2

Optimal Controls on Finite and Infinite Horizons: A Review

2.1 Introduction

In this chapter, important results on optimal controls are reviewed.

Optimal controls depend on the performance criterion that should reflect the designer's concept of good performance. Two important performance criteria are considered for optimal controls. One is for minimization and the other for minimaximization.

Both nonlinear and linear optimal controls are reviewed. First, the general results for nonlinear systems are introduced, particularly with the dynamic programming and a minimum principle. Then, the optimal controls for linear systems are obtained as a special case. Actually, linear quadratic and H_∞ optimal controls are introduced for both state feedback and output feedback controls. Tracking controls are also introduced for future use.

Optimal controls are discussed for free and fixed terminal states. The former may or may not have a terminal cost. In particular, a nonzero terminal cost for the free terminal state is called a free terminal cost in the subsequent chapters. In addition, a fixed terminal state is posed as a terminal equality constraint in the subsequent chapters. The optimal controls for the fixed terminal and nonzero reference case will be derived in this chapter. They are important for RHC. However, they are not common in the literature.

Linear optimal controls are transformed to SDP using LMIs for easier computation of the control laws. This numerical method can be useful for obtaining optimal controls in constrained systems, which will be discussed later.

Most results given in this chapter lay the foundation for the subsequent chapters on receding horizon controls.

Proofs are generally given in order to make our presentation in this book more self-contained, though they appear in the existing literature. H_2 filters and H_2 controls are important, but not used for subsequent chapters; thus, they are summarized without proof.

The organization of this chapter is as follows. In Section 2.2, optimal controls for general systems such as dynamic programming and the minimum principle are dealt with for both minimum and minimax criteria. In Section 2.3, linear optimal controls, such as the LQ control based on the minimum criterion and H_∞ control based on the minimax criterion, are introduced. In Section 2.4, the Kalman filter on the minimum criterion and the H_∞ filter on the minimax criterion are discussed. In Section 2.5, LQG control on the minimum criterion and the output feedback H_∞ control on the minimax criterion are introduced for output feedback optimal controls. In Section 2.6, the infinite horizon LQ and H_∞ control are represented in LMI forms. In Section 2.7, H_2 controls are introduced as a general approach for LQ control.

2.2 Optimal Control for General Systems

In this section, we consider optimal controls for general systems. Two approaches will be taken. The first approach is based on the minimization and the second approach is based on the minimaximization.

2.2.1 Optimal Control Based on Minimum Criterion

Consider the following discrete-time system:

$$x_{i+1} = f(x_i, u_i, i), \;\; x_{i_0} = x_0 \tag{2.1}$$

where $x_i \in \Re^n$ and $u_i \in \Re^m$ are the state and the input respectively, and may be required to belong to the given sets, i.e. $x_i \in \mathcal{X} \in \Re^n$ and $u_i \in \mathcal{U} \in \Re^m$.

A performance criterion with the free terminal state is given by

$$J(x_{i_0}, i_0, u) = \sum_{i=i_0}^{i_f-1} g(x_i, u_i, i) \;+\; h(x_{i_f}, i_f) \tag{2.2}$$

i_0 and i_f are the initial and terminal time. $g(\cdot,\cdot,\cdot)$ and $h(\cdot,\cdot)$ are specified scalar functions. We assume that i_f is fixed here for simplicity. Note that x_{i_f} is free for the performance criterion (2.2). However, x_{i_f} can be fixed. A performance criterion with the fixed terminal state is given by

$$J(x_{i_0}, i_0, u) = \sum_{i=i_0}^{i_f-1} g(x_i, u_i, i) \tag{2.3}$$

subject to

$$x_{i_f} = x_{i_f}^r \tag{2.4}$$

where $x_{i_f}^r$ is given.

Here, the optimal control problem is to find an admissible control $u_i \in \mathcal{U}$ for $i \in [i_0, i_f - 1]$ that minimizes the cost function (2.2) or (2.3) with the constraint (2.4).

The Principle of Optimality and Dynamic Programming

If S-a-D is the optimal path from S to D with the cost J^*_{SD}, then a-D is the optimal path from a to D with J^*_{aD}, as can be seen in Figure 2.1. This property is called *the principle of optimality.* Thus, an optimal policy has the property that, whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

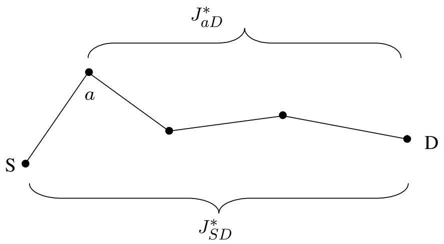


Fig. 2.1. Optimal path from S to D

Now, assume that there are allowable paths S-a-D, S-b-D, S-c-D, and S-d-D and optimal paths from a, b, c, and d to D are J^*_{aD}, J^*_{bD}, J^*_{cD}, and J^*_{dD} respectively, as can be seen in Figure 2.2. Then, the optimal trajectory that starts at S is found by comparing

$$\begin{aligned} J^*_{SaD} &= J_{Sa} + J^*_{aD} \\ J^*_{SbD} &= J_{Sb} + J^*_{bD} \\ J^*_{ScD} &= J_{Sc} + J^*_{cD} \\ J^*_{SdD} &= J_{Sd} + J^*_{dD} \end{aligned} \tag{2.5}$$

The minimum of these costs must be the one associated with the optimal decision at point S. Dynamic programming is a computational technique which extends the above decision-making concept to the sequences of decisions which together define an optimal policy and trajectory.

Assume that the final time i_f is specified. If we consider the performance criterion (2.2) subject to the system (2.1), the performance criterion of dynamic programming can be represented by

$$J(x_i, i, u) = g(x_i, u_i, i) + J^*(x_{i+1}, i+1), \quad i \in [i_0, i_f - 1] \tag{2.6}$$

$$\begin{aligned} J^*(x_i, i) &= \min_{u_\tau, \tau \in [i, i_f - 1]} J(x_i, i, u) \\ &= \min_{u_i} \{ g(x_i, u_i, i) + J^*(f(x_i, u_i, i), i+1) \} \end{aligned} \tag{2.7}$$

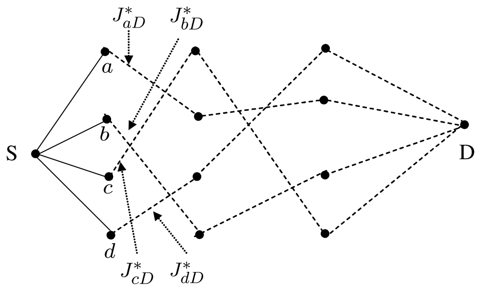


Fig. 2.2. Paths from S through a, b, c, and d to D

where

$$J^*(x_{i_f}, i_f) = h(x_{i_f}, i_f) \tag{2.8}$$

For the fixed terminal state, $J^*(x_{i_f}, i_f) = h(x_{i_f}, i_f)$ is fixed since x_{i_f} and i_f are constants.

It is noted that the dynamic programming method gives a closed-loop control, while the method based on the minimum principle considered next gives an open-loop control for most nonlinear systems.

Pontryagin's Minimum Principle

We assume that the admissible controls are constrained by some boundaries, since in realistic systems control constraints do commonly occur. Physically realizable controls generally have magnitude limitations. For example, the thrust of a rocket engine cannot exceed a certain value and motors provide a limited torque.

By definition, the optimal control u^* makes the performance criterion J a local minimum if

$$J(u) - J(u^*) = \triangle J \geq 0$$

for all admissible controls sufficiently close to u^*. If we let $u = u^* + \delta u$, the increment in J can be expressed as

$$\triangle J(u^*, \delta u) = \delta J(u^*, \delta u) + \text{higher order terms}$$

Hence, the necessary conditions for u^* to be the optimal control are

$$\delta J(u^*, \delta u) \geq 0$$

if u^* lies on the boundary during any portion of the time interval $[i_0,\ i_f]$ and

$$\delta J(u^*, \delta u) = 0$$

if u^* lies within the boundary during the entire time interval $[i_0,\ i_f]$.

We form the following augmented cost functional:

$$J_a = \sum_{i=i_0}^{i_f-1} \left\{ g(x_i, u_i, i) + p_{i+1}^T [f(x_i, u_i, i) - x_{i+1}] \right\} + h(x_{i_f}, i_f)$$

by introducing the Lagrange multipliers $p_{i_0}, p_{i_0+1}, \cdots, p_{i_f}$. For simplicity of the notation, we denote $g(x_i^*, u_i^*, i)$ by g and $f(x_i^*, u_i^*, i)$ by f respectively. Then, the increment of J_a is given by

$$\begin{aligned}
\triangle J_a = & \sum_{i=i_0}^{i_f-1} \Bigg\{ g(x_i^* + \delta x_i, u_i^* + \delta u_i, i) + [p_{i+1}^* + \delta p_{i+1}]^T \\
& \times [f(x_i^* + \delta x_i, u_i^* + \delta u_i, i) - (x_{i+1}^* + \delta x_{i+1}) \Bigg\} + h(x_{i_f}^* + \delta x_{i_f}, i_f) \\
& - \sum_{i=i_0}^{i_f-1} \left\{ g(x_i^*, u_i^*, i) + p_{i+1}^{*T} [f(x_i^*, u_i^*, i) - x_{i+1}^*] \right\} + h(x_{i_f}^*, i_f) \\
= & \sum_{i=i_0}^{i_f-1} \Bigg\{ \left[\frac{\partial g}{\partial x_i} \right]^T \delta x_i + \left[\frac{\partial g}{\partial u_i} \right]^T \delta u_i + p_{i+1}^{*T} \left[\frac{\partial f}{\partial x_i} \right]^T \delta x_i \\
& + p_{i+1}^{*T} \left[\frac{\partial f}{\partial u_i} \right]^T \delta u_i + \delta p_{i+1}^T f(x_i^*, u_i^*, i) \\
& - \delta p_{i+1}^T x_{i+1}^* - p_{i+1}^{*T} \delta x_{i+1} \Bigg\} + \left[\frac{\partial h}{\partial x_{i_f}} \right]^T \delta x_{i_f} \\
& + \text{higher order terms}
\end{aligned} \tag{2.9}$$

To eliminate δx_{i+1}, we use the fact

$$\sum_{i=i_0}^{i_f-1} p_{i+1}^{*T} \delta x_{i+1} = p_{i_f}^{*T} \delta x_{i_f} + \sum_{i=i_0}^{i_f-1} p_i^T \delta x_i$$

Since the initial state x_{i_0} is given, it is apparent that $\delta x_{i_0} = 0$ and p_{i_0} can be chosen arbitrarily. Now, we have

$$\begin{aligned}
\triangle J_a = & \sum_{i=i_0}^{i_f-1} \Bigg\{ \left[\frac{\partial g}{\partial x_i} - p_i^* + \frac{\partial f}{\partial x_i} p_{i+1}^* \right]^T \delta x_i + \left[\frac{\partial g}{\partial u_i} + \frac{\partial f}{\partial u_i} p_{i+1}^* \right]^T \delta u_i \\
& + \delta p_{i+1}^T \Big[f(x_i^*, u_i^*, i) - x_{i+1}^* \Big] \Bigg\} + \left[\frac{\partial h}{\partial x_{i_f}} - p_{i_f}^* \right]^T \delta x_{i_f} \\
& + \text{higher order terms}
\end{aligned}$$

Note that variable δx_i for $i = i_0 + 1, \cdots, i_f$ are all arbitrary. Define the function $\mathcal{H}$, called the *Hamiltonian*

$$\mathcal{H}(x_i, u_i, p_{i+1}, i) \triangleq g(x_i, u_i, i) + p_{i+1}^T f(x_i, u_i, i)$$

If the state equations are satisfied, and p_i^* is selected so that the coefficient of δx_i is identically zero, that is,

$$x_{i+1}^* = f(x_i^*, u_i^*, i) \tag{2.10}$$

$$p_i^* = \frac{\partial g}{\partial x_i} + \frac{\partial f}{\partial x_i} p_{i+1}^* \tag{2.11}$$

$$x_{i_0}^* = x_0 \tag{2.12}$$

$$p_{i_f}^* = \frac{\partial h}{\partial x_{i_f}} \tag{2.13}$$

then we have

$$\triangle J_a = \sum_{i=i_0}^{i_f-1} \left\{ \left[\frac{\partial \mathcal{H}}{\partial u}(x_i^*, u_i^*, p_i^*, i) \right]^T \delta u_i \right\} + \text{higher order terms}$$

The first-order approximation to the change in $\mathcal{H}$ caused by a change in u alone is given by

$$\left[\frac{\partial \mathcal{H}}{\partial u}(x_i^*, u_i^*, p_i^*, i) \right]^T \delta u_i \approx \mathcal{H}(x_i^*, u_i^* + \delta u_i, p_{i+1}^*, i) - \mathcal{H}(x_i^*, u_i^*, p_{i+1}^*, i)$$

Therefore,

$$\begin{aligned} \triangle J(u^*, \delta u) = & \sum_{i=i_0}^{i_f-1} \Big[\mathcal{H}(x_i^*, u_i^* + \delta u_i, p_{i+1}^*, i) - \mathcal{H}(x_i^*, u_i^*, p_{i+1}^*, i) \Big] \\ & + \text{higher order terms} \end{aligned} \tag{2.14}$$

If $u^* + \delta u$ is in a sufficiently small neighborhood of u^*, then the higher order terms are small, and the summation (2.14) dominates the expression for $\triangle J_a$. Thus, for u^* to be an optimal control, it is necessary that

$$\sum_{i=i_0}^{i_f-1} \Big[\mathcal{H}(x_i^*, u_i^* + \delta u_i, p_{i+1}^*, i) - \mathcal{H}(x_i^*, u_i^*, p_{i+1}^*, i) \Big] \geq 0 \tag{2.15}$$

for all admissible δu. We assert that in order for (2.15) to be satisfied for all admissible δu in the specified neighborhood, it is necessary that

$$\mathcal{H}(x_i^*, u_i^* + \delta u_i, p_{i+1}^*, i) - \mathcal{H}(x_i^*, u_i^*, p_{i+1}^*, i) \geq 0 \tag{2.16}$$

for all admissible δu_i and for all $i = i_0, \cdots, i_f$. In order to prove the inequality (2.15), consider the control

$$\begin{aligned} u_i &= u_i^*, \quad i \notin [i_1, i_2] \\ u_i &= u_i^* + \delta u_i, \quad i \in [i_1, i_2] \end{aligned} \tag{2.17}$$

where $[i_1, i_2]$ is a nonzero time interval, i.e. $i_1 < i_2$ and δu_i is an admissible control variation that satisfies $u^* + \delta u \in \mathcal{U}$.

Suppose that inequality (2.16) is not satisfied in the interval $[i_1, i_2]$ for the control described in (2.17). So, we have

$$\mathcal{H}(x_i^*, u_i, p_{i+1}^*, i) < \mathcal{H}(x_i^*, u_i^*, p_{i+1}^*, i)$$

in the interval $[i_1, i_2]$ and the following inequality is obtained:

$$\begin{aligned} &\sum_{i=i_0}^{i_f-1} \Big[\mathcal{H}(x_i^*, u_i, p_{i+1}^*, i) - \mathcal{H}(x_i^*, u_i^*, p_{i+1}^*, i)\Big] \\ &= \sum_{i=i_1}^{i_2} \Big[\mathcal{H}(x_i^*, u_i, p_{i+1}^*, i) - \mathcal{H}(x_i^*, u_i^*, p_{i+1}^*, i)\Big] < 0 \end{aligned}$$

Since the interval $[i_1, i_2]$ can be anywhere in the interval $[i_0, i_f]$, it is clear that if

$$\mathcal{H}(x_i^*, u_i, p_{i+1}^*, i) < \mathcal{H}(x_i^*, u_i^*, p_{i+1}^*, i)$$

for any $i \in [i_0, i_f]$, then it is always possible to construct an admissible control, as in (2.17), which makes $\triangle J_a < 0$, thus contradicting the optimality of the control u_i^*. Therefore, a necessary condition for u_i^* to minimize the functional J_a is

$$\mathcal{H}(x_i^*, u_i^*, p_{i+1}^*, i) \leq \mathcal{H}(x_i^*, u_i, p_{i+1}^*, i) \tag{2.18}$$

for all $i \in [i_0, i_f]$ and for all admissible controls. The inequality (2.18) indicates that *an optimal control must minimize the Hamiltonian*. Note that we have established a necessary, but not, in general, sufficient, condition for optimality. An optimal control must satisfy the inequality (2.18). However, there may be controls that satisfy the minimum principle that are not optimal.

We now summarize the principle results. In terms of the Hamiltonian, the necessary conditions for u_i^* to be an optimal control are

$$x_{i+1}^* = \frac{\partial \mathcal{H}}{\partial p_{i+1}}(x_i^*, u_i^*, p_{i+1}^*, i) \tag{2.19}$$

$$p_i^* = \frac{\partial \mathcal{H}}{\partial x}(x_i^*, u_i^*, p_{i+1}^*, i) \tag{2.20}$$

$$\mathcal{H}(x_i^*, u_i^*, p_{i+1}^*, i) \leq \mathcal{H}(x_i^*, u_i, p_{i+1}^*, i) \tag{2.21}$$

for all admissible u_i and $i \in [i_0, i_f - 1]$, and two boundary conditions

$$x_{i_0} = x_0, \quad p_{i_f}^* = \frac{\partial h}{\partial x_{i_f}}(x_{i_f}^*, i_f)$$

The above result is called Pontryagin's minimum principle. The minimum principle, although derived for controls in the given set $\mathcal{U}$, can also be applied to problems in which the admissible controls are not bounded. In this case, for u_i^* to minimize the Hamiltonian it is necessary (but not sufficient) that

$$\frac{\partial \mathcal{H}}{\partial u_i}(x_i^*, u_i^*, p_{i+1}^*, i) = 0, \quad i \in [i_0, i_f - 1] \tag{2.22}$$

If (2.22) is satisfied and the matrix

$$\frac{\partial^2 \mathcal{H}}{\partial u_i^2}(x_i^*, u_i^*, p_{i+1}^*, i)$$

is positive definite, then this is sufficient to guarantee that u_i^* makes J_a a local minimum. If the Hamiltonian can be expressed in the form

$$\mathcal{H}(x_i, u_i, p_{i+1}, i) = c_0(x_i, p_{i+1}, i) + \big[c_1(x_i, p_{i+1}, i)\big]^T u_i + \frac{1}{2} u_i^T R u_i$$

where $c_0(\cdot,\cdot,\cdot)$ and $c_1(\cdot,\cdot,\cdot)$ are a scalar and an $m \times 1$ vector function respectively, that do not have any term containing u_i, then (2.22) and $\partial^2\mathcal{H}/\partial u_i^2 > 0$ are necessary and sufficient for $\mathcal{H}(x_i^*, u_i^*, p_{i+1}^*, i)$ to be a global minimum.

For a fixed terminal state, δx_{i_f} in the last term of (2.9) is equal to zero. Thus, (2.13) is not necessary, which is replaced with $x_{i_f} = x_{i_f}^r$.

2.2.2 Optimal Control Based on Minimax Criterion

Consider the following discrete-time system:

$$x_{i+1} = f(x_i, u_i, w_i, i), \qquad x_{i_0} = x_0 \tag{2.23}$$

with a performance criterion

$$J(x_{i_0}, i_0, u, w) = \sum_{i=i_0}^{i_f-1} [g(x_i, u_i, w_i, i)] \; + \; h(x_{i_f}, i_f) \tag{2.24}$$

where $x_i \in \Re^n$ is the state, $u_i \in \Re^m$ is the input and $w_i \in \Re^l$ is the disturbance. The input and the disturbance are required to belong to the given sets, i.e. $u_i \in \mathcal{U}$ and $w_i \in \mathcal{W}$. Here, the fixed terminal state is not dealt with because the minimax problem in this case does not make sense.

The minimax criterion we are dealing with is related to a difference game. We want to minimize the performance criterion, while disturbances try to maximize one. A pair policies $(u, w) \in \mathcal{U} \times \mathcal{W}$ is said to constitute a saddle-point solution if, for all $(u, w) \in \mathcal{U} \times \mathcal{W}$,

$$J(x_{i_0}, i_0, u^*, w) \le J(x_{i_0}, i_0, u^*, w^*) \le J(x_{i_0}, i_0, u, w^*) \tag{2.25}$$

We may think that u^* is the best control, while w^* is the worst disturbance. The existence of these u^* and w^* is guaranteed by specific conditions.

The control u^* makes the performance criterion (2.24) a local minimum if

$$J(x_{i_0}, i_0, u, w) - J(x_{i_0}, i_0, u^*, w) = \triangle J \geq 0$$

for all admissible controls. If we let $u = u^* + \delta u$, the increment in J can be expressed as

$$\triangle J(u^*, \delta u, w) = \delta J(u^*, \delta u, w) + \text{higher order terms}$$

Hence, the necessary conditions for u^* to be the optimal control are

$$\delta J(u^*, \delta u, w) \geq 0$$

if u^* lies on the boundary during any portion of the time interval $[i_0,\ i_f]$ and

$$\delta J(u^*, \delta u, w) = 0$$

if u^* lies within the boundary during the entire time interval $[i_0,\ i_f]$.

Meanwhile, the disturbance w^* makes the performance criterion (2.24) a local maximum if

$$J(u, w) - J(u, w^*) = \triangle J \leq 0$$

for all admissible disturbances. Taking steps similar to the case of u^*, we obtain the necessary condition

$$\delta J(u, w^*, \delta w) \leq 0$$

if w^* lies on the boundary during any portion of the time interval $[i_0,\ i_f]$ and

$$\delta J(u, w^*, \delta w) = 0$$

if w^* lies within the boundary during the entire time interval $[i_0,\ i_f]$.

We now summarize the principle results. In terms of the Hamiltonian, the necessary conditions for u_i^* to be an optimal control are

$$\begin{aligned}
x_{i+1}^* &= \frac{\partial \mathcal{H}}{\partial p_{i+1}}(x_i^*, u_i^*, w_i^*, p_{i+1}^*, i) \\
p_i^* &= \frac{\partial \mathcal{H}}{\partial x_i}(x_i^*, u_i^*, w_i^*, p_{i+1}^*, i) \\
\mathcal{H}(x_i^*, u_i^*, w_i, p_{i+1}^*, i) &\leq \mathcal{H}(x_i^*, u_i^*, w_i^*, p_{i+1}^*, i) \leq \mathcal{H}(x_i^*, u_i, w_i^*, p_{i+1}^*, i)
\end{aligned}$$

for all admissible u_i and w_i on the $i \in [i_0, i_f - 1]$, and two boundary conditions

$$x_{i_0} = x_0, \quad p_{i_f}^* = \frac{\partial h}{\partial x_{i_f}}(x_{i_f}^*, i_f)$$

Now, a dynamic programming for minimaxization criterion is explained. Let there exist a function $J^*(x_i, i)$, $i \in [i_0, i_f - 1]$ such that

$$\begin{aligned} J^*(x_i, i) &= \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}} \left[g(x_i, u_i, w_i, i) + J^*(f(x_i, u_i, w_i, i), i+1) \right] \\ &= \max_{w \in \mathcal{W}} \min_{u \in \mathcal{U}} \left[g(x_i, u_i, w_i, i) + J^*(f(x_i, u_i, w_i, i), i+1) \right] \\ J^*(x_{i_f}, i_f) &= h(x_{i_f}, i_f) \end{aligned} \tag{2.26}$$

Then a pair of u and w that is generated by (2.26) provides a saddle point with the corresponding value given by $J^*(x_{i_0}, i_0)$.

