity and to the two given quantities ($1 : a = b : P$). To obtain this result, he extended the concept of ratio, without discussing whether that was admissible. For the four proportional quantities, he required not only the usual equality $A : B = C : D$, but in addition that if the first two terms were opposite to each other, the last two must be opposite, too (as well as the negated

[79] Only at a later occasion did Karsten give an explanation of his concept of quantity within the "general art of arithmetic," saying that letters "can be understood as entirely general signs, by which one comprehends numbers, lines, and generally every other kind of the quantities" (Karsten 1768, 73).

case); Karsten thus obtained four cases of permissible ratios (ibid., 76 f.), which precisely expressed the rule of signs (ibid., 79).

It shows Karsten's grasp of the consequences of changing the foundations of mathematics by changing one of the basic concepts that he did not only reflect on the transition to more general concepts for operations, but that he noted the necessity to change the concepts of relation as well, implementing this for the concept of equality, and for the relation between larger and smaller. Regarding the former, he stated that it could also be applied to negative quantities. It could be said "with some reason, [...] that every negative quantity is smaller than any positive one" of the same kind, a statement that enabled him to show how to calculate with such relations:

$$-a < +a,\ -2a < +a,\ -3a < +a;\ -2a < -a,\ -3a < -2a.$$

With regard to the respective positive quantity, it could thus also be said that a negative quantity *was smaller than nothing* (ibid., 78).

Of particular interest is Karsten's reflection on the concept of equality, because it already contains a clear formulation of the absolute value. For opposite quantities to be equal, not only equality of their "quantity" (their value) is required, but also equality of their "position" (their direction):

> Two opposite quantities can never be called identical quantities not even if they are equal considered for themselves, without paying attention to their oppositness. By no means $+a = -a$, although a signifies the same line and means the same quantity. Here, however, only those lines are identical which have both the same quantity and the same position; and generally those quantities are identical only if not only their *quantitative* amounts can be substituted, but when they are moreover not opposite to one another (ibid., 77 f.).

At the same time, this quotation shows how helpful a separate term to designate equality of quantities (like "absolute value") would have been.

In the part on solving equations of this textbook, Karsten presented the multiplicities of roots (ibid., 251 ff.). For coefficients in equations, he did not explicitly state that they can likewise be positive and negative, but his practice made clear that he held this to be self-evident.

On the basis of these three important and influential textbooks, the extended number concept, together with an understanding accordingly generalized of the foundations, was well anchored at least in Protestant Northern Germany by the second third of the eighteenth century. The wide acceptance of this generalization and its effect are illustrated by two new developments:

- for the first time, there was historical and comparative reflection on the development of the concept of negative quantities;
- the concept of negative quantities was taken up in philosophy with the intention of philosophically reflecting on the foundations of mathematics, and to apply the concept within philosophy itself as well.

We shall first discuss how historical and critical reflection set in, because this development for the first time raised the problem of how to justify the concept of number beyond the level of its former, more or less implicit, treatment in the elaboration of textbooks.

In the literature on the history of negative numbers, it is quite unknown that a German historical-critical account about the negative numbers was published as early as the eighteenth century. This is indeed impossible to see from the treatise's convoluted heading: *Ueber eine Stelle in Herrn Lamberts Briefwechsel, von verneinten und unmöglichen Wurzelgrößen*. This 80-page essay had been written by the W.J.G. Karsten already mentioned, and was published in 1786! Its key proposition is that the concepts of positive and negative quantities gradually emerged "by abstraction" in the course of generalizing the solving of numerical problems. After giving an account of contemporary positions, Karsten wrote an overview of the concepts of mathematicians before his time, introducing it by saying:

> The way algebraists of old choose to present the nature proper of numbers designated by (–) is indeed not as completely convincing as nowadays expected from an author who wishes to explain the foundations of mathematical science: we must take into account, however, that we now stand on the shoulders of the first inventors, which makes it easy for us to see farther than they did (Karsten 1786, 209).

The principal aspect under which Karsten examines and presents earlier concepts is to what extent negative quantities were accepted as real mathematical objects or the extent to which no status of quantities was attributed to them being understood as defects. For most of the authors who used expressions like *numeros absurdos* (Michael Stifel) or *racines fausses* (Descartes), Karsten concludes that they were using such expressions merely as willful terms (*Kunstwörter*) while actually having the correct thing in mind. Generally, Karsten sees an unequivocal trend toward the state of the art accepted in the mathematics of his time.

