2 Fuzzy Logic

2.1 Fuzzy Sets

Fuzzy set theory and its attendant *fuzzy logic* were developed by Lotfi Zadeh in 1965 to handle semantic and subjective ambiguity. In classical logic, the number 300 is an integer, whereas 300.7 is not. The same number, however, could be considered large, small, very large, very small, and so on depending on context and subjective opinion. Therefore, the number 300 could be considered large to a certain degree, very large to another, and so on. We then have various *linguistic values* of one *linguistic variable*, which are true to some degree. This degree, subjective as it may be, varies from 0 to 1.

In classical set theory, an element of a set either belongs or does not belong to the set. In fuzzy set theory, an element belongs with a *membership grade* in the interval [0, 1]. All membership grades together form the *membership function*. A classical set is often called *crisp* as opposed to fuzzy.

Definition 2.1. A set is a collection of elements or members. A set may be an element of another set.

Definition 2.2. Let X be a set of elements x. A fuzzy set A is a collection of ordered pairs $(x, \mu_A(x))$ for $x \in X$. X is called the *universe of discourse* and $\mu_A(x)$: $X \rightarrow [0, 1]$ is the membership function.

The function $\mu_A(x)$ provides the degree of fulfillment of x in X. When X is countable, the fuzzy set A is represented as

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \ldots + \mu_A(x_n)/x_n.$$

This is a common notation in the context of fuzzy sets. It simply states the elements x_i of X and the corresponding membership grades.

Example 2.1

Consider the temperature of a patient in degrees Celsius. Let $X = \{36.5, 37, 37.5, 38, 38.5, 39, 39.5\}$. The fuzzy set A = "High temperature" may be defined

$$A = \{\mu_A(x)/x \mid x \in X\}$$

$$= 0/36.5 + 0/37 + 0.1/37.5 + 0.5/38 + 0.8/38.5 + 1/39 + 1/39.5,$$

where the numbers 0, 0.1, 0.5, 0.8, and 1 express the degree to which the corresponding temperature is high.

Definition 2.3. The support of a fuzzy set A is the crisp set of all elements of X with nonzero membership in A, or symbolically

$$S(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

Example 2.2

Take Example 2.1. $S(A) = \{37.5, 38, 38.5, 39, 39.5\}$.

Definition 2.4. The set of all elements of X with membership in A at least α is called the α-level set, or symbolically

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

Definition 2.5. A fuzzy set A is said to be *convex* if the membership function is quasiconcave; that is, $\forall x_1, x_2 \in X$, and $\lambda \in [0, 1]$, the following is true:

$$\mu_A[\lambda x_1 + (1 - \lambda)x_2] \geq \min[\mu_A(x_1), \mu_A(x_2)].$$

Definition 2.6. The *height* of a fuzzy set A on X is defined as

$$h(A) = \sup_{x \in X} \mu_A(x).$$

If $h(A) = 1$, A is called *normal*, otherwise *subnormal*.

Definition 2.7. The *nucleus* of a fuzzy set A is the set of values x for which $\mu_A(x) = 1$.

Example 2.3

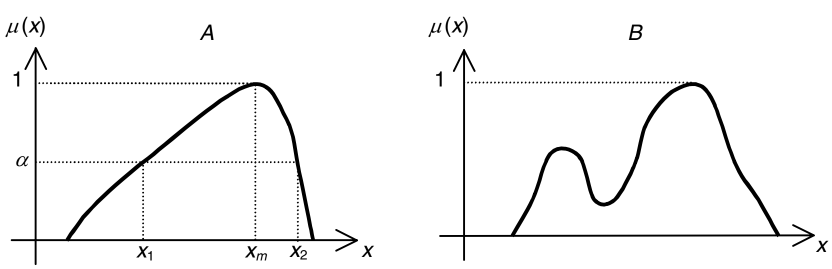

Figure 2.1. Convex-nonconvex fuzzy sets.

In Figure 2.1, A is convex but B is nonconvex. The α-level set of A is the set of $x \in [x_1, x_2]$, the height is $h(A) = 1$, and the nucleus is $\{x_m\}$.

Definition 2.8. A fuzzy number A is a fuzzy set in the reals R for which the following are true:

a. A is normal ($\exists x$: $\mu_A(x) = 1$)
b. A is convex
c. A is upper semicontinuous
d. A has bounded support

Example 2.4

The following are examples of fuzzy numbers.