2.3 Linear Optimal Control with State Feedback

2.3.1 Linear Quadratic Controls Based on Minimum Criterion

In this section, an LQ control in a tracking form for discrete time-invariant systems is introduced in a state-feedback form. We consider the following discrete time-invariant system:

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i \\ z_i &= C_z x_i \end{aligned} \tag{2.27}$$

There are two methods which are used to obtain the control of minimizing the chosen cost function. One is dynamic programming and the other is the minimum principle of Pontryagin. The minimum principle of Pontryagin and dynamic programming were briefly introduced in the previous section. In the method of dynamic programming, an optimal control is obtained by employing the intuitively appealing concept called the principle of optimality. Here, we use the minimum principle of Pontryagin in order to obtain an optimal finite horizon LQ tracking control (LQTC).

We can divide the terminal states into two cases. The first case is a free terminal state and the second case is a fixed terminal state. In the following, we will derive two kinds of LQ controls in a tracking form.

1) Free Terminal State

The following quadratic performance criterion is considered:

$$\begin{aligned} J(z^r, u.) = &\sum_{i=i_0}^{i_f-1} [(z_i - z_i^r)^T \bar{Q} (z_i - z_i^r) + u_i^T R u_i] \\ &+ [z_{i_f} - z_{i_f}^r]^T \bar{Q}_f [z_{i_f} - z_{i_f}^r] \end{aligned} \tag{2.28}$$

Here, $x_i \in \Re^n$, $u_i \in \Re^m$, $z_i \in \Re^p$, z_i^r, $\bar{Q} > 0$, $R > 0$, $\bar{Q}_f > 0$ are the state, the input, the controlled output, the command signal or the reference signal, the state weighting matrix, the input weighting matrix, and the terminal weighting matrix respectively. Here, $z^r_{i_0}$, $z^r_{i_0+1}$, $\cdots$, $z^r_{i_f}$ are command signals which are assumed to be available over the future horizon $[i_0, i_f]$.

For tracking problems, with $\bar{Q} > 0$ and $\bar{Q}_f > 0$ in (2.28), there is a tendency that $z_i \rightarrow z^r_i$. In order to derive the optimal tracking control which minimizes the performance criterion (2.28), it is convenient to express the performance criterion (2.28) with the state x_i instead of z_i. It is well known that for a given $p \times n$ $(p \leq n)$ full rank matrix C_z there always exist some $n \times p$ matrices L such that $C_z L = I_{p\times p}$. Let

$$x^r_i = L z^r_i \tag{2.29}$$

The performance criterion (2.28) is then rewritten as

$$\begin{aligned} J(x^r, u) = & \sum_{i=i_0}^{i_f-1} [(x_i - x^r_i)^T C_z^T \bar{Q} C_z (x_i - x^r_i) + u_i^T R u_i] \\ & + [x_{i_f} - x^r_{i_f}]^T C_z^T \bar{Q}_f C_z [x_{i_f} - x^r_{i_f}] \end{aligned} \tag{2.30}$$

The performance criterion (2.30) can be written as

$$\begin{aligned} J(x^r, u) = & \sum_{i=i_0}^{i_f-1} \left[(x_i - x^r_i)^T Q (x_i - x^r_i) + u_i^T R u_i \right] \\ & + (x_{i_f} - x^r_{i_f})^T Q_f (x_{i_f} - x^r_{i_f}) \end{aligned} \tag{2.31}$$

where

$$Q = C_z^T \bar{Q} C_z \quad \text{and} \quad Q_f = C_z^T \bar{Q}_f C_z \tag{2.32}$$

Q and Q_f in (2.31) can be independent design parameters ignoring the relation (2.32). That is, the matrices Q and Q_f can be positive definite, if necessary, though Q and Q_f in (2.32) are semidefinite when C_z is not of full rank. In this book, Q and Q_f in (2.31) are independent design parameters. However, whenever necessary, we will make some connections to (2.32).

We first form a Hamiltonian:

$$\mathcal{H}_i = [(x_i - x^r_i)^T Q (x_i - x^r_i) + u_i^T R u_i] + p^T_{i+1}[A x_i + B u_i] \tag{2.33}$$

where $i \in [i_0, i_f - 1]$. According to (2.20) and (2.2.1), we have

$$p_i = \frac{\partial \mathcal{H}_i}{\partial x_i} = 2Q(x_i - x^r_i) + A^T p_{i+1} \tag{2.34}$$

$$p_{i_f} = \frac{\partial h(x_{i_f}, i_f)}{\partial x_{i_f}} = 2Q_f(x_{i_f} - x^r_{i_f}) \tag{2.35}$$

where $h(x_{i_f}, i_f) = (x_{i_f} - x^r_{i_f})^T Q_f (x_{i_f} - x^r_{i_f})$.

A necessary condition for u_i to minimize $\mathcal{H}_i$ is $\frac{\partial \mathcal{H}_i}{\partial u_i} = 0$. Thus, we have

$$\frac{\partial \mathcal{H}_i}{\partial u_i} = 2Ru_i + B^T p_{i+1} = 0 \tag{2.36}$$

Since the matrix $\frac{\partial^2 \mathcal{H}_i}{\partial u_i^2} = 2R$ is positive definite and $\mathcal{H}_i$ is a quadratic form in u, the solution of (2.36) is an optimal control to minimize $\mathcal{H}_i$. The optimal solution u_i^* is

$$u_i^* = -\frac{1}{2} R^{-1} B^T p_{i+1} \tag{2.37}$$

If we assume that

$$p_i = 2K_{i,i_f} x_i + 2g_{i,i_f} \tag{2.38}$$

the solution to the optimal control problem can be reduced to finding the matrices K_{i,i_f} and g_{i,i_f}. From (2.35), the boundary conditions are given by

$$K_{i_f,i_f} = Q_f \tag{2.39}$$

$$g_{i_f,i_f} = -Q_f x_{i_f}^r \tag{2.40}$$

Substituting (2.27) into (2.38) and replacing u_i with (2.37), we have

$$\begin{aligned} p_{i+1} &= 2K_{i+1,i_f} x_{i+1} + 2g_{i+1,i_f} \\ &= 2K_{i+1,i_f}(Ax_i + Bu_i) + 2g_{i+1,i_f} \\ &= 2K_{i+1,i_f}(Ax_i - \frac{1}{2} BR^{-1}B^T p_{i+1}) + 2g_{i+1,i_f} \end{aligned} \tag{2.41}$$

Solving for p_{i+1} in (2.41) yields the following equation:

$$p_{i+1} = [I + K_{i+1,i_f} BR^{-1}B^T]^{-1}[2K_{i+1,i_f} Ax_i + 2g_{i+1,i_f}] \tag{2.42}$$

Substituting for p_{i+1} from (2.37), we can write

$$u_i^* = -R^{-1}B^T[I + K_{i+1,i_f} BR^{-1}B^T]^{-1}[K_{i+1,i_f} Ax_i + g_{i+1,i_f}] \tag{2.43}$$

What remains to do is to find K_{i,i_f} and g_{i,i_f}. If we put the equation (2.42) into the equation (2.34), we have

$$\begin{aligned} p_i &= 2Q(x_i - x_i^r) + A^T[I + K_{i+1,i_f} BR^{-1}B^T]^{-1}[2K_{i+1,i_f} Ax_i + 2g_{i+1,i_f}], \\ &= 2[Q + A^T(I + K_{i+1,i_f} BR^{-1}B^T)^{-1} K_{i+1,i_f} A]x_i \\ &\quad +2[-Qx_i^r + A^T(I + K_{i+1,i_f} BR^{-1}B^T)^{-1} g_{i+1,i_f}] \end{aligned} \tag{2.44}$$

The assumption (2.38) holds by choosing K_{i,i_f} and g_{i,i_f} as

$$
\begin{aligned}
K_{i,i_f} &= A^T[I + K_{i+1,i_f}BR^{-1}B^T]^{-1}K_{i+1,i_f}A + Q \\
&= A^T K_{i+1,i_f}A - A^T K_{i+1,i_f}B(R + B^T K_{i+1,i_f}B)^{-1}B^T K_{i+1,i_f}A \\
&\quad + Q \qquad (2.45) \\
g_{i,i_f} &= A^T[I + K_{i+1,i_f}BR^{-1}B^T]^{-1}g_{i+1,i_f} - Qx_i^r \qquad (2.46)
\end{aligned}
$$

where the second equality comes from

$$
[I + K_{i+1,i_f}BR^{-1}B^T]^{-1} = I - K_{i+1,i_f}B(R + B^T K_{i+1,i_f}B)^{-1}B^T \quad (2.47)
$$

using the matrix inversion lemma (A.2) in Appendix A. The optimal control derived until now is summarized in the following theorem.

Theorem 2.1. *In the system (2.27), the LQTC for the free terminal state is given as (2.43) for the performance criterion (2.31).* K_{i,i_f} *and* g_{i,i_f} *in (2.43) are obtained from Riccati Equation (2.45) and (2.46) with boundary condition (2.39) and (2.40).*

Depending on Q_f, K_{i,i_f} may be nonsingular (positive definite) or singular (positive semidefinite). This property will be important for stability and the inversion of the matrix K_{i,i_f} in coming sections.

For a zero reference signal, g_{i,i_f} becomes zero so that we have

$$
u_i^* = -R^{-1}B^T[I + K_{i+1,i_f}BR^{-1}B^T]^{-1}K_{i+1,i_f}Ax_i \quad (2.48)
$$

The performance criterion (2.31) associated with the optimal control (2.43) is given in the following theorem.

Theorem 2.2. *The optimal cost* $J^*(x_i)$ *with the reference value can be given*

$$
J^*(x_i) = x_i^T K_{i,i_f}x_i + 2x_i^T g_{i,i_f} + w_{i,i_f} \quad (2.49)
$$

where

$$
w_{i,i_f} = w_{i+1,i_f} + x_i^{rT}Qx_i^r - g_{i+1,i_f}^T B(B^T K_{i+1,i_f}B + R)^{-1}B^T g_{i+1,i_f} \quad (2.50)
$$

with boundary condition $w_{i_f,i_f} = x_{i_f}^{rT}Q_f x_{i_f}^r$.

Proof. A long and tedious calculation is required to obtain the optimal cost using the result of Theorem 2.1. Thus, we derive the optimal cost using dynamic programming, where the optimal control and the optimal cost are obtained simultaneously.

Let $J^*(x_{i+1})$ denote the optimal cost associated with the initial state x_{i+1} and the interval $[i+1, i_f]$. Suppose that the optimal cost $J^*(x_{i+1})$ is given as

$$
J^*(x_{i+1}) = x_{i+1}^T K_{i+1,i_f}x_{i+1} + 2x_{i+1}^T g_{i+1,i_f} + w_{i+1,i_f} \quad (2.51)
$$

where w_{i+1,i_f} will be determined later. We wish to calculate the optimal cost $J^*(x_i)$ from (2.51).

By applying the principle of optimality, $J^*(x_i)$ can be represented as follows:

$$J^*(x_i) = \min_{u_i} \left[(x_i - x_i^r)^T Q (x_i - x_i^r) + u_i^T R u_i + J^*(x_{i+1}) \right] \quad (2.52)$$

(2.52) can be evaluated backward by starting with the condition $J^*(x_{i_f}) = (x_{i_f} - x_{i_f}^r)^T Q_f (x_{i_f} - x_{i_f}^r)$.

Substituting (2.27) and (2.51) into (2.52), we have

$$\begin{aligned} J^*(x_i) &= \min_{u_i} \Big[(x_i - x_i^r)^T Q (x_i - x_i^r) + u_i^T R u_i + x_{i+1}^T K_{i+1,i_f} x_{i+1} \\ &\quad + 2 x_{i+1}^T g_{i+1,i_f} + w_{i+1,i_f} \Big] \qquad (2.53) \\ &= \min_{u_i} \Big[(x_i - x_i^r)^T Q (x_i - x_i^r) + u_i^T R u_i \\ &\quad + (Ax_i + Bu_i)^T K_{i+1,i_f} (Ax_i + Bu_i) + 2(Ax_i + Bu_i)^T g_{i+1,i_f} \\ &\quad + w_{i+1,i_f} \Big] \qquad (2.54) \end{aligned}$$

Note that $J^*(x_i)$ has a quadratic equation with respect to u_i and x_i. For a given x_i, the control u_i is chosen to be optimal according to (2.54). Taking derivatives of (2.54) with respect to u_i to obtain

$$\frac{\partial J^*(x_i)}{\partial u_i} = 2Ru_i + 2B^T K_{i+1,i_f} B u_i + 2B^T K_{i+1,i_f} A x_i + 2B^T g_{i+1,i_f} = 0$$

we have the following optimal control u_i:

$$\begin{aligned} u_i &= -(R + B^T K_{i+1,i_f} B)^{-1} [B^T K_{i+1,i_f} A x_i + B^T g_{i+1,i_f}] \qquad (2.55) \\ &= -L_{1,i} x_i + L_{2,i} g_{i+1,i_f} \qquad (2.56) \end{aligned}$$

where

$$L_{1,i} \triangleq [R + B^T K_{i+1,i_f} B]^{-1} B^T K_{i+1,i_f} A \quad (2.57)$$

$$L_{2,i} \triangleq -[R + B^T K_{i+1,i_f} B]^{-1} B^T \quad (2.58)$$

It is noted that the optimal control u_i in (2.56) is the same as (2.43) derived from the minimum principle. How to obtain the recursive equations of K_{i+1,i_f} and g_{i+1,i_f} is discussed later.

From definitions (2.57) and (2.58), we have the following relations:

$$\begin{aligned} A^T K_{i+1,i_f} B [R + B^T K_{i+1,i_f} B]^{-1} B^T K_{i+1,i_f} A &= A^T K_{i+1,i_f} B L_{1,i} \\ &= L_{1,i}^T B^T K_{i+1,i_f} A \\ &= L_{1,i}^T [R + B^T K_{i+1,i_f} B] L_{1,i} \end{aligned} \quad (2.59)$$

where the most left side is equivalent to the second term of Riccati Equation (2.45) and these relations are useful for representing the Riccati equation in terms of closed-loop system $A - BL_{1,i}$.

Substituting (2.56) into (2.54) yields

$$
\begin{aligned}
J^*(x_i) = {}& x_i^T \Big[Q + L_{1,i}^T R L_{1,i} + (A - BL_{1,i})^T K_{i+1,i_f} (A - BL_{1,i}) \Big] x_i \\
& + 2x_i^T \Big[-L_{1,i}^T R L_{2,i} g_{i+1,i_f} + (A - BL_{1,i})^T K_{i+1,i_f} B L_{2,i} g_{i+1,i_f} \\
& + (A - BL_{1,i})^T g_{i+1,i_f} - Q x_i^r \Big] + g_{i+1,i_f}^T L_{2,i}^T R L_{2,i} g_{i+1,i_f} \\
& + g_{i+1,i_f}^T L_{2,i}^T B^T K_{i+1,i_f} B L_{2,i} g_{i+1,i_f} + 2 g_{i+1,i_f}^T L_{2,i}^T B^T g_{i+1,i_f} + w_{i+1,i_f} \\
& + x_i^{rT} Q x_i^r
\end{aligned} \tag{2.60}
$$

where the terms are arranged according to the order of x_i. The quadratic terms with respect to x_i in (2.60) can be reduced to $x_i^T K_{i,i_f} x_i$ from Riccati Equation given by

$$
K_{i,i_f} = [A - BL_{1,i}]^T K_{i+1,i_f} [A - BL_{1,i}] + L_{1,i}^T R L_{1,i} + Q \tag{2.61}
$$

which is the same as (2.45) according to the relation (2.59).

The first-order coefficients with respect to x_i in (2.60) can be written as

$$
\begin{aligned}
& - L_{1,i}^T R L_{2,i} g_{i+1,i_f} + (A - BL_{1,i})^T K_{i+1,i_f} B L_{2,i} g_{i+1,i_f} \\
& + (A - BL_{1,i})^T g_{i+1,i_f} - Q x_i^r \\
& = -L_{1,i}^T R L_{2,i} g_{i+1,i_f} + A^T K_{i+1,i_f} B L_{2,i} g_{i+1,i_f} - L_{1,i}^T B^T K_{i+1,i_f} B L_{2,i} g_{i+1,i_f} \\
& + A^T g_{i+1,i_f} - L_{1,i}^T B^T g_{i+1,i_f} - Q x_i^r \\
& = -A^T [K_{i+1,i_f}^T B (R + B^T K_{i+1,i_f} B)^{-1} B^T - I] g_{i+1,i_f} - Q x_i^r \\
& = A^T [I + K_{i+1,i_f}^T B R^{-1} B]^{-1} g_{i+1,i_f} - Q x_i^r
\end{aligned}
$$

which can be reduced to g_{i,i_f} if it is generated from (2.46).

The terms without x_i in (2.60) can be written as

$$
\begin{aligned}
& g_{i+1,i_f}^T L_{2,i}^T R L_{2,i} g_{i+1,i_f} + g_{i+1,i_f}^T L_{2,i}^T B^T K_{i+1,i_f} B L_{2,i} g_{i+1,i_f} \\
& + 2 g_{i+1,i_f}^T L_{2,i}^T B^T g_{i+1,i_f} + w_{i+1,i_f} + x_i^{rT} Q x_i^r \\
& = g_{i+1,i_f}^T B [R + B^T K_{i+1,i_f} B]^{-1} B^T g_{i+1,i_f} \\
& - 2 g_{i+1,i_f}^T B [R + B^T K_{i+1,i_f} B]^{-1} B^T g_{i+1,i_f} + w_{i+1,i_f} + x_i^{rT} Q x_i^r \\
& = -g_{i+1,i_f}^T B [R + B^T K_{i+1,i_f} B]^{-1} B^T g_{i+1,i_f} + w_{i+1,i_f} + x_i^{rT} Q x_i^r
\end{aligned}
$$

which can be reduced to w_{i,i_f} if it is defined as (2.50). If g_{i,i_f} and w_{i,i_f} are chosen as (2.46) and (2.50), then $J^*(x_i)$ is in a form such as (2.51), i.e.

$$J^*(x_i) = x_i^T K_{i,i_f} x_i + 2x_i^T g_{i,i_f} + w_{i,i_f} \tag{2.62}$$

Now, we have only to find the boundary value of g_{i,i_f} and w_{i,i_f}. $J^*(x_{i_f})$ should be equal to the performance criterion for the final state. Thus, w_{i_f,i_f} and g_{i_f,i_f} should be chosen as $w_{i_f,i_f} = x_{i_f}^{rT} Q_f x_{i_f}^r$ and $g_{i_f,i_f} = -Q_f x_{i_f}^r$ so that we have

$$\begin{aligned} J^*(x_{i_f}) &= x_{i_f}^T K_{i_f,i_f} x_{i_f} + 2x_{i_f}^T g_{i_f,i_f} + w_{i_f,i_f} \\ &= x_{i_f}^T Q_f x_{i_f} - 2x_{i_f}^T Q_f x_{i_f}^r + x_{i_f}^{rT} Q_f x_{i_f}^r \\ &= (x_{i_f} - x_{i_f}^r)^T Q_f (x_{i_f} - x_{i_f}^r) \end{aligned}$$

This completes the proof. ∎

The result of Theorem 2.2 will be utilized only for zero reference signals in subsequent sections.

For positive definite Q_f and nonsingular matrix A, we can have another form of the above control (2.43). Let $\hat{P}_{i,i_f} = K_{i,i_f}^{-1}$ if the inverse of K_{i,i_f} exists. Then (2.45) can be represented by

$$\hat{P}_{i,i_f}^{-1} = A^T[I + \hat{P}_{i+1,i_f}^{-1} BR^{-1}B^T]^{-1} \hat{P}_{i+1,i_f}^{-1} A + Q \tag{2.63}$$

$$\hat{P}_{i,i_f} = \left\{ A^T[\hat{P}_{i+1,i_f} + BR^{-1}B^T]^{-1} A + Q \right\}^{-1} \tag{2.64}$$

Let $P_{i,i_f} = \hat{P}_{i,i_f} + BR^{-1}B^T$. Then

$$\begin{aligned} P_{i,i_f} &= (A^T P_{i+1,i_f}^{-1} A + Q)^{-1} + BR^{-1}B^T \\ &= A^{-1}(P_{i+1,i_f}^{-1} + A^{-T} Q A^{-1})^{-1} A^{-T} + BR^{-1}B^T \\ &= A^{-1}[I + P_{i+1,i_f} A^{-T} Q A^{-1}]^{-1} P_{i+1,i_f} A + BR^{-1}B^T \qquad (2.65) \\ g_{i,i_f} &= A^T[I + \hat{P}_{i+1,i_f}^{-1} BR^{-1}B^T]^{-1} g_{i+1,i_f} - Q x_i^r \\ &= A^T[\hat{P}_{i+1,i_f} + BR^{-1}B^T]^{-1} \hat{P}_{i+1,i_f} g_{i+1,i_f} - Q x_i^r \\ &= A^T P_{i+1,i_f}^{-1} (P_{i+1,i_f} - BR^{-1}B^T) g_{i+1,i_f} - Q x_i^r \qquad (2.66) \end{aligned}$$

with the boundary condition

$$P_{i_f,i_f} = Q_f^{-1} + BR^{-1}B^T \tag{2.67}$$

Using the following relation:

$$\begin{aligned} &-[I + R^{-1}B^T \hat{P}_{i+1,i_f}^{-1} B]^{-1} R^{-1} B^T \hat{P}_{i+1,i_f}^{-1} A x_i \\ &\qquad = -R^{-1}B^T \hat{P}_{i+1,i_f}^{-1} [I + BR^{-1}B^T \hat{P}_{i+1,i_f}^{-1}]^{-1} A x_i \\ &\qquad = -R^{-1}B^T [\hat{P}_{i+1,i_f} + BR^{-1}B^T]^{-1} A x_i \\ &\qquad = -R^{-1}B^T P_{i+1,i_f}^{-1} A x_i \end{aligned}$$

and

$$
\begin{aligned}
-[I + R^{-1}B^T\hat{P}^{-1}_{i+1,i_f}B]^{-1}R^{-1}Bg_{i+1,i_f} \\
&= R^{-1}B[I + \hat{P}^{-1}_{i+1,i_f}BR^{-1}B^T]^{-1}g_{i+1,i_f} \\
&= R^{-1}BP^{-1}_{i+1,i_f}(P_{i+1,i_f} - BR^{-1}B^T)g_{i+1,i_f}
\end{aligned}
$$

we can represent the control in another form:

$$
u_i^* = -R^{-1}B^TP^{-1}_{i+1,i_f}[Ax_i + (P_{i+1,i_f} - BR^{-1}B^T)g_{i+1,i_f}] \tag{2.68}
$$

where P_{i+1,i_f} and g_{i+1,i_f} are obtained from (2.65) and (2.66) with boundary conditions (2.67) and (2.40) respectively.

2) Fixed Terminal State

Here, the following performance criterion is considered:

$$
J(x^r, u) = \sum_{i=i_0}^{i_f-1}\left[(x_i - x_i^r)^TQ(x_i - x_i^r) + u_i^TRu_i\right] \tag{2.69}
$$

$$
x_{i_f} = x_{i_f}^r \tag{2.70}
$$

For easy understanding, we start off from a simple case.