Among the numerous authors Karsten studied, beginning with Stifel, Viète, Descartes, Schooten, and Newton, he basically criticized only Christian Wolff, for seeming to have misled "toward thinking [...] as if the negative was to be sought in the matter itself and not alone in the expression" (ibid., 241).

As eminent pioneers of the conception that is relevant for himself, Karsten, conversely, praised Hausen and Segner (ibid., 243). It is fascinating to see which non-German authors of his own eighteenth century Karsten mentions: while lightly praising the Englishman MacLaurin and slightly criticizing the Italian M.G. Agnesi, he makes no reference at all to any French author (with the exception of Claude Rabuel, as a commentator of Descartes). He is all the more severe, by contrast, is his wholesale condemnation of French authors, whom he accuses of adhering to an obsolete lack of rigor:

> Authors from abroad, in particular French, and even recent ones among them, continue to speak in this outdated manner, at least partially, but only when presenting the elements, not as conscientiously observing the necessary rigor as German authors do (ibid., 249 f.).

Undertaking historical research on how concepts have hitherto developed signifies a new level of metareflection that permits one to expect further impulses toward advances. An additional dimension of metareflection was opened some years after Karsten's publication, by an essay in a journal that, again as a first, realized a methodological–critical analysis of various conceptual approaches. Its author, Georg Simon Klügel (1739–1812), was Karsten’s successor to the chair at the university of Halle. Klügel pointed out that one of the essential methodological reasons for conceiving of negative numbers was the striving for generalization in mathematics, toward summarizing and concurrently treating and solving related problems, thus identifying the concept of negative numbers as a prime expression of the analytic method. At the same time, he critically examined views rejecting, or restricting, the use of this concept, and was thus able to identify these views as expressions of the synthetic method. Klügel focused on English authors as proponents of this method; he did not quote any French authors at all:

> The analytic differs from the synthetic method particularly in that the former embraces several cases in a single formula, while the synthetic discusses each case separately. The reason for this is that analysis expresses the connection of the quantities by equations, and that it uses the general properties of the equations, as well as the rules for connecting them to give the value of each quantity by those belonging together with it, or to develop their relations. According to the synthetic method, one must seek a separate path for each problem, having no other general formulae for calculating than the proportions together with their modifications, except for the propositions already found. One must therefore always make an effort to discover identical ratios. While the synthetic method avails itself of such propositions which state an equality, it does not use algebraic equations (Klügel 1795, 312 f.).

For Klügel, this striving for generalization was inescapable for mathematics:

> Moreover, it is necessary to present all related cases of a connection between quantities in one calculation, to economize on repetition, and to avoid a too cumbersome set of propositions, as well as in particular to survey all the differences in a formula at a glance (ibid., 313 f.).

In Greek mathematics, Klügel said, such striving for generalization had been just as nonexistent as it was among some of his contemporary “recent” mathematicians (ibid., 311). By these “recent” mathematicians, Klügel meant his English colleagues who were stuck with the Euclidean method, trying to avoid the occurrence of negative quantities by considering separate cases, and thus to prevent any generalization:[80]

> Things here are just as with the geometry of the ancients, and of the Englishmen imitating them, according to whom negative quantities will not occur in any proposition, since it is determined in any case what is a sum, or what is a difference, and since it can never be demanded, for a given difference, to subtract the whole from the part (ibid., 316).

Klügel warned in particular against adopting and using the new sign suggested “by the English” for indeterminate subtractions (precisely the sign ~

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[80] Klügel’s concept was contradictory, however, because he suggested, at the end of his paper, a method to avoid negative quantities and to operate only by absolute quantities (cf. below).

used later by Frend as well),[81] a sign intended to ensure that the result of subtractions was always positive. Noteworthy and instructive is Karsten's and Klügel's growing awareness of cultural difference, and in particular their tendency toward national distinction. Klügel added this warning: "I [find] it necessary to stand up against this innovation, and to make sure that German analysts will not fall for it" (ibid., 471).

Besides this onset of historical-critical reflection, the conceptual consequences of introducing negative numbers were analyzed and developed further in philosophy. Immanuel Kant pioneered this philosophy of mathematics with his 1763 essay "Attempt to introduce the conception of Negative Quantities into Philosophy." For his mathematical foundations, Kant explicitly drew on Kästner:

> Perhaps no one has determined in a more clear and definite way what is understood by negative quantities than the famous Mr. Professor Kästner, in whose hands everything becomes precise, comprehensible, and enjoyable (Kant 1968, 782).