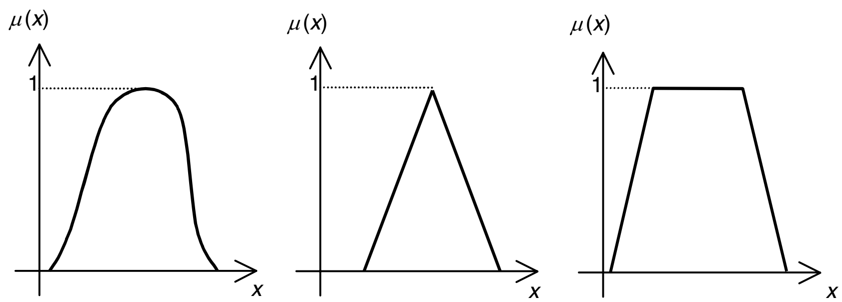

Figure 2.2. Fuzzy numbers.

2.2 Operations of Fuzzy Sets

The basic notions concerning operations on crisp sets will now be extended to fuzzy sets.

Definition 2.9. Two fuzzy sets A and B in X are equal if $\mu_A(x) = \mu_B(x)$, $\forall x \in X$. We write $A = B$.

Definition 2.10. A fuzzy set A in X is a subset of another fuzzy set B also in X if $\mu_A(x) \leq \mu_B(x)$, $\forall x \in X$.

The following definitions are concerned with the complement, the union, and the intersection of fuzzy sets as defined by Zadeh. It should be stressed that these definitions, intuitively appealing as they may be, are by no means unique because of the nature of fuzzy sets. Others have proposed different definitions.

Definition 2.11. The following membership functions are defined:

a. Complement $\overline{A}$ of a fuzzy set A in X

$$\mu_{\overline{A}} = 1 - \mu_A(x)\ ,\ x \in X.$$

b. Union $A \cup B$ of two fuzzy sets in X

$$\mu_{A \cup B} = \max[\mu_A(x),\ \mu_B(x)],\ x \in X.$$

c. Intersection $A \cap B$ of two fuzzy sets in X

$$\mu_{A \cap B} = \min[\mu_A(x),\ \mu_B(x)],\ x \in X.$$

Example 2.5

In the context of Example 2.1 let us define a new fuzzy set B = “Dangerous temperature” as $B = \{0/37.5,\ 0.1/38,\ 0.2/38.5,\ 0.5/39,\ 0.8/39.5,\ 1/40\}$. According to Definition 2.11, we have

$A \cup B$ = "High or dangerous temperature"

$= 0/36.5 + 0/37 + 0.1/37.5 + 0.5/38 + 0.8/38.5 + 1/39 + 1/39.5 + 1/40.$

$A \cap B$ = "High and dangerous temperature"

$= 0/36.5 + 0/37 + 0/37.5 + 0.1/38 + 0.2/38.5 + 0.5/39 +.0.8/39.5$

$+ 1/40.$

$\overline{A}$ = "Not high temperature"

$= 1/36.5 + 1/37 + 0.9/37.5 + 0.5/38 + 0.2/38.5 + 0/39 + 0/39.5.$

The definitions of an intersection and union can be developed from a more general point of view. An intersection may be defined via a *t-norm*.

Definition 2.12. A t-norm is a bivariate function t: $[0, 1]\times[0, 1]\rightarrow[0, 1]$ satisfying:

a. $t(0, 0) = 0$
b. $t(x, 1) = x$
c. $t(x, y) \le t(w, z)$ if $x \le w$ and $y \le z$ (monotonicity)
d. $t(x, y) = t(y, x)$ (symmetry)
e. $t[x, t(y, z)] = t[t(x, y), z]$ (associativity)

This definition provides the tools of combining two membership functions to find the membership function of $A \cap B$. For the union $A \cup B$, we have correspondingly the definition of the *t-conorm* or *s-norm*.

Definition 2.13. A t-conorm is a bivariate function c: $[0, 1]\times[0, 1]\rightarrow[0, 1]$ satisfying:

a. $c(1, 1) = 1$
b. $c(x, 0) = x$
c. $c(x, y) \le c(w, z)$ if $x \le w$ and $y \le z$ (monotonicity)
d. $c(x, y) = c(y, x)$ (symmetry)
e. $c[x, c(y, z)] = c[c(x, y), z]$ (associativity)

From these definitions, for two fuzzy sets $\mu_A(x)$ and $\mu_B(x)$, we obtain $\mu_{A\cap B}(x) = t[\mu_A(x), \mu_B(x)]$ and $\mu_{A\cup B}(x) = c[\mu_A(x), \mu_B(x)]$.

Example 2.6

The following are examples of t-norms and t-conorms.