Case 1: zero state weighting

Now, our terminal objective will be to make x_{i_f} match exactly the desired final reference state $x_{i_f}^r$. Since we are demanding that x_{i_f} be equal to a known desired $x_{i_f}^r$, the final state has no effect on the performance criterion (2.31). It is therefore redundant to include a final state weighting term in a performance criterion. Accordingly, we may as well set $Q_f = 0$.

Before we go to the general problem, we first consider a simple case for the following performance criterion:

$$
J_{i_0} = \frac{1}{2}\sum_{i=i_0}^{i_f-1} u_i^TRu_i \tag{2.71}
$$

where $Q = 0$. Observe that the weighting matrix for the state becomes zero.

As mentioned before, we require the control to drive x_{i_0} exactly to

$$
x_{i_f} = x_{i_f}^r \tag{2.72}
$$

using minimum control energy. The terminal condition can be expressed by

$$
x_{i_f} = A^{i_f-i_0}x_{i_0} + \sum_{i=i_0}^{i_f-1} A^{i_f-i-1}Bu_i = x_{i_f}^r \tag{2.73}
$$

We try to find the optimal control among ones satisfying (2.73). It can be seen that both the performance criterion and the constraint are expressed in terms of the control, not including the state, which makes the problem tractable. Introducing a Lagrange multiplier λ, we have

$$J_{i_0} = \frac{1}{2}\sum_{i=i_0}^{i_f-1} u_i^T R u_i + \lambda^T (A^{i_f-i_0}x_{i_0} + \sum_{i=i_0}^{i_f-1} A^{i_f-i-1}Bu_i - x_{i_f}^r) \quad (2.74)$$

Take the derivative on both sides of Equation (2.74) with respect to u_i to obtain

$$Ru_i + B^T(A^T)^{i_f-i-1}\lambda = 0 \quad (2.75)$$

Thus,

$$u_i = -R^{-1}B^T(A^T)^{i_f-i-1}\lambda \quad (2.76)$$

Substituting (2.76) into (2.73) and solving for λ yields

$$\lambda = -G_{i_0,i_f}^{-1}(x_{i_f}^r - A^{i_f-i_0}x_{i_0}) \quad (2.77)$$

where

$$G_{i_0,i_f} = \sum_{i=i_0}^{i_f-1} A^{i_f-i-1}BR^{-1}B^T(A^T)^{i_f-i-1}. \quad (2.78)$$

Actually, G_{i_0,i_f} is a controllability Gramian of the systems (2.27). In the case of controllable systems, G_{i_0,i_f} is guaranteed to be nonsingular if $i_f - i_0$ is more than or equal to the controllability index n_c.

The optimal open-loop control is given by

$$u_i^* = R^{-1}B^T(A^T)^{i_f-i-1}G_{i_0,i_f}^{-1}(x_{i_f}^r - A^{i_f-i_0}x_{i_0}) \quad (2.79)$$

It is noted that the open-loop control is defined for all $i \in [i_0,\ i_f - 1]$. Since i_0 is arbitrary, we can obtain the closed-loop control by replacing with i such as,

$$u_i^* = R^{-1}B^T(A^T)^{i_f-i-1}G_{i,i_f}^{-1}(x_{i_f}^r - A^{i_f-i}x_i) \quad (2.80)$$

It is noted that the closed-loop control can be defined only on i that is less than or equal $i_f - n_c$. After the time $i_f - n_c$, the open-loop control can be used, if necessary. In Figure 2.3, the regions of the closed- and open-loop control are shown respectively.

The above solutions can also be obtained with the formal procedure using the minimum principle, but it is given in a closed form from this procedure. Thus, the control after $i_f - n_c$ cannot be obtained.

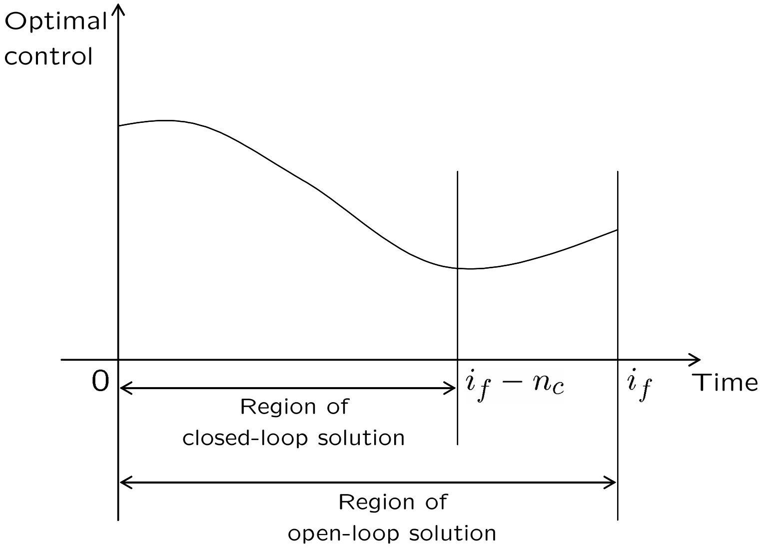


Fig. 2.3. Region of the closed-loop solution

Case 2: nonzero state weighting

We derive the optimal solution for the fixed terminal state from that for the free terminal state by setting $Q_f = \infty I$. We assume that A is nonsingular.

Since $K_{i_f,i_f} = Q_f = \infty I$, the boundary condition of P_{i,i_f} becomes

$$P_{i_f,i_f} = BR^{-1}B^T \tag{2.81}$$

from (2.67). From (2.65), we know that Equation (2.81) is satisfied with another terminal condition:

$$P_{i_f+1,i_f+1} = 0 \tag{2.82}$$

It is noted that P_{i+1,i_f} can be singular on $[i_f - n_c + 2, i_f]$. Therefore, g_{i,i_f} cannot be generated from (2.66) and the control (2.68) does not make sense. However, the control for the zero reference signal can be represented as

$$u_i^* = -R^{-1}B^T P_{i+1,i_f}^{-1} A x_i \tag{2.83}$$

where g_{i,i_f} is not necessary.

For nonzero reference signals we will take an approach called the sweep method. The state and costate equations are the same as those of the free terminal case:

$$u_i = -R^{-1}B^T p_{i+1} \tag{2.84}$$

$$x_{i+1} = Ax_i - BR^{-1}B^T p_{i+1} \tag{2.85}$$

$$p_i = Q(x_i - x_i^r) + A^T p_{i+1} \tag{2.86}$$

We try to find an optimal control to ensure $x_{i_f} = x^r_{i_f}$. Assume the following relation:

$$p_i = K_i x_i + M_i p_{i_f} + g_i \tag{2.87}$$

where we need to find S_i, M_i, and g_i satisfying the boundary conditions

$$\begin{aligned} K_{i_f} &= 0 \\ M_{i_f} &= I \\ g_{i_f} &= 0 \end{aligned}$$

respectively. Combining (2.85) with (2.87) yields the following optimal trajectory:

$$\begin{aligned} x_{i+1} = (I + BR^{-1}B^T K_{i+1})^{-1}(Ax_i - BR^{-1}B^T M_{i+1} p_{i_f} \\ - BR^{-1}B^T g_{i+1}) \end{aligned} \tag{2.88}$$

Substituting (2.87) into (2.86) provides

$$K_i x_i + M_i p_{i_f} + g_i = Q(x_i - x^r_i) + A^T[K_{i+1}x_{i+1} + M_{i+1}p_{i_f} + g_{i+1}] \tag{2.89}$$

Substituting x_{i+1} in (2.88) into (2.89) yields

$$\begin{aligned} &[-K_i + A^T K_{i+1}(I + BR^{-1}B^T K_{i+1})^{-1}A + Q]x_i + \\ &[-M_i - A^T K_{i+1}(I + BR^{-1}B^T K_{i+1})^{-1}BR^{-1}B^T M_{i+1} + A^T M_{i+1}]p_{i_f} + \\ &[-g_i + A^T g_{i+1} - A^T K_{i+1}(I + BR^{-1}B^T K_{i+1})^{-1}BR^{-1}B^T g_{i+1} - Qx^r_i] = 0 \end{aligned}$$

Since this equality holds for all trajectories x_i arising from any initial condition x_{i_0}, each term in brackets must vanish. The matrix inversion lemma, therefore, yields the Riccati equation

$$K_i = A^T K_{i+1}(I + BR^{-1}B^T K_{i+1})^{-1}A + Q \tag{2.90}$$

and the auxiliary homogeneous difference equation

$$\begin{aligned} M_i &= A^T M_{i+1} - A^T K_{i+1}(I + BR^{-1}B^T K_{i+1})^{-1}BR^{-1}B^T M_{i+1} \\ g_i &= A^T g_{i+1} - A^T K_{i+1}(I + BR^{-1}B^T K_{i+1})^{-1}BR^{-1}B^T g_{i+1} - Qx^r_i \end{aligned}$$

We assume that $x^r_{i_f}$ is a linear combination of x_i, p_{i_f}, and some specific matrix N_i for all i, i.e.

$$x^r_{i_f} = U_i x_i + S_i p_{i_f} + h_i \tag{2.91}$$

Evaluating for $i = i_f$ yields

$$U_{i_f} = I \tag{2.92}$$

$$S_{i_f} = 0 \tag{2.93}$$

$$h_{i_f} = 0 \tag{2.94}$$

Clearly, then

$$U_i = M_i^T \tag{2.95}$$

The left-hand side of (2.91) is a constant, so take the difference to obtain

$$0 = U_{i+1}x_{i+1} + S_{i+1}p_{i_f} + h_{i+1} - U_i x_i - S_i p_{i_f} - h_i \tag{2.96}$$

Substituting x_{i+1} in (2.88) into (2.96) and rearranging terms, we have

$$\begin{aligned}
&[U_{i+1}\{A - B(B^T K_{i+1}B + R)^{-1}B^T K_{i+1}A\} - U_i]x_i \\
&+ [S_{i+1} - S_i - U_{i+1}B(B^T K_{i+1}B + R)^{-1}B^T M_{i+1}]p_{i_f} \\
&+ h_{i+1} - h_i - U_{i+1}(I + BR^{-1}B^T K_{i+1})^{-1}BR^{-1}B^T g_{i+1} = 0
\end{aligned} \tag{2.97}$$

The first term says that

$$U_i = U_{i+1}\{A - B(B^T K_{i+1}B + R)^{-1}B^T K_{i+1}A\} \tag{2.98}$$

The second and third terms now yield the following recursive equations:

$$S_i = S_{i+1} - M_{i+1}^T B(B^T K_{i+1}B + R)^{-1}B^T M_{i+1} \tag{2.99}$$

$$h_i = h_{i+1} - M_{i+1}^T (I + BR^{-1}B^T K_{i+1})^{-1}BR^{-1}B^T g_{i+1} \tag{2.100}$$

We are now in a position to determine p_{i_f}. From (2.91), we have

$$p_{i_f} = S_{i_0}^{-1}(x_{i_f}^r - M_{i_0}^T x_{i_0} - h_{i_0}) \tag{2.101}$$

We can now finally compute the optimal control

$$u_i = -R^{-1}B^T[K_{i+1}x_{i+1} + M_{i+1}p_{i_f} + g_{i+1}] \tag{2.102}$$

by substituting (2.87) into (2.84).

u_i can be represented in terms of the current state x_i:

$$\begin{aligned}
u_i &= -R^{-1}B^T(I + K_{i+1}BR^{-1}B^T)^{-1}[K_{i+1}Ax_i + M_{i+1}p_{i_f} + g_{i+1}], \\
&= -R^{-1}B^T(I + K_{i+1}BR^{-1}B^T)^{-1}[K_{i+1}Ax_i + M_{i+1}S_{i_0}^{-1}(x_{i_f}^r \\
&\quad - M_{i_0}^T x_{i_0} - h_{i_0}) + g_{i+1}]
\end{aligned} \tag{2.103}$$

What we have done so far is summarized in the following theorem.

Theorem 2.3. *The LQTC for the fixed terminal state is given in (2.103). S_i, M_i, P_i, g_i, h_i are as follows:*

$$\begin{aligned}
K_i &= A^T K_{i+1}(I + BR^{-1}B^T K_{i+1})^{-1}A + Q \\
M_i &= A^T M_{i+1} - A^T K_{i+1}(I + BR^{-1}B^T K_{i+1})^{-1}BR^{-1}B^T M_{i+1} \\
S_i &= S_{i+1} - M_{i+1}^T B(B^T K_{i+1}B + R)^{-1}B^T M_{i+1} \\
g_i &= A^T g_{i+1} - A^T K_{i+1}(I + BR^{-1}B^T K_{i+1})^{-1}BR^{-1}B^T g_{i+1} - Qx_i^r \\
h_i &= h_{i+1} - M_{i+1}^T (I + BR^{-1}B^T K_{i+1})^{-1}BR^{-1}B^T g_{i+1}
\end{aligned}$$

where

$$K_{i_f} = 0, \; M_{i_f} = I, \; S_{i_f} = 0, \; g_{i_f} = 0, \; h_{i_f} = 0$$

∎

It is noted that the control (2.103) is a state feedback control with respect to the current state and an open-loop control with respect to the initial state, which looks somewhat awkward at first glance. However, if the receding horizon scheme is adopted, then we can obtain the state feedback control that requires only the current state, not other past states. That will be covered in the next chapter.

Replacing i_0 with i in (2.103) yields the following closed-loop control:

$$\begin{aligned} u_i = &-R^{-1}B^T(I + K_{i+1}BR^{-1}B^T)^{-1}[K_{i+1}Ax_i + M_{i+1}S_i^{-1} \\ &\times (x^r_{i_f} - M_i^T x_i - h_i) + g_{i+1}] \end{aligned} \tag{2.104}$$

where S_i is guaranteed to be nonsingular on $i \leq i_f - n_c$.

If Q in (2.103) becomes zero, then (2.103) is reduced to (2.79), which is left as a problem at the end of this chapter.

For the zero reference signal, g_i and h_i in Theorem 2.3 become zero due to $x_i^r = 0$. Thus, we have

$$u_i = -R^{-1}B^T(I + K_{i+1}BR^{-1}B^T)^{-1}[K_{i+1}A - M_{i+1}S_i^{-1}M_i^T]x_i \tag{2.105}$$

in the form of the closed-loop control. As seen above, it is a little complex to obtain the closed-form solution for the fixed terminal state problem with nonzero reference signals.

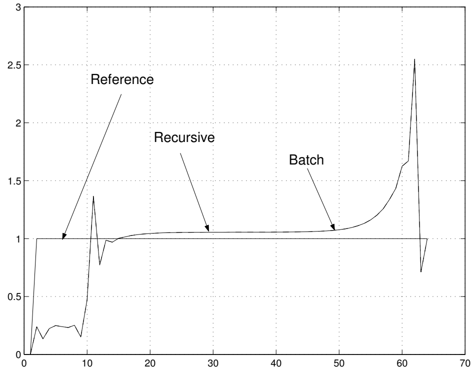
Example 2.1

The LQTC (2.103) with the fixed terminal state is a new type of a tracking control. It is demonstrated through a numerical example.

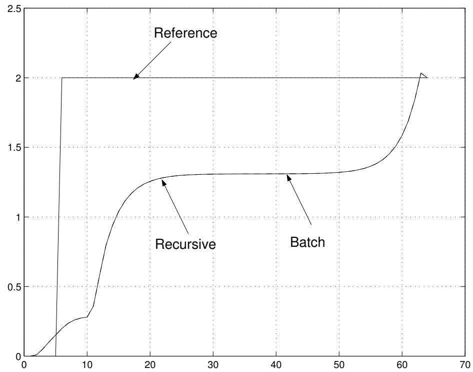
Consider the following state space model:

$$x_{k+1} = \begin{bmatrix} 0.013 & 0.811 & 0.123 \\ 0.004 & 0.770 & 0.096 \\ 0.987 & 0.903 & 0.551 \end{bmatrix} x_k + \begin{bmatrix} 0.456 \\ 0.018 \\ 0.821 \end{bmatrix} u_k \tag{2.106}$$

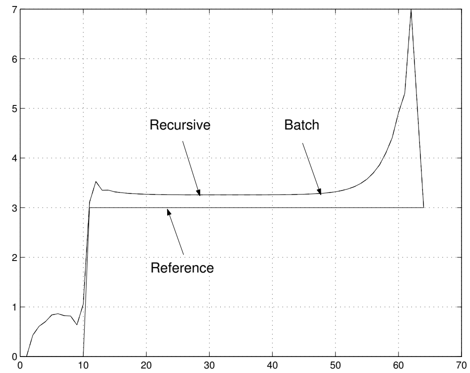
Q and R in the performance criterion (2.69) are set to $100I$ and I respectively. The reference signal and state trajectories can be seen in Figure 2.4 where the fixed terminal condition is met. A batch form solution for the fixed terminal state is given in Section 3.5, and its computation turns out to be the same as that of (2.103). ∎



(a) First component of the state



(b) Second component of the state



(c) Third component of the state

Fig. 2.4. State trajectory of Example 2.1

Infinite Horizon Case

If i_f goes to ∞ in (2.43), (2.45), and (2.46), an infinite horizon LQTC is given by

$$\begin{aligned} u_i^* &= -[I + R^{-1}B^T K_\infty B]^{-1} R^{-1} B^T [K_\infty A x_i + g_{i+1,\infty}] \\ &= -[R + B^T K_\infty B]^{-1} B^T [K_\infty A x_i + g_\infty] \end{aligned} \tag{2.107}$$

where K_∞ is a solution of the following algebraic Riccati Equation (ARE):

$$K_\infty = A^T [I + K_\infty B R^{-1} B^T]^{-1} K_\infty A + Q \tag{2.108}$$
$$= A^T K_\infty A - A^T K_\infty B [R + B K_\infty B^T]^{-1} B^T K_\infty A + Q \tag{2.109}$$

and g_∞ is given by

$$g_\infty = A^T [I + K_\infty B R^{-1} B^T]^{-1} g_\infty - Q \bar{x}^r \tag{2.110}$$

with a fixed reference signal $\bar{x}^r$. The stability and an existence of the solution to the Riccati equation are summarized as follows:

Theorem 2.4. *If (A, B) is controllable and $(A, Q^{\frac{1}{2}})$ is observable, the solution to Riccati Equation (2.108) is unique and positive definite, and the stability of u_i (2.107) is guaranteed.*

We can see a proof of Theorem 2.4 in much of the literature, e.g. in [Lew86a]. The conditions on controllability and observability in Theorem 2.4 can be weakened to the reachability and detectability.

Here, we shall present the return difference equality for the infinite horizon LQ control and introduce some robustness in terms of gain and phase margins. From the following simple relation:

$$\begin{aligned} K_\infty - A^T K_\infty A &= (z^{-1}I - A)^T K_\infty (zI - A) + (z^{-1}I - A)^T K_\infty A \\ &\quad + A^T K_\infty (zI - A) \end{aligned} \tag{2.111}$$

$K_\infty - A^T K_\infty A$ in (2.111) is replaced with $-A^T K_\infty B(B^T K_\infty B + R)^{-1} B^T K_\infty A - Q$ according to (2.109) to give

$$\begin{aligned} &(z^{-1}I - A)^T K_\infty (zI - A) + (z^{-1}I - A)^T K_\infty A + A^T K_\infty (zI - A) \\ &\qquad + A^T K_\infty B (B^T K_\infty B + R)^{-1} B^T K_\infty A = Q \end{aligned} \tag{2.112}$$

Pre- and post-multiply (2.112) by $B^T(z^{-1}I - A)^{-T}$ and $(zI - A)^{-1}B$ respectively to get

$$\begin{aligned} &B^T K_\infty B + B^T K_\infty A (zI - A)^{-1} B + B^T (z^{-1}I - A)^{-T} A^T K_\infty B \\ &+ B^T (z^{-1}I - A)^{-T} A^T K_\infty B (B^T K_\infty B + R)^{-1} B^T K_\infty A (zI - A)^{-1} B \\ &= B^T (z^{-1}I - A)^{-T} Q (zI - A)^{-1} B \end{aligned} \tag{2.113}$$

Adding R to both sides of (2.113) and factorizing it yields the following equation:

$$\begin{aligned} & B^T(z^{-1}I - A)^{-T}Q(zI - A)^{-1}B + R \\ &= [I + \mathcal{K}_\infty(z^{-1}I - A)^{-1}B]^T(B^TK_\infty B + R)[I + \mathcal{K}_\infty(zI - A)^{-1}B] \end{aligned} \quad (2.114)$$

where $\mathcal{K}_\infty = [R + B^TK_\infty B]^{-1}B^TK_\infty A$.

From Equation (2.114), we are in a position to check gain and phase margins for the infinite horizon LQ control. First, let $F(z)$ be $I+\mathcal{K}_\infty(zI-A)^{-1}B$, which is called a return difference matrix. It follows from (2.114) that

$$B^TK_\infty B + R = F^{-T}(z^{-1})[R + B^T(z^{-1}I - A)^{-T}Q(zI - A)^{-1}B]F^{-1}(z)$$

which implies that

$$\begin{aligned} \bar{\sigma}(B^TK_\infty B + R) \geq\; & \bar{\sigma}^2(F^{-1}(z)) \\ & \times \underline{\sigma}[R + B^T(z^{-1}I - A)^{-T}Q(zI - A)^{-1}B] \end{aligned} \quad (2.115)$$

Note that $\bar{\sigma}(M^TSM) \geq \underline{\sigma}(S)\bar{\sigma}^2(M)$ for $S \geq 0$. Recalling the two facts $\bar{\sigma}[F^{-1}(z)] = \underline{\sigma}^{-1}[F(z)]$ and $\bar{\sigma}(I - z^{-1}A) \leq 1 + \bar{\sigma}(A)$ for $|z| = 1$, we have

$$\begin{aligned} & \underline{\sigma}[R + B^T(z^{-1}I - A)^{-T}Q(zI - A)^{-1}B] \\ & \geq \underline{\sigma}(R)\underline{\sigma}[I + R^{-\frac{1}{2}}B^T(z^{-1}I - A)^{-T}Q(zI - A)^{-1}BR^{-\frac{1}{2}}] \\ & \geq \underline{\sigma}(R)[1 + \underline{\sigma}^{-1}(R)\underline{\sigma}^2(B)\underline{\sigma}(Q)\bar{\sigma}^{-2}(I - z^{-1}A)\alpha] \\ & \geq \frac{\underline{\sigma}(R)}{\bar{\sigma}(R)}[\bar{\sigma}(R) + \underline{\sigma}^2(B)\underline{\sigma}(Q)\{1 + \bar{\sigma}(A)\}^{-2}\alpha] \end{aligned} \quad (2.116)$$

where α is 1 when $p \leq q$ and 0 otherwise. Recall that the dimensions of inputs u_i and outputs y_i are p and q respectively. Substituting (2.116) into (2.115) and arranging terms yields

$$\begin{aligned} \underline{\sigma}^2[F(z)] &\geq \frac{\underline{\sigma}(R)/\bar{\sigma}(R)}{\bar{\sigma}(R) + \bar{\sigma}^2(B)\bar{\sigma}(K_\infty)}[\bar{\sigma}(R) + \underline{\sigma}^2(B)\underline{\sigma}(Q)\{1 + \bar{\sigma}(A)\}^{-2}\alpha] \\ &\triangleq R_f^2 \end{aligned} \quad (2.117)$$

Let a circle of radius R_f centered at $(-1, 0)$ be $C(-1, R_f)$. The Nyquist plot of the open-loop system of the optimal regulator lies outside $C(-1, R_f)$ for an SISO system, as can be seen in Figure 2.5; the guaranteed gain margins GM of a control are given by

$$(1 + R_f)^{-1} \leq GM \leq (1 - R_f)^{-1} \quad (2.118)$$

and the phase margins PM of the control are given by

$$-2\sin^{-1}(\frac{R_f}{2}) \leq PM \leq 2\sin^{-1}(\frac{R_f}{2}) \quad (2.119)$$

It is noted that margins for discrete systems are smaller than those for continuous systems, i.e. $0.5 \leq GM < \infty$, and $-\pi/3 \leq PM \leq \pi/3$.

