Preface

It is often said that mathematics and music go together, and that people with a special aptitude for mathematics often have similar gifts in music. Some music is very profound, and we find more in it at a second hearing. A similar point can be made about an understanding of mathematics.

In the world of music, there are *two* sets of people: active musicians who play musical instruments or sing, and the much larger set of passive musicians who listen to the sounds produced by members of the first set. However, in the world of mathematics, I contend that there is only *one* set of mathematicians: the active set. There are no such people as passive mathematicians. Of course, students attend mathematics lectures, and professional mathematicians take part in seminars in which a fellow mathematician discusses his or her research. But all who participate in such lectures and seminars, students and professional mathematicians alike, are active mathematicians, just as student musicians who attend a master class in their instrument are active musicians. In other words, there is nothing in the world of mathematics that corresponds to an audience in a concert hall, where the passive listen to the active. Happily, mathematicians are all *doers*, not spectators. *Doing* is much more fun than merely watching or listening, and I celebrate this in the title of this book.

Some very fine books have been written *about* mathematics. These give us the flavor of certain areas of mathematics, but they don't *make* us mathematicians. This book will introduce you to a range of topics in mathematics and will help you on your way to *becoming* a mathematician, even if you have only begun this journey. It is intended for senior students in high school and those who are beginning their study of mathematics at university level. There is a third category of readers that I have had very much in mind while writing this book. This is the large set of people who are not at school or university but are intellectually curious and active. If you are in this category, perhaps you have read that mathematics is more than just arithmetic and a little geometry, and want to know, what is it?

It seems to be generally accepted that the distribution of mathematical ability, like that of many other human characteristics, follows a bell-shaped curve (a Gaussian distribution), with comparatively few individuals at the two extremes. Relatively few individuals are either exceptionally poor or exceptionally good at mathematics. Thus many people have a much greater *potential* talent for mathematics than they think they have. As G. H. Hardy (1877–1947) writes in his fascinating book *A Mathematician's Apology* (see [11] in the References), "Most people are so frightened of the name of mathematics that they are ready, quite unaffectedly, to exaggerate their own mathematical stupidity." Many topics in mathematics are not immediately accessible; before we can understand them it is necessary to study some preliminary material. I have therefore chosen topics for this book that we can get into straight away.

Bertrand Russell (1872–1970) said that "A good notation has a subtlety and suggestiveness which at times make it almost seem like a live teacher." I agree with this wholeheartedly and have tried to explain carefully the notation that I use in this book. It is, in the main, notation that is widely used and is the common language of all mathematicians. I suppose that good notations have evolved by some kind of Darwinian natural selection process, and we need to be thoroughly familiar with mathematical notation in order to be able to construct or follow a mathematical argument.

With the possible exception of identical twins who have never been parted, and who have read all the same books and heard all the same things, we all have different backgrounds. So I can only try my best to guess, aided by my experience of writing and of lecturing, what I can best write in this book to get my message through to you. And on your part, you can try your best to understand me. I trust that our combined endeavors will be successful! But as you read this book, if some new notation puzzles you or if I fail to explain something to you adequately, I urge you to find someone who can help you lower the drawbridge and let you proceed into the "castle of mathematics." In writing the last sentence in more poetic language than is usual for me, I am mindful of the book *Drawbridge Up* (see reference [8]), in which the distinguished German poet and polymath Hans Magnus Enzensberger, who has a deep interest in mathematics, advances the thesis that mathematicians are often guilty of raising rather than lowering the drawbridge of understanding.

The earliest practitioners of the art of explaining mathematics are surely the Greeks, whose mathematics was developed during the millennium that began before the time of Pythagoras, in the sixth century BC. They introduced the concept of *proof* to mathematics. Thus, having identified the prime numbers, they did not just vaguely say that there are a lot of them; they *proved* that there is an infinite number of primes. Proofs are essential to mathematicians, and there are many proofs in this book. Greek mathematics was much concerned with geometry and number. At that time, numbers were regarded in a geometrical way, as lengths, or as ratios of lengths. This is very much the spirit of Chapter 1 of this book, which is concerned with a variety of results in number theory involving squares. It also includes material on Pythagoras's theorem, from the time of ancient Babylon and ancient China to a proof attributed to a nineteenth-century president of the United States. Chapter 1 concludes with a section on complex numbers.

Chapters 2 and 3 continue with the theme of numbers. Chapter 2 begins with a section on representing numbers in different bases, including base 2, which gives the binary system. It continues with a section on congruences, which involves algebraic ideas; a section on the fascinating arithmetic of continued fractions; and one on the Euclidean algorithm, whose origins are geometrical. Chapter 2 concludes with a section on the strange things that happen when we deal with infinite sets. This is based on the pioneering work of Georg Cantor (1845–1918) and introduces ideas that challenge our intuitive understanding of number.