Name	$t(x, y)$ (intersection)	$c(x, y)$ (union)
Algebraic product-sum	$x\,y$	$x + y - x\,y$
Hamacher product-sum	$\dfrac{xy}{x + y - xy}$	$\dfrac{x + y - 2xy}{1 - xy}$
Einstein product-sum	$\dfrac{xy}{1 + (1 - x)(1 - y)}$	$\dfrac{x + y}{1 + xy}$
Bounded difference product-sum	$\max(0, x + y - 1)$	$\min(1, x + y)$
Dubois-Prade $0 \le p \le 1$	$\dfrac{xy}{\max(x, y, p)}$	$1 - \dfrac{(1 - x)(1 - y)}{\max[(1 - x), (1 - y), p]}$
Minimum-maximum	$\min(x, y)$	$\max(x, y)$

It is worth noting that, contrary to what holds in set theory, when A is a fuzzy set in X, then $A \cup \overline{A} \neq X$ and $A \cap \overline{A} \neq \emptyset$ because it is not certain where A ends and $\overline{A}$ begins. This is the fundamental reason that places probability and fuzzy sets apart, although both handle uncertainty. Probability is suitable for a different kind of uncertainty than fuzzy sets, and in our opinion, the debate about which discipline is "better" or "correct" is rather beside the point. Each of them performs its own scientific function successfully within its capabilities and limitations. Below we outline some of the differences between probability and fuzzy sets.

1. In probability, an event is a crisp subset of a σ-algebra and the uncertainty revolves about the odds of its occurrence. For example, the probability of being 1.75 m tall, or $P(\text{height} = 1.75)$, concerns the frequency of the relevant event. In fuzzy set theory, events do not form σ-algebras. A pertinent question in this context would be "to what degree is 1.75 m tall?"

2. Given a probability space $(\Omega, \mathcal{F}, P)$, where Ω is the universe, $\mathcal{F}$ a σ-algebra of events, P a probability measure, and mutually exclusive events A_i, then by an axiom

$$P(\cup_i A_i) = \textstyle\sum_i P(A_i).$$

This does not happen in fuzzy set theory. A fuzzy measure in [0, 1] could be defined for a finite X, called a possibility measure Π, as follows:

a. $\Pi(\emptyset) = 0$
b. $\Pi(X) = 1$
c. $A \subset B \Rightarrow \Pi(A) \le \Pi(B)$
d. $\Pi(\cup_i A_i) = \sup_i \Pi(A_i)$

Obviously Π and P obey different rules.

3. Finally, although a membership function μ ranges in [0, 1], it does not share all the features of a probability distribution function $F(x)$, which are

$$F(-\infty) = 0,\ F(+\infty) = 1,$$

$$F(x) = F(x^{+}),$$

$$F(x_1) \leq F(x_2) \text{ if } x_1 < x_2,$$

or of a density function, which are

$$f(x) \geq 0,$$

$$\int_{-\infty}^{\infty} f(x)dx = 1 .$$

2.3 The Extension Principle

Functions in mathematics map points x_1, x_2, … in a set X to points in another set Y. Such mappings may occur between fuzzy sets X and Y using the *extension principle*. Let a function f that maps subsets of X into subsets of Y. If

$$A = \mu_1/x_1 + \mu_2/x_2 + \ldots + \mu_n/x_n,$$

then by the extension principle

$$B \triangleq f(A) = \mu_1/f(x_1) + \mu_2/f(x_2) + \ldots + \mu_n/f(x_n)$$

$$= \mu_1/y_1 + \mu_2/y_2 + \ldots + \mu_n/y_n$$

for $x_i \in X$ and $y_i = f(x_i) \in Y$. If the same y corresponds to more than one x_i's, then we use the maximum of the membership grades of the x_i's such that $y = f(x_1) = f(x_2) = \ldots = f(x_n)$; i.e.,

$$\mu_B(y) = \max[\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)].$$

If the function f is defined on vector spaces of proper dimensions, i.e., $B = f(A_1, A_2, \ldots, A_k)$, then

$$\mu_B(y) = \min[\mu_{A1}(x_1), \mu_{A2}(x_2), \ldots, \mu_{Ak}(x_k)],$$

which is the minimum of the membership grades of the values x_i that produce y. Furthermore, if

$$y = f(x_1, x_2, \ldots, x_k) = f(x_1', x_2', \ldots, x_k') = \ldots,$$

then

$$\mu_B(y) = \max_{\substack{(x_1, \ldots, x_k) \\ (x_1', \ldots, x_k') \\ \vdots}} \{\min[\mu_{A1}(x_1), \ldots, \mu_{Ak}(x_k)], \min[\mu_{A1}(x_1'), \ldots, \mu_{Ak}(x_k')], \ldots\}.$$

Example 2.7

Let $A = 0.5/x_1 + 0.2/x_2 + 0.7/x_3$ and $y = f(x_1) = f(x_2) = f(x_3)$. Then

$$B = \max(0.5, 0.2, 0.7)/y = 0.7/y.$$

Example 2.8

Let A be a fuzzy set in [2, 4] with the triangular membership function of Figure 2.3 and $y = \ln x$.