Kant's contribution to the foundations of mathematics consists in addressing the question always underlying the controversies on the status of negative numbers, the question what "nothing" means, in treating the nothing that occurs when opposite quantities are combined, in stating it as an explicit problem, and even in having proposed a solution for this problem. Kant's solution consists in differentiating between logic, or philosophy, and mathematics.[82] While an absolute nothing emerged in cases of a logical opposition or contradiction, a relative nothing resulted when two quantities mutually "canceled" one another, this relative nothing being the mathematical zero. Kant conceived of this differentiation conceptually as of the difference between a logical repugnancy and a real opposition:

> THINGS contrary to one another means that one cancels what the other effects. This contrariety is twofold, either logical by way of contradiction, or actual [real] *i.e.*, without a *démenti* [contradiction]. The former, the logical, is that which so far alone has claimed attention. It involves both assertion and denial of one and the same thing. The conclusion of this logical relationship is nothing at all (*nihil negativum irrepraesentabile*), as the Principle of Contradiction expresses it. [...] The second, the actual [real], is where two predicates of one thing conflict, but not by the principle of contradiction. Here also the one cancels what the other causes, but the result is something (*cogitabile*). [...] For the future we shall call this actual nothing, Zero = 0.

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81 Klügel did not reproduce the sign, but referred for this to a paper in volume 51 of the *Philosophical Transactions* (i.e., from 1760), without naming its author, cf. Section 2.10.3. above. In addition, Klügel mentioned that the Italian Antonio Cagnoli (1743–1816) had also used this sign in his trigonometry textbook of 1786.

82 Kant had been prompted to this by a treatise of the philosopher C.A. Crusius, in which Crusius in 1749—discussing Newton's contrasting pair of attraction and repulsion—had understood negative quantities as negations of quantities; hence, Crusius had claimed attraction like repulsion to be "positive" effects or positive causes (Kant 1968, 781; Crusius 1774, §295, 739 f.).
A detailed analysis of Kant's contribution was published by M. Wolff (M. Wolff 1981, 39–77).

> [...] It is easily discoverable that this Zero is a relative nothing, [since, namely, only a certain consequence does not occur] [...]. The *nihil negativum* is not therefore to be expressed by Zero, because Zero involves no *démenti* [contradiction]” (ibid., 783 f.; Engl. transl. by Irvine 1911, 117–119).[83]

By means of this conceptual differentiation, Kant successfully deontologized the “nothing” in mathematics, attributing to it the exclusively relational character of the zero. This relativization at the same time accepted that negative numbers enjoyed the mathematical status of real numbers. The important step away from metaphysics, and toward the program of algebraization of mathematics, this meant was rapidly adopted in Germany, whereas France and England did not follow suit for a long time to come.

In Germany, embedding the process of generalization in mathematics in the entire scientific debate as a whole contributed to a stronger acceptance of abstract concepts within the larger cultural context as well. Moses Mendelssohn’s change of view concerning negative numbers on the basis of his scientific correspondence provides an informative example of this, as has been shown by Hans Lausch (Lausch 1993). Mendelssohn (1729–1786), a culturally highly influential philosophical author and significant representative of Enlightenment in Germany, used algebraic modes of expression in one his essays on sensations as well. Comparing this essay’s three editions of 1755, 1761, and 1771 reveals differences of essence in the propositions made on negative quantities. While the 1755 text qualified isolated negative quantities as an “absurdity” (quoted from Lausch 1993, 25), obviously under the spell of Wolff’s textbooks, Mendelssohn’s debate with Thomas Abbt (1738–1766), a philosopher of Enlightenment with an interest in mathematics, made Mendelssohn modify this point in the second edition of 1761. Abbt had tried to convince Mendelssohn of the “reality” of the negative quantities. One might “conceive negative quantities also in abstracto and think of them as entities without positing anything positive, provided only they are based on a relation or a position, or on something similar” (ibid., 28). After this, Mendelssohn deleted the qualification of absurdity, proposing the concept of the two modes of being instead:

> With respect to magnitude, a negative quantity is not distinct from a positive one at all, but it is distinct with respect to the operation which is to be executed with this quantity (ibid., 29).

In 1771, eventually, Mendelssohn referred to Kant’s treatise quoted above, adding as an afterthought: “Herr Abbt is thus right in stating that a negative quantity is something just as real as a positive quantity” (ibid., 35).

Due to the wide dissemination of mathematical culture in Germany during the last third of the eighteenth century, more and more mathematical textbooks were published, and it is no longer feasible to discuss all these here. Generally, it can

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83 David Irvine did not translate Kant’s preface. Some words used by him are peculiar, so that the literal translations are given in square brackets, such as the half sentence, which he omitted to translate.

be said that concepts adhering to, or enhancing, those proposed by Kästner or Karsten prevailed.