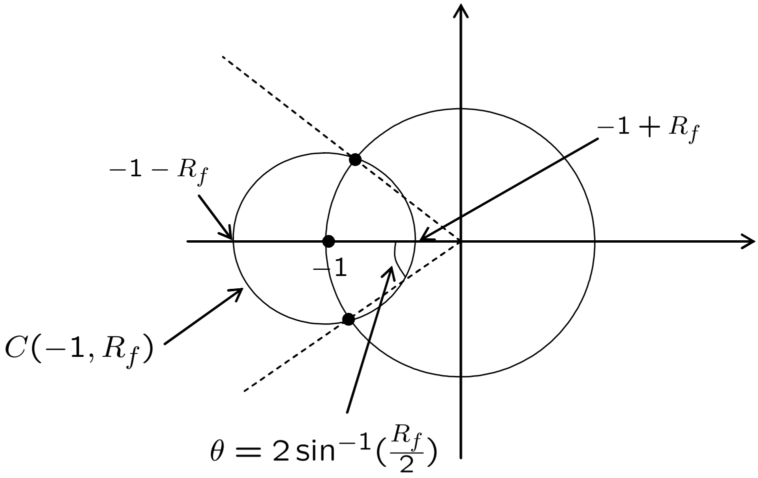


Fig. 2.5. Nyquist plot

2.3.2 H_∞ Control Based on Minimax Criterion

In this subsection we derive an H_∞ tracking control (HTC) for discrete time-invariant systems in a state-feedback form. Consider the following discrete time-invariant system:

$$\begin{aligned} x_{i+1} &= Ax_i + B_w w_i + Bu_i \\ \hat{z}_i &= \begin{bmatrix} Q^{\frac{1}{2}} x_i \\ R^{\frac{1}{2}} u_i \end{bmatrix} \end{aligned} \tag{2.120}$$

where $x_i \in \Re^n$ denotes the state, $w_i \in \Re^l$ the disturbance, $u_i \in \Re^m$ the control input, and $\hat{z}_i \in \Re^{q+n}$ the controlled variable which needs to be regulated. The H_∞ norm of $T_{\hat{z}w}(e^{jw})$ can be represented as

$$\begin{aligned} \|T_{\hat{z}w}(e^{jw})\|_\infty &= \sup_{w_i} \bar{\sigma}(T_{\hat{z}w}(e^{jw})) = \sup_{w_i} \frac{\sum_{i=i_0}^{\infty} [x_i^T Q x_i + u_i^T R u_i]}{\sum_{i=i_0}^{\infty} w_i^T w_i} \\ &= \sup_{\|w_i\|_2 = 1} \sum_{i=i_0}^{\infty} [x_i^T Q x_i + u_i^T R u_i] = \gamma^{*2} \end{aligned} \tag{2.121}$$

where $T_{\hat{z}w}(e^{jw})$ is a transfer function from w_i to $\hat{z}_i$ and $\bar{\sigma}(\cdot)$ is the maximum singular value. $\hat{z}_i$ in (2.120) is chosen to make a quadratic cost function as (2.121).

The H_∞ norm of the systems is equal to the induced L_2 norm. The H_∞ control is obtained so that the H_∞ norm is minimized with respect to u_i.

However, it is hard to achieve an optimal H_∞ control. Instead of the above performance criterion, we can introduce a suboptimal control such that

$\|T_{\hat{z}w}(e^{jw})\|_\infty < \gamma^2$ for some positive gamma γ^2 $(> \gamma^{*2})$. For $\|T_{\hat{z}w}(e^{jw})\|_\infty < \gamma^2$ we will obtain a control so that the following inequality is satisfied:

$$\frac{\sum_{i=i_0}^{\infty}[x_i^T Q x_i + u_i^T R u_i]}{\sum_{i=i_0}^{\infty} w_i^T w_i} < \gamma^2 \tag{2.122}$$

for all w_i. Observe that the gain from $\|w_i\|_2^2$ to $\|\hat{z}_i\|_2^2$ in (2.122) is always less than γ, so that the maximum gain, i.e. H_∞ norm, is also less than γ^2.

From simple algebraic calculations, we have

$$\sum_{i=i_0}^{\infty}[x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T w_i] < 0 \tag{2.123}$$

from (2.122). Since the inequality (2.123) should be satisfied for all w_i, the value of the left side of (2.123) should be always negative, i.e.

$$\sup_{w_i}\left\{\sum_{i=i_0}^{\infty}[x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T w_i]\right\} < 0 \tag{2.124}$$

In order to check whether the feasible solution to (2.124) exists, we try to find out a control minimizing the left side of the inequality (2.124) and the corresponding optimal cost. If this optimal cost is positive, then we cannot obtain the control satisfying the H_∞ norm. Unlike an LQ control, the fixed terminal state is impossible in H_∞ controls. We focus only on the free terminal state.

1) Free Terminal State

When dealing with the finite horizon case, we usually include a weighting matrix for the terminal state, such as

$$\max_{w_i}\left\{\sum_{i=i_0}^{i_f-1}\left[x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T w_i\right] + x_{i_f}^T Q_f x_{i_f}\right\} \; < \; 0 \tag{2.125}$$

A feasible solution u_i in (2.125) can be obtained from the following difference game problem:

$$\min_u \max_w \; J(u,w) \tag{2.126}$$

where

$$J(u,w) = \sum_{i=i_0}^{i_f-1}\left[x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T w_i\right] + x_{i_f}^T Q_f x_{i_f} \tag{2.127}$$

Note that the initial state is assumed to be zero in H_∞ norm in (2.121). However, in the difference game problem (2.126)–(2.127), the nonzero initial state can be handled.

Until now, the regulation problem has been considered. If a part of the state should be steered according to the given reference signal, we can consider

$$\begin{aligned} J(u,w) &= \sum_{i=i_0}^{i_f-1} [\,(z_i - z_i^r)^T \bar{Q}(z_i - z_i^r) + u_i R u_i - \gamma^2 w_i R_w w_i] \\ &\quad + (z_{i_f} - z_{i_f}^r)^T \bar{Q}_f (z_{i_f} - z_{i_f}^r) \end{aligned}$$

instead of (2.127). Here, $z_i = C_z x_i$ is expected to approach z_i^r.

It is well known that for a given $p \times n (p \leq n)$ full rank matrix C_z there always exist some $n \times p$ matrices L such that $C_z L = I_{p \times p}$. For example, we can take $L = C_z^T (C_z C_z^T)^{-1}$. Let $x_i^r = L y_i^r$. $J(u,w)$ is rewritten as

$$\begin{aligned} J(u,w) &= \sum_{i=i_0}^{i_f-1} \left[(x_i - x_i^r)^T Q (x_i - x_i^r) + u_i^T R u_i - \gamma^2 w_i^T R_w w_i \right] \\ &\quad + (x_{i_f} - x_{i_f}^r)^T Q_f (x_{i_f} - x_{i_f}^r) \end{aligned} \tag{2.128}$$

where $Q = C_z^T \bar{Q} C_z$ and $Q_f = C_z^T \bar{Q}_f C_z$ or Q_f and Q are independent design parameters.

For the optimal solution, we first form the following Hamiltonian:

$$\begin{aligned} \mathcal{H}_i &= [(x_i - x_i^r)^T Q (x_i - x_i^r) + u_i^T R u_i - \gamma^2 w_i^T R_w w_i] \\ &\quad + p_{i+1}^T (A x_i + B_w w_i + B u_i), \quad i = i_0, \cdots, i_f - 1 \end{aligned}$$

The necessary conditions for u_i and w_i to be the saddle points are

$$x_{i+1} = \frac{\partial \mathcal{H}}{\partial p_{i+1}} = A x_i + B_w w_i + B u_i \tag{2.129}$$

$$p_i = \frac{\partial \mathcal{H}}{\partial x_i} = 2Q(x_i - x_i^r) + A^T p_{i+1} \tag{2.130}$$

$$0 = \frac{\partial \mathcal{H}}{\partial u_i} = 2 R u_i + B^T p_{i+1} \tag{2.131}$$

$$0 = \frac{\partial \mathcal{H}}{\partial w_i} = -2\gamma^2 R_w w_i + B_w^T p_{i+1} \tag{2.132}$$

$$p_{i_f} = \frac{\partial h(x_{i_f})}{\partial x_{i_f}} = 2 Q_f (x_{i_f} - x_{i_f}^r) \tag{2.133}$$

where

$$h(x_{i_f}) = (x_{i_f} - x_{i_f}^r)^T Q_f (x_{i_f} - x_{i_f}^r) \tag{2.134}$$

Assume

$$p_i = 2 M_{i,i_f} x_i + 2 g_{i,i_f} \tag{2.135}$$

From (2.129), (2.131), and (2.135), we have

$$\begin{aligned}\frac{\partial \mathcal{H}_i}{\partial u_i} &= 2Ru_i + B^T p_{i+1} \\ &= 2Ru_i + 2B^T M_{i+1,i_f} x_{i+1} + 2B^T g_{i+1,i_f} \\ &= 2Ru_i + 2B^T M_{i+1,i_f}[Ax_i + B_w w_i + Bu_i] + 2B^T g_{i+1,i_f}\end{aligned}$$

Therefore,

$$\frac{\partial^2 \mathcal{H}_i}{\partial u_i^2} = 2R + 2B^T M_{i+1,i_f} B$$

It is apparent that $\frac{\partial^2 \mathcal{H}_i}{\partial u_i^2} > 0$ for $i = i_0, \cdots, i_f - 1$. Similarly, we have

$$\frac{\partial \mathcal{H}_i}{\partial w_i} = -2\gamma^2 R_w w_i + 2B_w^T M_{i+1,i_f}[Ax_i + B_w w_i + Bu_i] + 2B_w^T g_{i+1,i_f}$$

Therefore,

$$\frac{\partial^2 \mathcal{H}_i}{\partial w_i^2} = -2\gamma^2 R_w + 2B_w^T M_{i+1,i_f} B_w$$

From these, the difference game problem (2.126) for the performance criterion (2.128) has a unique solution if and only if

$$R_w - \gamma^{-2} B_w^T M_{i+1,i_f} B_w > 0, \; i = i_0, \cdots, i_f - 1 \tag{2.136}$$

We proceed to obtain the optimal solution w_i^* and u_i^*. Eliminating u_i and w_i in (2.129) using (2.131) and (2.132) yields

$$x_{i+1} = Ax_i + \frac{1}{2}(-BR^{-1}B^T + \gamma^{-2} B_w R_w^{-1} B_w^T) p_{i+1} \tag{2.137}$$

From (2.135) and (2.137) we obtain

$$\begin{aligned}p_{i+1} &= 2M_{i+1,i_f} x_{i+1} + 2g_{i+1,i_f} \\ &= 2M_{i+1,i_f} Ax_i + M_{i+1,i_f}(-BR^{-1}B^T + \gamma^{-2} B_w R_w^{-1} B_w^T) p_{i+1} + 2g_{i+1,i_f}\end{aligned}$$

Therefore,

$$p_{i+1} = 2[I + M_{i+1,i_f}(BR^{-1}B^T - \gamma^{-2} B_w R_w^{-1} B_w^T)]^{-1}(M_{i+1,i_f} Ax_i + g_{i+1,i_f})$$

Let

$$\Lambda_{i+1,i_f} = I + M_{i+1,i_f}(BR^{-1}B^T - \gamma^{-2} B_w R_w^{-1} B_w^T) \tag{2.138}$$

Then p_{i+1} is rewritten as

$$p_{i+1} = 2\Lambda_{i+1,i_f}^{-1}[M_{i+1,i_f} Ax_i + g_{i+1,i_f}] \tag{2.139}$$

If we substitute (2.139) into (2.130), then we obtain

$$\begin{aligned}p_i &= 2Q(x_i - x_i^r) + 2A^T \Lambda_{i+1,i_f}^{-1}[M_{i+1,i_f} Ax_i + g_{i+1,i_f}] \\ &= 2[A^T \Lambda_{i+1,i_f}^{-1} M_{i+1,i_f} A + Q]x_i + 2A^T \Lambda_{i+1,i_f}^{-1} g_{i+1,i_f} - 2Qx_i^r\end{aligned}$$

Therefore, from (2.133) and the assumption (2.135), we have

$$M_{i,i_f} = A^T \Lambda_{i+1,i_f}^{-1} M_{i+1,i_f} A + Q \tag{2.140}$$
$$M_{i_f,i_f} = Q_f \tag{2.141}$$

and

$$g_{i,i_f} = A^T \Lambda_{i+1,i_f}^{-1} g_{i+1,i_f} - Q x_i^r \tag{2.142}$$
$$g_{i_f,i_f} = -Q_f x_{i_f}^r \tag{2.143}$$

for $i = i_0, \cdots, i_f - 1$. From (2.136), we have

$$I - \gamma^{-2} R_w^{-\frac{1}{2}} B_w^T M_{i+1,i_f} B_w R_w^{-\frac{1}{2}} > 0 \tag{2.144}$$
$$I - \gamma^{-2} M_{i+1,i_f}^{\frac{1}{2}} B_w R_w^{-1} B_w^T M_{i+1,i_f}^{\frac{1}{2}} > 0 \tag{2.145}$$

where the second inequality comes from the fact that $I - SS^T > 0$ implies $I - S^T S > 0$. $\Lambda_{i+1,i_f}^{-1} M_{i+1,i_f}$ in the right side of (2.140) can be written as

$$\begin{aligned}
&\Lambda_{i+1,i_f}^{-1} M_{i+1,i_f} \\
&= \left[I + M_{i+1,i_f}(BR^{-1}B^T - \gamma^{-2} B_w R_w^{-1} B_w^T) \right]^{-1} M_{i+1,i_f} \\
&= M_{i+1,i_f}^{\frac{1}{2}} \left[I + M_{i+1,i_f}^{\frac{1}{2}}(BR^{-1}B^T - \gamma^{-2} B_w R_w^{-1} B_w^T) M_{i+1,i_f}^{\frac{1}{2}} \right]^{-1} M_{i+1,i_f}^{\frac{1}{2}} \\
&\geq 0
\end{aligned}$$

where the last inequality holds because of (2.145). Therefore, M_{i,i_f} generated by (2.140) is always nonnegative definite.

From (2.131) and (2.132), the H_∞ controls are given by

$$u_i^* = -R^{-1} B^T \Lambda_{i+1,i_f}^{-1} [M_{i+1,i_f} A x_i + g_{i+1,i_f}] \tag{2.146}$$
$$w_i^* = \gamma^{-2} R_w^{-1} B_w^T \Lambda_{i+1,i_f}^{-1} [M_{i+1,i_f} A x_i + g_{i+1,i_f}] \tag{2.147}$$

It is noted that u_i^* is represented by

$$u_i^* = H_i x_i + v_i$$

where

$$H_i = -R^{-1} B^T \Lambda_{i+1,i_f}^{-1} M_{i+1,i_f} A$$
$$v_i = -R^{-1} B^T \Lambda_{i+1,i_f}^{-1} g_{i+1,i_f}$$

Here, H_i is the feedback gain matrix and v_i can be viewed as a command signal.

The optimal cost can be represented as

$$J^* = x_i^T M_{i,i_f} x_i + 2x_i^T g_{i,i_f} + h_{i,i_f} \tag{2.148}$$

where M_{i,i_f} and g_{i,i_f} are defined as (2.140) and (2.142) respectively. The derivation for h_{i,i_f} is left as an exercise.

The saddle-point value of the difference game with (2.128) for a zero reference signal is given as

$$J^*(x_i, i, i_f) = x_i^T M_{i,i_f} x_i \tag{2.149}$$

Since M_{i,i_f} is nonnegative definite, the saddle-point value (2.149) is nonnegative.

To conclude, the solution of the HTC problem can be reduced to finding M_{i,i_f} and g_{i,i_f} for $i = i_0, \cdots, i_f - 1$. The Riccati solution M_{i,i_f} is a symmetric matrix, which can be found by solving (2.140) backward in time using the boundary condition (2.141). In a similar manner, g_{i,i_f} can be found by solving (2.142) backward in time using the boundary condition (2.143).

For a regulation problem, i.e. $x_i^r = 0$, the control (2.146) and the disturbance (2.147) can also be represented as

$$\begin{bmatrix} u_i^* \\ w_i^* \end{bmatrix} = -R_{c,i}^{-1} \begin{bmatrix} B^T \\ B_w^T \end{bmatrix} M_{i+1,i_f} A x_i \tag{2.150}$$

$$M_{i,i_f} = A^T M_{i+1,i_f} A - A^T M_{i+1,i_f} \begin{bmatrix} B & B_w \end{bmatrix} R_{c,i}^{-1} \begin{bmatrix} B^T \\ B_w^T \end{bmatrix} M_{i+1,i_f} A$$

where

$$R_{c,i} = \begin{bmatrix} B^T \\ B_w^T \end{bmatrix} M_{i+1,i_f} \begin{bmatrix} B & B_w \end{bmatrix} + \begin{bmatrix} R & 0 \\ 0 & -\gamma^2 R_w \end{bmatrix}$$

It is observed that optimal solutions u^* and w^* in (2.150) look like an LQ solution.

For a positive definite Q_f and a nonsingular matrix A, we can have another form of the control (2.146) and the disturbance (2.147). Let

$$\Pi = BR^{-1}B - \gamma^{-2} B_w R_w^{-1} B_w^T \tag{2.151}$$

It is noted that M_{i,i_f} is obtained from K_{i,i_f} of the LQ control by replacing $BR^{-1}B^T$ by Π. If M_{i,i_f} is nonsingular at $i \leq i_f$, then there exists the following quantity:

$$P_{i,i_f} = M_{i,i_f}^{-1} + \Pi$$

In terms of P_{i,i_f}, (2.146) and (2.147) are represented as

$$u_i^* = -R^{-1} B^T P_{i+1,i_f}^{-1} [A x_i + (P_{i+1,i_f} - \Pi) g_{i+1,i_f}] \tag{2.152}$$

$$w_i^* = \gamma^{-2} R_w^{-1} B_w^T P_{i+1,i_f}^{-1} [A x_i + (P_{i+1,i_f} - \Pi) g_{i+1,i_f}] \tag{2.153}$$

where

$$P_{i,i_f} = A^{-1}P_{i+1,i_f}[I + A^{-1}QA^{-1}P_{i+1,i_f}]^{-1}A^{-1} + \Pi \tag{2.154}$$

and

$$g_{i,i_f} = -A^T P_{i+1,i_f}^{-1}(P_{i+1,i_f} - \Pi)g_{i+1,i_f} - Qx_i^r \tag{2.155}$$

with

$$P_{i_f,i_f} = M_{i_f,i_f}^{-1} + \Pi = Q_f^{-1} + \Pi > 0, \quad g_{i_f,i_f} = -Q_f x_{i_f}^r \tag{2.156}$$

Here, Q_f must be nonsingular.

Note that P_{i,i_f}^{-1} is well defined only if M_{i,i_f} satisfies the condition

$$R_w - \gamma^{-2}B_w^T M_{i+1,i_f} B_w > 0 \tag{2.157}$$

which is required for the existence of the saddle-point.

The terminal weighting matrix Q_f cannot be arbitrarily large, since M_{i,i_f} generated from the large $M_{i_f,i_f} = Q_f$ is also large and thus the inequality condition (2.136) may not be satisfied. That is why the terminal equality constraint for case of the RH H_∞ control does not make sense.

Infinite Horizon Case

From the finite horizon H_∞ control of a form (2.150), we now turn to the infinite horizon H_∞ control, which is summarized in the following theorem.

Theorem 2.5. *Suppose that (A, B) is stabilizable and $(A, Q^{\frac{1}{2}})$ is observable. For the infinite horizon performance criterion*

$$\inf_{u_i} \sup_{w_i} \sum_{i=i_0}^{\infty} \left[x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T R_w w_i \right] \tag{2.158}$$

the H_∞ control and the worst-case disturbance are given by

$$\begin{aligned} \begin{bmatrix} u_i^* \\ w_i^* \end{bmatrix} &= -R_{c,\infty}^{-1} \begin{bmatrix} B^T \\ B_w^T \end{bmatrix} M_\infty A x_i \\ M_\infty &= A^T M_\infty A - A^T M_\infty \begin{bmatrix} B & B_w \end{bmatrix} R_{c,\infty}^{-1} \begin{bmatrix} B^T \\ B_w^T \end{bmatrix} M_\infty A \end{aligned} \tag{2.159}$$

where

$$R_{c,\infty} = \begin{bmatrix} B^T \\ B_w^T \end{bmatrix} M_\infty \begin{bmatrix} B & B_w \end{bmatrix} + \begin{bmatrix} R & 0 \\ 0 & -\gamma^2 R_w \end{bmatrix}$$

if and only if the following conditions are satisfied:

(1) there exists a solution M_∞ satisfying (2.159);
(2) the matrix

$$A - \begin{bmatrix} B & B_w \end{bmatrix} R_{c,\infty}^{-1} \begin{bmatrix} B^T \\ B_w^T \end{bmatrix} M_\infty A \tag{2.160}$$

is stable;
(3) the numbers of the positive and negative eigenvalues of $\begin{bmatrix} R & 0 \\ 0 & -\gamma^2 R_w \end{bmatrix}$ *are the same as those of $R_{c,\infty}$;*
(4) $M_\infty \geq 0$.

We can see a proof of Theorem 2.5 in much of the literature including textbooks listed at the end of this chapter. In particular, its proof is made on the Krein space in [HSK99].

2.4 Optimal Filters

2.4.1 Kalman Filter on Minimum Criterion

Here, we consider the following stochastic model:

$$x_{i+1} = Ax_i + Bu_i + Gw_i \tag{2.161}$$
$$y_i = Cx_i + v_i \tag{2.162}$$

At the initial time i_0, the state x_{i_0} is a Gaussian random variable with a mean $\bar{x}_{i_0}$ and a covariance P_{i_0}. The system noise $w_i \in \Re^p$ and the measurement noise $v_i \in \Re^q$ are zero-mean white Gaussian and mutually uncorrelated. The covariances of w_i and v_i are denoted by Q_w and R_v respectively, which are assumed to be positive definite matrices. We assume that these noises are uncorrelated with the initial state x_{i_0}.

In practice, the state may not be available, so it should be estimated from measured outputs and known inputs. Thus, a state estimator, called a filter, is needed. This filter can be used for an output feedback control. Now, we will seek a derivation of a filter which estimates the state x_i from measured data and known inputs so that the error between the real state and the estimated state is minimized. When the filter is designed, the input signal is assumed to be known, and thus it is straightforward to handle the input signal.

A filter, called the Kalman filter, is derived for the following performance criterion:

$$E[(x_i - \hat{x}_{i|i})^T(x_i - \hat{x}_{i|i})|Y_i] \tag{2.163}$$

where $\hat{x}_{i|j}$ is denoted by the estimated value at time i based on the measurement up to j and $Y_i = [y_{i_0}, \cdots, y_i]^T$. Note that $\hat{x}_{i|i}$ is a function of Y_i. $\hat{x}_{i+1|i}$

and $\hat{x}_{i|i}$ are often called a predictive estimated value and a filtered estimated value respectively.

