6

Geometrical Constructions

Let no one enter who does not know geometry.

Plato (427–347 BC)

Written above the door of Plato's Academy in Athens.

6.1 Ruler and Compasses

Euclid circa 300 BC gave a systematic account of the geometry known at the time, beginning with certain basic concepts, known as *axioms*. These axioms are to be thought of as initial assumptions on which the geometry of Euclid depend. His axioms are as follows:

1. *A straight line may be drawn from any point to any other point.*
2. *A finite straight line may be extended continuously in a straight line.*
3. *A circle may be drawn with any center and any radius.*
4. *All right angles are equal to one another.*
5. *If a straight line meets any two other straight lines so as to make the two interior angles on one side of it together less than two right angles, the other straight lines, if extended indefinitely, will meet on that side on which the angles are less than two right angles.*

At the heart of ancient Greek geometry are the "ruler and compass" constructions Two implements were used, a *ruler* for drawing straight lines and a *pair of compasses* for drawing circles. The ruler is simply a straight edge with no markings on it. It is used only for drawing straight lines, and not for measuring lengths. The compasses consist of two arms connected by a movable joint. At the end of one arm there is a sharp point that is placed at the center of the circle to be drawn. There is a a pencil at the end of the other arm, which can be moved to change the radius of the circle. Set against their many successes in this area of geometry, there were a few constructions by ruler and compass that the Greeks kept vainly struggling to achieve. After more than two millennia of experience of ruler and compass constructions, mathematicians at last attained a fuller understanding of the limitations of these methods and were able to *prove* that these long-standing unsolved classical problems were truly unsolvable. However, it is not difficult, after a little experimenting, to rediscover for ourselves some of the more obvious constructions that *can* be carried out by ruler and compasses.

Construction 1 *Draw a line that is perpendicular to a given line at a given point A.*

We use the compasses to mark two points B and C on the given line, equally spaced on either side of A. See Figure 6.1. We then draw equal arcs centered at B and C and label their point of intersection D . Then AD meets BC at right angles. To see this, note that in the triangles ADB and ADC, $|AB| = |AC|$, $|DB| = |DC|$, and DA is common to both triangles. (We write $|AB|$ to denote the *length* of the line segment AB.) Thus the two triangles are congruent. Finally, the angles DAB and DAC must both be right angles, since they are equal and their sum is equal to two right angles. ■

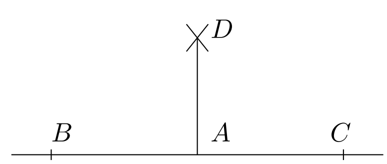


FIGURE 6.1. Draw a line that is perpendicular to the horizontal line at A.

Construction 2 *Draw a line through a given point A that is perpendicular to a given line that does not pass through A.*

With center A we use the compasses to mark off two points, B and C, on the line. (Draw a diagram.) Then we draw equal arcs centered at B and C to meet at a point D that is on the other side of the given line from A.

Let AD and BC intersect at the point E. We deduce that the triangles ABD and ACD are congruent, and that triangles ABE and ACE are congruent. Thus the angles AEB and AEC are equal and so must both be right angles. ■

Construction 3 *Find the midpoint of a given line segment BC.*

This construction is similar to Construction 2. With any radius greater than half the distance BC we draw arcs of the same length, centered at B and C, to intersect at a point A on one side of BC and at a point D on the other side of BC. Then we can verify that E, the point where AD and BC intersect, is the midpoint of BC, and that AE is the *perpendicular bisector* of BC. ■

Construction 4 *Draw a line through a given point A that is parallel to a given line.*

We begin by using Construction 2 to obtain points B and C on the given line and the midpoint D of BC. Then AD is perpendicular to BC. Next, using the ruler, we extend the line DA and use the compasses to find a point E on the extended line DA such that A is the midpoint of DE. Finally we construct the perpendicular bisector of the line DE. This line passes through A and, being perpendicular to DE, must be parallel to the original line BC. ■

Construction 5 *Given two lengths a and b, with $a > b$, construct the lengths $a + b$ and $a - b$.*

These constructions are very simple. We use the compasses to mark off the lengths $|OA| = a$ and $|AB| = b$, as shown in the diagram on the left of Figure 6.2. Then the length $|OB|$ equals $a + b$. The construction of $a - b$ is shown in the diagram on the right of Figure 6.2. ■

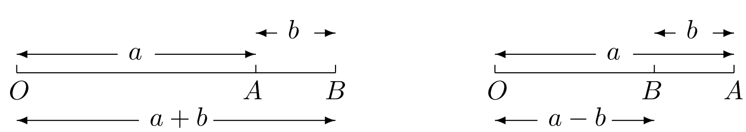
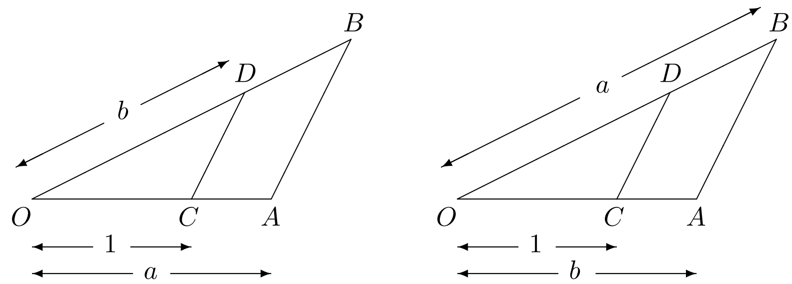


FIGURE 6.2. Construction of $a + b$ and $a - b$.

Construction 6 *Given two lengths a and b and a unit length, construct the lengths ab and a/b.*



$|OB| = ab$ $\qquad$ $|OD| = a/b$

FIGURE 6.3. Construction of ab and a/b.

In Figure 6.3 the two diagrams are identical, apart from the assignment of the segments that have lengths a, b, and 1. The lines AB and CD are parallel, and thus the triangles OAB and OCD are similar. (The angle between OA and OB is chosen arbitrarily.) It follows from the similarity of triangles OAB and OCD that

$$\frac{|OD|}{|OC|} = \frac{|OB|}{|OA|}. \tag{6.1}$$

In the diagram on the left of Figure 6.3, $|OA| = a$, $|OD| = b$, and $|OC| = 1$, and it then follows from (6.1) that

$$\frac{b}{1} = \frac{|OB|}{a},$$

and hence $|OB| = ab$. In the diagram on the right of Figure 6.3, $|OB| = a$, $|OA| = b$, and $|OC| = 1$, and it then follows from (6.1) that

$$\frac{|OD|}{1} = \frac{a}{b},$$

so that $|OD| = a/b$. ■

Remark 6.1.1 Through the repeated use of Constructions 5 and 6, we can construct any length that can be obtained by beginning with a finite number of given lengths and carrying out a finite number of applications of addition, subtraction, multiplication, and division. ■

Construction 7 *Divide a given line segment into n equal parts.*

Figure 6.4 illustrates the case $n = 3$. We begin with the line AB, and

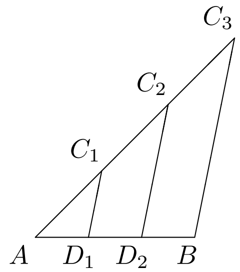


FIGURE 6.4. Divide the line segment AB into three equal parts.

draw a second line through A, making any angle with AB. Beginning at A we mark off n equal segments on the second line, AC_1, C_1C_2, and so on, ending with the segment C_{n-1}, C_n. We join C_n to B. Then, using Construction 4, through each point C_1, C_2, and so on, up to C_{n-1}, we draw a line parallel to the line C_nB. We see that the triangles AC_1D_1, AC_2D_2, and so on, are all similar. Thus in Figure 6.4 we have

$$\frac{|AD_2|}{|AD_1|} = \frac{|AC_2|}{|AC_1|} = 2 \qquad \text{and} \qquad \frac{|AB|}{|AD_1|} = \frac{|AC_3|}{|AC_1|} = 3.$$

We can obviously extend this argument if $n > 3$. This justifies the construction. ■

Construction 8 *Bisect a given angle.*

Let the given angle be denoted by angle BAC, where $|BA| = |AC|$. We use the compasses to construct a point D such that $|BD| = |CD|$. (See Figure 6.5.) Then the line AD bisects the angle BAC.

We observe that the triangles DAB and DAC have three corresponding sides equal. Thus they are congruent, and so the angles DAB and DAC are equal. ■

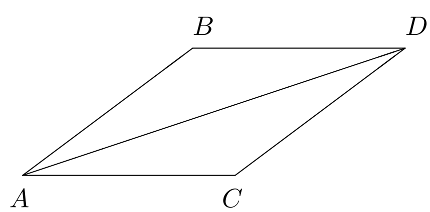


FIGURE 6.5. The line DA bisects the angle BAC.

Construction 9 *Inscribe a circle inside a given triangle.*

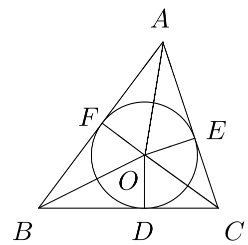


FIGURE 6.6. The incircle for the triangle ABC.

Using Construction 8, draw the lines that bisect angles ABC and ACB, and denote their point of intersection by O. (See Figure 6.6.) From O, draw perpendiculars to each side of the triangle, and let them meet BC, CA, and AB at D, E, and F, respectively. Then O is the center of the inscribed circle and its radius is $|OD|$, which is equal to $|OE|$ and $|OF|$.

The inscribed circle is called the *incircle* of triangle ABC, and O is called the *incenter*. Let us compare the triangles OFB and ODB. The angles OBF and OBD are equal, and the angles OFB and ODB are also equal, both being right angles. Since also the triangles OFB and ODB have the common side OB, the two triangles are congruent, and thus $|OF| = |OD|$. Similarly, we can show that the triangles ODC and OEC are congruent, and deduce that $|OD| = |OE|$. Thus we can draw a circle with center O, and radius $|OD| = |OE| = |OF|$. The sides of the triangle, BC, CA, and AB meet the radii OD, OE, and OF, respectively, at right angles. We say that the sides of the triangle are tangents to the circle. Finally, let us compare the triangles OEA and OFA. They are right-angled triangles with two corresponding sides equal, since the side OA is common to both triangles and $|OE| = |OF|$. Hence, by Pythagoras's theorem (see Section 1.2), we have

$$|AE|^2 = |OA|^2 - |OE|^2 = |OA|^2 - |OF|^2 = |AF|^2.$$

Since $|AE| = |AF|$, the triangles OEA and OFA are congruent, and so OA bisects the angle BAC. Thus the bisectors of the three angles of a triangle are *concurrent*, that is, they meet in a common point. ■

Construction 10 *Draw a circle that passes through three points that do not lie in a straight line.*

Let the three points be denoted by A, B, and C. Using Construction 3, we draw the perpendicular bisectors of AB and BC. These two perpendicular bisectors must intersect at some point, say O. Since $|OA| = |OB| = |OC|$, the points A, B, and C lie on a circle whose center is O. Since O is equidistant from C and A, it lies on the perpendicular bisector of CA, and thus the three perpendicular bisectors are concurrent at O. ■

It is clear from Construction 10 that three points A, B, and C not on a straight line determine a *unique* circle that passes through all three points. This is called the *circumcircle* of the triangle ABC, and its center is called the *circumcenter*. In general, a given fourth point, D, will not lie on the circle that passes through A, B, and C. A quadrilateral $ABCD$ for which, unusually, all four points lie on a circle, is called a *cyclic quadrilateral*.

Construction 11 *Given a line segment of length* 1, *construct a line segment whose length is the golden ratio*, $\alpha = \frac{1}{2}(\sqrt{5}+1)$.

Let $|AB| = 1$. We use Construction 1 to create the square $ABCD$, as depicted in Figure 6.7, and use Construction 3 to bisect the line CD at E. Then, by Pythagoras's theorem (see Section 1.2) we have

$$|EB|^2 = |BC|^2 + |CE|^2 = 1 + \frac{1}{4} = \frac{5}{4},$$

and hence $|EB| = \frac{1}{2}\sqrt{5}$. We now use the compasses, centered on E, to draw a circle of radius $|EB|$ to cut the extended line DC at F. Then

$$|DF| = |DE| + |EF| = |DE| + |EB| = \frac{1}{2} + \frac{1}{2}\sqrt{5} = \alpha. \qquad \blacksquare$$

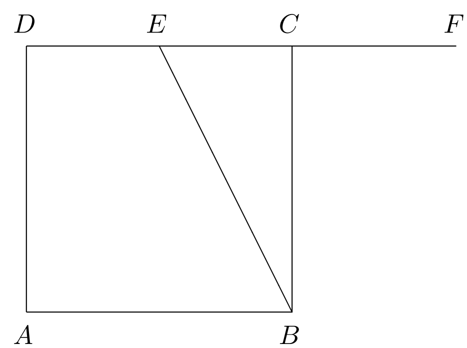


FIGURE 6.7. Construction of the golden ratio $|DF|/|DC|$.