One very informative example, however, shall nevertheless be quoted here, since it shows how the philosophical reflection, in particular Kant's, contributed to the emergence of a separate foundational discipline of mathematics. The example is J.G.E. Maaß's (1766–1823) textbook *Grundriß der reinen Mathematik*. Maaß was professor for philosophy at the University of Halle, who lectured on mathematics side by side with Klügel.

In this textbook, Maaß understood *logic* as the basis for the mathematical operations. In particular, the concept of oppositeness arose from logic. Maaß thus presented logic even before arithmetic proper, saying that two other relations with regard to quality could occur with numbers beside quantitative relations: having the same direction (*Einstimmigkeit*) or oppositeness. The quality of oppositeness signifies that numbers "cancel" each other if one wants to unite them. The quantity considered as canceled is called positive, and the canceling quantity negative (Maaß 1796, 18). After having established these foundations, Maaß introduced the basic arithmetic operations (ibid., 23 ff.).

The authors analyzed up to now were all active at Protestant universities in Northern Germany. A last example may show, however, that the Catholic south was not exempt from this process of generalization: a *Handbuch der Elementar-Arithmetik in Verbindung mit der Elementar-Algebra*, published in 1804 by Andreas Metz (1767-1839), first a *Gymnasium* teacher, after 1798 professor for philosophy at the University of Würzburg, and after 1805 for mathematics as well. As Metz explained in its preface, his textbook intended to integrate the modern theories from Segner's, Kästner's, Karsten's, and Schultz's textbooks.[84] Indeed, he inserted a general part on the foundations of arithmetic before his arithmetic proper, introducing, among other notions, the concepts of homogeneity and heterogeneity of quantities. Metz explained the concept of number to be an object of "general pure *Mathesis*" (Metz 1804, 3). According to his view of opposite quantities, these were not to be considered as homogeneous in themselves; they had first to be transformed into "unanimous" quantities: he hence understood assets and debts to be different qualities, but debts, for example, could be regarded as a negative asset and therefore as homogeneous quantities as well (ibid., 49). He also established the purely algebraic proposition

$$-7 < -3 \text{ (ibid., 53).}$$

He justified the rule of signs by logic; positing the negative negatively meant to posit it positively (ibid., 59).

In Germany, too, however, the concepts did not develop "free of contradictions," nor even continuously, say, in the sense of a progressive algebraization. The unresolved problems concerning the extension of the basic operations, and concerning the relation between the concept of number and of

84 For Johann Schultz, see below in this Section.

function, also challenged both concepts established and the relation between algebra and geometry.

Johann Andreas Christian Michelsen (1749–1797), mathematics teacher ("professor") at the *Berlinisch-Cölnisches Gymnasium* in Berlin after 1778, provides the first instance of such "diverging" views. He not only authored numerous textbooks on elementary mathematics, but also translated several of Euler's eminent textbooks from the Latin into German: the *Introductio* (1788) and the *Differential Calculus* (1790). Michelsen supplemented these translations by extensive annotations and annexes also criticizing some of Euler's central ideas, attempting to prove alternative concepts. The spirit of this critique and his revisions completely correspond to his own fixation on the idea of striving for rigor in the foundations of mathematics, and of attaining it.

In his textbook *Buchstabenrechnung und Algebra* (1788), he presented opposite quantities so as to show no obvious discrepancy to the now habitual form in Germany (Michelsen 1788a, 58 ff.). In reflections on foundations laid down in his *Gedanken über den gegenwärtigen Zustand der Mathematik* (1789), and in his translations of Euler, however, he presented a divergent concept; his propositions are noteworthy not only as an adaptation of d'Alembert 's view on negative numbers, but also as a response to Euler's view of the logarithm function.

Like d'Alembert, Michelsen postulated the generality of functions, i.e., requiring that all functions should be defined for all number areas and not be subjected to any restrictions in their domain of definition. Hence, in a mode analogous to that of d'Alembert, Michelsen rejected in principle Euler's restriction of the logarithm function to positive real numbers; rather, it should be defined for all real numbers, and for imaginary numbers as well (Michelsen 1788b, 522). In addition, he accepted only *one* value for the function in every case. As the reason for Euler's alleged "difficulties" with the logarithm function, Michelsen attributed to him "not sufficiently clear and determined concepts of positive and negative numbers" (ibid., 525).[85]

His alternative concept was precisely an explication of d'Alembert's ideas, which had partially remained implicit as yet (cf. above, Section 2.9.3). Particularly noteworthy is the fact that Michelsen had been the first to extensively declare absolute numbers to constitute the basic concept and to

