2 Examples

Example is the school of mankind, and they will learn at no other.
Edmund Burke (1729–1797)

This section, like the previous section, is organised into 19 groups:

2.1 Trigonometry
2.2 Circles
2.3 Triangles
2.4 Quadrilaterals
2.5 Polygons
2.6 Three-dimensional objects
2.7 Coordinate systems
2.8 Vectors
2.9 Quaternions
2.10 Transformations
2.11 Two-dimensional straight lines
2.12 Lines and circles
2.13 Second degree curves
2.14 Three-dimensional straight lines
2.15 Planes
2.16 Lines, planes and spheres
2.17 Three-dimensional triangles
2.18 Parametric curves and patches
2.19 Second degree surfaces in standard form

The following examples illustrate how geometric formulas are used in practice. Hopefully, the reader will see the advantages of using unit vectors, and the difference between using parametric equations and the general form of line equations and plane equations. There is no one strategy that overall is superior to another – much will depend upon the context.

Vectors

Vector notation provides a very compact way of expressing the solution to a geometric problem. For example, the formula for calculating the intersection of a line and plane is given by

$$\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$$

where $$\lambda = \frac{-(\mathbf{n} \bullet \mathbf{t} + d)}{\mathbf{n} \bullet \mathbf{v}}$$

The position vector **p** identifies a point *P* where the line intersects the plane. Therefore, the coordinates of *P* are given by

$$x_p = x_t + \lambda x_v$$

$$y_p = y_t + \lambda y_v$$

$$z_p = z_t + \lambda z_v$$

This sort of 'coordinate unpacking' is used throughout the examples in this section.

2.1 Trigonometry

Examples of cofunction identities

$$\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right) = \cos\beta \qquad \sin 30^\circ = \cos 60^\circ = 0.5$$

$$\tan\alpha = \cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta \qquad \tan 45^\circ = \frac{1}{\tan 45^\circ} = 1$$

$$\csc\alpha = \sec\left(\frac{\pi}{2} - \alpha\right) = \sec\beta \qquad \frac{1}{\sin 30^\circ} = \frac{1}{\cos 60^\circ} = 2$$

Examples of even–odd identities

$$\sin(-\alpha) = -\sin\alpha \qquad \sin(-30^\circ) = -\sin 30^\circ = -0.5$$

$$\cos(-\alpha) = \cos\alpha \qquad \cos(-60^\circ) = \cos 60^\circ = 0.5$$

$$\tan(-\alpha) = -\tan\alpha \qquad \tan(-45^\circ) = -\tan 45^\circ = -1$$

Examples of Pythagorean identities

$$\sin^2\alpha + \cos^2\alpha = 1 \qquad \sin^2 30^\circ + \cos^2 30^\circ = \tfrac{1}{4} + \tfrac{3}{4} = 1$$

$$1 + \tan^2\alpha = \sec^2\alpha \qquad 1 + \tan^2 45^\circ = \frac{1}{\cos^2 45^\circ} = 2$$

$$1 + \cot^2\alpha = \csc^2\alpha \qquad 1 + \cot^2 45^\circ = \frac{1}{\sin^2 45^\circ} = 2$$

Examples of compound angle identities

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(10^\circ + 20^\circ) = \sin 10^\circ \cos 20^\circ + \cos 10^\circ \sin 20^\circ = 0.5$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(10^\circ + 50^\circ) = \cos 10^\circ \cos 50^\circ - \sin 10^\circ \sin 50^\circ = 0.5$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(20^\circ + 25^\circ) = \frac{\tan 20^\circ + \tan 25^\circ}{1 - \tan 20^\circ \tan 25^\circ} = 1$$

Examples of double-angle identities

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\quad$ $\sin 30° = 2\sin 15°\cos 15° = 0.5$

$\cos 2\alpha = 1 - 2\sin^2\alpha$ $\quad$ $\cos 60° = 1 - 2\sin^2 30° = 0.5$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ $\quad$ $\cos 60° = \cos^2 30° - \sin^2 30° = 0.5$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\beta} \qquad \tan 45° = \frac{2\tan 22.5°}{1-\tan^2 22.5°} = 1$$

Examples of multiple-angle identities

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ $\quad$ $\sin 30° = 3\sin 10° - 4\sin^3 10° = 0.5$