Of all the regular polygons, the equilateral triangle and the regular hexagon are the easiest to construct.

Construction 12 *Construct a regular hexagon in a circle of radius* 1.

We choose a point A_1 on the circumference of the circle, and use the compasses with radius 1 and center A_1 to draw an arc that cuts the circle at a point A_2. See Figure 6.8, where O marks the center of the circle. We next center the compasses on A_2 and, with the same radius, draw an arc

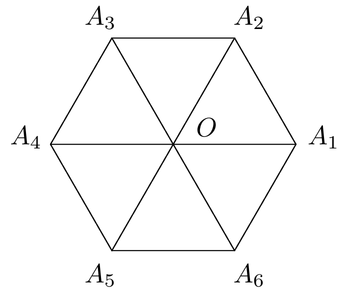


FIGURE 6.8. Construction of the regular hexagon.

that cuts the circumference at a point A_3, as shown in Figure 6.8. We repeat this process three more times to complete the construction. We note that the figure contains six congruent equilateral triangles, namely OA_1A_2, OA_2A_3, and four others. If we pick out every second point of the hexagon $A_1A_2A_3A_4A_5A_6$, we obtain an equilateral triangle. ∎

Construction 13 *Construct a regular decagon in a circle of radius* 1.

Let O be the center of a circle of radius 1, let $|OA| = |OB| = 1$ and let AB be a chord of a regular decagon, the regular polygon with ten sides. Thus the angle AOB is $\frac{\pi}{5}$. See Figure 6.9. The two remaining angles in triangle OAB are equal, and must both be $\frac{2\pi}{5}$, since the sum of the angles in a triangle is equal to 2π. The point C is chosen on OA so that $|BA| = |BC|$. Therefore, the two angles BCA and BAC are both $\frac{2\pi}{5}$, and consequently angle CBA is $\frac{\pi}{5}$. This means that angle CBO is also $\frac{\pi}{5}$, and so $|CB| = |CO|$. Thus the triangles AOB and ABC are both similar to the triangle FAG in Figure 3.7, which depicts the pentagram. On comparing the two similar triangles in Figure 6.9, we have

$$\frac{|AB|}{|AC|} = \frac{x}{y} = \frac{x}{1-x} = \frac{|OA|}{|AB|} = \frac{1}{x}. \tag{6.2}$$

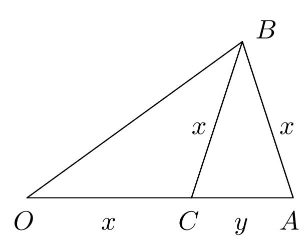


FIGURE 6.9. Construction of the regular decagon.

As we found in Section 3.3 (see (3.39)),

$$\frac{x}{y} = \alpha = \frac{1}{2}\left(\sqrt{5}+1\right),$$

the golden ratio. Since $y = 1 - x$, we find that

$$x = \frac{\alpha}{\alpha+1} = \frac{\alpha^2-1}{\alpha+1} = \alpha - 1.$$

We can obtain the golden ratio α by ruler and compasses using Construction 11, and thus we can construct $x = \alpha - 1$, the length of the side of a regular decagon in a circle of radius 1. The construction of the decagon is then easily completed. We can obtain the regular pentagon by connecting every second point of the decagon, and can also construct the star of Pythagoras, which we discussed in Section 3.3. ■

Given two positive numbers a and b, the number

$$A(a,b) = \frac{1}{2}(a+b) \tag{6.3}$$

is called their *arithmetic mean*. If $0 < b \leq a$, we have

$$2b \leq a + b \leq 2a.$$

Thus

$$b \leq \frac{1}{2}(a+b) \leq a, \tag{6.4}$$

so that $A(a,b)$ lies between a and b. We define the *geometric mean* of a and b as

$$G(a,b) = \sqrt{ab}. \tag{6.5}$$

If $0 < b \leq a$ we can show that $0 < b^2 \leq ab \leq a^2$, and thus

$$0 < b \leq \sqrt{ab} \leq a, \tag{6.6}$$

so that $G(a,b)$ lies between a and b. If we compute the arithmetic and geometric means of several pairs of positive numbers a and b, we always find that $A(a,b) \geq G(a,b)$. To verify this, consider

$$\frac{1}{2}(a+b) - \sqrt{ab} = \frac{1}{2}\left(a - 2\sqrt{ab} + b\right) = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0. \tag{6.7}$$

Let $x = \sqrt{ab}$, so that $ab = x^2$. On dividing throughout by bx, we obtain

$$\frac{a}{x} = \frac{x}{b}, \tag{6.8}$$

and we say that the geometric mean x is the *mean proportional* of a and b. I will mention only one more mean, although the list of means is endless. This is the *harmonic mean* of a and b, defined by

$$H(a,b) = \frac{2ab}{a+b}. \tag{6.9}$$

The arithmetic, geometric, and harmonic means have been studied since at least the time of Pythagoras, in the sixth century BC. If we divide both the numerator and the denominator of the fraction on the right of (6.9) by ab, we obtain

$$H(a,b) = 2\Big/\left(\frac{1}{a} + \frac{1}{b}\right).$$

Thus the harmonic mean is the reciprocal of the arithmetic mean of $1/a$ and $1/b$, that is

$$H(a,b) = 1\Big/A\left(\frac{1}{a}, \frac{1}{b}\right). \tag{6.10}$$

It is also not hard to verify that

$$\frac{H(a,b)}{G(a,b)} = \frac{G(a,b)}{A(a,b)}. \tag{6.11}$$

Thus

$$G(a,b) = \big(A(a,b)H(a,b)\big)^{1/2},$$

so that the geometric mean of a and b is itself the geometric mean of the arithmetic and harmonic means of a and b.

Construction 14 *Construct the arithmetic and geometric means of two line segments AB and BC.*

In Book III of his *Mathematical Collection*, Pappus of Alexandria (third century AD) gives the construction that is shown in Figure 6.10. This depicts the case $|AB| > |BC|$. The arithmetic mean is easily found by using Construction 3 to find O, the midpoint of AC. Then both $|AO|$ and $|OC|$ give the required arithmetic mean of $|AB|$ and $|BC|$. Next, using Construction 1, we draw a perpendicular to AC, passing through B, to meet the semicircle with diameter AC at the point D. Then $|BD|$ is the geometric mean of $|AB|$ and $|BC|$. The proof relies on similar right-angled triangles, and we first show that the angle ADC is a right angle. We argue that in triangle OAD, $|OA| = |OD|$, both being radii of the semicircle that passes through A, D, and C. Thus the angles OAD and ODA are equal, and we will denote them by α. Similarly, in triangle OCD, since OC and OD are radii of the semicircle, the angles OCD and ODC are equal, and we will denote them by β. Since the sum of the angles of triangle ADC, which equals two right angles, is also equal to $2\alpha + 2\beta$, it follows that $\alpha + \beta$ is a right angle. Thus angle ADC is a right angle. It follows that the triangles ADC, ABD, and DBC, having corresponding angles equal, are similar right-angled triangles. We deduce from the similarity of the two smaller triangles that

$$\frac{|AB|}{|BD|} = \frac{|BD|}{|BC|}, \tag{6.12}$$

so that $|BD|$, being the mean proportional of $|AB|$ and $|BC|$, is their geometric mean. ■

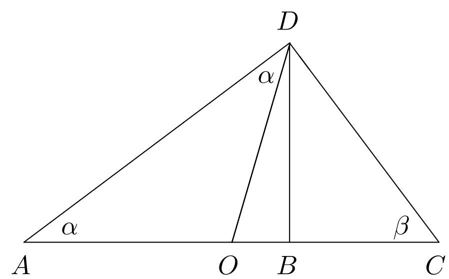


FIGURE 6.10. $|OD|$ is the arithmetic mean of $|AB|$ and $|BC|$, and $|BD|$ is their geometric mean.

Construction 15 *Construct the harmonic mean of two line segments AB and BC.*

This construction is also given by Pappus in Book III of his *Mathematical Collection*. We amend Figure 6.10 by drawing a line from B, perpendicular to OD, meeting OD at the point E, as in Figure 6.11. Then $|ED|$ is the harmonic mean of $|AB|$ and $|BC|$. To justify this, we observe that the triangles DEB and DBO are similar, and thus

$$\frac{|ED|}{|DB|} = \frac{|DB|}{|OD|}.$$

This shows that $|DB|$, which is the geometric mean of $|AB|$ and $|BC|$, is a mean proportional to $|ED|$ and $|OD|$, and since $|OD|$ is the arithmetic mean of $|AB|$ and $|BC|$, it follows from (6.11) that $|ED|$ is the harmonic mean of $|AB|$ and $|BC|$. ■

We observe from Figure 6.11 that

$$|OD| > |BD| > |ED|,$$

showing an ordering of the arithmetic, geometric, and harmonic means. This may be justified algebraically (see Problem 6.1.7).

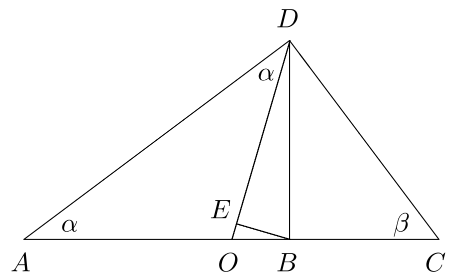


FIGURE 6.11. $|ED|$ is the harmonic mean of $|AB|$ and $|BC|$.

Problem 6.1.1 To construct a regular octagon, draw a circle with center O, and inscribe a square $ABCD$ in it, giving four vertices of the octagon. (One way is to draw two diameters of the circle that are mutually perpendicular.) Use Construction 3 to find E, the midpoint of AB, and show that the point where OE cuts the circle gives a fifth vertex of the octagon. Finally, construct the three remaining vertices of the octagon.

Problem 6.1.2 Construct an equilateral triangle and a regular pentagon that have one vertex in common within a circle of radius 1. (See Constructions 12 and 13.) Hence construct a regular polygon with 15 sides.

Problem 6.1.3 If $0 < b \leq a$, verify that $0 < b^2 \leq ab \leq a^2$ and hence show that

$$b \leq \sqrt{ab} \leq a.$$

Problem 6.1.4 Show that for the arithmetic mean,

$$A(\lambda a, \lambda b) = \lambda A(a, b),$$

where λ, a, and b are any positive numbers. Any mean satisfying this property is called *homogeneous*. Show that the geometric and harmonic means are also homogeneous.

Problem 6.1.5 In Figure 6.11, in which we have $|AB| > |BC|$, prove that $|ED| > |BC|$.

Problem 6.1.6 Deduce from (6.9) that if $0 < b \leq a$, then

$$b \leq H(a, b) \leq a.$$

Problem 6.1.7 Verify that if $0 < b \leq a$,

$$b \leq H(a, b) \leq G(a, b) \leq A(a, b) \leq a,$$

where A, G, and H denote the arithmetic, geometric, and harmonic means.

6.2 Unsolvable Problems

In Section 6.1 we saw how to construct regular n-sided polygons for $n = 3$, 4, 5, 6, 8, and 10, that is, the equilateral triangle, the square, and the regular pentagon, hexagon, octagon, and decagon. However, neither the regular *heptagon*, which has seven sides, nor the regular *nonagon*, which has nine sides, can be constructed using ruler and compasses. Heron of Alexandria found an approximate construction for the heptagon, and the famous painter, engraver, *and* mathematician Albrecht Dürer (1471–1528) found an approximate construction for the nonagon. (See Eves [9].) It may seem surprising that the regular nonagon cannot be constructed by ruler and compasses. We can begin by constructing an equilateral triangle, say ABC, in a circle with center O. Then we could complete the construction by trisecting the angles AOB, BOC, and COA. Alas, we cannot do that! For apart from some special cases, including a right angle, we cannot trisect an angle using ruler and compasses. The trisection of an angle is one of the three famous unsolved problems of Greek mathematics.