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85 Karsten, who in 1786 analyzed the controversy about the logarithms of negative numbers in a detailed treatise, criticized Leibniz, like Bernoulli and Euler, for having "premised the concept of what a negative quantity or a negative number was as being entirely known," while d'Alembert had been the only one to state that the concept of negative number needed to be clarified beforehand. Actually, d'Alembert's explanation of this concept, Karsten said, was "just as little satisfying" (Karsten 1786, 294 f.). Karsten reviewed d'Alembert's arguments against Euler in great detail, essentially rejecting them. In particular, he sharply criticized d'Alembert's definition of the logarithm via general ratios as too unspecific (ibid., 333 ff.). He emphasized that it was a common motivation of numerous such authors to obtain that roots of a negative quantity be negative, and not imaginary (ibid., 328).

propagate the distinction between absolute and positive numbers as necessary (e.g., Michelsen 1789, 20). Michelsen understood numbers as sets of units; absolute numbers formed the basis, as repeatedly taken unit (Michelsen 1788b, 525). For him, the positive and the negative numbers presented "a special kind of concrete numbers" (ibid., 526). He declared the set of their parts, the absolute number, to be the "essential" property, whereas their additional designation as positive or negative numbers constituted "something random" or external (ibid).

Michelsen thus attained, like d'Alembert, the goal set for the context of the logarithm function, of maintaining the logarithm as everywhere defined and of obtaining only real values at the same time. For the negative numbers, in this same context, were virtually identical with the positive, because one just had to modify their "absolute number by adding a random determination" (ibid., 527). Like d'Alembert, Michelsen only posited his view, but did not apply it operatively, nor develop it systematically in a textbook, say, for algebra.

It is telling for Michelsen's awareness of the problems of founding the number concept that he not only repeatedly pointed out the necessity of differentiating between absolute and positive numbers, but also always emphasized, every time he discussed the problem of how to conceptualize opposite quantities, that one of the basic problems consists in how exactly to grasp an adequate conception of multiplication (cf. Michelsen 1789, 18). In his discussion of the conceptual problems, however, Michelsen focused on application to geometrical quantities; for him, the multiplier, if it was a quantity at all, did not become transformed into a scalar by tricks or ad hoc definitions; rather, the product should contain its elements, and in this way he attained, by multiplication, quantities of higher dimension: the product of two lines to yield a rectangle, and that of three lines to yield a parallelepiped.[86] In his algebra textbook, however, Michelsen had not defined multiplication other than as was standard in Germany since Hausen,: as a ratio (Michelsen 1788a, 65), although he had declared the application of this multiplication concept to opposite quantities to be dubious in his critical *Gedanken...* (Michelsen 1789, 37). Michelsen thus remained quite inconsistent in his efforts at establishing rigorous foundations.

An almost immediate response to Michelsen came from Johann Schultz (1739–1805). Schultz, preacher to the Prussian Court and after 1786 mathematics professor at the University of Königsberg, close friend of Kant and the latter's mathematical adviser (cf. Schubring 1982), not only developed noteworthy conceptions on the infinite, but also contributed to the foundations of the number concept.

In his quite influential textbook *Anfangsgründe der reinen Mathesis* (1790), he began with numbers as the fundamental idea. More explicitly than other

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86 Michelsen went as far as to geometrically admit—to my knowledge, as the first since Viète—a "product by more" lines than three, thus admitting higher-dimensional products! He stressed that such a product has to be considered "as a quantity of just so many dimensions, not representable in intuition" (Michelsen 1789, 176).

authors, he stressed that absolute numbers constituted the basic concept: numbers by themselves were neither positive nor negative, and nor were they opposite to one another. Furthermore, he emphasized that only "unanimous" quantities, i.e., quantities of the same kind, were comparable. Opposite quantities were inhomogeneous in themselves and had to be rendered unanimous, of the same kind, before one were able to operate on them (Schultz 1790, 120 ff.).