$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$ $\quad$ $\cos 60° = 4\cos^3 20° - 3\cos 20° = 0.5$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha} \qquad \tan 45° = \frac{3\tan 15° - \tan^3 15°}{1-3\tan^2 15°} = 1$$

$\sin 4\alpha = 4\sin\alpha\cos\alpha - 8\sin^3\alpha\cos\alpha$ $\quad$ $\sin 30° = 4\sin 7.5°\cos 7.5° - 8\sin^3 7.5°\cos 7.5° = 0.5$

$\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$ $\quad$ $\cos 60° = 8\cos^4 15° - 8\cos^2 15° + 1 = 0.5$

$$\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1-6\tan^2\alpha + \tan^4\alpha} \qquad \tan 60° = \frac{4\tan 15° - 4\tan^3 15°}{1-6\tan^2 15° + \tan^4 15°} = 1.732051$$

$\sin 5\alpha = 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha$ $\quad$ $\sin 30° = 16\sin^5 6° - 20\sin^3 6° + 5\sin 6° = 0.5$

$\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$ $\quad$ $\cos 60° = 16\cos^5 12° - 20\cos^3 12° + 5\cos 12° = 0.5$

$$\tan 5\alpha = \frac{5\tan\alpha - 10\tan^3\alpha + \tan^5\alpha}{1-10\tan^2\alpha + 5\tan^4\alpha} \qquad \tan 45° = \frac{5\tan 9° - 10\tan^3 9° + \tan^5 9°}{1-10\tan^2 9° + 5\tan^4 9°} = 1$$

Functions of the half-angle

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \sin 30° = \pm\sqrt{\frac{1-\cos 60°}{2}} = \pm 0.5$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \cos 60° = \pm\sqrt{\frac{1+\cos 120°}{2}} = \pm 0.5$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \qquad \tan 45° = \pm\sqrt{\frac{1-\cos 90°}{1+\cos 90°}} = \pm 1$$

Functions converting to the half-angle tangent form

$$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \qquad \sin 30° = \frac{2\tan 15°}{1+\tan^2 15°} = 0.5$$

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \qquad \cos 60° = \frac{1-\tan^2 30°}{1+\tan^2 30°} = 0.5$$

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} \qquad \tan 45° = \frac{2\tan 22.5°}{1-\tan^2 22.5°} = 1$$

Relationships between sums of functions

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \sin 30° + \sin 30° = 2\sin 30°\cos 0° = 1$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \qquad \sin 60° - \sin 30° = 2\cos 45°\sin 15° = 0.366$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \cos 60° + \cos 60° = 2\cos 60°\cos 0° = 1$$

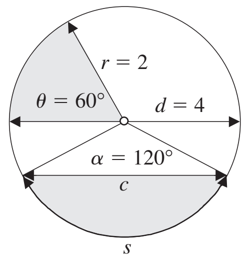
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \qquad \cos 60° - \cos 30° = -2\sin 45°\sin 15° = -0.366$$

$$\tan\alpha + \tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} \qquad \tan 45° + \tan 45° = \frac{\sin 90°}{\cos 45°\cos 45°} = 2$$

$$\tan\alpha - \tan\beta = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} \qquad \tan 60° - \tan 45° = \frac{\sin 15°}{\cos 60°\cos 45°} = 0.732$$

2.2 Circles

Example: Properties of circles



Circle

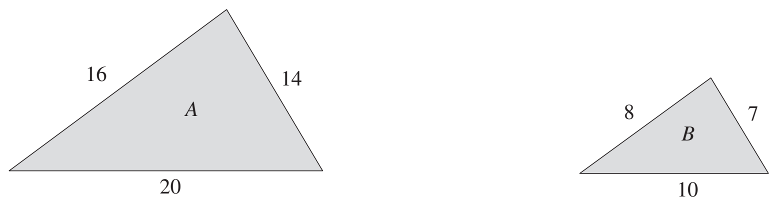
Area of circle	$A = \pi r^2$	$A = \pi 2^2 = 12.57$
Perimeter	$C = \pi d$	$C = \pi 4 = 12.57$
Length of arc	$s = \dfrac{\alpha°}{360°}\pi d$	$s = \dfrac{120°}{360°}\pi 4 = 4.19$
Area of sector	$\dfrac{\theta°}{360°}\pi r^2$	$\dfrac{60°}{360°}\pi 4 = 2.09$
Area of segment	$\dfrac{r^2}{2}\left(\alpha^{[rad]} - \sin\alpha^{[rad]}\right)$	$\dfrac{4}{2}\left(\dfrac{2}{3}\pi - \dfrac{\sqrt{3}}{2}\right) = 2.46$
Length of chord	$c = 2r\sin\dfrac{\alpha}{2}$	$c = 4\sin 60° = 3.46$