From Appendix C.1 we have the optimal filter

$$\hat{x}_{i|i} = E[x_i|Y_i] \tag{2.164}$$

We first obtain a probability density function of x_i given Y_i and then find out the mean of it.

By the definition of the conditional probability, we have

$$p(x_i|Y_i) = \frac{p(x_i, Y_i)}{p(Y_i)} = \frac{p(x_i, y_i, Y_{i-1})}{p(y_i, Y_{i-1})} \tag{2.165}$$

The numerator in (2.165) can be represented in terms of the conditional expectation as follows:

$$\begin{aligned} p(x_i, y_i, Y_{i-1}) &= p(y_i|x_i, Y_{i-1})p(x_i, Y_{i-1}) \\ &= p(y_i|x_i, Y_{i-1})p(x_i|Y_{i-1})p(Y_{i-1}) \\ &= p(y_i|x_i)p(x_i|Y_{i-1})p(Y_{i-1}) \end{aligned} \tag{2.166}$$

where the last equality comes from the fact that Y_{i-1} is redundant information if x_i is given. Substituting (2.166) into (2.165) yields

$$\begin{aligned} p(x_i|Y_i) = \frac{p(y_i|x_i)p(x_i|Y_{i-1})p(Y_{i-1})}{p(y_i, Y_{i-1})} &= \frac{p(y_i|x_i)p(x_i|Y_{i-1})p(Y_{i-1})}{p(y_i|Y_{i-1})p(Y_{i-1})} \\ &= \frac{p(y_i|x_i)p(x_i|Y_{i-1})}{p(y_i|Y_{i-1})} \end{aligned} \tag{2.167}$$

For the given Y_i, the denominator $p(y_i|Y_{i-1})$ is fixed. Two conditional probability densities in the numerator of Equation (2.167) can be evaluated from the statistical information. For the given x_i, y_i follows the normal distribution, i.e. $y_i \sim \mathcal{N}(Cx_i, R_v)$. The conditional probability $p(x_i|Y_{i-1})$ is also normal. Since $E[x_i|Y_{i-1}] = \hat{x}_{i|i-1}$ and $E[(x_i - \hat{x}_{i|i-1})(x_i - \hat{x}_{i|i-1})^T|Y_{i-1}] = P_{i|i-1}$, $p(x_i|Y_{i-1})$ is a normal probability function, i.e. $\mathcal{N}(\hat{x}_{i|i-1}, P_{i|i-1})$. Therefore, we have

$$\begin{aligned} p(y_i|x_i) &= \frac{1}{\sqrt{(2\pi)^m|R_v|}} \exp\left\{-\frac{1}{2}[y_i - Cx_i]^T R_v^{-1}[y_i - Cx_i]\right\} \\ p(x_i|Y_{i-1}) &= \frac{1}{\sqrt{(2\pi)^n|P_{i|i-1}|}} \exp\left\{-\frac{1}{2}[x_i - \hat{x}_{i|i-1}]^T P_{i|i-1}^{-1}[x_i - \hat{x}_{i|i-1}]\right\} \end{aligned}$$

from which, using (2.167), we find that

$$\begin{aligned} p(x_i|Y_i) = \mathcal{C} \exp\{-\frac{1}{2}[y_i - Cx_i]R_v^{-1}[y_i - Cx_i]\} \times \\ \exp\{-\frac{1}{2}[x_i - \hat{x}_{i|i-1}]P_{i|i-1}^{-1}[x_i - \hat{x}_{i|i-1}]\} \end{aligned} \tag{2.168}$$

where $\mathcal{C}$ is the constant involved in the denominator of (2.167).

We are now in a position to find out the mean of $p(x_i|Y_i)$. Since the Gaussian probability density function has a peak value at the average, we will find x_i that sets the derivative of (2.168) to zero. Thus, we can obtain the following equation:

$$-2C^T R_v^{-1}(y_i - Cx_i) + 2P_{i|i-1}^{-1}(x_i - \hat{x}_{i|i-1}) = 0 \tag{2.169}$$

Denoting the solution x_i to (2.169) by $\hat{x}_{i|i}$ and arranging terms give

$$\begin{aligned}
\hat{x}_{i|i} &= (I + P_{i|i-1}C^T R_v^{-1} C)^{-1}\hat{x}_{i|i-1} \\
&\quad + (I + P_{i|i-1}C^T R_v^{-1} C)^{-1} P_{i|i-1} C^T R_v^{-1} y_i && (2.170)\\
&= [I - P_{i|i-1}C^T (CP_{i|i-1}C^T + R_v)^{-1} C]\hat{x}_{i|i-1} \\
&\quad + P_{i|i-1}C^T (R_v + CP_{i|i-1}C^T)^{-1} y_i && (2.171)\\
&= \hat{x}_{i|i-1} + K_i(y_i - C\hat{x}_{i|i-1}) && (2.172)
\end{aligned}$$

where

$$K_i \triangleq P_{i|i-1}C^T (R_v + CP_{i|i-1}C^T)^{-1} \tag{2.173}$$

$\hat{x}_{i+1|i}$ can be easily found from the fact that

$$\begin{aligned}
\hat{x}_{i+1|i} &= E[x_{i+1}|Y_i] = AE[x_i|Y_i] + GE[w_i|Y_i] + Bu_i \\
&= A\hat{x}_{i|i} + Bu_i && (2.174)\\
&= A\hat{x}_{i|i-1} + AK_i(y_i - C\hat{x}_{i|i-1}) + Bu_i && (2.175)
\end{aligned}$$

$P_{i+1|i}$ can be obtained recursively from the error dynamic equations.

Subtracting x_i from both sides of (2.172) yields the following error equation:

$$\tilde{x}_{i|i} = [I - K_i C]\tilde{x}_{i|i-1} - K_i v_i \tag{2.176}$$

where $\tilde{x}_{i|i} \triangleq \hat{x}_{i|i} - x_i$ and $\tilde{x}_{i|i-1} = \hat{x}_{i|i-1} - x_i$. From (2.175) and (2.161), an additional error equation is obtained as

$$\tilde{x}_{i+1|i} = A\tilde{x}_{i|i} - Gw_i \tag{2.177}$$

From (2.176) and (2.177), $P_{i|i}$ and $P_{i+1|i}$ are represented as

$$\begin{aligned}
P_{i|i} &= (I - K_i C)P_{i|i-1}(I - K_i C)^T + K_i R_v K_i = (I - K_i C)P_{i|i-1} \\
P_{i+1|i} &= AP_{i|i}A^T + GQ_w G^T \\
&= AP_{i|i-1}A^T + GQ_w G^T \\
&\quad - AP_{i|i-1}C^T (R_v + CP_{i|i-1}C^T)^{-1} CP_{i|i-1}A^T && (2.178)
\end{aligned}$$

The initial values $\hat{x}_{i_0|i_0-1}$ and $P_{i_0|i_0-1}$ are given by $E[x_{i_0}]$ and $E[(\hat{x}_{i_0} - x_{i_0})(\hat{x}_{i_0} - x_{i_0})^T]$, which are *a priori* knowledge.

The Kalman filter can be represented as follows:

$$\hat{x}_{i+1|i} = A\hat{x}_{i|i-1} + AP_iC^T(CP_iC^T + R_v)^{-1}(y_i - C\hat{x}_{i|i-1}) \quad (2.179)$$

where

$$\begin{aligned} P_{i+1} &= AP_iA^T - AP_iC^T(R_v + CP_iC^T)^{-1}CP_iA^T + GQ_wG^T \\ &= A[I + P_iC^TR_v^{-1}C]^{-1}P_iA^T + GQ_wG^T \end{aligned} \quad (2.180)$$

with the given initial condition P_{i_0}. Note that P_i in (2.179) is used instead of $P_{i|i-1}$.

Throughout this book, we use the predicted form $\hat{x}_{i|i-1}$ instead of filtered form $\hat{x}_{i|i}$. For simple notation, $\hat{x}_{i|i-1}$ will be denoted by $\hat{x}_i$ if necessary.

If the index i in (2.180) goes to ∞, then the infinite horizon or steady-state Kalman filter is given by

$$\hat{x}_{i+1|i} = A\hat{x}_{i|i-1} + AP_\infty C^T(CP_\infty C^T + R_v)^{-1}(y_i - C\hat{x}_{i|i-1}) \quad (2.181)$$

where

$$\begin{aligned} P_\infty &= AP_\infty A^T - AP_\infty C^T(R_v + CP_\infty C^T)^{-1}CP_\infty A^T + GQ_wG^T \\ &= A[I + P_\infty C^TR_v^{-1}C]^{-1}P_\infty A^T + GQ_wG^T \end{aligned} \quad (2.182)$$

As in LQ control, the following theorem gives the result on the condition for the existence of P_∞ and the stability for the infinite horizon Kalman filter.

Theorem 2.6. *If (A, G) is controllable and (A, C) is observable, then there is a unique positive definite solution P_∞ to the ARE (2.182). Additionally, the steady-state Kalman filter is asymptotically stable.*

We can see a proof of Theorem 2.6 in much of the literature including textbooks listed at the end of this chapter. In Theorem 2.6, the conditions on controllability and observability can be weakened to the reachability and detectability.

2.4.2 H_∞ Filter on Minimax Criterion

Here, an H_∞ filter is introduced. Consider the following systems:

$$\begin{aligned} x_{i+1} &= Ax_i + B_ww_i + Bu_i \\ y_i &= Cx_i + D_ww_i \\ z_i &= C_zx \end{aligned} \quad (2.183)$$

where $x_i \in \Re^n$ denotes states, $w_i \in \Re^l$ disturbance, $u_i \in \Re^m$ inputs, $y_i \in \Re^p$ measured outputs, and $z_i \in \Re^q$ estimated values. $B_wD_w^T = 0$ and $D_wD_w^T =$

I are assumed for simple calculation. In the estimation problem, the input control has no effect on the design of the estimator, so that B in (2.183) is set to zero and added later .

Our objective is to find a linear estimator $\hat{x}_i = T(y_{i_0}, y_{i_0+1}, \cdots, y_{i-1})$ so that $e_i = z_i - \hat{z}_i$ satisfies the following performance criterion:

$$\sup_{w_i \neq 0} \frac{\sum_{i=i_0}^{i_f} e_i^T e_i}{\sum_{i=i_0}^{i_f} w_i^T w_i} < \gamma^2 \tag{2.184}$$

From the system (2.183), we obtain the following state-space realization that has inputs $[w_i^T \ \hat{z}_i^T]^T$ and outputs $[e_i^T \ y_i^T]$ as

$$\begin{bmatrix} x_{i+1} \\ e_i \\ y_i \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & 0 \\ \hline C_z & 0 & -I \\ C & D_w & 0 \end{array}\right] \begin{bmatrix} x_i \\ w_i \\ \hat{z}_i \end{bmatrix} \tag{2.185}$$

under which we try to find the filter represented by

$$\hat{z}_i = T(y_{i_0}, y_{i_0+1}, \cdots, y_{i-1}) \tag{2.186}$$

The adjoint system of (2.185) can be represented as

$$\begin{bmatrix} \tilde{x}_i \\ \tilde{w}_i \\ \tilde{\hat{z}}_i \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & 0 \\ \hline C_z & 0 & -I \\ C & D_w & 0 \end{array}\right]^T \begin{bmatrix} \tilde{x}_{i+1} \\ \tilde{e}_i \\ \tilde{y}_i \end{bmatrix} = \left[\begin{array}{c|cc} A^T & C_z^T & C^T \\ \hline B_w^T & 0 & D_w^T \\ 0 & -I & 0 \end{array}\right] \begin{bmatrix} \tilde{x}_{i+1} \\ \tilde{e}_i \\ \tilde{y}_i \end{bmatrix} \tag{2.187}$$

where $\tilde{x}_{i_f+1} = 0$ and $i = i_f, i_f - 1, \cdots, i_0$. Observe that the input and the output are switched. Additionally, time indices are arranged in a backward way. The estimator that we try to find out is changed as follows:

$$\tilde{y}_i = \tilde{T}(\tilde{\hat{z}}_{i_f}, \tilde{\hat{z}}_{i_f-1}, \cdots, \tilde{\hat{z}}_{i+1}) \tag{2.188}$$

where $\tilde{T}(\cdot)$ is the adjoint system of $T(\cdot)$. Now we are in a position to apply the H_∞ control theory to the above H_∞ filter problem.

The state feedback H_∞ control is obtained from the following adjoint system:

$$\tilde{x}_i = A^T \tilde{x}_{i+1} + C^T \tilde{y}_i + \tilde{e}_i \tag{2.189}$$
$$\tilde{w}_i = B_w^T \tilde{x}_{i+1} + D_w^T \tilde{y}_i \tag{2.190}$$
$$\tilde{\hat{z}}_i = -\tilde{e}_i \tag{2.191}$$
$$\tilde{y}_i = \tilde{T}(\tilde{\hat{z}}_{i_f}, \tilde{\hat{z}}_{i_f-1}, \cdots, \tilde{\hat{z}}_{i+1}) \tag{2.192}$$

From the above system, the $\tilde{y}_i$ and $\tilde{w}_i$ are considered as an input and controlled output respectively. It is noted that time indices are reversed, i.e. we goes from the future to the past.

The controller $\tilde{T}(\cdot)$ can be selected to bound the cost:

$$\max_{\|\tilde{e}_i\|_{2,[0,i_f]}\neq 0} \frac{\sum_{i=i_0}^{i_f} \tilde{w}_i^T \tilde{w}_i}{\sum_{i=i_0}^{i_f} \tilde{e}_i^T \tilde{e}_i} < \gamma^2 \tag{2.193}$$

According to (2.146) and the correspondence

$$Q \longleftarrow B_w B_w^T,\ R \longleftarrow I,\ R_w \longleftarrow I,\ A^T \longleftarrow A,\ B \longleftarrow C^T,\ B_w \longleftarrow C_z$$

the resulting controller and the worst case $\tilde{e}_i$ are given:

$$\tilde{y}_i = -L_{i,i_0}^T \tilde{x}_{i+1},\ \tilde{e}_i = -N_{i,i_0}^T \tilde{x}_{i+1} \tag{2.194}$$

where

$$L_{i,i_0}^T = C\Gamma_{i,i_0}^{-1} S_{i,i_0} A^T,\ N_{i,i_0}^T = -C_z \Gamma_{i,i_0}^{-1} S_{i,i_0} A^T \tag{2.195}$$

$$S_{i+1,i_0} = A S_{i,i_0} \Gamma_{i,i_0}^{-1} A^T + B_w B_w^T \tag{2.196}$$

$$\Gamma_{i,i_0} = I + (C^T C - \gamma^{-2} C_z^T C_z) S_{i,i_0} \tag{2.197}$$

with $S_{i_0,i_0} = 0$. In Figure 2.6, controls using Riccati solutions are represented in forward and backward ways. The state-space model for the controller is given as

$$\tilde{x}_i = A^T \tilde{x}_{i+1} - C^T L_{i,i_0}^T \tilde{x}_{i+1} + C_z^T \tilde{e}_i \tag{2.198}$$

$$\tilde{y}_i = -L_{i,i_f}^T \tilde{x}_{i+1}, \tag{2.199}$$

which can be represented as

$$\begin{aligned} \begin{bmatrix} \tilde{x}_i \\ \tilde{y}_i \end{bmatrix} &= \left[\begin{array}{c|c} A^T - C^T L_{i,i_0}^T & C_z^T \\ \hline -L_{i,i_0}^T & 0 \end{array}\right] \begin{bmatrix} \tilde{x}_{i+1} \\ \tilde{e}_i \end{bmatrix} \\ &= \left[\begin{array}{c|c} A^T - C^T L_{i,i_0}^T & -C_z^T \\ \hline -L_{i,i_0}^T & 0 \end{array}\right] \begin{bmatrix} \tilde{x}_{i+1} \\ \tilde{\tilde{z}}_i \end{bmatrix} \end{aligned} \tag{2.200}$$

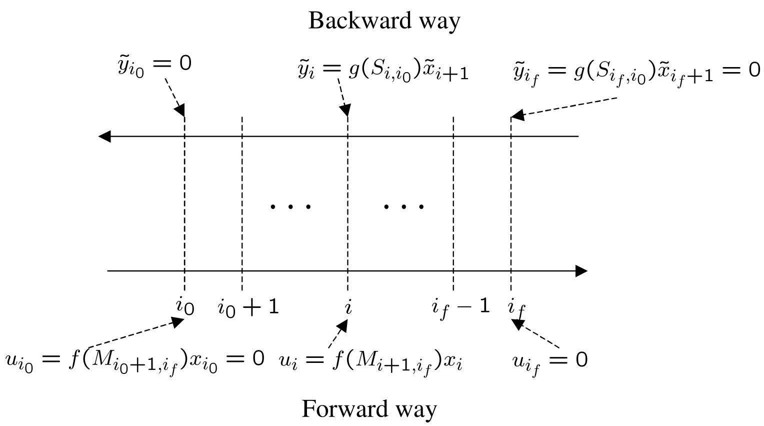


Fig. 2.6. Computation directions for H_∞ filter

which is a state-space realization for the control (2.188). The adjoint system of (2.200) is as follows:

$$\begin{bmatrix} \hat{\eta}_{i+1} \\ \hat{z}_i \end{bmatrix} = \left[\begin{array}{c|c} A^T - C^T L_{i,i_0}^T & -C_z^T \\ \hline -L_{i,i_0}^T & 0 \end{array}\right]^T \begin{bmatrix} \hat{\eta}_i \\ y_i \end{bmatrix} = \left[\begin{array}{c|c} A - L_{i,i_0} C & -L_{i,i_0} \\ \hline -C_z & 0 \end{array}\right] \begin{bmatrix} \hat{\eta}_i \\ y_i \end{bmatrix}$$

which is a state-space realization for the filter (2.186). Rearranging terms, replacing $-\hat{\eta}_i$ with $\hat{x}_i$, and adding the input into the estimator equation yields the H_∞ filter

$$\hat{z}_i = C_z \hat{x}_i, \;\; \hat{x}_{i+1} = A\hat{x}_i + Bu_i + L_{i,i_0}(y_i - C\hat{x}_i) \tag{2.201}$$

The H_∞ filter can also be represented as follows:

$$\hat{z}_i = C_z \hat{x}_i, \;\; \hat{x}_{i+1} = A\hat{x}_i + AS_{i,i_0} \begin{bmatrix} C^T & C_z^T \end{bmatrix} R_{f,i}^{-1} \begin{bmatrix} y - C\hat{x}_i \\ 0 \end{bmatrix} \tag{2.202}$$

$$S_{i+1,i_0} = AS_{i,i_0}A^T - AS_{i,i_0} \begin{bmatrix} C^T & C_z^T \end{bmatrix} R_{f,i}^{-1} \begin{bmatrix} C \\ C_z \end{bmatrix} S_{i,i_0} A^T + B_w B_w^T \tag{2.203}$$

where

$$R_{f,i} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} C \\ C_z \end{bmatrix} S_{i,i_0} \begin{bmatrix} C^T & C_z^T \end{bmatrix}$$

It is observed that the H_∞ filter of the form (2.202) and (2.203) looks like the Kalman filter.

From the finite horizon H_∞ filter of the form (2.202) and (2.203), we now turn to the infinite horizon H_∞ filter. If the index i goes to ∞, the infinite horizon H_∞ filter is given by

$$\hat{z}_i = C_z \hat{x}_i, \;\; \hat{x}_{i+1} = A\hat{x}_i + AS_\infty \begin{bmatrix} C^T & C_z^T \end{bmatrix} R_{f,\infty}^{-1} \begin{bmatrix} y - C\hat{x}_i \\ 0 \end{bmatrix} \tag{2.204}$$

$$S_\infty = AS_\infty A^T - AS_\infty \begin{bmatrix} C^T & C_z^T \end{bmatrix} R_{f,\infty}^{-1} \begin{bmatrix} C \\ C_z \end{bmatrix} S_\infty A^T + B_w B_w^T \tag{2.205}$$

where

$$R_{f,\infty} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} C \\ C_z \end{bmatrix} S_\infty \begin{bmatrix} C^T & C_z^T \end{bmatrix} \tag{2.206}$$

As in the infinite horizon H_∞ control, the following theorem gives the result on the condition for the existence of S_∞ and stability for the infinite horizon H_∞ filter.

Theorem 2.7. *Suppose that (A, B) is stabilizable and $(A, Q^{\frac{1}{2}})$ is observable. For the following infinite horizon performance criterion:*

$$\max_{w_i \neq 0} \frac{\sum_{i=i_0}^\infty e_i^T e_i}{\sum_{i=i_0}^\infty w_i^T w_i} < \gamma^2 \tag{2.207}$$

the H_∞ filter (2.204) exists if and only if the following things are satisfied:

(1) there exists a solution S_∞ satisfying (2.205);
(2) the matrix

$$A - AS_\infty \begin{bmatrix} C^T & C_z^T \end{bmatrix} R_{f,\infty}^{-1} \begin{bmatrix} C \\ C_z \end{bmatrix} \tag{2.208}$$

is stable;
(3) the numbers of the positive and negative eigenvalues of $R_{f,\infty}$ in (2.206) are the same as those of the matrix $\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix}$;
(4) $S_\infty \geq 0$.

We can see a proof of Theorem 2.7 in a number of references. The H_∞ filter in Theorem 2.7 is obtained mostly by using the duality from the H_∞ control, e.g. in [Bur98]. The Krein space instead of the Hilbert space is used to derive H_∞ filters in [HSK99].

2.4.3 Kalman Filters on Minimax Criterion

We assume that Q_w and R_v are unknown, but are bounded above as follows:

$$Q_w \leq Q_o, \ R_v \leq R_o \tag{2.209}$$

The Kalman filter can be derived for the minimax performance criterion given by

$$\min_{L_i} \max_{Q_w \leq Q_o, R_v \leq R_o} E[(x_i - \hat{x}_{i|i-1})(x_i - \hat{x}_{i|i-1})^T]$$

From the error dynamics

$$\tilde{x}_{i+1|i} = [A - L_i C]\tilde{x}_{i|i-1} + \begin{bmatrix} G & -L_i \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix}$$

where $\tilde{x}_{i|i-1} = x_i - \hat{x}_{i|i-1}$, the following equality between the covariance matrices at time $i+1$ and i is satisfied:

$$P_{i+1} = [A - L_i C]P_i[A - L_i C]^T + \begin{bmatrix} G & L_i \end{bmatrix} \begin{bmatrix} Q_w & 0 \\ 0 & R_v \end{bmatrix} \begin{bmatrix} G^T \\ L_i^T \end{bmatrix} \tag{2.210}$$

As can be seen in (2.210), P_i is monotonic with respect to Q_w and R_v, so that taking Q_w and R_v as Q_o and R_o we have

$$P_{i+1} = [A - L_i C]P_i[A - L_i C]^T + \begin{bmatrix} G & L_i \end{bmatrix} \begin{bmatrix} Q_o & 0 \\ 0 & R_o \end{bmatrix} \begin{bmatrix} G^T \\ L_i^T \end{bmatrix} \tag{2.211}$$

It is well known that the right-hand side of (2.211) is minimized for the solution to the following Riccati equation:

$$P_{i+1} = -AP_iC^T(R_o + CP_iC^T)^{-1}CP_iA^T + AP_iA^T + GQ_oG^T$$

where L_i is chosen as

$$L_i = AP_iC^T(R_o + CP_iC^T)^{-1} \tag{2.212}$$

It is noted that (2.212) is the same as the Kalman gain with Q_o and R_o.