It is particularly interesting that Schultz inferred consistently from all those manifold debates about how to multiply negative quantities that multiplying quantities by quantities must be systematically *excluded*, and only multiplying quantities with numbers admitted. Schultz presented a systematic deduction of this restriction to multiplication, also visualizing it by differentiating according to signs. For purposes of multiplication, he distinguished between multiplier and multiplicand. Only a number is permissible as multiplier, the latter being expressed by letters m, n, etc., whereas quantities designated by letters a, b, etc. were permissible as multiplicands, in the sense of *quanta* (Schultz 1790, 61). Since Schultz wrote the multiplier on the right, he consistently symbolized multiplication in the following form:

$$a \times n.$$

By means of these forms, Schultz also discussed the rule of signs, by always returning to a multiplication by an absolute number. Thus he first transformed, say, $-a \times +m$ into $-a \times m$ and then interpreted the product $-a \times m$ as a quantity to be subtracted (ibid., 127 ff.). According to Schultz's view, multiplying negative quantities by one another was not possible, the possible operation only being to multiply a negative quantity by a number (ibid., 130). Nevertheless, he attempted a proof of the rule of signs that implied a *petitio principii* (ibid., 131). It was left to Förstemann (1817) to elaborate a really consistent separation of quantities from numbers (cf. Chapter VII).

How d'Alembert's ideas had been taken up in Germany, perhaps conveyed by Michelsen's publications, and how absolute numbers were differentiated from positive numbers, can be elicited from some of Klügel's writings. In an annex to his 1795 essay containing his reflections on how to methodologically justify negative numbers (cf. above), he demonstrated, quite in contrast to the generality of method required in the rest of his text, how to avoid "the concept of opposite quantities," by restricting oneself to "*absolute* quantity," that is by assuming "that the *Subtrahendus* is less than the quantity from which the subtraction is made" (Klügel 1795, 479). If the *Subtrahendus* was larger, however, a "different case" was present "from that for which the calculation was intended" (ibid). In such a "different" case, the conditions and with them the algebraic signs concerned should be changed in the "initial equations," according, in fact, to d'Alembert's and Bézout's conception. It is quite characteristic that applying this approach in the annex, contradicting his general views in the main text and excluding opposite quantities, consisted in declaring the logarithms of negative numbers to be real numbers, by virtually identifying positive numbers with negative ones: “The *Logarithm of a denied number* is identical with the logarithm of the same number regarded as positive; [...]” (ibid., 481).

Actually, Klügel only postulated this view and did not try to present the operative use of logarithms conceived in such a way. In his entry of 1805 on opposite quantities in the highly influential *Mathematisches Wörterbuch* he edited, he did not take up this “French” concept (Klügel 1805, 104 ff.).[87] While agreeing without problem to operative concretizations, he alluded to his persisting epistemological reservations against negative quantities in the entry “equation” of the same encyclopedia. He declared that “the negative roots serve to unite two equations differing only in their algebraic signs, into a single equation” (ibid., 364). This assumes hence an epistemological distinction of the positive from the negative, which are only afterwards composed to an external whole for purposes of operating.

Klügel’s “alternative” positions illustrate a conflict that was basic for the stage of the generalization process discussed here. Does the more general meaning constitute the simplest version of the concepts, that which is subject to the least number of conditions, or is the more general meaning that stage of degree of conceptual development which permits the largest number of applications?

This conflict was explicitly reflected in a series of publications in 1799 and 1800 by a mathematician otherwise unknown; these treatises illustrate that it was impossible to solve the problem of how to justify negative numbers within the frame of an understanding of mathematics already directionally developing toward algebraization but still being determined by the traditional concept of quantities. Analyzing these writings will hence conclude our part on the development in Germany—before the genuine transition to algebraization. The author of this treatise in a series comprising three parts was Peter Johann Hecker (1747–1835). Born as a son of a parson, he was a student at the renowned *Realschule* directed by his uncle J.J. Hecker in Berlin. Later he studied theology in Halle, and mathematics with Segner. Beginning as a mathematics teacher in Berlin, he was appointed as Karsten’s successor at the tiny University of Bützow in 1778. In 1789, upon the dissolution of this university and its union with that of Rostock, he changed to that university.[88] Published by him as the university rector, as a *Programmschrift* for the three occasions of Christmas 1799, and Easter and Pentecost 1800, his reflections about the operations on opposite quantities were to remain his first and only mathematical publication, but one

87 In his textbook on arithmetic and algebra of 1792, Klügel did not introduce opposite quantities at all, mentioning negative quantities only marginally—in his section about progressions—without explaining them operatively, in Wolff’s sense, as an indication of a “deficit” that one can represent, say, by a debt (Klügel 1792, 50 f.).

88 I owe these hitherto unknown biographical data to Peter Strassberg’s dissertation, *Peter Johann Hecker - Mathematiker und Kalendermacher? [...]*, Universität Rostock 1988, which Wolfgang Engel was so kind to place at my disposal in November 1988.

testifying to his acute penetration of contemporary foundational problems. Noteworthy in Hecker's approach—besides his profound knowledge of the pertinent and recent literature—is his inclusion of a mathematical field that had barely been tackled until then in the debates about negative numbers despite the fact that it constituted one of its major areas of application: trigonometry. Hecker attempted to draw on this *application* for *justifying* the concept of negative numbers.