2.3 Triangles

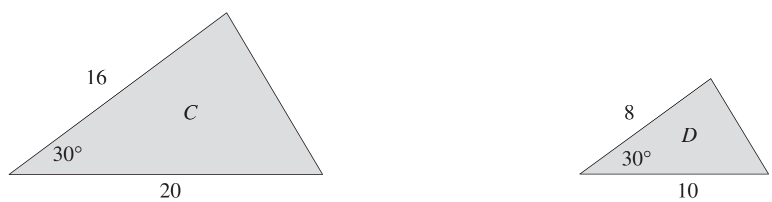
2.3.1 Checking for similar triangles

Triangles A and B are similar because three corresponding sides are in the same ratio:

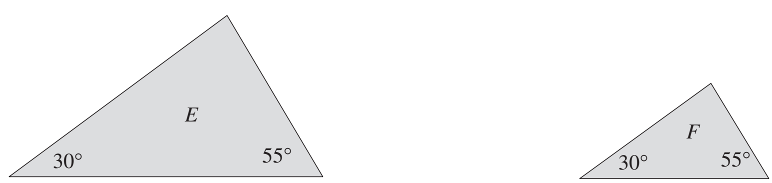
$$\frac{20}{10} = \frac{16}{8} = \frac{14}{7} = 2$$



Triangles C and D are similar because two corresponding sides are in the same ratio, and the included angles are equal: $\frac{20}{10} = \frac{16}{8} = 2$ and the included angles equal 30°.



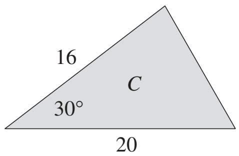
Triangles E and F are similar because two corresponding angles are equal.

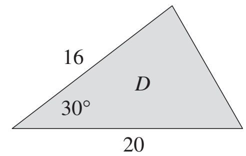


2.3.2 Checking for congruent triangles

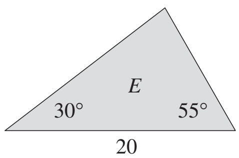
Triangles A and B are congruent because three corresponding sides are equal.

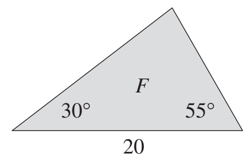
Triangles C and D are congruent because two corresponding sides are equal, and the included angles are equal.





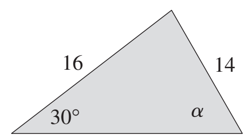
Triangles E and F are congruent because one side and the adjoining angles are equal.





2.3.3 Solving the angles and sides of a triangle

Use the sine rule to find angle α.

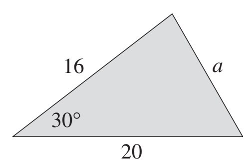


$$\frac{16}{\sin\alpha} = \frac{14}{\sin 30°}$$

$$\sin\alpha = \frac{16}{14}\sin 30°$$

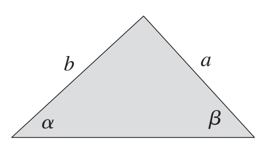
$$\alpha = \sin^{-1}\left(\frac{16}{14}\sin 30°\right) = 34.85°$$

Use the cosine rule to find side a.



$$a^2 = 20^2 + 16^2 - 2 \times 20 \times 16 \cos 30°$$
$$a^2 = 400 + 256 - 720 \cos 30°$$
$$a = 5.7$$

Use the tangent rule to find side b.