Chapter 3 includes a study of the Fibonacci numbers, which are very simply defined, have many fascinating properties, and satisfy a large number of identities. They were introduced by Leonardo of Pisa eight centuries ago in his *Liber Abaci*, published in 1202. I describe how these numbers are related to the golden section number, involving ideas that go back to Pythagoras in the sixth century BC.

Chapter 4 is concerned with prime numbers. The Greek mathematician Eratosthenes devised a method for finding primes that has been applied in our own era, using the computer. It is sometimes said that mathematics is not an experimental subject. This is not true! Mathematicians often use the evidence of lots of examples to help form a conjecture, and this is an experimental approach. Having formed a conjecture about what might be true, the next task is to try to *prove* it. Thus, from near the beginning of the nineteenth century, the great mathematician C. F. Gauss (1777–1855) assembled lots of numerical evidence about how many primes there are in the first n numbers, where n is large. As I describe in Chapter 4, Gauss formed a conjecture about the *density* of primes, and a proof *was* found by others before the end of the nineteenth century. This chapter also describes special methods used in finding really large primes, and includes a discussion of the famous Riemann hypothesis. Although there is a massive amount of evidence to support the Riemann hypothesis, a proof has eluded mathematicians for about 150 years.

It is said that the early study of probability was prompted by questions put to mathematicians by gamblers who wanted to know the likelihood of various events that depend on chance. Probability theory is part of the content of Chapter 5, which is concerned with combinatorial mathematics: the number of ways of making choices. In this chapter I give a combinatorial interpretation of the Fibonacci numbers, providing a method for finding and verifying Fibonacci identities that supplements the more usual methods pursued in Chapter 3.

We return to geometry in Chapter 6, where we look at some of the geometrical constructions that were created by Greek mathematicians, and discuss some (like the trisection of an angle) that defeated them and were shown to be impossible two millennia later. This chapter includes a section on properties of the triangle, many of which were known to the ancient Greeks, and some which have been discovered since the nineteenth century. Chapter 6 also includes material on coordinate geometry, developed chiefly by Descartes. This makes it possible to solve certain *geometrical* problems by *algebraic* methods. You will already be familiar with the cube, but do you know that it is one of the five Platonic solids? I discuss these in the last section of Chapter 6, and also the thirteen Archimedean solids, which include one that inspired a widely used design of soccer balls.

By the nineteenth century some mathematicians had become aware that the study of certain symmetries could be developed using algebraic methods, a most exciting algebra in which ba could be different from ab. This led to the topic of group theory, which is discussed in Chapter 7. It is remarkable that such a glorious construct as group theory, in which mathematicians are still researching, is founded on four simple properties that can be stated in a few lines. I hope you will find this chapter rewarding, even if it proves to be more taxing than some of the other chapters. I have made it the last chapter for historical reasons and because it may require a more sustained effort on your part than the earlier chapters. This chapter ends with an algebraic interpretation of the Platonic solids. Here you will discover how even the humble regular tetrahedron (the Platonic solid that has four equilateral triangular faces) contains much more mathematics than one might suppose.

In writing a book of this length, with the aims stated above, I had to decide what to include and what to omit. As I have already said, I made a selection of varied topics that we can get into quickly, without lengthy preliminaries. These topics are largely independent, so that your understanding of a chapter is not seriously impaired by any (temporary!) difficulties with another chapter. However, I have drawn attention to connections between the various topics to show that they are indeed all constituent parts of one whole body of mathematics. One topic that I have omitted is the differential and integral calculus, which, although not especially difficult, does require some introductory material, such as a rigorous treatment of limits, making it inappropriate for a book such as this. Nonetheless, I have used the concept of limit a few times in the book, and rely on the reader's intuitive understanding of this concept. It is important in mathematics to appreciate when we have a rigorous proof, and when we have only an imprecise, intuitive understanding. I try at all times in this book to make it clear when I am *not* giving a proof.

One of the great charms of mathematics is its timeless quality. Results in mathematics that were obviously exciting and interesting to our predecessors in every era, from the time of ancient Babylon onwards, have the same fascination for us today. Mathematics seems to enjoy eternal youth. As a friend of Anthony says of Cleopatra in Shakespeare's *Anthony and Cleopatra*, "Age cannot wither her, nor custom stale her infinite variety." What is more, mathematics continues to develop, with no end in sight. Indeed, even within areas of the subject that have been studied for a long time, there are unsolved problems.

I hope you will enjoy *working* through this book, and that you will be happy at your work. You don't need to read it all at once. It is harder to read than a novel. Also, you don't need to read the chapters in the order in which they appear in the book. At a first reading, skim through the book and dip into it here and there. See what attracts you. If you get stuck, don't spend too long before finding someone who will help you. There are many avenues I haven't explored, even within the limited number of topics that I have discussed. Therefore, at some stage you may find yourself thinking that something or other is obviously true, and wonder why I haven't mentioned it. Then perhaps a little later, having learned something you didn't know before, *by your own efforts*, you will know that you are a mathematician. Good luck!

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