Hecker criticized the presentation of the concept of opposite numbers in the (German) textbooks, since they did not make clear this elementary topic to the students. His first objection was that many authors subsumed opposite quantities virtually only by accident under a "superior" concept. One had to suppose, he said, that quantities stop being opposite as soon as one no longer considered them under the alignment to this superior respect (Hecker 1799, 6 f.). Hecker did not understand his own critique as general; essentially, however, it aimed at a transition from the substantialist concept of quantities to the abstract concept of number.

His second criticism concerned the fact that the meaning of the basic operations in arithmetic was being treated differently from this meaning in algebra. The most frequent way of presenting them was to define the basic operations "at the beginning of arithmetic" for positive numbers, resulting in addition signifying real increase and subtraction likewise real decrease, but to assign another meaning to the operations later upon treating opposite quantities, resulting in subtraction often being executed by addition. Then, the signs + and – were also attributed new, additional meanings (ibid., 9 f.). Hecker quoted Segner's and Karsten's textbooks as examples of how to avoid this contradiction. These textbooks explicitly distinguished between addition, subtraction, etc. in arithmetic and general addition, subtraction, etc. in the "general art of arithmetic." This approach caused less difficulty for the learner, Hecker claimed, since the "concepts for these kinds of calculating were only extended [...], and not changed" (ibid., 11).

Here, the keyword of the *extension of the number field* was formulated for the first time. Hecker, however, was not satisfied with this solution. Instead of changing, or extending, the concepts later, he said that they could be taught from the very beginning in their more general form. This was not more difficult for the beginner, he claimed, and arithmetic and the general art of arithmetic were not divergent sciences; rather, the latter was a part of the former (ibid., 11 f.). To attain readily teachable presentations of that kind, however, required solving a conceptual problem, that of finding a general notion of the operation of multiplication. As Hecker stated, the common understanding of multiplication in arithmetic as a repeated addition contradicted the practice of multiplying quantities. The explicit conception, he said, was,

> that the multiplier generally must be a number, namely an abstract (*unbenannte*) number, and that multiplication be possible in no other case. [...] It will always remain obscure to the beginner whence it came that while one could add and

> subtract, e.g., lines and lines, and divide them by one another, one could not multiply them? (ibid., 14).

Hecker attached the following note to the word "multiply": "At least not without changing the concept of multiplication." Hecker established in clearer and more explicit terms than all his predecessors that the problem of the rigorous justification of negative numbers was linked to finding an adequate definition of multiplication. In particular, he stressed that he did not consider its common definition via reiteration of the unit to be promising, since inferring the multiplier from the unit was not possible if the latter was a negative number (ibid., 16).

In his three essays, Hecker attempted to develop general concepts for the four basic operations, which were be applicable to negative numbers as well. In the end, he failed upon grappling with multiplication, since he fell back on the concept of geometrical quantities for this purpose. Despite his clear diagnosis that the basic operations had hitherto not been introduced in a sufficiently simple and general manner, Hecker considerably complicated the conceptual construction and did not succeed in solving the problem proper.

An additional element of his conceptual construction was that he added, as the "most natural," a "theory of proportions" to those "doctrines which must be taught first in pure mathematics," before he felt able to develop the doctrine of opposite quantities (Hecker 1800a, 12). Hecker needed such a general theory of proportions as foundation, since his goal was not only to introduce multiplication as a proportion, as Segner and Karsten had done, but likewise all the four basic operations. Thus, he introduced addition and subtraction as forms of an arithmetic ratio, and multiplication and division as forms of a geometric ratio (ibid., 13 ff.). Even for this general introduction of the operations, notwithstanding the fact that it relied on quantities, Hecker was unable to avoid assigning an exceptional position to multiplication. Since the multiplier constituted the "exponent" of a geometric ratio, it had inevitably to be "an abstract number," in order to ensure that the product of a line with, say, a line was excluded (ibid., 20 f.). Hecker thus had to note that his intended general multiplication could not be introduced in an elementary way, but only after the general theory of proportions (Hecker 1800b, 29).