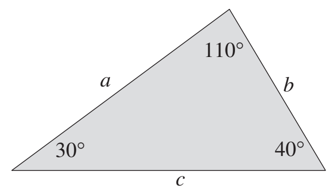
$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{\alpha+\beta}{2}\right)}{\tan\left(\frac{\alpha-\beta}{2}\right)}$$

$$a = 3 \qquad \alpha = 36.87° \qquad \beta = 53.13°$$

$$\frac{3+b}{3-b} = \frac{\tan 45°}{\tan(-8.13°)} = \frac{1}{-0.14285} = -7$$

$$3 + b = -7(3 - b) \quad \therefore b = 4$$

Given $a - b$ use Mollweide's rule to find side c.



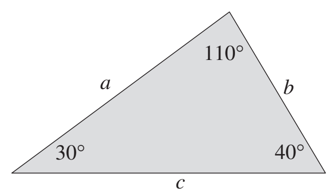
$$\frac{a-b}{c} = \frac{\sin\left(\dfrac{\alpha-\beta}{2}\right)}{\cos\left(\dfrac{\gamma}{2}\right)}$$

$$a - b = 2 \qquad \alpha = 40° \qquad \beta = 30° \qquad \gamma = 110°$$

$$\frac{2}{c} = \frac{\sin 5°}{\cos 55°} = 0.15195$$

$$c = 13.162$$

Given $a + b$ use Newton's rule to find side c.



$$\frac{a+b}{c} = \frac{\cos\left(\dfrac{\alpha-\beta}{2}\right)}{\sin\left(\dfrac{\gamma}{2}\right)}$$

$$a + \mathrm{b} = 16 \qquad \alpha = 40° \qquad \beta = 30° \qquad \gamma = 110°$$

$$\frac{16}{c} = \frac{\cos 5°}{\sin 55°} = 1.21613$$

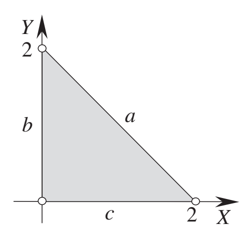
$$c = 13.15648$$

2.3.4 Calculating the area of a triangle

Use Heron's formula to calculate the area of a triangle.

$$a = \sqrt{8} \qquad b = 2 \qquad c = 2$$

Semiperimeter $s = \dfrac{\sqrt{8}+2+2}{2} = 2 + \sqrt{2}$

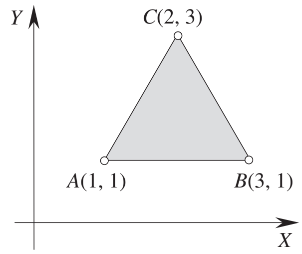


$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{(2+\sqrt{2})(2+\sqrt{2}-\sqrt{8})\sqrt{2}\sqrt{2}}$$

$$\text{Area} = 2$$

Use a determinant to calculate the area of a triangle.



$$\text{Area } \triangle ABC = \tfrac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}$$

$$\text{Area } = \tfrac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix}$$

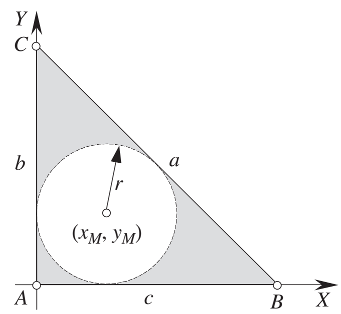
$$= \tfrac{1}{2}(1+2+9-3-3-2) = 2$$

Reversing the vertex order:

$$\text{Area } = \tfrac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix} = \tfrac{1}{2}(3+3+2-1-2-9) = -2$$

2.3.5 The center and radius of the inscribed and circumscribed circles for a triangle

Calculate the center of the inscribed circle for triangle ABC.



$$a = \sqrt{8} \qquad b = 2 \qquad c = 2$$

$$A = (0,0) \qquad B = (2,0) \qquad C = (0,2)$$

$$x_M = \frac{ax_A + bx_B + cx_C}{a+b+c}$$

$$y_M = \frac{ay_A + by_B + cy_C}{a+b+c}$$

$$x_M = \frac{\sqrt{8}\times 0 + 2\times 2 + 2\times 0}{\sqrt{8}+2+2} = \frac{4}{4+\sqrt{8}}$$

$$y_M = \frac{\sqrt{8}\times 0 + 2\times 0 + 2\times 2}{\sqrt{8}+2+2} = \frac{4}{4+\sqrt{8}}$$