2.5 Output Feedback Optimal Control

Before moving to an output feedback control, we show that a quadratic performance criterion for deterministic systems with no disturbances can be represented in a square form.

Lemma 2.8. *A quadratic performance criterion can be represented in a perfect square expression for any control,*

$$
\begin{aligned}
&\sum_{i=i_0}^{i_f-1} \left[x_i^T Q x_i + u_i^T R u_i\right] + x_{i_f}^T Q_f x_{i_f} \\
&= \sum_{i=i_0}^{i_f-1} \{-\mathcal{K}_i x_i + u_i\}^T [R + B^T K_i B] \{-\mathcal{K}_i x_i + u_i\} + x_{i_0}^T K_{i_0} x_{i_0} \qquad (2.213)
\end{aligned}
$$

where $\mathcal{K}_i$ is defined in

$$
\mathcal{K}_i \triangleq (B^T K_{i+1} B + R_v)^{-1} B^T K_{i+1} A \qquad (2.214)
$$

and K_i is the solution to Riccati Equation (2.45).

Proof. Now note the simple identity as

$$
\sum_{i=i_0}^{i_f-1} [x_i K_i x_i^T - x_{i+1} K_{i+1} x_{i+1}^T] = x_{i_0}^T K_{i_0} x_{i_0} - x_{i_f}^T K_{i_f} x_{i_f}
$$

Then, the second term of the right-hand side can be represented as

$$
x_{i_f}^T K_{i_f} x_{i_f} = x_{i_0}^T K_{i_0} x_{i_0} - \sum_{i=i_0}^{i_f-1} [x_i K_i x_i^T - x_{i+1} K_{i+1} x_{i+1}^T]
$$

Observe that the quadratic form for x_{i_f} is written in terms of the x_i on $[i_0 \quad i_f - 1]$. Substituting the above equation into the terminal performance criterion yields

$$
\begin{aligned}
&\sum_{i=i_0}^{i_f-1} \left[x_i^T Q x_i + u_i^T R u_i\right] + x_{i_f}^T Q_f x_{i_f} \\
&= \sum_{i=i_0}^{i_f-1} \left[x_i^T Q x_i + u_i^T R u_i\right] + x_{i_0}^T K_{i_0} x_{i_0} - \sum_{i=i_0}^{i_f-1} [x_i K_i x_i^T - x_{i+1} K_{i+1} x_{i+1}^T] \qquad (2.215)
\end{aligned}
$$

If x_{i+1} is replaced with $Ax_i + Bu_i$, then we have

$$
\begin{aligned}
&\sum_{i=i_0}^{i_f-1} \Big[x_i^T (Q - K_i + A^T K_i A) x_i + u_i^T B^T K_i A x_j \\
&\quad + x_i^T A^T K_{i+1} B u_i + u_i^T (R + B^T K_i B) u_i \Big] + x_{i_0}^T K_{i_0} x_{i_0} \qquad (2.216)
\end{aligned}
$$

If K_i satisfies (2.45), then the square completion is achieved as (2.213), This completes the proof. ∎

2.5.1 Linear Quadratic Gaussian Control on Minimum Criterion

Now, we introduce an output feedback LQG control. A quadratic performance criterion is given by

$$J = \sum_{i=i_0}^{i_f-1} E\left[x_i^T Q x_i + u_i^T R u_i \middle| y_{i-1}, y_{i-2}, \cdots, y_{i_0}\right]$$
$$+ E\left[x_{i_f}^T Q_f x_{i_f} \middle| y_{i_f-1}, y_{i_f-2}, \cdots, y_{i_0}\right] \tag{2.217}$$

subject to $u_i = f(y_{i-1}, \cdots, y_{i_0})$. Here, the objective is to find a controller u_i that minimizes (2.217). From now on we will not include the condition part inside the expectation for simplicity. Before obtaining the LQG control, as in the deterministic case (2.213), it is shown that the performance criterion (2.217) can be represented in a square form.

Lemma 2.9. *A quadratic performance criterion can be represented in a perfect square expression for any control,*

$$E\left[\sum_{i=i_0}^{i_f-1}\left[x_i^T Q x_i + u_i^T R u_i\right] + x_{i_f}^T Q_f x_{i_f}\right]$$
$$= E\left[\sum_{i=i_0}^{i_f-1}\{-\mathcal{K}_i x_i + u_i\}^T[R + B^T K_i B]\{-\mathcal{K}_i x_i + u_i\}\right]$$
$$+ \operatorname{tr}\left[\sum_{i=i_0}^{i_f-1} K_i G Q_w G^T\right] + E\left[x_{i_0}^T K_{i_0} x_{i_0}\right] \tag{2.218}$$

where $\mathcal{K}_i$ is defined in

$$\mathcal{K}_i \triangleq (B^T K_{i+1} B + R)^{-1} B^T K_{i+1} A \tag{2.219}$$

and K_i is the solution to Riccati Equation (2.45).

Proof. The relation (2.215) holds even for stochastic systems (2.161)-(2.162). Taking an expectation on (2.215), we have

$$E\left[\sum_{i=i_0}^{i_f-1}\left[x_i^T Q x_i + u_i^T R u_i\right] + x_{i_f}^T Q_f x_{i_f}\right]$$
$$= E\left[\sum_{i=i_0}^{i_f-1}\left[x_i^T Q x_i + u_i^T R u_i\right] + x_{i_0}^T K_{i_0} x_{i_0}\right.$$
$$\left. - \sum_{i=i_0}^{i_f-1}\left[x_i K_i x_i^T - x_{i+1} K_{i+1} x_{i+1}^T\right]\right] \tag{2.220}$$

Replacing x_{i+1} with $Ax_i + Bu_i + Gw_i$ yields

$$E\left[\sum_{i=i_0}^{i_f-1}\left[x_i^T(Q - K_i + A^TK_iA)x_i + u_i^TB^TK_iAx_j\right.\right.$$
$$\left.\left. + x_i^TA^TK_{i+1}Bu_i + u_i^T(R + B^TK_iB)u_i\right]\right] + E\left[x_{i_0}^TK_{i_0}x_{i_0}\right]$$
$$+ \operatorname{tr}\left[\sum_{i=i_0}^{i_f-1} K_iGQ_wG^T\right] \tag{2.221}$$

If K_i satisfies (2.140), then the square completion is achieved as (2.218). This completes the proof. ∎

Using Lemma 2.9, we are now in a position to represent the performance criterion (2.217) in terms of the estimated state. Only the first term in (2.221) is dependent on u_i. So, we consider only this term. Let $\hat{x}_{i|i-1}$ be denoted by $\hat{x}_{i|i-1} = E[x_i|y_{i-1}, y_{i-2}, \cdots, y_{i_0}]$. According to (C.2) in Appendix C, we can change the first term in (5.69) to

$$E\sum_{i=i_0}^{i_f-1}(\mathcal{K}_ix_i + u_i)^T\hat{R}_i(\mathcal{K}_ix_i + u_i) = \sum_{i=i_0}^{i_f-1}(\mathcal{K}_i\hat{x}_i + u_i)^T\hat{R}_i(\mathcal{K}_i\hat{x}_i + u_i)$$
$$+ \operatorname{tr}\sum_{i=i_0+1}^{i_f}\hat{R}_i^{\frac{1}{2}}\mathcal{K}_i\tilde{P}_i\mathcal{K}_i^T\hat{R}_i^{\frac{1}{2}} \tag{2.222}$$

where

$$\hat{R}_i \triangleq R + B^TK_iB \tag{2.223}$$

and $\tilde{P}_i$ is the variance between $\hat{x}_{i|i-1}$ and x_i. Note that $\mathcal{K}_i\hat{x}_{i|i-1} + u_i = E[\mathcal{K}_ix_i + u_i \mid y_{i-1}, y_{i-2}, \cdots, y_{i_0}]$ and $\operatorname{tr}(\hat{R}_i\mathcal{K}_i\tilde{P}_i\mathcal{K}_i^T) = \operatorname{tr}(\hat{R}_i^{\frac{1}{2}}\mathcal{K}_i\tilde{P}_i\mathcal{K}_i^T\hat{R}_i^{\frac{1}{2}})$.

We try to find the optimal filter gain L_i making the following filter minimizing P_i:

$$\hat{x}_{i+1|i} = A\hat{x}_{i|i-1} + L_i(y_i - C\hat{x}_{i|i-1}) + Bu_i \tag{2.224}$$

Subtracting (2.161) from (2.224), we have

$$\tilde{x}_{i+1|i} = \hat{x}_{i+1|i} - x_{i+1} = (A - L_iC)\tilde{x}_{i|i-1} + L_iv_i - Gw_i \tag{2.225}$$

which leads to the following equation:

$$\tilde{P}_{i+1} = (A - L_iC)\tilde{P}_i(A - L_iC)^T + L_iR_vL_i^T + GQ_wG^T \tag{2.226}$$

where $\tilde{P}_i$ is the covariance of $\tilde{x}_{i|i-1}$. As can be seen in (2.226), $\tilde{P}_i$ is independent of u_i, so that $\tilde{P}_i$ and u_i can be determined independently. $\tilde{P}_{i+1}$ in (2.226) can be written as

$$\begin{aligned}\tilde{P}_{i+1} &= \Big[L_i(C\tilde{P}_iC^T + R_v) - A\tilde{P}_i\Big](C\tilde{P}_iC^T + R)^{-1}\Big[L_i(C\tilde{P}_iC^T + R_v) - A\tilde{P}_i\Big]^T \\ &\quad + GQ_wG^T + A\tilde{P}_i(C\tilde{P}_iC^T + R_v)^{-1}\tilde{P}_iA^T \\ &\geq GQ_wG^T + A\tilde{P}_i(C\tilde{P}_iC^T + R_v)^{-1}\tilde{P}_iA^T \end{aligned} \tag{2.227}$$

where the equality holds if $L_i = AP_iC^T(R_v + CP_iC^T)^{-1}$.

It can be seen in (2.227) that the covariance P_i generated by the Kalman filter is optimal in view that the covariance $\tilde{P}_i$ of any linear estimator is larger than P_i of the Kalman filter, i.e. $P_i \leq \tilde{P}_i$. This implies that $\mathrm{tr}(P_i) \leq \mathrm{tr}(\tilde{P}_i)$, leading to $\mathrm{tr}(\hat{R}_i^{\frac{1}{2}}K_iP_iK_i^T\hat{R}_i^{\frac{1}{2}}) \leq \mathrm{tr}(\hat{R}_i^{\frac{1}{2}}K_i\tilde{P}_iK_i^T\hat{R}_i^{\frac{1}{2}})$. Thus, the $\hat{x}_{i|i-1}$ minimizing (2.222) is given by the Kalman filter as follows:

$$\hat{x}_{i+1|i} = A\hat{x}_{i|i-1} + [AP_iC^T(R_v + CP_iC^T)^{-1}](y_i - C\hat{x}_{i|i-1}) + Bu_i \tag{2.228}$$

$$P_{i+1} = GQ_wG^T + AP_i(CP_iC^T + R_v)^{-1}P_iA^T \tag{2.229}$$

with the initial state mean $\hat{x}_{i_0}$ and the initial covariance P_{i_0}. Thus, the following LQG control minimizes the performance criterion:

$$u_i^* = -(B^TK_{i+1}B + R)^{-1}B^TK_{i+1}A\hat{x}_{i|i-1} \tag{2.230}$$

Infinite Horizon Linear Quadratic Gaussian Control

We now turn to the infinite horizon LQG control. It is noted that, as the horizon N gets larger, (2.217) also becomes larger and finally blows up. So, the performance criterion (2.217) cannot be applied as it is to the infinite horizon case. In a steady state for the infinite horizon case, we may write

$$\min_{u_i} J = \min_{u_i} E[x_i^TQx_i + u_i^TRu_i] \tag{2.231}$$

$$= \min_{u_i} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}(\mathcal{T}(e^{j\omega})\mathcal{T}^*(e^{j\omega}))\ d\omega \tag{2.232}$$

where $\mathcal{T}(e^{j\omega})$ is the transfer function from w_i and v_i to u_i and x_i. The infinite horizon LQG control is summarized in the following theorem.

Theorem 2.10. *Suppose that (A, B) and (A, G) are controllable and $(A, Q^{\frac{1}{2}})$ and (A, C) are observable. For the infinite horizon performance criterion (2.232), the infinite horizon LQG control is given by*

$$u_i^* = -(B^TK_\infty B + R)^{-1}B^TK_\infty A\hat{x}_{i|i-1} \tag{2.233}$$

where

$$\begin{aligned} K_\infty &= A^TK_\infty A - A^TK_\infty B[R + BK_\infty B^T]^{-1}B^TK_\infty A + Q \\ \hat{x}_{i+1|i} &= A\hat{x}_{i|i-1} + [AP_\infty C^T(R_v + CP_\infty C^T)^{-1}](y_i - C\hat{x}_{i|i-1}) + Bu_i \\ P_\infty &= GQ_wG^T + AP_\infty(CP_\infty C^T + R_v)^{-1}P_\infty A^T \end{aligned}$$

We can see a proof of Theorem 2.10 in many references, e.g. in [Bur98, Lew86b]. The conditions on controllability and observability in Theorem 2.10 can be weakened to the reachability and detectability.

2.5.2 Output Feedback H_∞ Control on Minimax Criterion

Now, we derive the output feedback H_∞ control. The result of the previous H_∞ filter will be used to obtain the output feedback H_∞ control. First, the performance criterion is transformed in perfect square forms with respect to the optimal control and disturbance.

Lemma 2.11. *H_∞ performance criterion can be represented in a perfect square expression for arbitrary control.*

$$\begin{aligned}
&\sum_{i=i_0}^{i_f-1}\left[x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T R_w w_i\right] + x_{i_f}^T Q_f x_{i_f} \\
&= \sum_{i=i_0}^{i_f-1}\left[(u_i - u_i^*)^T \mathcal{V}_i (u_i - u_i^*) - \gamma^2 (w_i - w_i^*)^T \mathcal{W}_i (w_i - w_i^*)\right] \\
&+ x_{i_0}^T M_{i_0} x_{i_0}
\end{aligned} \tag{2.234}$$

where w_i^* and u_i^* are given as

$$w_i^* = (\gamma^2 R_w - B_w^T M_{i+1} B_w)^{-1} B_w^T M_{i+1}(Ax_i + Bu_i) \tag{2.235}$$

$$u_i^* = -R^{-1} B^T M_{i+1}[I + (BR^{-1}B^T - \gamma^{-2} B_w R_w^{-1} B_w^T) M_{i+1}]^{-1} A x_i \tag{2.236}$$

$$\mathcal{V}_i = R + B^T M_{i+1}(I - \gamma^{-2} B_w R_w^{-1} B_w^T M_{i+1})^{-1} B$$

$$\mathcal{W}_i = \gamma^2 R_w - B_w^T M_{i+1} B_w$$

and M_i shortened for M_{i,i_f} is given in (2.140).

Proof. Recalling the simple identity as

$$\sum_{i=i_0}^{i_f-1}[x_i M_i x_i^T - x_{i+1} M_{i+1} x_{i+1}^T] = x_{i_0}^T M_{i_0} x_{i_0} - x_{i_f}^T M_{i_f} x_{i_f}$$

we have

$$x_{i_f}^T M_{i_f} x_{i_f} = x_{i_f}^T Q_f x_{i_f} = x_{i_0}^T M_{i_0} x_{i_0} - \sum_{i=i_0}^{i_f-1}[x_i^T M_i x_i - x_{i+1}^T M_{i+1} x_{i+1}] \tag{2.237}$$

By substituting (2.237) into the final cost $x_{i_f}^T Q_f x_{i_f}$, the H_∞ performance criterion of the left-hand side in (2.234) can be changed as

$$
\begin{aligned}
&\sum_{i=i_0}^{i_f-1} \left[x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T R_w w_i\right] + x_{i_f}^T Q_f x_{i_f} \\
&= \sum_{i=i_0}^{i_f-1} \left[x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T R_w w_i\right] + x_{i_0}^T M_{i_0} x_{i_0} \\
&- \sum_{i=i_0}^{i_f-1} [x_i^T M_i x_i - x_{i+1}^T M_{i+1} x_{i+1}] \\
&= \sum_{i=i_0}^{i_f-1} \Big[x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T R_w w_i - x_i^T M_i x_i \\
&+ x_{i+1}^T M_{i+1} x_{i+1}\Big] + x_{i_0}^T M_{i_0} x_{i_0}
\end{aligned} \tag{2.238}
$$

Now, we try to make terms inside the summation represented in a perfect square form. First, time variables are all changed to i. Next, we complete the square with respect to w_i and u_i respectively.

Terms in the summation (2.238) can be arranged as follows:

$$
\begin{aligned}
&x_{i+1}^T M_{i+1} x_{i+1} - x_i^T M_i x_i + x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T R_w w_i \\
&+ x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T R_w w_i \\
&= A_{o,i}^T M_{i+1} A_{o,i} + w_i^T B_w^T M_{i+1} A_{o,i} + A_{o,i}^T M_{i+1} B_w w_i \\
&- w_i^T (\gamma^2 R_w - B_w^T M_{i+1} B_w) w_i + x_i^T (-M_i + Q) x_i + u_i^T R u_i
\end{aligned} \tag{2.239}
$$

where $A_{o,i} = A x_i + B u_i$. Terms including w_i in (2.239) can be arranged as

$$
\begin{aligned}
&w_i^T B_w^T M_{i+1} A_{o,i} + A_{o,i}^T M_{i+1} B_w w_i - w_i^T (\gamma^2 R_w - B_w^T M_{i+1} B_w) w_i \\
&= -W_i^T (\gamma^2 R_w - B_w^T M_{i+1} B_w)^{-1} W_i \\
&+ A_{o,i}^T M_{i+1} B_w (\gamma^2 R_w - B_w^T M_{i+1} B_w)^{-1} B_w^T M_{i+1} A_{o,i}
\end{aligned} \tag{2.240}
$$

where $W_i = (\gamma^2 R_w - B_w^T M_{i+1} B_w) w_i - B_w^T M_{i+1} A_{o,i}$. After completing the square with respect to disturbance w_i, we try to do that for the control u_i. Substituting (2.240) into (2.239) yields

$$
\begin{aligned}
&-W_i^T (\gamma^2 R_w - B_w^T M_{i+1} B_w)^{-1} W_i \\
&+ A_{o,i}^T M_{i+1} B_w (\gamma^2 R_w - B_w^T M_{i+1} B_w)^{-1} B_w^T M_{i+1} A_{o,i} \\
&+ A_{o,i}^T M_{i+1} A_{o,i} + x_i^T (-M_i + Q) x_i + u_i^T R u_i \\
&= -W_i^T (\gamma^2 R_w - B_w^T M_{i+1} B_w)^{-1} W_i + A_{o,i}^T M_{i+1} (I - \gamma^{-2} B_w R_w^{-1} B_w^T M_{i+1})^{-1} \\
&\times A_{o,i} + x_i^T (-M_i + Q) x_i + u_i^T R u_i
\end{aligned} \tag{2.241}
$$

where the last equality comes from

$$
P(I - RQ^{-1}R^T P)^{-1} = P + PR(Q - R^T P R)^{-1} R^T P
$$

for some matrix P, R, and Q.

The second, the third, and the fourth terms in the right-hand side of (2.241) can be factorized as

$$\begin{aligned} &(Ax_i + Bu_i)^T M_{i+1} \mathcal{Z}_i^{-1}(Ax_i + Bu_i) + x_i^T(-M_i + Q)x_i + u_i^T R u_i \\ &= u_i^T[R + B^T M_{i+1} \mathcal{Z}_i^{-1} B]u_i + u_i^T B^T M_{i+1} \mathcal{Z}_i^{-1} A x_i + x_i^T A^T M_{i+1} \mathcal{Z}_i^{-1} B u_i \\ &+ x_i^T[A^T M_{i+1} \mathcal{Z}_i^{-1} A - M_i + Q]x_i = U_i^T\left[R + B^T M_{i+1} \mathcal{Z}_i^{-1} B\right]^{-1} U_i \end{aligned} \quad (2.242)$$

where

$$\begin{aligned} U_i &= \left[R + B^T M_{i+1} \mathcal{Z}_i^{-1} B\right] u_i + B^T M_{i+1} \mathcal{Z}_i^{-1} A x_i \\ \mathcal{Z}_i &= I - \gamma^{-2} B_w R_w^{-1} B_w^T M_{i+1} \end{aligned}$$

and the second equality comes from the Riccati equation represented by

$$\begin{aligned} M_i &= A^T M_{i+1}(I + (BR^{-1}B^T - \gamma^{-2} B_w R_w^{-1} B_w^T) M_{i+1})^{-1} A + Q \\ &= A^T M_{i+1}(\mathcal{Z}_i + BR^{-1}B^T M_{i+1})^{-1} A + Q \\ &= A^T M_{i+1}\left[\mathcal{Z}_i^{-1} - \mathcal{Z}_i^{-1} BR^{-1} B^T (M_{i+1} \mathcal{Z}_i^{-1} BR^{-1} B^T + I)^{-1} M_{i+1} \mathcal{Z}_i^{-1}\right] A \\ &+ Q \end{aligned}$$

or

$$\begin{aligned} &A^T M_{i+1} \mathcal{Z}_i^{-1} A - M_i + Q \\ &= A^T M_{i+1} \mathcal{Z}_i^{-1} BR^{-1} B^T (M_{i+1} \mathcal{Z}_i^{-1} BR^{-1} B^T + I)^{-1} M_{i+1} \mathcal{Z}_i^{-1} A \\ &= A^T M_{i+1} \mathcal{Z}_i^{-1} B (B^T M_{i+1} \mathcal{Z}_i^{-1} B + R)^{-1} B^T M_{i+1} \mathcal{Z}_i^{-1} A \end{aligned}$$

This completes the proof. ∎

Note that by substituting (5.154) into (5.153), w_i^* can be represented as

$$w_i^* = \gamma^{-2} R_w^{-1} B_w^T M_{i+1}[I + (BR^{-1}B^T - \gamma^{-2} B_w R_w^{-1} B_w^T) M_{i+1}]^{-1} A x_i$$

which is of very similar form to u_i^*.

For the zero initial state, the inequality

$$\sum_{i=i_0}^{i_f-1}\left[x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T R_w w_i\right] + x_{i_f}^T Q_f x_{i_f} < 0 \quad (2.243)$$

guarantees the bound on the following ∞ norm:

$$\sup_{\|w_i\|_{2,[0,i_f]} \neq 0} \frac{\sum_{i=i_0}^{i_f-1}(u_i - u_i^*)^T \mathcal{V}_i (u_i - u_i^*)}{\sum_{i=i_0}^{i_f-1}(w_i - w_i^*)^T \mathcal{W}_i (w_i - w_i^*)} < \gamma^2 \quad (2.244)$$

where u_i^* and w_i^* are defined in (5.154) and (5.153) respectively. Here, $R_w = I$, $D_w B_w^T = 0$, and $D_w D_w^T = I$ are assumed for simple calculation.