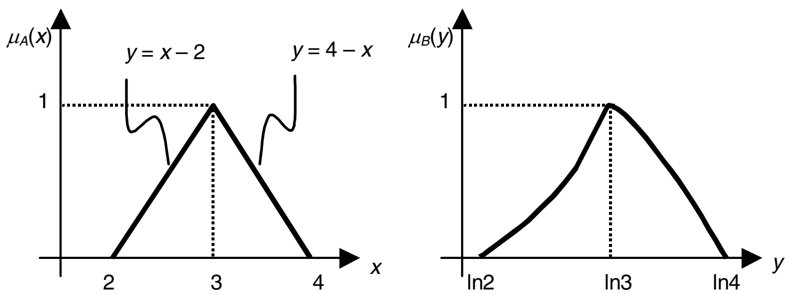

Figure 2.3. Membership functions of Example 2.8.

Then we obtain $x = e^y$ and

$$\mu_B(y) = \begin{cases} e^y - 2 & \text{if } \ln 2 \le y \le \ln 3, \\ 4 - e^y & \text{if } \ln 3 \le y \le \ln 4. \end{cases}$$

Example 2.9

Now we have two fuzzy sets

$$A_1 = 0.1/x_1 + 0.4/x_2 + 0.8/x_3$$

$$A_2 = 0.6/x_1' + 1/x_2'$$

and a function f that maps x_i and x_i' into y_i as follows:

$$y_1 = f(x_1, x_1') = f(x_1, x_2') = f(x_3, x_2'),$$

$$y_2 = f(x_2, x_1'),$$

$$y_3 = f(\mathrm{x}_2, x_2') = f(x_3, x_1'),$$

or in matrix form

$$\begin{array}{cc} & \begin{array}{cc} x_1' & x_2' \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} y_1 & y_1 \\ y_2 & y_3 \\ y_3 & y_1 \end{bmatrix} \end{array}.$$

Then

$$\mu_B(y_1) = \max[\min(0.1, 0.6), \min(0.1, 1), \min(0.8, 1)] = 0.8,$$

$$\mu_B(y_2) = \max[\min(0.4, 0.6)] = 0.4,$$

$$\mu_B(y_3) = \max[\min(0.4,1), \min(0.8, 0.6)] = 0.6.$$

Therefore, $B = 0.8/y_1 + 0.4/y_2 + 0.6/y_3$.

2.4 Linguistic Variables

Loosely speaking, a *linguistic variable* is a variable "whose values are words or sentences in a natural or artificial language," as Zadeh has put it. Take, for example the concept "Height," which can be seen as a linguistic variable with values "very tall," "tall," "not tall," "average," "short," "very short," and so on. To each of these values, we may assign a membership function. Let the height range over a region [0, 230 cm] and assume that the linguistic terms are governed by a given set of rules. Then we define formally a linguistic variable.

Definition 2.14. A linguistic variable is a 4-tuple (T, X, G, M), where

T is a set of natural language terms called *linguistic values*
X is a universe of discourse
G is a context free grammar used to generate elements of T
M is a mapping from T to the fuzzy subsets of X

Example 2.10

In the example above,

$$T = \{\text{very tall, tall, } \ldots\},\ X = [0, 230]$$

and M for tall:

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le 170, \\ \dfrac{x-170}{15} & \text{if } 170 < x \le 185, \\ 1 & \text{if } 185 < x. \end{cases}$$

Linguistic variables are fundamental when we want to represent knowledge in approximate reasoning. Often the meaning of a term needs to be modified.