In Book IV of his *Elements*, Euclid described the construction of regular polygons of 3, 4, 5, 6, and 15 sides. We can extend these constructions by carrying out repeated bisections, using the same device as we employed in Problem 6.1.1 to construct the regular octagon from the square. Thus we can construct regular polygons with 2^n sides for any $n \geq 2$, or $m \times 2^n$ sides, where $m = 3,\ 5,$ or 15, and n is any nonnegative integer. Since the time of Euclid, the only substantial addition to our knowledge of the ruler and compass construction of regular polygons was made by C. F. Gauss, who had the last word on this topic. Gauss showed that the construction of a regular polygon with a prime number of sides p is possible if and only if p is a prime number of the form $f_n = 2^{2^n} + 1$, for $n \geq 0$. These are the Fermat numbers, defined in (4.22), and we obtain the prime values $3,\ 5,\ 17,\ 257,$ and $65{,}537,$ corresponding to $n = 0$, 1, 2, 3, and 4. As was stated in Section 4.3, no other Fermat primes are known. (A star-like figure based on the regular polygon with 17 sides is inscribed on the plinth of Gauss's statue in Braunschweig, the city of his birth.)

The second of the three famous unsolved problems of Greek mathematics is the duplication of the cube. It is easy to construct a square whose area is twice that of a given square. For if we have a square of side a, its area is a^2, and we see from Pythagoras's theorem that its diagonal is of length $\sqrt{2}a$. Using ruler and compasses, we can construct a square on this diagonal whose area is $2a^2$. Analogously, given a cube of side a, can we construct a number b such that $b^3 = 2a^3$? Hippocrates of Chios showed in the fifth century BC that this is equivalent to finding numbers b and c such that

$$\frac{2a}{c} = \frac{c}{b} = \frac{b}{a}. \tag{6.13}$$

For on multiplying the first equality in (6.13) by bc, and the second equality by ab, we find that (6.13) is equivalent to

$$2ab = c^2 \quad \text{and} \quad ac = b^2. \tag{6.14}$$

The second equation in (6.14) is equivalent to

$$a^2c^2 = b^4. \tag{6.15}$$

On multiplying the first equation in (6.14) by a^2, and using (6.15), we obtain

$$2a^3b = b^4,$$

which indeed reduces to

$$b^3 = 2a^3. \tag{6.16}$$

The numbers b and c sought by Hippocrates, as in (6.13), are said to be mean proportionals to the numbers a and $2a$. Although the construction of *one* mean proportional to two numbers (their geometric mean) is easy, the construction of two mean proportionals defied the considerable ingenuity of generations of Greek mathematicians, and surely this included some of the cleverest people who have ever existed.

The last of the three classical unsolved problems of ancient Greek mathematics is called the squaring of the circle. By their construction of the geometric mean, the Greeks had shown how to construct a square whose area is equal to the area of a given rectangle. They also tried, in vain, to construct a square whose area is the same as that of a given circle. You may think that this seems too ambitious, since unlike the perimeter of a rectangle, the perimeter of a circle is a curve, and so its area is much more difficult to reconcile with that of a square. However, the greatest of the Greek mathematicians, Archimedes of Syracuse (287–212 BC), showed that the area of a segment of a parabola can be expressed as a rational multiple of the area of a certain triangle. Since the the circle appears to be a simpler object than the parabola, it is understandable if this encouraged the belief that the squaring of the circle was achievable.

Let us now give a brief account of *conic sections*. We begin with a straight line ℓ that intersects a given vertical line at a point O. (See Figure 6.12.) A *right circular cone* is the three-dimensional figure that is created by rotating ℓ around the vertical line. Both ℓ and the vertical line are infinite in length, and thus the cone extends to infinity both above and below the central point O, which is called the *vertex* of the cone. In everyday language, the word "cone" is commonly used to denote just one half of the figure we have just described. The vertical line in Figure 6.12 is called the *axis* of the cone. Every line that passes through O and lies on the surface of the cone, like the line ℓ, is called a *generator*. The cone is determined uniquely by the angle that ℓ makes with the axis. This angle, which we will take to be less than a right angle, is denoted by α in Figure 6.12.

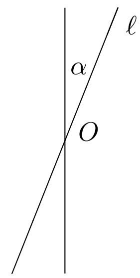


FIGURE 6.12. A right circular cone is generated by rotating the straight line ℓ around the vertical line.

Conic sections, as the name suggests, are the curves that are obtained from sections of the cone by planes. Consider a plane that cuts the axis of the cone at an angle $\beta \leq \frac{\pi}{2}$. The simplest and not so interesting case is that of the plane cutting through the vertex. In this case, if $\beta < \alpha$, the plane cuts the cone in a pair of generators, and if $\beta = \alpha$, the plane cuts the cone in a single generator. If $\beta > \alpha$, the plane obviously cuts the cone only at O. Now let us consider the curves that are obtained when a plane forming an angle β with the axis does not pass through the vertex O. If $\beta < \alpha$, the plane cuts both halves of the cone and we obtain a *hyperbola*, which thus has two branches. If $\beta = \alpha$, the plane is parallel to a generator of the cone. It cuts only one half of the cone and the resulting curve of intersection is the *parabola*. If $\beta > \alpha$, the plane cuts the cone through one half only, and we obtain an *ellipse*. In particular, if $\beta = \frac{\pi}{2}$ we obtain the *circle* as a special case of an ellipse.

Archimedes used a most ingenious construction to prove that the area of a segment of a parabola is equal to $\frac{4}{3}$ times the area of a triangle whose base is the same as the length of the parabolic segment and that has the same height. Figure 6.13 shows the parabolic segment and the triangle ABC that has the same base and height as the segment. Archimedes constructed two further triangles, shown as triangles ADB and BEC in Figure 6.13. The point D is chosen so that the triangle ADB has the same height as the parabolic segment with base AB. Similarly, triangle BEC, with base BC, has the same height as the parabolic segment with base BC. Archimedes proved that the sum of the areas of triangles ADB and BEC is one quarter of the area of triangle ABC, and a fine account of this proof is given in Edwards [7]. Archimedes continued this process, constructing four triangles with bases AD, DB, BE, and EC, whose combined area is one-quarter of the combined areas of triangles ADB and BEC, and so is $\frac{1}{16}$ of the area of triangle ABC. Archimedes thought of this construction being continued indefinitely, and proved that the area of the original parabolic segment that is not covered by one of his triangles tends to zero. He deduced that

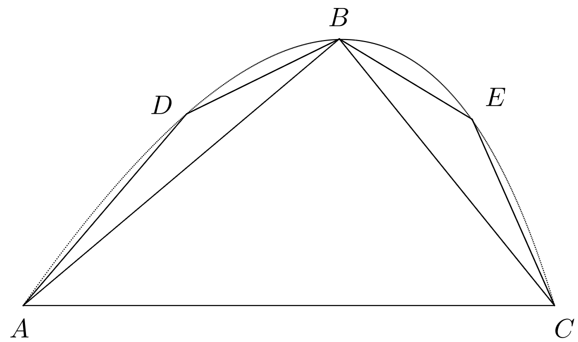


FIGURE 6.13. The area of the parabolic segment is $\frac{4}{3}$ times the area of the triangle ABC.

the area of the parabolic segment ABC is S times the area of the triangle ABC, where

$$S = 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots. \tag{6.17}$$

This is an infinite geometric series, and we see from (4.19) that $S = \frac{4}{3}$, thus verifying this wonderful result of Archimedes.

By the nineteenth century, some areas of mathematics had been developed that were not known to the ancient Greek mathematicians. It was one of these, the theory of equations, that led to proofs that the three famous long-standing problems of Greek mathematics are insoluble. Consider the polynomial equation

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0, \tag{6.18}$$

where the coefficients a_0, $a_1, \ldots, a_n$ are all real and a_0 is nonzero. This equation is easily solved if $n = 1$, when it is called a linear equation. When $n = 2$ we have a quadratic equation, whose complete solution is discussed in Section 1.5. Some quadratic equations were solved by Babylonian mathematicians as early as the second millennium BC. When $n = 3$, 4, and 5 we have a cubic, quartic, and quintic equation, respectively. It was not until the sixteenth century AD that the general cubic and quartic equations were solved. When I say "solved," I mean that solutions of the equation can be obtained by carrying out a finite number of arithmetical operations, specifically additions, subtractions, multiplications, divisions, and the extraction of square roots, cube roots, and so on, beginning with the coefficients a_0, $a_1, \ldots, a_n$. By the early eighteenth century, P. Ruffini (1765–1822) and N. H. Abel (1802–1829) independently proved that the general quintic cannot be solved, in the sense defined above. Although it may seem rather negative to show that something cannot be done, this was a very important achievement in the history of mathematics.

Recall from Definition 2.5.2 that a complex number that is a solution of an equation of the form (6.18) where the coefficients are all integers is called algebraic, and a complex number that is not a solution of such an equation is called transcendental. The breakthrough in the long quest to settle the three most famous unsolved ruler and compass problems came with the establishing of the following two key results in the theory of equations, which are stated in Eves [9].

Theorem 6.2.1 Beginning with a line segment of unit length, the length of any line segment constructed by any sequence of operations using ruler and compasses is an algebraic number. ■

Theorem 6.2.2 Beginning with a line segment of unit length, it is impossible to construct by any sequence of operations using ruler and compasses a line segment whose length is a solution of a cubic equation that has integer coefficients but has no rational solution. ■

As we will see, Theorem 6.2.2 settles the question of the duplication of the cube in the negative. In (6.16) let us write $a = 1$ and $b = x$. Then we need to construct a number x such that $x^3 - 2 = 0$. If there were a rational solution of this equation, we could write it as

$$x = \frac{p}{q}, \quad \text{where} \quad \frac{p^3}{q^3} = 2,$$

where p and q are positive integers, and we can assume that p and q have no common factor greater than 1. Thus

$$p^3 = 2q^3, \tag{6.19}$$

and so p must be even. Let us write $p = 2p_1$ in (6.19), so that

$$8p_1^3 = 2q^3,$$

and thus

$$4p_1^3 = q^3,$$

showing that q must be even. Notice that, beginning with the assumption that the equation $x^3 - 2 = 0$ has a solution of the form $x = p/q$, where p and q have no common factor greater than 1, we have shown that p and q have the common factor 2. This shows that our assumption is untenable, and that this cubic equation with integer coefficients does not have a rational solution. It follows from Theorem 6.2.2 that the duplication of the cube cannot be achieved by a ruler and compass construction.

Theorem 6.2.2 can also be used to show that not every angle can be trisected by using a ruler and compass construction. One needs only a little knowledge of trigonometry to follow a proof of this. See Eves [9].

The area of a circle of radius r is πr^2. Thus the area of a circle of unit radius is π. To "square the circle," we need to construct a square with side of length $\sqrt{\pi}$. In 1882, C. L. F. Lindemann (1852–1939) proved that π is transcendental, following up the work of C. Hermite (1822–1901), who showed in 1873 that e is transcendental. Thus, by Theorem 6.2.1, π cannot be constructed by a ruler and compass construction. Now if $\sqrt{\pi}$ were constructible by ruler and compasses, we could redraw Figure 6.10, beginning with $|BD| = \sqrt{\pi}$ and $|BC| = 1$. Then, since $|BD|$ is the geometric mean of $|AB|$ and $|BC|$, it follows that $|AB| = \pi$. Thus if $\sqrt{\pi}$ could be constructed by ruler and compasses, so could π, which is not constructible. This shows the impossibility of squaring the circle by using a ruler and compass construction.

Problem 6.2.1 Verify from (4.19) that

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots,$$

and hence show how we can get as close as we wish to trisecting an angle using ruler and compasses by using repeated bisections of an angle. Note that this does *not* show that we can trisect an angle using a *finite* number of ruler and compass operations.

6.3 Properties of the Triangle

Consider three line segments whose lengths are a, b, and c, where $a \geq b \geq c$. It is obvious that these three segments can be fitted together to form a triangle if and only if $b + c > a$, and if a triangle can be formed, it must be unique. Thus if we have two triangles whose sides are a, b, and c, they must be congruent. Suppose we have two triangles that both have sides b and c, and have the same angle between these sides. We say that the two triangles have two sides and the included angle equal. Then we can fit one triangle on top of the other, and it is clear that the third sides must be equal. Therefore, having two sides and the included angle equal is a second condition that gives congruent triangles. There is a third congruence condition, two triangles being congruent if they have the same angles and have one corresponding side equal. For again we can verify that they are congruent by fitting one triangle on top of the other. In practice, we need only verify that two pairs of angles are equal, since equality of the third angles follows from the result we discuss in the next paragraph.