According to (2.244), we should design u_i so that the H_∞ norm between the weighted disturbance deviation

$$\triangle w_i \triangleq \mathcal{W}_i^{\frac{1}{2}} w_i - \mathcal{W}_i^{\frac{1}{2}} (\gamma^2 I - B_w^T M_{i+1} B_w)^{-1} B_w^T M_{i+1} (A x_i + B u_i)$$

and the weighted control deviation

$$\triangle u_i \triangleq \mathcal{V}_i^{\frac{1}{2}} u_i + \mathcal{V}_i^{\frac{1}{2}} R^{-1} B^T M_{i+1} [I + (B R^{-1} B^T - \gamma^{-2} B_w B_w^T) M_{i+1}]^{-1} A x_i$$

is minimized. By using $\triangle w_i$, we obtain the following state-space model:

$$\begin{aligned} x_{i+1} &= A_{a,i} x_i + B_{a,i} u_i + B_w \mathcal{W}_i^{-\frac{1}{2}} \triangle w_i \\ y_i &= C x_i + D_w \mathcal{W}_i^{-\frac{1}{2}} \triangle w_i \end{aligned} \tag{2.245}$$

where $A_{a,i} = A + B_w(\gamma^2 I - B_w^T M_{i+1} B_w)^{-1} B_w^T M_{i+1} A$, and $B_{a,i} = B + B_w(\gamma^2 I - B_w^T M_{i+1} B_w)^{-1} B_w^T M_{i+1} B$. Note that $D_w B_w^T = 0$ is assumed as mentioned before.

The performance criterion (2.244) for the state-space model (2.245) is just one for the H_∞ filter that estimates u_i^* with respect to $\triangle w_i$ by using measurements y_i. This is a similar structure to (2.183). Note that the variable z_i in (2.183) to be estimated corresponds to u_i^*. Using the result of H_∞ filters, we can think of this as finding out the output feedback H_∞ control u_i by obtaining the estimator of u_i^*.

All derivations require long and tedious algebraic calculations. In this book, we just summarize the final result. The output feedback H_∞ control is given by

$$\begin{aligned} u_i &= -K_{of,i} \hat{x}_i \\ K_{of,i} &= R^{-1} B^T M_{i+1} [I + (B R^{-1} B^T - \gamma^{-2} B_w B_w^T) M_{i+1}]^{-1} A \\ \hat{x}_{i+1} &= A_{a,i} \hat{x}_i + L_{of,i} \begin{bmatrix} 0 \\ y_i - C \hat{x}_i \end{bmatrix} + B u_i \end{aligned}$$

where M_i, i.e. M_{i,i_f}, is given in (2.140) and $L_{of,i}$ is defined as

$$\begin{aligned} L_{of,i} &= \left(A_{a,i} S_{of,i} \left[-K_{of,i}^T \; C^T \right] - \gamma^2 B_w \left[\bar{S}_i \;\; 0 \right] \right) R_{of,i}^{-1} \\ S_{of,i+1} &= A_{a,i} S_{of,i} A_{a,i}^T - \gamma^2 B_w W_i^{-1} B_w^T - L_{of,i} R_{of,i} L_{of,i}^T \\ R_{of,i} &= \begin{bmatrix} -\gamma^2 Z_i^{-1} & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} -K_{of,i} \\ C \end{bmatrix} S_{of,i} \left[-K_{of,i}^T \; C^T \right] \end{aligned}$$

$$\begin{bmatrix} W_i^{-1} & \bar{S}_i \\ \bar{S}_i^T & Z_i^{-1} \end{bmatrix} = \left(\begin{bmatrix} -\gamma^2 I & 0 \\ 0 & R \end{bmatrix} + \begin{bmatrix} B_w^T \\ B^T \end{bmatrix} M_{i+1} \left[B_w \; B \right] \right)^{-1}$$

with the initial conditions $M_{i_f,i_f} = Q_f$ and $S_{i_0} = 0$.

Infinite Horizon Output Feedback H_∞ Control

Now we introduce the infinite horizon H_∞ output feedback control in the following theorem:

Theorem 2.12. *Infinite horizon H_∞ output feedback control is composed of*

$$\begin{aligned} u_i &= -K_{of,\infty}\hat{x}_i \\ K_{of,\infty} &= R^{-1}B^T M_\infty[I + (BR^{-1}B^T - \gamma^{-2}B_w B_w^T)M_\infty]^{-1}A \\ \hat{x}_{i+1} &= A_{a,\infty}\hat{x}_i + L_{of,\infty}\begin{bmatrix} y_i - C\hat{x}_i \\ 0 \end{bmatrix} + Bu_i \end{aligned}$$

where

$$\begin{aligned} A_{a,\infty} &= A + B_w(\gamma^2 I - B_w^T M_\infty B_w)^{-1}B_w^T M_\infty A \\ L_{of,\infty} &= \left(A_{a,\infty}S_{of,\infty}\begin{bmatrix} -K_{of,\infty}^T & C^T \end{bmatrix} - \gamma^2 B_w\begin{bmatrix} \bar{S}_\infty & 0 \end{bmatrix}\right)R_{of,\infty}^{-1} \\ S_{of,\infty} &= A_{a,\infty}S_{of,\infty}A_{a,\infty}^T - \gamma^2 B_w W_\infty^{-1}B_w^T - L_{of,\infty}R_{of,\infty}L_{of,\infty}^T \\ R_{of,\infty} &= \begin{bmatrix} -\gamma^2(Z_\infty)^{-1} & 0 \\ 0 & I_p \end{bmatrix} + \begin{bmatrix} -K_{of,\infty} \\ C \end{bmatrix} P_i \begin{bmatrix} -K_{of,\infty}^T & C^T \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} W_\infty^{-1} & \bar{S}_\infty \\ \bar{S}_\infty^T & Z_\infty^{-1} \end{bmatrix} = \left(\begin{bmatrix} -\gamma^2 I_l & 0 \\ 0 & R \end{bmatrix} + \begin{bmatrix} B_w^T \\ B^T \end{bmatrix} M_\infty \begin{bmatrix} B_w & B \end{bmatrix}\right)^{-1}$$

and achieves the following specification

$$\frac{\sum_{i=i_0}^{\infty} x_i^T R x_i + u_i^T Q u_i}{\sum_{i=i_0}^{\infty} w_i^T w_i} < \gamma^2$$

if and only if there exists solutions $M_\infty \geq 0$ satisfying (2.159) and $S_{of,\infty} \geq 0$ such that

(1) $A - \begin{bmatrix} B & B_w \end{bmatrix}\left(\begin{bmatrix} R & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} B^T \\ B_w^T \end{bmatrix} M_\infty \begin{bmatrix} B & B_w \end{bmatrix}\right)^{-1}\begin{bmatrix} B^T \\ B_w^T \end{bmatrix} M_\infty A$ *is stable.*

(2) The numbers of the positive and negative eigenvalues of the two following matrices are the same:

$$\begin{bmatrix} R & 0 \\ 0 & -\gamma^2 I_l \end{bmatrix}, \quad \begin{bmatrix} R & 0 \\ 0 & -\gamma^2 I_l \end{bmatrix} + \begin{bmatrix} B^T \\ B_w^T \end{bmatrix} M_\infty \begin{bmatrix} B & B_w \end{bmatrix} \tag{2.246}$$

(3) $A_{a,\infty} - L_{of,\infty}\begin{bmatrix} C \\ (I + B^T M_\infty B)^{\frac{1}{2}} K_{of,\infty} \end{bmatrix}$ *is stable.*

(4) The numbers of the positive and negative eigenvalues of $\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_m \end{bmatrix}$ *are the same as those of the following matrix:*

$$\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_m + T \end{bmatrix} + \begin{bmatrix} C \\ X^{\frac{1}{2}} K_{of,\infty} \end{bmatrix} S_{of,\infty} \begin{bmatrix} C^T & K_{of,\infty}^T X^{\frac{1}{2}} \end{bmatrix} \tag{2.247}$$

where $T = X^{-\frac{1}{2}} B^T M_\infty Z^{-1} M_\infty B X^{-\frac{1}{2}}$, $Z = I - B^T M_\infty (I + BB^T M_\infty)^{-1} B$, *and* $X = I + B^T M_\infty B$.

We can see a proof of Theorem 2.12 in [HSK99]. It is shown in [Bur98] that output feedback H_∞ control can be obtained from a solution to an estimation problem.

2.6 Linear Optimal Controls via Linear Matrix Inequality

In this section, optimal control problems for discrete linear time-invariant systems are reformulated in terms of linear matrix inequalities (LMIs). Since LMI problems are convex, it can be solved very efficiently and the global minimum is always found. We first consider the LQ control and then move to H_∞ control.

2.6.1 Infinite Horizon Linear Quadratic Control via Linear Matrix Inequality

Let us consider the infinite horizon LQ cost function as follows:

$$J_\infty = \sum_{i=0}^{\infty} \left\{ x_i^T Q x_i + u_i^T R u_i \right\},$$

where $Q > 0, R > 0$. It is noted that, unlike the standard LQ control, Q is positive-definite. The nonsingularity of Q is required to solve an LMI problem. We aim to find the control u_i which minimizes the above cost function. The main attention is focused on designing a linear optimal state-feedback control, $u_i = Hx_i$. Assume that $V(x_i)$ has the form

$$V(x_i) = x_i^T K x_i, \;\; K > 0$$

and satisfies the following inequality:

$$V(x_{i+1}) - V(x_i) \le -[x_i^T Q x_i + u_i^T R u_i] \tag{2.248}$$

Then, the system controlled by u_i is asymptotically stable and $J_\infty \le V(x_0)$. With $u_i = Hx_i$, the inequality (2.248) is equivalently rewritten as

$$x_i^T (A + BH)^T K (A + BH) x_i - x_i^T K x_i \le -x_i^T [Q + H^T R H] x_i \tag{2.249}$$

From (2.249), it is clear that (2.248) is satisfied if there exists H and K such that

$$(A + BH)^T K (A + BH) - K + Q + H^T R H \le 0 \tag{2.250}$$

Instead of directly minimizing $x_0^T K x_0$, we take an approach where its upper bound is minimized. For this purpose, assume that there exists $\gamma_2 > 0$ such that

$$x_0^T K x_0 \le \gamma_2 \tag{2.251}$$

Now the optimal control problem for given x_0 can be formulated as follows:

$$\min_{\gamma_2, K, H} \gamma_2 \quad \text{subject to (2.250) and (2.251)}$$

However, the above optimization problem does not seem easily solvable because the matrix inequalities (2.250) and (2.251) are not of LMI forms. In the following, matrix inequalities (2.250) and (2.251) are converted to LMI conditions. First, let us turn to the condition in (2.250), which can be rewritten as follows:

$$-K + \begin{bmatrix} (A+BH)^T & H^T & I \end{bmatrix} \begin{bmatrix} K^{-1} & 0 & 0 \\ 0 & R^{-1} & 0 \\ 0 & 0 & Q^{-1} \end{bmatrix}^{-1} \begin{bmatrix} (A+BH) \\ H \\ I \end{bmatrix} \le 0$$

From the Schur complement, the above inequality is equivalent to

$$\begin{bmatrix} -K & (A+BH)^T & H^T & I \\ (A+BH) & -K^{-1} & 0 & 0 \\ H & 0 & -R^{-1} & 0 \\ I & 0 & 0 & -Q^{-1} \end{bmatrix} \le 0 \tag{2.252}$$

Also from the Schur complement, (2.251) is converted to

$$\begin{bmatrix} \gamma_2 & x_0^T \\ x_0 & K^{-1} \end{bmatrix} \ge 0 \tag{2.253}$$

Pre- and post-multiply (2.252) by $\operatorname{diag}\{K^{-1}, I, I, I\}$. It should be noted that this operation does not change the inequality sign. Introducing new variables $Y \triangleq HK^{-1}$ and $S \triangleq K^{-1}$, (2.252) is equivalently changed into

$$\begin{bmatrix} -S & (AS+BY)^T & Y^T & S \\ (AS+BY) & -S & 0 & 0 \\ Y & 0 & -R^{-1} & 0 \\ S & 0 & 0 & -Q^{-1} \end{bmatrix} \le 0 \tag{2.254}$$

Furthermore, (2.253) is converted to

$$\begin{bmatrix} \gamma_2 & x_0^T \\ x_0 & S \end{bmatrix} \ge 0 \tag{2.255}$$

Now that (2.254) and (2.255) are LMI conditions, the resulting optimization problem is an infinite horizon control, which is represented as follows:

$$\min_{\gamma_2, Y, S} \gamma_2$$
$$\text{subject to (2.254) and (2.255)}$$

Provided that the above optimization problem is feasible, then $H = YS^{-1}$ and $K = S^{-1}$.

2.6.2 Infinite Horizon H_∞ Control via Linear Matrix Inequality

Consider the system

$$x_{i+1} = Ax_i + Bu_i \tag{2.256}$$
$$z_i = C_z x_i + D_{zu} u_i \tag{2.257}$$

where A is a stable matrix. For the above system, the well-known bounded real lemma (BRL) is stated as follows:

Lemma 2.13 (Bounded Real Lemma). *Let $\gamma > 0$. If there exists $X > 0$ such that*

$$\begin{bmatrix} -X^{-1} & A & B & 0 \\ A^T & -X & 0 & C_z^T \\ B^T & 0 & -\gamma W_u & D^T \\ 0 & C & D_{zu} & -\gamma W_z^{-1} \end{bmatrix} < 0 \tag{2.258}$$

then

$$\frac{\sum_{i=i_0}^{\infty} z_i^T W_z z_i}{\sum_{i=i_0}^{\infty} u_i^T W_u u_i} < \gamma^2 \tag{2.259}$$

where u_i and z_i are governed by the system (2.256) and (2.257).

Proof. The inequality (2.259) is equivalent to

$$J_{zu} = \sum_{i=0}^{\infty} \left\{ z_i^T W_z z_i - \gamma^2 u_i^T W_u u_i \right\} < 0 \tag{2.260}$$

Let us take $V(x)$ as follows:

$$V(x) = x^T K x, \quad K > 0$$

Respectively adding and subtracting $\sum_{i=0}^{\infty} \{V(x_{i+i}) - V(x_i)\}$ to and from J_{zu} in (2.260), does not make any difference to J_{zu}. Hence, it follows that

$$J_{zu} = \sum_{i=0}^{\infty} \left\{ z_i^T W_z z_i - \gamma^2 u_i^T W_u u_i + V(x_{i+1}) - V(x_i) \right\} + V(x_0) - V(x_\infty)$$

Since x_0 is assumed to be zero and $V(x_\infty) \geq 0$, we have

$$J_{zu} \leq \sum_{i=0}^{\infty} \{ z_i^T W_z z_i - \gamma^2 u_i^T W_u u_i + V(x_{i+1}) - V(x_i) \}$$

Furthermore,

$$
\begin{aligned}
&\sum_{i=0}^{\infty}\{z_i^T W_z z_i - \gamma^2 u_i^T W_u u_i + V(x_{i+1}) - V(x_i)\} \\
&= \sum_{i=0}^{\infty}\Big\{[C_z x_i + D_{zu}u_i]^T W_z [C_z x_i + D_{zu}u_i] - \gamma^2 u_i^T W_u u_i \\
&\qquad + [Ax_i + Bu_i]^T K [Ax_i + Bu_i] - x_i^T K x_i\Big\} \\
&= \sum_{i=0}^{\infty}\left\{\begin{bmatrix} x_i \\ u_i \end{bmatrix}^T \Lambda \begin{bmatrix} x_i \\ u_i \end{bmatrix}\right\}
\end{aligned}
$$

where

$$
\Lambda \triangleq \begin{bmatrix} -K + A^T K A + C_z^T W_z C_z & A^T K B + C_z^T W_z D_{zu} \\ B^T K A + D_{zu}^T W_z C_z & B^T K B + D_{zu}^T W_z D_{zu} - \gamma^2 W_u \end{bmatrix} \tag{2.261}
$$

Hence, if the 2-by-2 block matrix Λ in (2.261) is negative definite, then $J_{zu} < 0$ and equivalently the inequality (2.259) holds.

The 2-by-2 block matrix Λ can be rewritten as follows:

$$
\Lambda = \begin{bmatrix} -K & 0 \\ 0 & -\gamma^2 W_u \end{bmatrix} + \begin{bmatrix} A^T & C_z^T \\ B^T & D_{zu}^T \end{bmatrix}\begin{bmatrix} K & 0 \\ 0 & W_z \end{bmatrix}\begin{bmatrix} A & B \\ C_z & D_{zu} \end{bmatrix}
$$

From the Schur complement, the negative definiteness of Λ is guaranteed if the following matrix equality holds:

$$
\begin{bmatrix} -K & 0 & A^T & C_z^T \\ 0 & -\gamma^2 W_u & B^T & D_{zu}^T \\ A & B & -K^{-1} & 0 \\ C_z & D_{zu} & 0 & -W_z^{-1} \end{bmatrix} < 0 \tag{2.262}
$$

Define Π as follows:

$$
\Pi \triangleq \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}
$$

Pre- and post-multiplying (2.262) by Π^T and Π respectively does not change the inequality sign. Hence, the condition in (2.262) is equivalently represented by

$$
\begin{bmatrix} -K^{-1} & A & B & 0 \\ A^T & -K & 0 & C_z^T \\ B^T & 0 & -\gamma^2 W_u & D_{zu}^T \\ 0 & C_z & D_{zu} & -W_z^{-1} \end{bmatrix} < 0 \tag{2.263}
$$

Pre- and post-multiplying (2.263) by $\mathrm{diag}\{\sqrt{\gamma}I, \sqrt{\gamma}^{-1}I, \sqrt{\gamma}^{-1}I, \sqrt{\gamma}I\}$ and introducing a change of variables such that $X \triangleq \frac{1}{\sqrt{\gamma}}K$, the condition in (2.263) is equivalently changed to (2.258). This completes the proof. ∎

Using the BRL, the LMI-based H_∞ control problem can be formulated. Let us consider the system

$$\begin{aligned} x_{i+1} &= Ax_i + B_w w_i + Bu_i, \ x_0 = 0 \\ z_i &= C_z x_i + D_{zu} u_i \end{aligned}$$

As in the LMI-based LQ problem, the control is constrained to have a state-feedback, $u_i = Hx_i$. With $u_i = Hx_i$, the above system is rewritten as follows:

$$\begin{aligned} x_{i+1} &= [A + BH]x_i + B_w w_i, \quad x_0 = 0 \\ z_i &= [C_z + D_{zu} H]x_i \end{aligned}$$

According to the BRL, H which guarantees $\|G_{cl}(z)\|_\infty < \gamma_\infty$ should satisfy, for some $X > 0$,

$$\begin{bmatrix} -X^{-1} & (A+BH) & B_w & 0 \\ (A+BH)^T & -X & 0 & (C_z + D_{zu}H)^T \\ B_w^T & 0 & -\gamma_\infty I & 0 \\ 0 & (C_z + D_{zu}H) & 0 & -\gamma_\infty I \end{bmatrix} < 0 \qquad (2.264)$$

where $G_{cl}(z) = [C_z + D_{zu}H](zI - A - BH)^{-1}B_w$. Pre- and post-multiplying (2.264) by $\mathrm{diag}\{I, X^{-1}, I, I\}$ and introducing a change of variables such that $S_\infty \triangleq X^{-1}$ and $Y \triangleq HX^{-1}$ lead to

$$\begin{bmatrix} -S_\infty & (AS_\infty + BY) & B_w & 0 \\ (AS_\infty + BY)^T & -S_\infty & 0 & (C_z S_\infty + D_{zm} Y)^T \\ B_w^T & 0 & -\gamma_\infty I & 0 \\ 0 & (C_z S_\infty + D_{zu} Y) & 0 & -\gamma_\infty I \end{bmatrix} < 0 \quad (2.265)$$

Provided that the above LMI is feasible for some given γ_∞, H_∞ state-feedback control guaranteeing $\|G_{cl}(z)\|_\infty < \gamma_\infty$ is given by

$$H = YS_\infty^{-1}$$

In this case, we can obtain the infinite horizon H_∞ control via LMI, which minimizes γ_∞ by solving the following optimization problem:

$$\min_{\gamma_\infty, Y, S_\infty} \ \gamma_\infty \quad \text{subject to (2.265)}$$

2.7 * H_2 Controls

Since LQ regulator and LQG control problems are studied extensively in this book, H_2 controls and H_2 filters are introduced in limited problems and are only briefly summarized without proofs in this section.

To manipulate more general problems, it is very useful to have a general system with the input, the disturbance, the controlled output, and the measure output given by

$$
\begin{aligned}
x_{i+1} &= Ax_i + B_w w_i + Bu_i \\
y_i &= Cx_i + D_w w_i \\
z_i &= C_z x_i + D_{zw} w_i + D_{zu} u_i
\end{aligned} \tag{2.266}
$$

The standard H_2 problem is to find a proper, real rational controller $u(z) = K(z)y(z)$ which stabilizes the closed-loop system internally and minimizes the H_2 norm of the transfer matrix T_{zw} from w_i to z_i.

The H_2 norm can be represented as

$$
\begin{aligned}
\|T_{zw}(e^{jw})\|_2^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\{T_{zw}^*(e^{jw})T_{zw}(e^{jw})\}dw = \sum_{k=i_0}^{\infty} \mathrm{tr}\{H_{k-i_0}^* H_{k-i_0}\} \\
&= \sum_{l=1}^{m}\sum_{i=-\infty}^{\infty} z_i^{lT} z_i^l
\end{aligned} \tag{2.267}
$$

where

$$
T_{zw}(e^{jw}) = \sum_{k=0}^{\infty} H_k e^{-jwk}
$$

and A^* is a complex conjugate transpose of A and z^l is an output resulting from applying unit impulses to lth input. From (2.267), we can see that the H_2 norm can be obtained from applying unit impulses to each input. We should require the output to settle to zero before applying an impulse to the next input. In the case of single input systems, the H_2 norm is obtained by the driving unit impulse once, i.e. $\|T_{zw}(e^{jw})\|_2 = \|z\|_2$.

The H_2 norm can be given another interpretation for stochastic systems. The expected power in the error signal z_i is then given by

$$
\begin{aligned}
E\{z_i^T z_i\} = \mathrm{tr}[E\{z_i z_i^T\}] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\{T_{zw}(e^{jw})T_{zw}^*(e^{jw})\}\ dw \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\{T_{zw}^*(e^{jw})T_{zw}(e^{jw})\}\ dw
\end{aligned}
$$

where the second equality comes from Theorem C.4 in Appendix C.

Thus, by minimizing the H_2 norm, the output (or error) power of the generalized system, due to a unit intensity white noise input, is minimized.

It is noted that for the given system transfer function

$$
G(z) \triangleq \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(zI - A)^{-1}B + D
$$

$\|G(z)\|_2$ is obtained by

$$
\|G(z)\|_2^2 = \mathrm{tr}(D^T D + B^T L_o B) = \mathrm{tr}(DD^T + CL_c C^T) \tag{2.268}
$$

where L_c and L_o are the controllability and observability Gramians

$$AL_cA^T - L_c + BB^T = 0 \tag{2.269}$$
$$A^TL_oA - L_o + C^TC = 0 \tag{2.270}$$

The H_2 norm has a number of good mathematical and numerical properties, and its minimization has important engineering implications. However, the H_2 norm is not an induced norm and does not satisfy the multiplicative property.

It is assumed that the following things are satisfied for the system (2.266):

(i) (A, B) is stabilizable and (C, A) is detectable,
(ii) D_{zu} is of full column rank with $\begin{bmatrix} D_{zu} & D_\perp \end{bmatrix}$ unitary and D_w is full row with $\begin{bmatrix} D_w \\ \tilde{D}_\perp \end{bmatrix}$ unitary,
(iii) $\begin{bmatrix} A - e^{j\theta}I & B \\ C_z & D_{zu} \end{bmatrix}$ has full column rank for all $\theta \in [0\ 2\pi]$,
(iv) $\begin{bmatrix} A - e^{j\theta}I & B_w \\ C & D_w \end{bmatrix}$ has full rank for all $\theta \in [0\ 2\pi]$.