For introducing the opposite quantities themselves, Hecker integrated an even more extensive concept field: the concepts of variable and of function (ibid., 8 ff.). His intention in doing so was to conceptually improve Segner's and Karsten's attempts at justifying negative quantities by alluding to a continuous transition of a variable quantity by decrease—across its vanishing at zero—until it assumed the opposite value. For this purpose, Hecker used the application of opposite quantities in trigonometry, with trigonometric functions. Since each of the two arcs *a* and *b* could assume, for the trigonometric functions of sine and of cosine, "all possible values, by increase or by decrease," it was legitimate, he said, to consider "them as *variable* quantities, and the quantity sought as a *function* of both" (ibid., 9). For Hecker, the trigonometric functions provided a suitable model for conceptually grasping the notion of opposite quantities, namely as "transitions from decreasing to increasing values via their vanishing" (ibid.).

Rather far from his intention proper of introducing the basic operations in a way so simple and so general that they could be lucidly taught to beginners, Hecker now had even included parts of analysis into the foundations of arithmetic to be able to introduce negative numbers. Indeed, Hecker had not only demanded that opposite quantities be understood in such a manner that "the given quantities of a problem are to be considered as *variable quantities*," and that "the quantities sought [be regarded] as a *function* of them" (ibid., 12); he had moreover premised the validity of the *law of continuity* for securing the effective transition to the opposite value (ibid., 15). Hecker was able to legitimize this further additional assumption only by an ontological appeal to geometrical quantities (ibid).

While Hecker's treatises certainly constitute a culmination point in the reflection on foundations, and in the context of concept fields of mathematics concerned with justifying negative numbers, they also clearly demonstrate that no further substantial progress was achievable within the scope of the traditional concept of quantity, and that a breakthrough would necessitate finding some other way.

2.11. Looking Back

The problem of elaborating a coherent mathematical status for negative numbers developed over long periods of time, assuming at last a quite fundamental character because—or although—negative numbers became an important operative tool in an increasing number of concept fields in mathematics. The problem originated from the generalization of only one operation, subtraction, and it occurred in almost all of the great cultures. It challenged the traditional first understanding of mathematics, its first "paradigm" in Kuhn's terms, its understanding of being a science of quantities: of quantities that, while being abstracted to attain some autonomy from objects of the real world, continued at the same time to be epistemologically legitimized by the latter. The various cultures succeeded over a long time in finding various auxiliary constructions that permitted them to remain within the existing paradigm. Where the geometrical concept of quantity and the synthetic method prevailed, this could be obtained by considering individual cases. This meant that operations had to be limited to subtractive quantities. The fact that this meant that generalization had to be given up in treating problems became itself a problem if the operations of multiplication and division were to be extended to this new kind of quantity.

It is typical that it was still possible, during the period of exclusive dominance of the geometric concept of method and quantity, to legitimize operations by limitation to subtractive quantities, and that insurmountable conceptual conflicts began to become manifest only as the independent function of *algebra* started to grow toward the end of the Middle Ages and the beginning of modern times in Europe. The massive increase of algebraic methods which were relatively independent of geometric methods gave an impetus toward expanding the operations of subtraction and multiplication (division being treated as a corollary to multiplication because it functioned analogously), but this tested the limits of the paradigm of quantity.

In three countries of Europe, different mathematical cultures gradually evolved in modern times, which at the same time developed different approaches in dealing with the problem of generalization. Due to the efforts of the Malebranche group, respectively of Wallis and Newton, the path toward algebraization first seemed to be open both in France and in England. Contributing to the *querelle des anciens et des modernes*, they confidently propagated the superiority of modernity. This development, however, was ended by a rupture, obviously a side effect of Berkeley's attack against Newton's foundation of analysis. In England and Scotland, this rupture led to a return to the traditional concept of quantity and to the synthetic method. The prevailing attitude in France was ambivalent in not denying mathematical legitimacy to negative quantities on the one hand while at the same time suggesting a return to the synthetic method because of the necessary consideration of individual cases. In Germany, the concept of opposite quantities was elaborated for the most part without ruptures. Here again, however, algebraic generalization of operations and maintaining the quantity paradigm became increasingly irreconcilable, leading to eclectic "epicycles."

Acceptance, respectively rejection, of the zero adhered widely to the same patterns as those of the negative quantities, since legitimizing the zero via the concept of quantity was analogously problematical.

Euler's radical algebraization, which was not tied in with national cultures, met with violent resistance in many of them, leading to first cases of fundamentalist exaggerations.

All paths of solution within the prevailing paradigm having been exhausted, it was suggestive to assume that only radical solutions by different means would be able to deal with the "Gordian knot." Carnot's (1801/1803) and Förstemann's (1817) approaches were such radical steps.

The distinctive ways the concept of number was conceived of would take effect on the mathematization of limit concepts in analysis.

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