Examples of modifiers are the following:

$$\mu_{\text{VERY}} = [\mu_A(x)]^2,$$

$$\mu_{\text{MORE OR LESS}} = \sqrt{\mu_A(x)}\,,$$

$$\mu_{\text{INDEED}} = \begin{cases} 2[\mu_A(x)]^2 & \text{if } 0 \le \mu_A(x) < 0.5, \\ 1 - 2[1 - \mu_A(x)]^2 & \text{if } 0.5 < \mu_A(x) \le 1. \end{cases}$$

2.5 Fuzzy Reasoning

A queue is observed, and the conclusion "the queue is positive small" is derived. This conclusion may be formally written as "s is PS" by choosing a symbol s for queue size and a symbol PS for "positive small." Experience has shown that in fuzzy control, a large number of linguistic variables can be represented by seven linguistic values: NB (negative big), NM (negative medium), NS (negative small), ZO (zero), PS (positive small), PM (positive medium), and PB (positive big). A common domain for these values is the *standard* domain $[-6, 6]$ or the *normalized* one $[-1, 1]$. A large number of control problems can be solved efficiently over these domains.

The proposition "s is PS" is called *atomic* and assumes a certain membership grade, say $\mu_{\mathrm{PS}} = 0.4$. Atomic propositions together with *connectives* such as AND, OR, NOT, or IF-THEN form *compound* propositions. For example, the expressions

$$\text{IF } X \text{ is } A\text{, THEN } X \text{ is } B,$$

$$X \text{ is } A \text{ OR } B,$$

and so on are compound propositions.

The connective AND corresponds to logical *conjunction* "X is $A \cap B$" where A and B are fuzzy sets and the appropriate membership function is $\mu_{A \cap B}$. Similarly OR corresponds to *disjunction* "X is $A \cup B$" and $\mu_{A \cup B}$ and NOT corresponds to "X is $\bar{A}$" and $\mu_{\bar{A}}$.

Now consider two queues in parallel with queue lengths s_1 and s_2 and one server with variable service rates. An experienced operator decides in terms of natural language "if the queue size s_1 is large and the queue size s_2 is also large, then the server should run at a high rate." This statement can be written

$$\text{IF } s_1 \text{ is PB AND } s_2 \text{ is PB, THEN } r \text{ is PB.}$$

This proposition has the form

$$\text{IF (antecedents) THEN (consequents)}$$

and is called a *fuzzy conditional* or *fuzzy if-then* production rule.

2.6 Rules of Inference

Classical logic is based on tautologies of the following type (we use "$\wedge$" for "AND," "$\vee$" for "OR," and "$\rightarrow$" for "implies").

1. *Modus ponens*

Premise: A is true
Implication: if A then B
Conclusion: B is true

Symbolically: $[A \wedge (A \rightarrow B)] \rightarrow B$

2. *Modus tollens*

Premise: not B is true
Implication: if A then B
Conclusion: not A is true

Symbolically: $[\overline{B} \wedge (A \rightarrow B)] \rightarrow \overline{A}$

3. *Syllogism*

Premise: "if A then B" is true
Implication: if B then C
Conclusion: "if A then C" is true

Symbolically: $[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$

Such rules can be generalized in the context of fuzzy logic. Two common rules of approximate reasoning are the *Generalized Modus Ponens* (GMP) and the *Compositional Rule of Inference* (CRI). Let A, A', B, and B' be fuzzy sets and X, Y be linguistic variables. Then we define

GMP Premise: X is A'
Implication: if X is A, then Y is B
Conclusion: y is B'

Example 2.11

GMP Premise: a student has a very high IQ
Implication: if a student has a high IQ, then he is academically good
Conclusion: The student is academically very good

The compositional rule of inference is a special case of the generalized modus ponens and has the form

CRI Premise: X is A'
Implication: $X\ R\ Y$ (X is related to Y)
Conclusion: y is B'

Here R substitutes for "if-then."

Example 2.12

CRI Premise: Jim is tall
Implication: Jim is somewhat taller than George
Conclusion: George is rather tall

2.7 Mamdani Implication

The meaning of "if-then" rules is represented by relevant membership functions. As expected, there is a long list of ways to represent the meaning of "if-then" rules. They are all subjective, but their efficacy depends on the application.

In fuzzy control, the most commonly used and the most efficient implication is called *Mamdani implication*. It is defined by

$$\mu_C(x, y) = \min[\mu_A(x), \mu_B(y)]$$

for the rule *if X is A, then Y is B*. In the sequel, we shall see numerous applications of the Mamdani implication in practical control problems. Here we give a simple example.

Example 2.13

Let

$$A = 0.2/x_1 + 0.3/x_2 + 0.4/x_3,$$

$$B = 0.1/y_1 + 0.2/y_2 + 0.6/y_3 + 0.7/y_4.$$

The following table summarizes the Mamdani implication for the rule *if X is A, then Y is B*:

	y_1	y_2	y_3	y_4
x_1	0.1	0.2	0.2	0.2
x_2	0.1	0.2	0.3	0.3
x_3	0.1	0.2	0.4	0.4

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