Let $X_2 \geq 0$ and $Y_2 \geq 0$ be the solutions to the following Riccati equations:

$$A_x^*(I + X_2BB^T)^{-1}X_2A_x - X_2 + C_z^TD_\perp D_\perp^TC_z = 0 \tag{2.271}$$
$$A_y(I + Y_2C^TC)^{-1}Y_2A_y^T - Y_2 + B_wD_\perp^TD_\perp B_w^T = 0 \tag{2.272}$$

where

$$A_x = A - BD_{zu}^TC_z, \ A_y = A - B_wC_w^TC \tag{2.273}$$

Note that the stabilizing solutions exist by the assumptions (iii) and (iv). The solution to the standard H_2 problem is given by

$$\hat{x}_{i+1} = (\hat{A}_2 - BL_0C)\hat{x}_i - (L_2 - BL_0)y_i \tag{2.274}$$
$$u_i = (F_2 - L_0C)\hat{x}_i + L_0y_i \tag{2.275}$$

where

$$\begin{aligned} F_2 &= -(I + B^TX_2B)^{-1}(B^TX_2A + D_{zu}^TC_z) \\ L_2 &= -(AY_2C^T + B_wC_w^T)(I + CY_2C^T)^{-1} \\ L_0 &= (F_2Y_2C^T + F_2C_w^T)(I + CY_2C^T)^{-1} \\ \hat{A}_2 &= A + BF_2 + L_2C \end{aligned}$$

The well-known LQR control problems can be seen as a special H_2 problem. The standard LQR control problem is to find an optimal control law $u \in l_2[0\ \infty]$ such that the performance criterion $\sum_{i=0}^{\infty} z_i^Tz_i$ is minimized in the following system with the impulse input $w_i = \delta_i$:

$$\begin{aligned} x_{i+1} &= Ax_i + x_0 w_i + Bu_i \\ z_i &= C_z x_i + D_{zu} u_i \\ y_i &= x_i \end{aligned} \tag{2.276}$$

where w_i is a scalar value, $C_z^T D_{zu} = O$, $C_z^T C_z = Q$, and $D_{zu}^T D_{zu} = R$. Note that B_w in (2.266) corresponds to x_0 in (2.276). Here, the H_2 performance criterion becomes an LQ criterion.

The LQG control problem is an important special case of the H_2 optimal control for the following system:

$$\begin{aligned} x_{i+1} &= Ax_i + \begin{bmatrix} GQ_w^{\frac{1}{2}} & O \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix} + Bu_i \\ y_i &= Cx_i + \begin{bmatrix} O & R_v^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix} \\ z_i &= C_z x_i + D_{zu} u_i \end{aligned} \tag{2.277}$$

where $C_z^T D_{zu} = O$, $C_z^T C_z = Q$, and $D_{zu}^T D_{zu} = R$. The LQG control is obtained so that the H_2 norm of the transfer function from w and v_i to z is minimized. It is noted that, according to (C.14) in Appendix C, the performance criterion (2.267) for the system (2.277) can be considered by observing the steady-state mean square value of the controlled output

$$E\left[\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} z_i^T z_i\right] = E\left[\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \left(x_i^T Q x_i + u_i^T R u_i\right)\right]$$

when the white Gaussian noises with unit power are applied. It is noted that w_i and v_i can be combined into one disturbance source w_i as in (2.266).

The H_2 filter problem can be solved as a special case of the H_2 control problem. Suppose a state-space model is described by the following:

$$\begin{aligned} x_{i+1} &= Ax_i + B_w w_i \\ y_i &= Cx_i + D_w w_i \end{aligned} \tag{2.278}$$

The H_2 filter problem is to find an estimate $\hat{x}_i$ of x_i using the measurement of y_i so that the H_2 norm from w_i to $x_i - \hat{x}_i$ is minimized. The filter has to be causal so that it can be realized.

The H_2 filter problem can be regarded as the following control problem:

$$\begin{aligned} x_{i+1} &= Ax_i + B_w w_i + 0 \times \hat{x}_i \\ z_i &= x_i - \hat{x}_i \\ y_i &= Cx_i + D_w w_i \end{aligned} \tag{2.279}$$

where the following correspondences to (2.266) hold

$$\begin{aligned} u_i &\longleftarrow \hat{x}_i \\ B &\longleftarrow 0 \\ C_z &\longleftarrow I \\ D_{zu} &\longleftarrow -I \end{aligned}$$

H_2/H_∞ Controls Based on Mixed Criteria

Each performance criterion has its own advantages and disadvantages, so that there are trade-offs between them. In some cases we want to adopt two or more performance criteria simultaneously in order to satisfy specifications. In this section, we introduce two kinds of controls based on mixed criteria. It is noted that an LQ control is a special case of H_2 controls. Here, the LQ control is used for simplicity.

1. Minimize the H_2 norm for a fixed guaranteed H_∞ norm such that

$$\min_{\gamma_2, Y, S} \gamma_2 \tag{2.280}$$

subject to

$$\begin{bmatrix} \gamma_2 & x_0^T \\ x_0 & S \end{bmatrix} \geq 0 \tag{2.281}$$

$$\begin{bmatrix} -S & (AS+BY)^T & Y^T & S \\ (AS+BY) & -S & 0 & 0 \\ Y & 0 & -R^{-1} & 0 \\ S & 0 & 0 & -Q^{-1} \end{bmatrix} \leq 0 \tag{2.282}$$

$$\begin{bmatrix} -S & (AS+BY) & B_w & 0 \\ (AS+BY)^T & -S & 0 & (C_zS+D_{zu}Y)^T \\ B_w^T & 0 & -\gamma_\infty I & 0 \\ 0 & (C_zS+D_{zu}Y) & 0 & -\gamma_\infty I \end{bmatrix} < 0 \tag{2.283}$$

From Y and S, the state feedback gain is obtained, i.e. $H = YS^{-1}$.

2. Minimize the H_∞ norm for a fixed guaranteed H_2 norm such that

$$\min_{\gamma_\infty, Y, S} \gamma_\infty \tag{2.284}$$

subject to
(2.281), (2.282), (2.283).

The state feedback gain is obtained from Y and S, i.e. $H = YS^{-1}$.

2.8 References

The material presented in this chapter has been established for a long time and is covered in several excellent books. The subject is so large that it would be a considerable task to provide comprehensive references.

Therefore, in this chapter, some references will be provided so that it is enough to understand the contents.

Dynamic programming and the minimum principle of Section 2.2 are discussed in many places. For general systems, dynamic programming and the minimum principle of Pontryagin appeare in [BD62, BK65, Bel57] and in [PBGM62] respectively. For a short review, [Kir70] is a useful reference for the minimum criterion in both dynamic programming and the minimum principle. For a treatment of the minimax criterion, see [Bur98] for the minimax principle and [BB91] [BLW91] [KIF93] for dynamic programming. Readers interested in rigorous mathematics are referred to [Str68].

The literature on LQ controls is vast and old. LQR for tracking problems as in Theorems 2.1, 2.2, and 2.4 is via dynamic programming [AM89] and via the minimum principle [BH75] [KS72]. In the case of a fixed terminal with a reference signal, the closed-loop solution is first introduced in this book as in Theorem 2.3.

The H_∞ control in Section 2.3.2 is closely related to an LQ difference game. The books by [BH75] and [BO82] are good sources for results on game theories. [BB91] is a book on game theories that deals explicitly with the connections between game theories and H_∞ control. The treatment of the finite horizon state feedback H_∞ control in this book is based on [LAKG92].

The Kalman filters as in Theorem 2.6 can be derived in many ways for stochastic systems. The seminal papers on the optimal estimation are [KB60] and [KB61].

The perfect square expression of the quadratic cost function in Lemmas 2.8 and 2.9 appeared in [Lew86b, Lew86a]. A bibliography of LQG controls is compiled by [MG71]. A special issue of the *IEEE Transactions on Automatic Control* was devoted to LQG controls in 1971 [Ath71]. Most of the contents about LQG in this book originate from the text of [Lew86b]. The LQG separation theorem appeared in [Won68]. The perfect square expression of the H_∞ cost function in Theorem 2.11 appeared in [GL95].

Even though finite horizon and infinite horizon LQ controls are obtained analytically from a Riccati approach, we also obtain them numerically from an LMI in this book, which can be useful for constrained systems. Detailed treatments of LMI can be found in [BGFB94, GNLC95]. The LMIs for LQ and H_∞ controls in Sections 2.6.1 and 2.6.2 appear in [GNLC95]. The bounded real lemma in Section 2.6.2 is investigated in [Yak62], [Kal63], and [Pop64], and also in a book by [Bur98].

The general H_2 problem and its solution are considered well in the frequency domain in [ZDG96]. This work is based on the infinite horizon. In [BGFB94], H_2 and H_∞ controls are given in LMI form, which can be used for the H_2/H_∞ mixed control. The work by [BH89] deals with the problem requiring the minimization of an upper bound on the H_2 norm under an H_∞ norm constraint.

2.9 Problems

2.1. Consider the system

$$x_{i+1} = \alpha x_i + u_i - \frac{u_i^2}{M - x_i} \tag{2.285}$$

where $0 < \alpha < 1$ and $x_{i_0} < M$. In particular, we want to maximize

$$J = \beta^{i_f - i_0} c x_{i_f} + \sum_{i=i_0}^{i_f - 1} [p x_i - u_i] \beta^{i - i_0} \tag{2.286}$$

where $p > 0$, $0 < \beta < 1$, and $c > 0$. Find an optimal control u_i so that (2.286) is maximized.

2.2. Consider a nonlinear system $x_{k+1} = f(x_k, u_k)$ with constraints given by $\phi(x_k) \geq 0$ and the performance criterion (2.2).

(1) Show that above $\phi(x_k) \geq 0$ can be represented by an extra state variable $x_{n+1,k+1}$ such as

$$x_{n+1,k+1} = x_{n+1,k} + \{[\phi_1(x_k)]^2 \tilde{u}(-\phi_1(x_k)) + \cdots + [\phi_l(x_k)]^2 \tilde{u}(-\phi_l(x_k))\}$$

where $\tilde{u}(\cdot)$ is a unit function given by $\tilde{u}(x) = 1$ only for $x > 0$ and 0 otherwise with

$$x_{n+1,i_0} = 0, \quad x_{n+1,i_f} = 0$$

(2) Using the minimum principle, find the optimal control so that the system

$$x_{1,k+1} = 0.4 x_{2,k} \tag{2.287}$$
$$x_{2,k+1} = -0.2 x_{2,k} + u_k \tag{2.288}$$

is to be controlled to minimize the performance criterion

$$J = \sum_{k=0}^{3} 0.5 [x_{1,k}^2 + x_{2,k}^2 + u_k^2] \tag{2.289}$$

The control and states are constrained by

$$-1 \leq u_k \leq 1 \tag{2.290}$$
$$-2 \leq x_{2,k} \leq 2 \tag{2.291}$$

2.3. Suppose that a man has his initial savings S and lives only on interest that comes from his savings at a fixed rate. His current savings x_k are therefore governed by the equation

$$x_{k+1} = \alpha x_k - u_k \tag{2.292}$$

where $\alpha > 1$ and u_k denotes his expenditure. His immediate enjoyment due to expenditure is $u_k{}^{\frac{1}{2}}$. As time goes on, the enjoyment is diminished as fast as β^k, where $|\beta| < 1$. Thus, he wants to maximize

$$J = \sum_{k=0}^{N} \beta^k u_k^{\frac{1}{2}} \tag{2.293}$$

where S, α, and β are set to 10, 1.8, and 0.6, respectively. Make simulations for three kinds of planning based on Table 1.1. For the long-term planning, use $N = 100$. For the periodic and short-term plannings, use $N = 5$ and the simulation time is 100. Using the minimum principle, find optimal solutions u_i analytically, not numerically.

2.4. Consider the following general nonlinear system:

$$x_{i+1} = f(x_i) + g(x_i)w_i, \;\; z_i = h(x_i) + J(x_i)w_i \tag{2.294}$$

(1) If there exists a nonnegative function $V : \Re^n \to \Re$ with $V(0) = 0$ such that for all $w \in \Re^p$ and $k = 0, 1, 2 \cdots$

$$V(x_{k+1}) - V(x_k) \leq \gamma^2 \|w_k\|^2 - \|z_k\|^2 \tag{2.295}$$

show that the following inequality is satisfied:

$$\sum_{k=0}^{N} ||z_k||^2 \leq \gamma^2 \sum_{k=0}^{N} ||w_k||^2$$

Conversely, show that a nonnegative function $V : \Re^n \to \Re$ with $V(0) = 0$ exists if the H_∞ norm of the system is less than γ^2.
(2) Suppose that there exists a positive definite function $V(x)$ satisfying

$$g^T(0)\frac{\partial^2 V}{\partial^2 x}(0)g(0) + J^T(0)J(0) - \gamma^2 < 0 \tag{2.296}$$

$$\begin{aligned} 0 &= V(f(x) + g(x)\alpha(x)) - V(x) \\ &+\frac{1}{2}(||h(x) + J(x)\alpha(x)|| - \gamma^2||\alpha(x)||) \end{aligned} \tag{2.297}$$

where $\alpha(x)$ is a unique solution of

$$\frac{\partial V}{\partial \alpha(x)}(x)g(x) + \alpha(x)^T(J^T(x)J(x) - \gamma^2) = -h^T(x)J(x) \tag{2.298}$$

and the systems is observable, i.e.

$$z_k|_{w_k=0} = h(x_k) = 0 \to \lim_{k\to\infty} x_k = 0 \tag{2.299}$$

Show that the system $x_{i+1} = f(x_i)$ is stable.

2.5. We consider a minimum time performance criterion in which the objective is to steer a current state into a specific target set in minimum time.

For the system

$$x_{i+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_i + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_i \tag{2.300}$$

the performance criterion is given as

$$J = \sum_{i=i_0}^{i_f-1} 1 = i_f - i_0 \tag{2.301}$$

where $|u_i| \leq 1$. Find the control u_i to bring the state from the initial point $x_{i_0} = [1 \ 4]^T$ to the origin in the minimum time.

2.6. An optimal investment plan is considered here. Without any external investment, the manufacturing facilities at the next time $k+1$ decrease in proportion to the manufacturing facilities at the current time k. In order to increase the manufacturing facilities, we should invest money. Letting x_k and u_k be the manufacturing facilities at time k and the investment at the time k respectively, we can construct the following model:

$$x_{k+1} = \alpha x_k + \gamma u_k \tag{2.302}$$

where $|\alpha| < 1$, $\gamma > 0$, and x_0 are given. Assume that manufacturing facilities are worth the value proportional to the investment and the product at the time k is proportional to the manufacturing facilities at time k. Then, the profit can be represented as

$$J = \beta x_N + \sum_{i=0}^{N-1} (\beta x_i - u_i) \tag{2.303}$$

The investment is assumed to be nonnegative and bounded above, i.e. $0 \leq u_i \leq \bar{u}$. Obtain the optimal investment with respect to α, β, γ, and N.

2.7. Let a free body obey the following dynamics:

$$y_{i+1} = y_i + v_i \tag{2.304}$$

$$v_{i+1} = v_i + u_i \tag{2.305}$$

with y_i the position and v_i the velocity. The state is $x_i = [y_i \ v_i]^T$. Let the acceleration input u_i be constrained in magnitude by

$$|u_i| \leq 1 \tag{2.306}$$

Suppose the objective is to determine a control input to bring any given initial state (y_{i_0}, v_{i_0}) to the origin so that

$$\begin{bmatrix} y_{i_f} \\ v_{i_f} \end{bmatrix} = 0 \tag{2.307}$$

The control should use minimum fuel, so let

$$J(i_0) = \sum_{i=i_0}^{i_f} |u_i| \tag{2.308}$$

(1) Find the minimum-fuel control law to drive any x_{i_0} to the origin in a given time $N = i_f - i_0$ if $|u_i| \le 1$.
(2) Draw the phase-plane trajectory. N, y_{i_0}, and v_{i_0} are set to 35, 10, and 10 respectively.

2.8. Consider a performance criterion with $Q = R = 0$ and a positive definite Q_f. The control can be given (2.43)–(2.45) with the inverse replaced by a pseudo inverse.

(1) Show that the solution to Riccati Equation (3.47) can be represented as

$$K_{i_f-k} = (Q_f^{\frac{1}{2}} A^k)^T \left[I - Q_f^{\frac{1}{2}} W_k (Q_f^{\frac{1}{2}} W_k)^T [Q_f^{\frac{1}{2}} W_k (Q_f^{\frac{1}{2}} W_k)^T]^\dagger\right] (Q_f^{\frac{1}{2}} A^k). \tag{2.309}$$

where $W_k = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{k-1}B]$ for $k = 1, 2, ..., N-1$ with $W_0 = 0$ and $A^\dagger$ is a pseudo inverse of A.
(2) In the deadbeat control problem, we desire that $x_{i_f} = 0$; this can happen only if a performance criterion is equal to zero, i.e. if $K_{i_0} = 0$. Show that K_{i_0} can be zero if the following condition holds for some k:

$$\mathrm{Im}(A^k) \subset \mathrm{Im}(W_k) \tag{2.310}$$

where $\mathrm{Im}(M)$ is the image of the matrix M.
(3) Show that (2.310) is satisfied if the system is controllable.

2.9. Consider the following performance criterion for the system (2.27):

$$J(x^r, u) = \sum_{i=i_0}^{i_f-1} \begin{bmatrix} x_i \\ u_i \end{bmatrix}^T \begin{bmatrix} Q & M \\ M^T & R \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} + x_{i_f}^T Q_f x_{i_f}$$

where

$$Q = Q^T \ge 0, \quad \begin{bmatrix} Q & M \\ M^T & R \end{bmatrix} \ge 0, \quad R = R^T > 0$$

Show that the optimal control is given as

$$\begin{aligned} u_i &= -K_i x_i \\ K_i &= (B^T S_{i+1} B + R)^{-1} (B^T S_{i+1} A + M^T) \\ S_i &= A^T S_{i+1} A - (B^T S_{i+1} A + M^T)^T (B^T S_{i+1} B + R)^{-T} (B^T S_{i+1} A + M^T) \\ &\quad + Q \end{aligned}$$

where $S_{i_f} = Q_f$.

2.10. Show that the general tracking control (2.103) is reduced to the simpler tracking control (2.79) if Q in (2.103) becomes zero.

2.11. Consider the minimum energy performance criterion given by

$$J = \sum_{i=i_0}^{i_f-1} u_i^T R u_i \tag{2.311}$$

for the system (2.27). Find the control u_i that minimizes (2.311) and satisfies the constraints $|u_i| \leq \bar{u}$ and $x_{i_f} = 0$.

2.12. Consider an optimal control problem on $[i_0 \ i_f]$ for the system (2.27) with the LQ performance criterion

$$J(x, u) = \sum_{i=i_0}^{i_f-1} (x_i^T Q x_i + u_i^T R u_i) \tag{2.312}$$

(1) Find the optimal control u_k subject to $Cx_{i_f} + b = 0$.
(2) Find the optimal control u_k subject to $x_{i_f}^T P x_{i_f} \leq 1$, where P is a symmetric positive definite matrix.

2.13. Consider the following performance criterion for the system (2.120):

$$J(x^r, u) = \sum_{i=i_0}^{i_f-1} \left\{ \begin{bmatrix} x_i \\ u_i \end{bmatrix}^T \begin{bmatrix} Q & M \\ M^T & R \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} - \gamma^2 w_i^T w_i \right\} + x_{i_f}^T Q_f x_{i_f},$$

where

$$Q = Q^T \geq 0, \quad \begin{bmatrix} Q & M \\ M^T & R \end{bmatrix} \geq 0, \quad R = R^T > 0. \tag{2.313}$$

Derive the H_∞ control.

2.14. Derive the last term h_{i,i_f} in the optimal cost (2.148) associated with the H_∞ control.

2.15. Consider the stochastic model (2.161) and (2.162) where w_i and v_i are zero-mean, white noise sequences with variance given by

$$\mathrm{E}\left\{ \begin{bmatrix} w_i \\ v_i \end{bmatrix} \begin{bmatrix} w_j^T & v_j^T \end{bmatrix} \right\} = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12} & \Xi_{22} \end{bmatrix} \delta_{i-j}$$

(1) Show that (2.161) and (2.162) are equivalent to the following model:

$$\begin{aligned} x_{i+1} &= \bar{A} x_i + B u_i + G \Xi_{12} \Xi_{22}^{-1} y_i + G \xi_i \\ y_i &= C x_i + v_i \end{aligned}$$

where

$$\bar{A} = A - G \Xi_{12} \Xi_{22}^{-1} C, \quad \xi_i = w_i - \Xi_{12} \Xi_{22}^{-1} v_i$$

(2) Find $\mathrm{E}\{\xi_i v_i^T\}$.
(3) Show that the controllability and observability of the pairs $\{\bar{A}, B\}$ and $\{\bar{A}, C\}$ are guaranteed by the controllability and observability of the pairs $\{A, B\}$ and $\{A, C\}$, respectively.
(4) Find the Kalman filter $\hat{x}_{k+1|k}$.

2.16. Consider the following system:

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xi_k$$
$$y_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_k + \theta_k$$

where ξ_k and θ_k are zero-mean Gaussian white noises with covariance 1 and μ.

(1) Express the covariance of the state estimation error P_k as a function of μ.
(2) Calculate the gain matrix of the Kalman filter.
(3) Calculate and plot the poles and zeros of the closed-loop system.

2.17. Consider the LQG problem

$$\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} + \sqrt{\rho} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{2.314}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.315}$$

for the following performance criterion:

$$J = \sum_{i=i_0}^{i_f - 1} \rho(x_1 + x_2)^2 + u_i^2 \tag{2.316}$$

Discuss the stability margin, such as gain and phase margins, for steady-state control.

2.18. Consider a controllable pair $\{A, B\}$ and assume A does not have unit-circle eigenvalues. Consider also arbitrary matrices $\{Q, S, R\}$ of appropriate dimensions and define a Popov function

$$S_y(z) = \begin{bmatrix} B^T(zI - A^T)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - A)^{-1}B \\ I \end{bmatrix} \tag{2.317}$$

where the central matrix is Hermitian but may be indefinite. The KYP lemma[KSH00] can be stated as follows.
The following three statements are all equivalent:

(a) $S_y(e^{jw}) \geq 0$ for all $w \in [-\pi \ , \ \pi]$.

(b) There exists a Hermitian matrix P such that

$$\begin{bmatrix} Q - P + A^T PA & S + A^T PB \\ S^T + B^T PA & R + B^T PB \end{bmatrix} \geq 0 \tag{2.318}$$

(c) There exist an $n \times n$ Hermitian matrix P, a $p \times p$ matrix $R_e \geq 0$, and an $n \times p$ matrix K_p, such that

$$\begin{bmatrix} Q - P + A^T PA & S + A^T PB \\ S^T + B^T PA & R + B^T PB \end{bmatrix} = \begin{bmatrix} K_p \\ I \end{bmatrix} R_e \begin{bmatrix} K_p^T & I \end{bmatrix} \tag{2.319}$$

Derive the bounded real lemma from the above KYP lemma and compare it with the LMI result in this section.

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