One of the most basic properties of a triangle is that the sum of its three angles is 2π, which is equal to two right angles. A proof of this is given in Problem 1.2.1. Another proof, which is equivalent to this, can be realized by cutting out a paper copy of the triangle and folding it, as shown in Figure

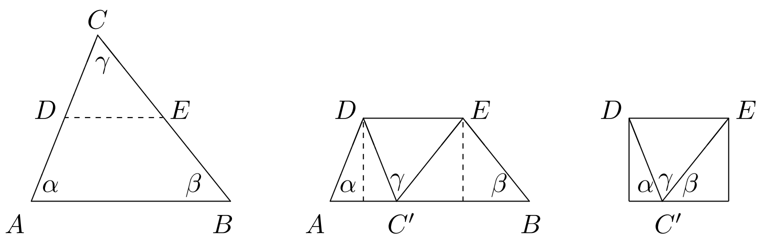


FIGURE 6.14. The sum of the angles in a triangle equals 2π.

6.14. We fold the triangle along a line that bisects two of the sides (see the dotted line DE in the left-hand diagram in Figure 6.14), so that the vertex denoted by C meets the side AB at C'. The line DE is parallel to AB, since the triangles CDE and CAB are similar, and hence the angles CDE and CAB are equal. In the middle diagram we see that the triangle DAC' is isosceles, since $|DA| = |DC'|$, and so the angles DAC' and $DC'A$ are equal. Similarly, the angles $EC'B$ and EBC' are equal. We next fold the triangle along the two vertical dotted lines shown in the middle diagram in Figure 6.14, so that the vertices A and B both coincide with C'. Then we see in the right-hand diagram in Figure 6.14 that the three angles of the original triangle ABC add up to 2π, or two right angles.

We proved in Construction 10 that the perpendicular bisectors of the three sides of a triangle are concurrent. The point of concurrence, which we will denote by A, is called the circumcenter. It is the center of a circle, the circumcircle, that passes through the vertices of the triangle.

In Construction 9 we showed that the bisectors of the three angles of a triangle are concurrent. This point of concurrence, which we will denote by O, is called the *incenter*. It is the center of a circle, the incircle, that is inscribed in the triangle, touching all three sides.

There are two other very well known points of concurrence in a triangle, which I will justify in the next section. One is the orthocenter, which we will denote by B, where the three perpendiculars from the vertices of the triangle to the opposite sides meet. (These perpendiculars are also called the *altitudes* of the triangle.) The other is the *centroid*, which we will denote by C, where the three *medians* of the triangle intersect. A median is a line joining a vertex of the triangle to the midpoint of the opposite side. The points A, B, and C are displayed in Figure 6.15. In this figure, A_1, A_2, and A_3 denote the vertices of the triangle, B_1, B_2, and B_3 denote the points where the perpendiculars from the vertices of the triangle meet the opposite sides, and C_1, C_2, and C_3 denote the midpoints of the sides of the triangle. The figure displays the three lines that are concurrent at B, and those that are concurrent at C are shown as dotted lines. The three lines that are concurrent at A are omitted from the figure for the sake of clarity,

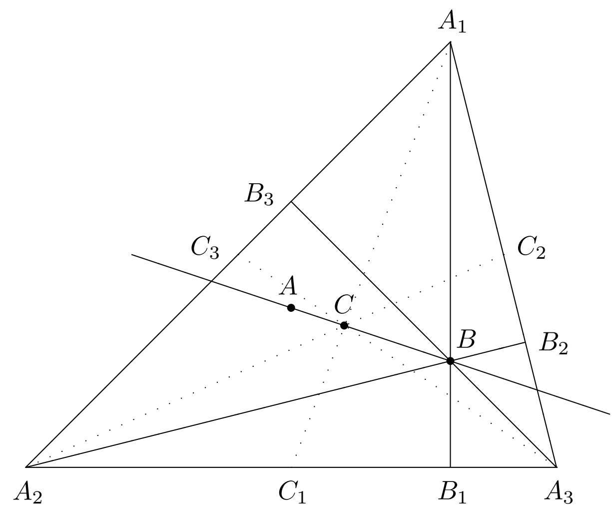


FIGURE 6.15. The circumcenter A, the orthocenter B, and the centroid C all lie on the Euler line.

and especially to help us see that the three points A, B, and C all lie on the same line. The line that contains A, B, and C is called the Euler line, after Leonhard Euler. The centroid trisects each median, meaning that

$$\frac{|CC_1|}{|A_1C_1|} = \frac{|CC_2|}{|A_2C_2|} = \frac{|CC_3|}{|A_3C_3|} = \frac{1}{3}.$$

Amazingly, the centroid also trisects AB, for we have

$$\frac{|AC|}{|AB|} = \frac{1}{3}.$$

We will verify both of these trisection properties of the centroid in the next section.

In Construction 9 we saw how given a triangle ABC we can draw the incircle, which touches BC, CA, and AB on the inside of the triangle. By making a simple modification of this process, we can construct three *excircles* that touch the line segments BC, CA, and AB, or their extensions, *outside* the triangle. Figure 6.16 shows one of these excircles. We bisect the angles DAC and ECA, and let the bisectors intersect at the point O. Then, using the same argument as we used in Construction 9, we see that O is equidistant from the lines BD, BE, and AC. Thus OB is the bisector of the angle DBE.

As we have already seen, there are many interesting and surprising properties possessed by all triangles, and some are more difficult to justify than others. It is not surprising that many of these properties have been known

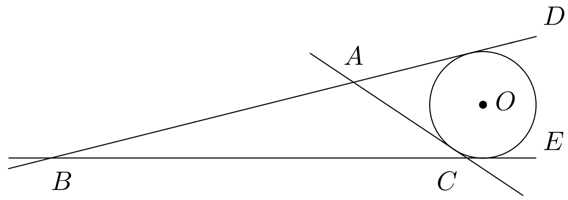


FIGURE 6.16. One of the three excircles of triangle ABC.

since the early part of the millennium of ancient Greek mathematics, and it is easy to imagine how they would have been discovered and rediscovered many times in an age when ruler and compasses were everyday tools of mathematicians. If we draw several arbitrarily chosen triangles and find every time that, for example, the perpendiculars from the vertices of the triangle to the opposite sides are concurrent, it is natural to believe that the result must hold for every triangle. Then we cannot rest until we find a proof!

One most unusual property of the triangle is named after Napoleon Bonaparte (1769–1821). Since Napoleon was a keen amateur geometer, some historians of mathematics believe that he did indeed discover the result, although at least some of its ingredients were known long before his time. We begin with any triangle ABC and draw an equilateral triangle on each of its sides, on the outside of the triangle, as in Figure 6.17. Then the following statements all hold.

1. The centroids of the three equilateral triangles are themselves the vertices of an equilateral triangle.

2. The three lines AA_1, BB_1, and CC_1 are collinear, meeting at a point P, called the *isogonic center* of the triangle ABC.

3. The three line segments AA_1, BB_1, and CC_1 are of equal length, and

$$|AA_1| = |BB_1| = |CC_1| = |PA| + |PB| + |PC|. \tag{6.20}$$

4. The angles B_1PC_1, C_1PA_1, and A_1PB_1 are all equal, and thus each has the value $\frac{2\pi}{3}$.

5. The circumcircles of the three equilateral triangles all pass through the isogonic center P.

In response to a challenge by Pierre de Fermat, E. Torricelli (1608–1647) found the isogonic center as the solution to the problem of finding a point in the plane of a triangle that minimizes the sum of its distances from the three vertices. Obviously Torricelli's result was obtained several generations

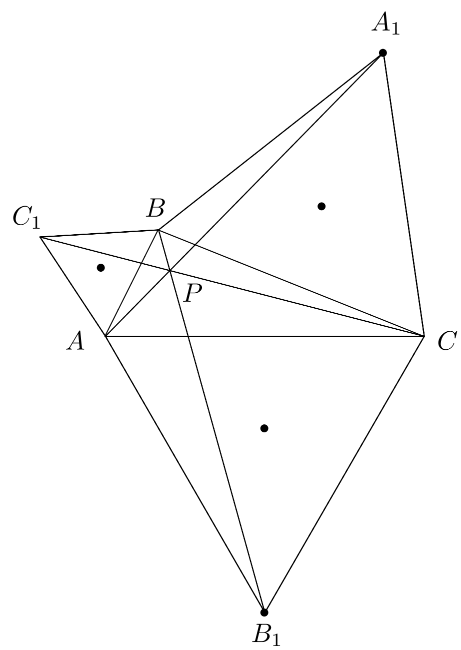


FIGURE 6.17. Napoleon's theorem.

before Napoleon's time. Eves [9] states that the isogonic center was the first notable point of the triangle to be discovered since the era of ancient Greek mathematics. Figure 6.17 gives a ruler and compasses construction of the isogonic center.

In the early nineteenth century, a further set of properties of the triangle was obtained by Karl Feuerbach (1800–1834). This is embodied in the *nine-point circle* theorem, which P. J. Davis [5] says goes back, in part, to J. V. Poncelet (1788–1867) in 1820. I now state this theorem and will omit the proof.

Theorem 6.3.1 The following nine points, related to the triangle with vertices A_1, A_2, and A_3, all lie on the same circle, and are shown in Figure 6.18:

1. The points B_1, B_2, and B_3 where the three altitudes from the vertices of the triangle meet the opposite sides.
2. The midpoints of the three sides of the triangle, C_1, C_2, and C_3.
3. The points D_1, D_2, and D_3, the midpoints along the altitudes from the vertices of the triangle to the orthocenter B. ■

There are other notable points that are on this circle. For example, the nine-point circle touches the incircle of the triangle $A_1A_2A_3$, say at E_0,

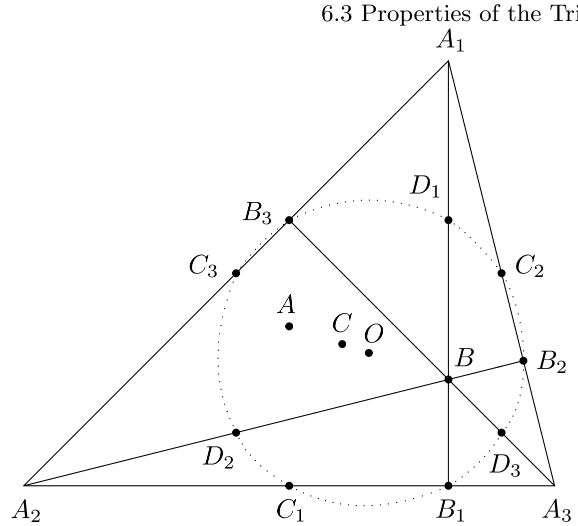


FIGURE 6.18. The nine-point circle.

and all three excircles of the triangle, say at E_1, E_2, and E_3, and thus the four points E_0, E_1, E_2, and E_3 also lie on the nine-point circle.

Let O denote the midpoint of the line joining A, the circumcenter of the triangle, and B, the orthocenter of the triangle. Then, in Figure 6.18, the triangles OBD_1 and ABA_1 are similar, since they have the common angle ABA_1 and

$$\frac{|OB|}{|AB|} = \frac{|D_1B|}{|A_1B|} = \frac{1}{2}.$$

It follows from the similarity of the triangles that

$$\frac{|D_1O|}{|A_1A|} = \frac{1}{2}. \tag{6.21}$$

Since $|A_1A| = |A_2A| = |A_3A|$, we see that $|D_1O| = |D_2O| = |D_3O|$, showing that O is the center of the circle that passes through D_1, D_2, and D_3. This defines the nine-point circle, and it follows from (6.21) that its radius is half that of the circumcircle of triangle $A_1A_2A_3$. Note that O, the center of the nine-point circle, lies on the Euler line.

Problem 6.3.1 Draw Figure 6.17 for the special case in which the triangle ABC is itself an equilateral triangle.

Problem 6.3.2 In Figure 6.17 the equilateral triangles are drawn on the sides of triangle ABC, on the outside of the triangle. Investigate what happens if you draw the same three triangles on the other sides of BC, CA, and AB.

Problem 6.3.3 Draw any quadrilateral, and draw a square on each of its sides, on the outside of the quadrilateral. Finally, draw the two line segments obtained by joining the centers of the squares on opposite sides of the quadrilateral. What can you observe about these two line segments?

Problem 6.3.4 Consider Problem 6.3.3 again, for the special case in which the quadrilateral is a parallelogram.

6.4 Coordinate Geometry

In the seventeenth century geometry was reborn, with the introduction of the new and powerful methods of *coordinate geometry*. These methods are credited to René Descartes (1596–1650) and his contemporary Pierre de Fermat, although the underlying ideas have earlier origins. Descartes realized that every point in the plane can be uniquely defined by its distance plus direction from each of two axes set at right angles. Figure 6.19 shows the horizontal x-axis and the vertical y-axis, marked off at integer points. Each point in the plane of the diagram has an ordered pair of x and y coordinates. These are called Cartesian coordinates, named after Descartes, to distinguish them from other coordinate systems.

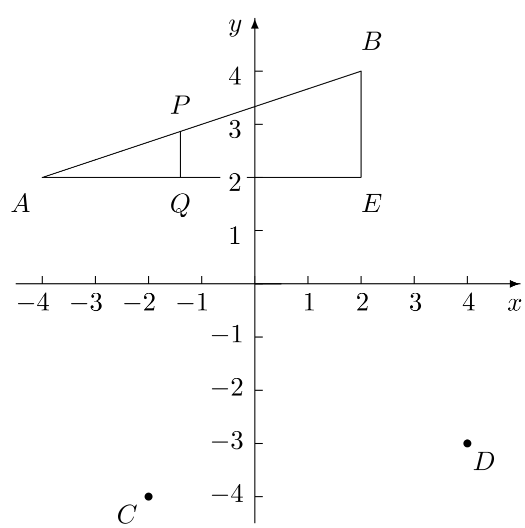


FIGURE 6.19. Cartesian coordinates.

In Figure 6.19, A has coordinates $(-4, 2)$, B has coordinates $(2, 4)$, and the points C and D have coordinates $(-2, -4)$ and $(4, -3)$, respectively.

In triangle ABE, the lines AE and BE are parallel to the x- and y-axes, respectively. Thus E is the point $(2, 2)$. Let P be the point on the straight line AB with coordinates (x, y). The point Q is vertically below P, and is on the line AE. Therefore, Q has coordinates $(x, 2)$. There must be a relation between the two coordinates of the point P, and we will now determine what it is. Since the triangles APQ and ABE are similar, we have

$$\frac{|PQ|}{|AQ|} = \frac{|BE|}{|AE|},$$

which gives

$$\frac{y-2}{x-(-4)} = \frac{4-2}{2-(-4)} = \frac{2}{6} = \frac{1}{3}.$$

On multiplying the latter equation throughout by $3(x+4)$, we obtain

$$3(y-2) = x+4,$$

which we can write in the form

$$x - 3y + 10 = 0. \tag{6.22}$$

Thus the coordinates (x, y) of all points that lie on the line AB satisfy (6.22), which is called the equation of the line AB. We can verify that the coordinates of A and B, namely $(-4, 2)$ and $(2, 4)$, both satisfy (6.22). This gives a reassuring check on our calculations. Every straight line has an equation of the form

$$ax + by + c = 0. \tag{6.23}$$

In particular, the x-axis has the equation $y = 0$, and the y-axis has the equation $x = 0$. Lines of the form $by + c = 0$ are parallel to the x-axis, and lines of the form $ax + c = 0$ are parallel to the y-axis. Apart from lines that are parallel to the y-axis, where $b = 0$ in (6.23), we can divide by b to recast (6.23) in the form

$$y = mx + d, \tag{6.24}$$

where $m = -a/b$ and $d = -c/b$. In (6.24) the constant m is called the *gradient* or *slope* of the straight line. It corresponds to the ratio $|BE|/|AE|$ for the line AB. If in Figure 6.19 we let $A = (x_1, y_1)$ and $B = (x_2, y_2)$, we find that the gradient of AB is

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{6.25}$$

Note that the value of m is unchanged if we interchange (x_1, y_1) and (x_2, y_2), as we should expect. If the gradient is positive, the line slopes upwards from left to right, and if the gradient is negative, the line slopes downwards from left to right. The greater the magnitude of the gradient, the steeper the slope. The number d in (6.24) is called the *intercept*. It marks the point on the y-axis where it is cut by the straight line.

In Figure 6.10 the lines AD and DC meet at right angles. The gradient of the line AD is the positive number $|DB|/|AB|$, and the gradient of DC is the negative number $-|DB|/|BC|$. Thus the *product* of these gradients is equal to -1, since we saw from the similar triangles ABD and DBC (see (6.12)) that $|DB|^2$ is equal to the product of $|AB|$ and $|BC|$. We deduce that, excluding lines that are parallel to the axes, the product of the gradients of any two lines that are perpendicular is equal to -1. For example, as we see from (6.25), the line AC in Figure 6.19 has gradient -3 and is therefore perpendicular to the line AB, which has gradient $1/3$.

Example 6.4.1 Let P and Q have coordinates (x_1, y_1) and (x_2, y_2), and let R have coordinates (x_2, y_1). With this choice of the points P, Q, and R, we see that the angle PRQ is a right angle. (Draw a diagram.) It then follows from Pythagoras's theorem that $|PQ|^2 = |PR|^2 + |QR|^2$, and thus

$$|PQ|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2. \tag{6.26}$$

On taking the square root we obtain a simple expression for the *distance* $|PQ|$ in terms of the Cartesian coordinates of P and Q. ■

Example 6.4.2 Let $ax + by + c = 0$ denote the straight line that joins the points P_1 and P_2, with coordinates (x_1, y_1) and (x_2, y_2), respectively. By this we mean not just the finite line segment that lies between P_1 and P_2, but the line that connects the points and extends indefinitely in both directions. Then we have

$$ax_1 + by_1 + c = 0 \tag{6.27}$$

and

$$ax_2 + by_2 + c = 0. \tag{6.28}$$

If we multiply (6.27) by λ, (6.28) by $1 - \lambda$, and add the two resulting equations, we obtain

$$a(\lambda x_1 + (1 - \lambda)x_2) + b(\lambda y_1 + (1 - \lambda)y_2) + c = 0.$$

This last equation shows that every point (x, y), where

$$x = \lambda x_1 + (1 - \lambda)x_2, \qquad y = \lambda y_1 + (1 - \lambda)y_2,$$

lies on the straight line $ax + by + c = 0$. The values of λ between 0 and 1 give the points that lie between P_1 and P_2. In particular, the value $\lambda = \frac{1}{2}$ gives the midpoint of P_1P_2. Can you prove this? ■

Example 6.4.3 Let us find the distance between two parallel lines l_1 and l_2 whose equations are

$$ax + by + c_1 = 0 \quad \text{and} \quad ax + by + c_2 = 0, \tag{6.29}$$

where $c_1 \neq c_2$. One way of solving this problem is to begin by choosing any point, say, P_1, on the line l_1. Next, we construct the line l_3 that passes through P_1 and is perpendicular to l_1. Let the lines l_3 and l_2 intersect at the point Q_1. Then $|P_1Q_1|$ is the distance between l_1 and l_2.

If $a \neq 0$, we can put $y = 0$ in the equation for l_1 in (6.29), and thus choose

$$P_1 = \left(-\frac{c_1}{a}, 0\right). \tag{6.30}$$

Then the line l_3 has gradient b/a, contains P_1, and has equation

$$-bx + ay - \frac{bc_1}{a} = 0. \tag{6.31}$$

On solving equation (6.31) and the second equation in (6.29) simultaneously, we find that Q_1, the point of intersection of l_3 and l_2, is given by

$$Q_1 = \left(\frac{-c_2a^2 - c_1b^2}{a\left(a^2 + b^2\right)}, \frac{b(c_1 - c_2)}{a^2 + b^2}\right). \tag{6.32}$$

Finally, we use (6.26) to obtain from (6.30) and (6.32) that the distance between the lines l_1 and l_2 is

$$|P_1Q_1| = \frac{|c_1 - c_2|}{\left(a^2 + b^2\right)^{1/2}}. \tag{6.33}$$

It is easily verified that (6.33) also holds when $a = 0$.

We can derive (6.33) in a more interesting way. First, let us assume that l_1 and l_2 are not parallel to the x-axis or the y-axis. In Figure 6.20 the parallel lines l_1 and l_2 cut the x-axis at the two points P_1 and P_2, respectively. Let $|P_1P_2| = X$, and let us write $d = |P_1Q_1|$, the distance between the lines l_1 and l_2. It then follows from triangle $Q_1P_1P_2$ that

$$\frac{|P_1Q_1|}{|P_1P_2|} = \frac{d}{X} = \sin\theta, \tag{6.34}$$

where θ is the angle that each line l_1 and l_2 makes with the x-axis. We can see that each line l_1 and l_2 makes an angle $\frac{\pi}{2} - \theta$ with the y-axis. Then if Y denotes the distance between the two points where the lines l_1 and l_2 cut the y-axis, we similarly obtain

$$\frac{d}{Y} = \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta. \tag{6.35}$$

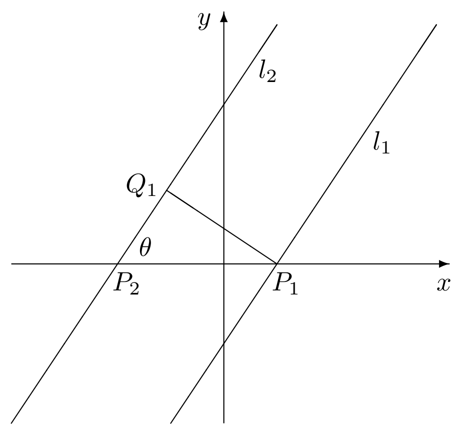


FIGURE 6.20. $P_1Q_1 = d$ is the distance between the parallel lines l_1 and l_2.

Since $\cos^2\theta + \sin^2\theta = 1$, we deduce from (6.34) and (6.35) that

$$\frac{d^2}{X^2} + \frac{d^2}{Y^2} = 1,$$

which we can write in the form

$$\frac{1}{d^2} = \frac{1}{X^2} + \frac{1}{Y^2},$$

so that

$$d = \frac{XY}{(X^2+Y^2)^{1/2}}. \tag{6.36}$$

Now let l_1 and l_2 be represented by the equations in (6.29). If $a \neq 0$, the lines l_1 and l_2 cut the x-axis where $x = -c_1/a$ and $x = -c_2/a$, respectively. If $b \neq 0$, l_1 and l_2 cut the y-axis where $y = -c_1/b$ and $y = -c_2/b$. Thus when $a \neq 0$ and $b \neq 0$ we obtain

$$X = \left|\frac{c_1 - c_2}{a}\right| \quad \text{and} \quad Y = \left|\frac{c_1 - c_2}{b}\right|,$$

and if we substitute these values into (6.36), we obtain (6.33). ■

Let the point A_j have coordinates (x_j, y_j), for $j = 1, 2$, and 3. Then the point C_1 with coordinates $(\frac{1}{2}(x_2+x_3), \frac{1}{2}(y_2+y_3))$ is the midpoint of A_2A_3. (See Example 6.4.2.) If we take λ times the coordinates of A_1 plus $1-\lambda$ times the coordinates of C_1, we get a point on the median A_1C_1. In particular, the choice of $\lambda = \frac{1}{3}$ gives the point

$$C = \left(\tfrac{1}{3}(x_1+x_2+x_3), \tfrac{1}{3}(y_1+y_2+y_3)\right). \tag{6.37}$$

We can show in the same way that C lies also on the other two medians, A_2C_2 and A_3C_3. Thus coordinate geometry has given us a simple proof that the medians of a triangle are concurrent at the centroid C, and that C trisects each of the medians.

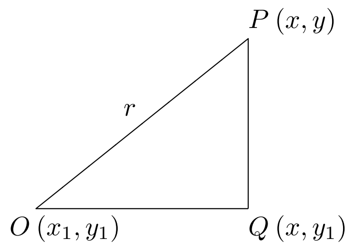


FIGURE 6.21. Derivation of the equation of a circle.

To obtain the equation of a circle, consider the triangle OPQ in Figure 6.21, where O is the fixed point with coordinates (x_1, y_1), P is any point (x, y) whose distance from O is r, and Q is the point where the vertical line through P intersects the horizontal line through O. Although Figure 6.21 shows P as lying above and to the right of O, it can lie anywhere on the circle with center at O and with radius r. Then, since the triangle OPQ has a right angle at Q, it follows from Pythagoras's theorem that $|OQ|^2 + |PQ|^2 = |OP|^2$. Since $|OQ|^2 = (x - x_1)^2$, $|PQ|^2 = (y - y_1)^2$, and $|OP|^2 = r^2$, we obtain

$$(x - x_1)^2 + (y - y_1)^2 = r^2. \tag{6.38}$$

The circle with center at (x_1, y_1) and radius r is the set of all points (x, y) whose coordinates satisfy (6.38). Thus the coordinate geometry of Descartes allows us to turn geometrical problems into algebraic problems.

Let us find the coordinates of the circumcenter A of the triangle $A_1A_2A_3$, where A_j has coordinates (x_j, y_j), for $j = 1, 2$, and 3. We can assume that y_1, y_2, and y_3 are all different. For if two of these three numbers were equal, we could rotate the triangle so that the new y coordinates were all different. Since the line A_2A_3 has gradient $(y_2 - y_3)/(x_2 - x_3)$, any line perpendicular to A_2A_3 has gradient $-(x_2 - x_3)/(y_2 - y_3)$. The equation of the line that passes through C_1, the midpoint of A_2A_3, and is perpendicular to A_2A_3 is

$$\frac{y - \frac{1}{2}(y_2 + y_3)}{x - \frac{1}{2}(x_2 + x_3)} = -\left(\frac{x_2 - x_3}{y_2 - y_3}\right). \tag{6.39}$$

Similarly, the equation of the line that passes through C_2, the midpoint of A_3A_1, and is perpendicular to A_3A_1 is

$$\frac{y - \frac{1}{2}(y_3 + y_1)}{x - \frac{1}{2}(x_3 + x_1)} = -\left(\frac{x_3 - x_1}{y_3 - y_1}\right). \tag{6.40}$$

Notice that we can change (6.39) into (6.40) by changing the suffixes 1, 2, and 3 in *cyclic* order, so that 1 becomes 2, 2 becomes 3, and 3 becomes 1. If we express the line in (6.39) in the form

$$y = m_1 x + d_1, \tag{6.41}$$

we find that

$$m_1 = -\left(\frac{x_2 - x_3}{y_2 - y_3}\right), \quad d_1 = \frac{1}{2}(y_2 + y_3) + \frac{(x_2 + x_3)(x_2 - x_3)}{2(y_2 - y_3)}. \tag{6.42}$$

By permuting the suffixes in (6.41) and (6.42) cyclically, we can recast the equation of the straight line in (6.40) in the form

$$y = m_2 x + d_2, \tag{6.43}$$

where

$$m_2 = -\left(\frac{x_3 - x_1}{y_3 - y_1}\right), \quad d_2 = \frac{1}{2}(y_3 + y_1) + \frac{(x_3 + x_1)(x_3 - x_1)}{2(y_3 - y_1)}. \tag{6.44}$$

Let (x_A, y_A) denote the coordinates of the point where the lines defined by (6.41) and (6.43) intersect. We see that

$$m_1 x_A + d_1 = m_2 x_A + d_2,$$

so that

$$x_A = -\left(\frac{d_1 - d_2}{m_1 - m_2}\right). \tag{6.45}$$

We now wish to express x_A directly in terms of the coordinates of the vertices of the triangle $A_1A_2A_3$. For the denominator of the fraction on the right of (6.45), we obtain from (6.42) and (6.44) that

$$m_1 - m_2 = \frac{-\Delta_{xy}}{(y_2 - y_3)(y_3 - y_1)}, \tag{6.46}$$

where

$$\Delta_{xy} = x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2). \tag{6.47}$$

Note that Δ_{xy} must be nonzero, since the gradients m_1 and m_2 are different. For the numerator on the right of (6.45), a little work shows that

$$d_2 - d_1 = \frac{\Delta_y - \{x_1^2(y_2 - y_3) + x_2^2(y_3 - y_1) + x_3^2(y_1 - y_2)\}}{2(y_2 - y_3)(y_3 - y_1)}, \tag{6.48}$$

where

$$\Delta_y = (y_2 - y_3)(y_3 - y_1)(y_1 - y_2). \tag{6.49}$$

Then, on combining our results in (6.45), (6.46), and (6.48), we find that the x coordinate x_A of the circumcenter A of the triangle $A_1A_2A_3$ is

$$x_A = \frac{\{x_1^2(y_2 - y_3) + x_2^2(y_3 - y_1) + x_3^2(y_1 - y_2)\} - \Delta_y}{2\Delta_{xy}}. \tag{6.50}$$

The y coordinate, y_A, obtained by interchanging x and y in (6.50), is

$$y_A = \frac{\{y_1^2(x_2 - x_3) + y_2^2(x_3 - x_1) + y_3^2(x_1 - x_2)\} - \Delta_x}{2\Delta_{yx}}. \tag{6.51}$$

Notice that the coordinates x_A and y_A, which we derived by finding the point of intersection of the perpendicular bisectors of A_2A_3 and A_3A_1, are symmetric in x_1, x_2, and x_3, and also in y_1, y_2, and y_3. Thus we would obtain the same point (x_A, y_A) as the point of intersection of the perpendicular bisectors of A_3A_1 and A_1A_2. This confirms algebraically what we found (much more easily) in Construction 10, that the perpendicular bisectors of the sides of a triangle are concurrent.

Using the same approach, we can similarly find the point of intersection of the perpendiculars from A_1 to A_2A_3 and A_2 to A_3A_1. Let this point have coordinates (x_B, y_B). We find that

$$x_B = \frac{\Delta_y - \{x_2x_3(y_2 - y_3) + x_3x_1(y_3 - y_1) + x_1x_2(y_1 - y_2)\}}{\Delta_{xy}}, \tag{6.52}$$

and y_B, obtained by interchanging x and y in (6.52), is

$$y_B = \frac{\Delta_x - \{y_2y_3(x_2 - x_3) + y_3y_1(x_3 - x_1) + y_1y_2(x_1 - x_2)\}}{\Delta_{yx}}. \tag{6.53}$$

We observe that the coordinates (x_B, y_B), like (x_A, y_A), are symmetric in x_1, x_2, and x_3, and also in y_1, y_2, and y_3. This *proves* that the three perpendiculars from the vertices of a triangle to the opposite sides are concurrent, and the equations (6.52) and (6.53) give the coordinates of the point of concurrence B, the orthocenter of the triangle $A_1A_2A_3$.

A comparison of (6.50) and (6.52) prompts the observation that we can eliminate Δ_y by adding twice x_A to x_B. On carrying out this calculation, we obtain

$$2x_A + x_B = \frac{X_1(y_2 - y_3) + X_2(y_3 - y_1) + X_3(y_1 - y_2)}{\Delta_{xy}}, \tag{6.54}$$

where

$$X_1 = x_1^2 - x_2x_3, \quad X_2 = x_2^2 - x_3x_1, \quad X_3 = x_3^2 - x_1x_2.$$

We can express

$$X_j = x_j(x_1 + x_2 + x_3) - (x_2x_3 + x_3x_1 + x_1x_2), \quad \text{for} \quad 1 \le j \le 3,$$

and we note that

$$(x_2x_3 + x_3x_1 + x_1x_2)\{(y_2 - y_3) + (y_3 - y_1) + (y_1 - y_2)\} = 0,$$

since the second factor on the left of the latter equation is zero. Thus the numerator on the right of (6.54) can be recast in the form

$$(x_1 + x_2 + x_3)\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}.$$

Since

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = \Delta_{xy},$$

(6.54) greatly simplifies to give

$$2x_A + x_B = x_1 + x_2 + x_3.$$

We can treat the y coordinates similarly, and so obtain

$$\frac{2}{3}x_A + \frac{1}{3}x_B = x_C, \quad \frac{2}{3}y_A + \frac{1}{3}y_B = y_C, \tag{6.55}$$

say, where $x_C = \frac{1}{3}(x_1+x_2+x_3)$ and $y_C = \frac{1}{3}(y_1+y_2+y_3)$ are the coordinates of the centroid of the triangle. We have shown that the circumcenter A, the orthocenter B, and the centroid C all lie on the same straight line, called the Euler line, and that C divides the line segment AB in the ratio $1 : 2$; that is, C is one-third of the way from A to B. Although these properties of the points A, B, and C were not *proved* until the eighteenth century, it is difficult to believe that they were not known empirically to at least some individuals among the many practitioners of the art of ruler and compass constructions two thousand years earlier.

Our verification of the above facts concerning the Euler line illustrates a general point about coordinate geometry. Although it can be aesthetically pleasing to use, it sometimes leaves us none the wiser about the underlying mathematics. It can seem as mechanical in its operation as a ruler and compass construction. Yet, and this shows its importance, it can provide us with *proofs* of geometrical results.

Fermat showed (see Edwards [7]) that if an equation of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{6.56}$$

is satisfied by any point with Cartesian coordinates (x, y), then it is the equation of an ellipse, hyperbola, parabola, or pair of straight lines. Thus, just as these curves are unified geometrically in that they are all sections of a cone, they are also unified algebraically by (6.56).

Problem 6.4.1 Consider a triangle ABC. Let D denote the midpoint of BC, and let D, A, B, and C have coordinates $(0, 0)$, (x_1, y_1), $(-x_2, 0)$, and $(x_2, 0)$, respectively. Hence show that $|AB|^2 + |AC|^2 = 2(|AD|^2 + |BD|^2)$, which we obtained using other methods in Problem 1.2.5.

Problem 6.4.2 Verify that $\Delta_{yx} = -\Delta_{xy}$, where Δ_{xy} is defined in (6.47), and Δ_{yx} is obtained by interchanging x and y in (6.47).

Problem 6.4.3 Verify that the equation of the straight line that is parallel to the line l with equation $ax + by + c = 0$ and passes through the point $P = (x_1, y_1)$ is $ax + by - ax_1 - by_1 = 0$. Deduce from (6.33) that the distance d from the point P to the line l is given by

$$d = \frac{|ax_1 + by_1 + c|}{(a^2 + b^2)^{1/2}}.$$

Problem 6.4.4 Consider the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with $a > b > 0$. Write $x = au/b$, and $y = v$, and verify that the above equation becomes

$$u^2 + v^2 = b^2,$$

the equation of a circle with center $u = 0$, $v = 0$, and radius b. Thus the first equation defines a curve in the coordinates x and y that is a circle "stretched" in the x direction. This is an ellipse.

6.5 Regular Polyhedra

In Section 6.1 we discussed regular polygons. The analogue of a polygon in three dimensions is a *polyhedron*. Just as a polygon is constructed by fixing line segments together, a polyhedron is constructed by fixing polygons together. A polyhedron that is constructed by fixing together a number of copies of the same *regular* polygon in a fully symmetric way is called a *regular polyhedron*. The plural of polyhedron is *polyhedra*, and the regular polyhedra are also called the Platonic solids. These are discussed in Euclid's *Elements*, and the best known regular polyhedron is the *cube*, which has six square *faces*, twelve *edges*, and eight *vertices*. Figure 6.22 displays a cube on the left and its *net*, the cross-shaped diagram in the middle of Figure 6.22, constructed from six squares. We can cut out this shape from a piece of cardboard and fold it to make a cube. (It is useful to augment the net by adding some extra pieces to help hold the cube together. We can coat the extra pieces with glue and fold them, out of sight, behind faces of the cube.) Coxeter [4] states that Leonardo da Vinci (1452–1519) made skeletal models of polyhedra, using strips of wood for their edges and leaving their faces to be imagined. When a model of this kind is viewed from just outside the center of one face, this face is seen as a large polygon

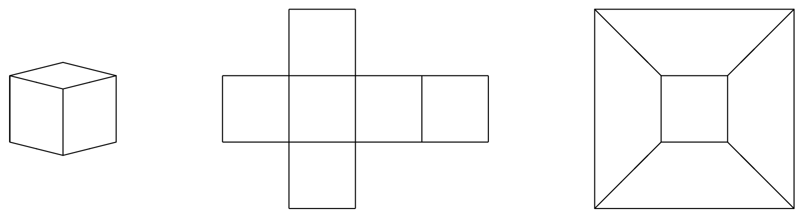


FIGURE 6.22. The cube, its net, and its Schlegel diagram.

with all the other faces filling its interior. Such a view of the cube is shown on the right of Figure 6.22. It is called a *Schlegel* diagram.

We will use the notation $\{n, m\}$ to denote a regular polyhedron that is constructed from polygons with n sides, with m such polygons meeting around each vertex. Note that we must have $n \geq 3$ and $m \geq 3$. In this notation the cube is denoted by $\{4, 3\}$. As we saw in (3.34), each angle of a regular polygon with n sides is $(1-2/n)\pi$. This angle increases with n and has the value $2\pi/3$ when $n = 6$. Thus three regular hexagons fit together precisely, lying flat on the plane, as shown in Figure 6.23.

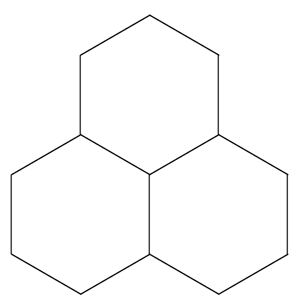


FIGURE 6.23. The plane can be covered with hexagons.

It follows that there is no regular polyhedron constructed from polygons with six or more sides. Incidentally, Figure 6.23 shows that the whole plane can be covered by regular hexagons of the same size. We can express such a covering of the plane by the notation $\{6, 3\}$. It is easy to verify that there are only two other regular polygons that can be used to cover the plane, namely the equilateral triangle and the square, and we will denote these coverings by $\{3,6\}$ and $\{4,4\}$, respectively. Coverings of the plane, such as $\{6,3\}$, $\{3,6\}$, and $\{4,4\}$, are also called plane *tessellations* or *tilings*.

The above analysis shows that the only possible regular polyhedra are those that are constructed from equilateral triangles, squares, and regular pentagons. First, let us consider regular polyhedra that are constructed from equilateral triangles, whose angles are all $\pi/3$. The only possible regular polyhedra $\{3,m\}$ are those for which

$$m \cdot \frac{\pi}{3} < 2\pi, \quad \text{with} \quad m \geq 3,$$

which gives $m = 3, 4$ or 5. Second, when $n = 4$, we need to consider regular polyhedra that are constructed from squares, whose angles are all $\pi/2$. The only possible regular polyhedra $\{4,m\}$ are those for which

$$m \cdot \frac{\pi}{2} < 2\pi, \quad \text{with} \quad m \geq 3.$$

The only solution is $m = 3$, giving the cube, which we have already studied in Figure 6.22. The only other possible regular polyhedra are those of the form $\{5,m\}$, involving regular pentagons, whose angles (see (3.34)) are all $3\pi/5$. In this case we require that

$$m \cdot \frac{3\pi}{5} < 2\pi, \quad \text{with} \quad m \geq 3,$$

and the only solution is $m = 3$. Thus there are only *five* possible regular polyhedra, namely

$$\{3,3\}, \quad \{3,4\}, \quad \{3,5\}, \quad \{4,3\} \quad \{5,3\}. \tag{6.57}$$

We know that the $\{4,3\}$ case does indeed give a regular polyhedron (the cube), and we will now follow up the other four possibilities defined in (6.57). As we will see, all five of the above configurations $\{n,m\}$ yield regular polyhedra.

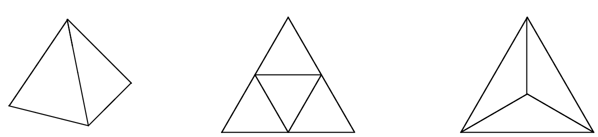


FIGURE 6.24. The regular tetrahedron, its net, and its Schlegel diagram.

Let us call a regular polygon with n vertices an n-gon. Consider any of the cases $\{n,m\}$ defined in (6.57). We begin by fixing together m n-gons so that they meet at a vertex. This creates further vertices. If possible, we

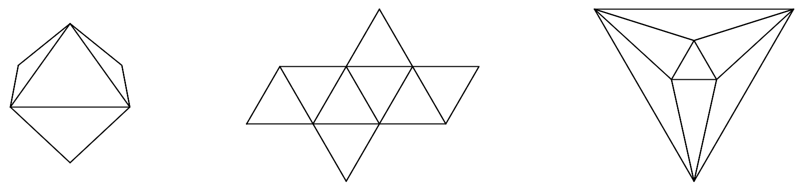


FIGURE 6.25. The regular octahedron, its net, and its Schlegel diagram.

add more n-gons, so that there are m at each new vertex. We continue this process to see whether, ultimately, we do obtain a regular polyhedron $\{n, m\}$. The reader may find it helpful to apply this constructive method to obtain the cube, since it is already familiar to us.

For the case $\{3, 3\}$, we begin by fixing together three equilateral triangles around a point P. There are now three other vertices, say A, B, and C. The three edges BC, CA, and AB are all equal, and we can fix a fourth equilateral triangle to ABC to complete a regular polyhedron. This is called the *regular tetrahedron* which, together with its net and its Schlegel diagram, is depicted in Figure 6.24. The regular tetrahedron can also be described, with less precision, as a pyramid on a triangular base.

Let us pursue the second case in (6.57), namely $\{3, 4\}$. We begin by joining together four equilateral triangles around a point P. There are now four other vertices, say A, B, C, and D. By symmetry, the vertices A, B, C, and D must lie in the same plane in a square. Thus the configuration $PABCD$ is a pyramid on a square base. Following the procedure described above, we can complete the polyhedron that is displayed in Figure 6.25, together with its net and Schlegel diagram. This is the *octahedron*, whose name means that it has eight faces. An octahedron can also be constructed by gluing together two copies of the pyramid $PABCD$.

Let us consider the cube again, and join the center of each face to the centers of all four neighboring faces. This construction gives an octahedron whose vertices are the centers of the six faces of the cube. Conversely, if we join the center of each face of an octahedron to the centers of all three neighboring faces, we obtain a cube. The vertices of the cube are the centers of the faces of the octagon. Thus there is a correspondence between the faces of a cube and the vertices of an octahedron, and a correspondence between the vertices of a cube and the faces of an octahedron. We say that each of these two polyhedra is the *dual* of the other. On applying the same process to the tetrahedron, joining the center of each face to the centers of all neighboring faces (that is, to *all* faces), we obtain another tetrahedron. We say that the tetrahedron is its own dual.

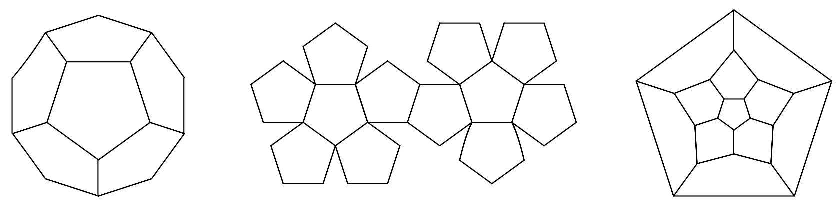


FIGURE 6.26. The regular dodecahedron, its net, and its Schlegel diagram.

The two remaining configurations defined in (6.57) are the most glorious. The polyhedron $\{5, 3\}$, involving the regular pentagon, is depicted in Figure 6.26, together with its net and its Schlegel diagram. It is called the regular dodecahedron, meaning twelve faces. The polyhedron $\{3, 5\}$, with five equilateral triangles around each vertex, is shown in Figure 6.27, together with its net and its Schlegel diagram. This is the regular icosahedron, meaning twenty faces. Note that the dodecahedron and icosahedron are duals. The net in Figure 6.27 illustrates the fact that $\{3, 6\}$ provides a tessellation of the plane, to which we referred above. Indeed, the whole net itself in Figure 6.27 provides a tessellation of the plane.

The basic data about the regular polyhedra are summarized in Table 6.1, where F, E, and V denote the number of faces, edges, and vertices, respectively. Note that the value of F for a given regular polyhedron is the same as the value of V for its dual, and the dual of $\{n, m\}$ is $\{m, n\}$. Using the appropriate ruler and compass constructions in Section 6.1, we can construct the nets for the regular polyhedra. The reader will find it instructive and satisfying to make models of all five regular polyhedra.

Name	$\{n, m\}$	F	E	V
tetrahedron	$\{3, 3\}$	4	6	4
cube	$\{4, 3\}$	6	12	8
octahedron	$\{3, 4\}$	8	12	6
dodecahedron	$\{5, 3\}$	12	30	20
icosahedron	$\{3, 5\}$	20	30	12

TABLE 6.1. The five Platonic solids.

From the definition of duality given above, the dual of a given Platonic solid is found on interchanging the values of F and V in Table 6.1. Thus, as already stated, the tetrahedron is its own dual, the cube and the octahedron are duals, and the dodecahedron and the icosahedron are duals.

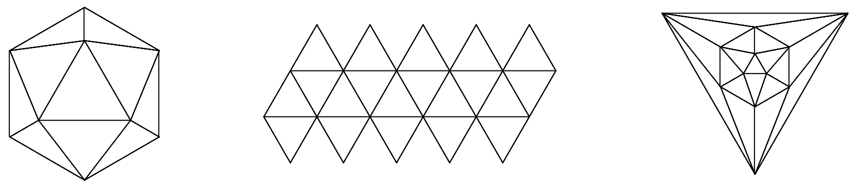


FIGURE 6.27. The regular icosahedron, its net, and its Schlegel diagram.

In Section 3.3 we considered the star of Pythagoras, or pentagram, which can be constructed by extending the sides of a regular pentagon. We can do this for any polygon with five or more sides to obtain star-like figures. For example, if we begin with a regular hexagon, we obtain the star of David. Analogously, we can obtain star-like polyhedra if we extend faces of polyhedra. These are called *stellated* polyhedra, *stella* being the Latin word for star. It is obvious that we cannot stellate the tetrahedron or the cube. However, we can construct a *stellated octahedron*. Although this figure was known much earlier, it was studied by Johannes Kepler (1571–1630). It consists of a regular octahedron with a tetrahedron glued to each face, and can be viewed as two intersecting tetrahedra. Of the many known stellated polyhedra, I will mention only two more, both of which were also studied by Kepler, and are derived from the regular dodecahedron and regular icosahedron. To obtain the first of these, we begin with a regular dodecahedron and extend the edges around each face to give a pentagram. This gives the figure called the *small stellated dodecahedron*. To obtain the other figure we begin with the regular icosahedron, and consider any vertex P. We then take the pentagon formed from the five neighboring vertices of P and replace it by a pentagram. If we do this for each of the 12 vertices P, we obtain the figure that is called the *great stellated dodecahedron*.

In Book V of his *Mathematical Collection*, Pappus of Alexandria discusses a set of polyhedra that, like the regular polyhedra, are constructed from regular polygons. In this case, the polygons do not all have the same number of sides. However, there is the same configuration of polygons around each vertex. These are called the *semiregular* polyhedra. Pappus attributes their discovery to Archimedes, and they are also called the Archimedean solids. We denote a semiregular polyhedron by $(n_1, n_2, \ldots, n_m)$, where n_1, $n_2, \ldots, n_m$, not all equal, denote the number of sides in the polygons, taken in cyclic order around each vertex. For example, $(3, 4, 3, 4)$ denotes the semiregular polyhedron that has four regular polygons, with sides 3, 4, 3, and 4, around each vertex. This is called the *cuboctahedron*. It is obtained by chopping off each of the eight corners of a cube by joining together the midpoints of adjoining edges. If we chop off the corners of the cube so that each of its six faces becomes a regular octagon, we obtain the polyhedron $(3, 8, 8)$, called the *truncated cube*. We could also use this notation to describe the regular polyhedra and tessellations of the plane. For example, $(4, 4, 4)$ denotes the cube, while $(3, 3, 3, 3, 3, 3)$, $(4, 4, 4, 4)$, and $(6, 6, 6)$ denote the tessellations of the plane described above.

For any Archimedean solid $(n_1, n_2, \ldots, n_m)$, the condition that the sum of the angles around each vertex must be less than 2π yields the inequality

$$\sum_{j=1}^{m}\left(1-\frac{2}{n_j}\right)\pi < 2\pi, \quad \text{where} \quad m \ge 3 \quad \text{and every} \quad n_j \ge 3,$$

and we can deduce that this is equivalent to

$$\sum_{j=1}^{m}\frac{1}{n_j} > \frac{1}{2}m - 1, \quad \text{where} \quad m \ge 3 \quad \text{and every} \quad n_j \ge 3. \tag{6.58}$$

The inequality (6.58) also applies to the regular polyhedra, where each $n_j = n$. In this case we find that (6.58) is equivalent to the identity

$$\frac{1}{m} + \frac{1}{n} > \frac{1}{2}, \quad \text{where} \quad m, n \ge 3. \tag{6.59}$$

In our discussion of the regular polyhedra, we effectively worked through the inequality (6.59) for the cases $n = 3$, $n = 4$, and $n = 5$ separately, and found that every feasible solution of (6.59) yields a regular polyhedron. In contrast, not all solutions of (6.58) correspond to semiregular polyhedra. For example, with $m = 3$, $n_1 = n_2 = 3$ and $n_3 = n$ in (6.58), we find that although the inequality holds for all values of n, none of the solutions with $n > 3$ yields a polyhedron. I state without proof that of all the solutions of (6.58) where the n_j are not all equal, only *thirteen* solutions yield polyhedra. All thirteen Archimedean solids $(n_1, n_2, \ldots, n_m)$ are listed in Table 6.2, together with their names and numbers of faces, edges, and vertices. The number of faces of each of the different constituent polygons is given in square brackets after the number of faces in column F. For example, the truncated cube has 14 faces (8 triangles and 6 octagons). The Archimedean solid with the smallest number of faces (4 triangles and 4 hexagons) is the truncated tetrahedron. This is obtained by taking a regular tetrahedron and removing four small regular tetrahedra, each including a vertex of the large tetrahedron and whose edges are one-third of the length of the edges of the large tetrahedron. The design of many soccer balls in current use is based on the truncated icosahedron, $(5, 6, 6)$, which has 12 pentagons and 20 hexagons. The Archimedean solid with most faces is the snub dodecahedron, with 80 triangular and 12 pentagonal faces. The great rhombicosidodecahedron is the Archimedean solid with the most edges and the most vertices, and it also has the longest name! Making models of all the Archimedean solids would make a worthy class project.

$(n_1, n_2, \ldots, n_m)$, name	F	E	V
$(3, 6, 6)$, truncated tetrahedron	$8\ [4, 4]$	18	12
$(3, 8, 8)$, truncated cube	$14\ [8, 6]$	36	24
$(3, 10, 10)$, truncated dodecahedron	$32\ [20, 12]$	90	60
$(4, 6, 6)$, truncated octahedron	$14\ [6, 8]$	36	24
$(4, 6, 8)$, great rhombicuboctahedron	$26\ [12, 8, 6]$	48	24
$(4, 6, 10)$, great rhombicosidodecahedron	$62\ [30, 20, 12]$	180	120
$(5, 6, 6)$, truncated icosahedron	$32\ [12, 20]$	90	60
$(3, 4, 3, 4)$, cuboctahedron	$14\ [8, 6]$	24	12
$(3, 5, 3, 5)$, icosidodecahedron	$32\ [20, 12]$	60	30
$(3, 4, 4, 4)$, small rhombicuboctahedron	$26\ [8, 18]$	48	24
$(3, 4, 5, 4)$, small rhombicosidodecahedron	$62\ [20, 30, 12]$	120	60
$(3, 3, 3, 3, 4)$, snub cube	$38\ [32, 6]$	60	24
$(3, 3, 3, 3, 5)$, snub dodecahedron	$92\ [80, 12]$	150	60

TABLE 6.2. The thirteen Archimedean solids.

We can also look for solutions of (6.58) where the inequality is replaced by an equality. Some of these solutions, but not *all*, give tessellations of the plane. I state without proof that there are eight of these. They are called Archimedean tessellations and are listed in Table 6.3. The last two both have 3 triangles and 2 squares around each vertex, but taken in a different order. Of all eight tessellations, my favorite is $(3, 3, 4, 3, 4)$.

$(3, 12, 12)$	$(4, 6, 12)$	$(4, 8, 8)$	$(3, 6, 3, 6)$
$(3, 4, 6, 4)$	$(3, 3, 3, 3, 6)$	$(3, 3, 3, 4, 4)$	$(3, 3, 4, 3, 4)$

TABLE 6.3. The eight Archimedean tessellations.

There is an infinite number of ways of tessellating the plane, and we have looked only at some of those that involve regular polygons. Let us call any shape that can be used repeatedly to tessellate the plane a *motif*. We can obviously use any parallelogram as a motif, or a cross constructed from five squares, or a triangle of any shape. The last one is obvious, because we can put two identical triangles together to make a parallelogram. However, it is not so obvious that any quadrilateral can be used as a motif. We can find motifs for the Archimedean tessellations. Motifs for $(3, 3, 4, 3, 4)$ and $(3, 4, 6, 4)$ are given in Figure 6.28.

Let us begin with the tessellation $(4, 4, 4, 4)$. It is obvious that we can amend every square in the same way so that we still have a tessellation of the plane. A very simple example is given in Figure 6.29.

A segment has been removed from the left side of the square and added to the right side. If I had chosen an appropriate triangle as the amending segment in Figure 6.29, I could have changed the square into a parallelogram.

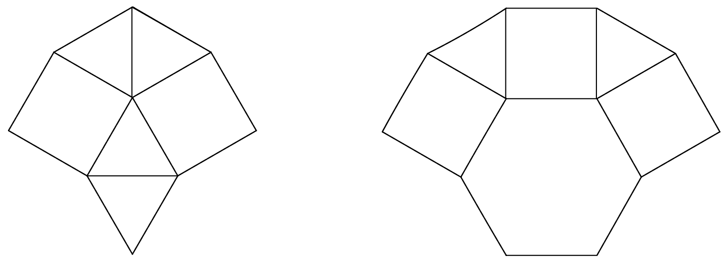


FIGURE 6.28. Motifs for the tessellations $(3,3,4,3,4)$ and $(3,4,6,4)$.

We can also amend the top and bottom edges of the square. Similarly, we can create amended versions of other well-known tessellations, such as $(3,3,3,3,3,3)$. This process is one of the keys to understanding the many very beautiful and fascinating tessellations of the plane that were created by the artist and mathematician M. C. Escher (1898–1972).

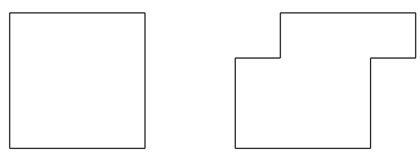


FIGURE 6.29. An amended square that still tessellates the plane.

Roger Penrose (born 1931) invented tessellations that are constructed from two particular quadrilaterals, called a kite and a dart, and involve the golden section. The kite and dart are obtained by dissecting the rhombus $ABCD$ in Figure 6.30. The sides of the rhombus are all of length $\frac{1}{2}\left(\sqrt{5}+1\right)$, $|AE| = |EC| = |BE| = 1$, and the angle ABC is $2\pi/5$.

It can be proved that the kite and dart can be used to cover the plane in an infinite number of ways that are not periodic. This means that if we were able to make a transparency of such a tessellation, there is no way we could move it, without rotating it, so that it matched the tessellation. More material on tessellations can be found in Wells [17].

If we examine Table 6.2 we observe that for every Archimedean solid, F plus V is approximately equal to E. More precisely, we see that

$$F - E + V = 2, \tag{6.60}$$

and an inspection of Table 6.1 reveals that (6.60) holds also for the five Platonic solids. Although the identity (6.60) is often named after Euler and Descartes, it is hard to believe that it was not known to Pappus of

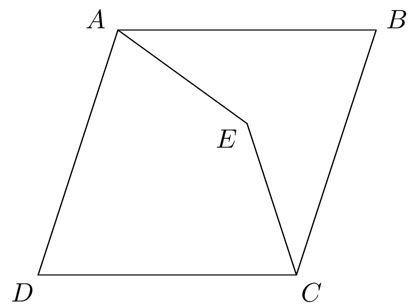


FIGURE 6.30. The rhombus $ABCD$ dissected into Penrose's kite and dart.

Alexandria and to Archimedes. If we begin with a sphere, choose $n \geq 1$ vertices on its surface, and join them up in any way that creates a map of polygonal-shaped countries, then the number of faces, edges, and vertices always satisfies (6.60), and will continue to do so if we stretch or contract any of the line segments. In this way we obtain any polyhedron that is said to be *simply connected*, meaning that it has no holes in it. Beginning with such a polyhedron, let us stretch it, pull the assemblage of faces, edges, and vertices off the sphere and flatten it out to give a map of polygons on the plane that has same number of vertices and edges as before, but has one face fewer. The resulting map will be equivalent to the Schlegel diagram in the case of the Platonic solids.

Let us now consider a map of this kind. We will show that its number of faces, edges, and vertices satisfies the equation $F - E + V = 1$. Since we "lost" a face when we removed the assemblage from the sphere, this is equivalent to proving that (6.60) holds for any polyhedron. To construct such a diagram, we will begin with the empty plane and place one vertex on it. At this stage we have no faces, no edges, and one vertex, and thus $F - E + V = 1$. We then build up the required map by adding one edge at a time. Unless we are completing a polygon, each time we add one edge, we add one vertex. However, if we complete a polygon by adding an edge, we add one face. In either case, we do not change the value of $F - E + V$ with the addition of an edge, and this proves that $F - E + V = 1$ when we have completed the construction of the chosen map. We have thus proved the following theorem.

Theorem 6.5.1 If F, E, and V denote the number of faces, edges, and vertices of a given simply connected polyhedron, then $F - E + V = 2$. ■

For each Platonic solid $\{n, m\}$ we see that if we cut out every face, we would have nF edges. However, each edge of the Platonic solid connects two faces, and so $nF = 2E$. Similarly, each vertex is on m edges, and each edge connects two vertices. It follows that $mV = 2E$, and thus we have

$$nF = 2E = mV. \tag{6.61}$$

Beginning with (6.61) and the identity $F - E + V = 2$ we can derive explicit values of F, E, and V for the platonic solids. See Problem 6.5.6.

Problem 6.5.1 Show that an octahedron and four tetrahedra can be fixed together to form a larger tetrahedron, and that three-dimensional space can be filled using tetrahedra and octahedra.

Problem 6.5.2 Cut out copies of the appropriate regular polygons and explore the eight Archimedean tessellations given in Table 6.3.

Problem 6.5.3 Construct a dual of the $(3, 3, 4, 3, 4)$ tessellation by joining the centers of adjacent polygons. This is called the Cairo tessellation. Observe that it has a pentagonal motif that has four sides of one length and one shorter side.

Problem 6.5.4 Begin with the $(3, 3, 4, 3, 4)$ tessellation whose triangles and squares have sides of length 1, and construct its dual, as in Problem 6.5.3. Show that the pentagon that occurs in the Cairo tessellation has sides, taken in order, whose lengths are a, a, b, a, and b, where $a = \frac{1}{2}$ and $b = \frac{1}{\sqrt{2}}$. Show also that the angle enclosed by the two adjacent sides of length $\frac{1}{2}$ is $\frac{\pi}{3}$, and that the four other angles are $\frac{5\pi}{12}$.

Problem 6.5.5 Show that the dual tessellation of every Archimedean tessellation is composed of repetitions of the same polygon.

Problem 6.5.6 Use (6.61) to express E and V in terms of F and so deduce from (6.60) that

$$F = \frac{4m}{2m + 2n - mn}.$$

Show also that

$$E = \frac{2mn}{2m + 2n - mn},$$

and

$$V = \frac{4n}{2m + 2n - mn}.$$

Note that interchanging m and n in the above expressions corresponds to interchanging F and V, and leaves E unchanged, as we should expect from duality.

Problem 6.5.7 Deduce from the expressions obtained for F, E, and V in Problem 6.5.6 that $2m + 2n - mn > 0$. Show that the latter inequality is equivalent to (6.59), and is also equivalent to

$$(m - 2)(n - 2) < 4.$$

Show that since we must have $m \geq 3$ and $n \geq 3$, this last inequality has only five solutions and that these correspond to the five Platonic solids.

Problem 6.5.8 For the Platonic solid $\{n, m\}$, let δ denote the amount by which the sum of the angles at a vertex falls short of 2π. (Thus $\delta = \pi$ for the tetrahedron and $\delta = \pi/5$ for the dodecahedron.) Show that

$$\delta = 2\pi - m\left(1 - \frac{2}{n}\right)\pi,$$

and deduce that $\delta = 4\pi/V$, where V is the number of vertices in the Platonic solid.

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