Position of the center

$$x_M = 0.5858 \qquad y_M = 0.5858$$

Calculate the radius of the inscribed circle for triangle ABC.

$$s = \frac{\sqrt{8}+2+2}{2} = \sqrt{2}+2$$

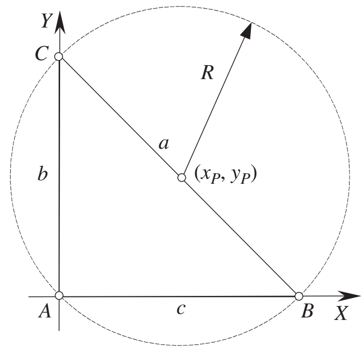
$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$r = \sqrt{\frac{(\sqrt{2}+2-\sqrt{8})(\sqrt{2})(\sqrt{2})}{\sqrt{2}+2}}$$

$$r = 2-\sqrt{2} = 0.5858$$

$$r = 0.5858 \qquad x_M = 0.5858 \qquad y_M = 0.5858$$

Calculate the radius of the circumscribed circle for triangle ABC.



$$a = \sqrt{8} \qquad b = 2 \qquad c = 2$$

$$A = (0,0) \qquad B = (2,0) \qquad C = (0,2)$$

$$R = \frac{abc}{4\times \text{Area}\,\triangle ABC} = \frac{\sqrt{8}\times 2\times 2}{4\times 2} = \sqrt{2}$$

Calculate the center of the circumscribed circle for triangle ABC.

$$x_P = x_A + \frac{R}{abc}\begin{vmatrix} y_{AC} & b^2 \\ y_{AB} & c^2 \end{vmatrix}$$

$$y_P = y_A + \frac{R}{abc}\begin{vmatrix} b^2 & x_{AC} \\ c^2 & x_{AB} \end{vmatrix}$$

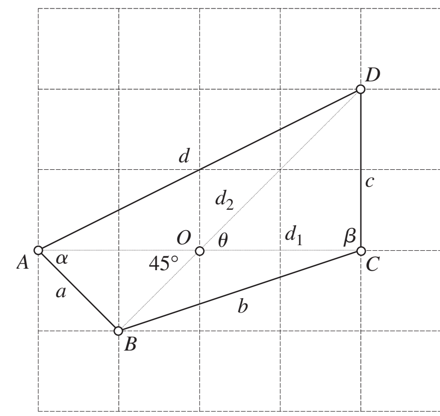
$$x_P = \frac{\sqrt{2}}{\sqrt{8}\times 2\times 2}\begin{vmatrix} 2 & 4 \\ 0 & 4 \end{vmatrix} = 1$$

$$y_P = \frac{\sqrt{2}}{\sqrt{8}\times 2\times 2}\begin{vmatrix} 4 & 0 \\ 4 & 2 \end{vmatrix} = 1$$

$$R = \sqrt{2} \qquad x_p = 1 \qquad y_P = 1$$

2.4 Quadrilaterals

Example: Calculate the area of a quadrilateral.



$$a = \sqrt{2}$$
$$b = \sqrt{10}$$
$$c = 2$$
$$d = \sqrt{20}$$
$$AC = d_1 = 4$$
$$BD = d_2 = \sqrt{18}$$
$$s = \frac{a+b+c+d}{2} = 5.5243$$

By inspection

$$\triangle ABO = 1$$
$$\triangle BCO = 1$$
$$\triangle CDO = 2$$
$$\triangle DAO = 2$$

therefore Area $ABCD = 6$.

Here are four ways of computing the area:

$$\text{Area} = \frac{d_1 d_2}{2} \sin\theta = \frac{4\sqrt{18}}{2} \sin 45° = 6\sqrt{2}\,\frac{\sqrt{2}}{2} = 6$$

$$\text{Area} = \tfrac{1}{4}(b^2 + d^2 - a^2 - c^2)\tan\theta = \tfrac{1}{4}(10 + 20 - 2 - 4)\tan 45° = 6$$

$$\text{Area} = \tfrac{1}{4}\sqrt{4d_1^2 d_2^2 - (b^2 + d^2 - a^2 - c^2)^2}$$
$$= \tfrac{1}{4}\sqrt{4 \times 16 \times 18 - (10 + 20 - 2 - 4)^2} = 6$$

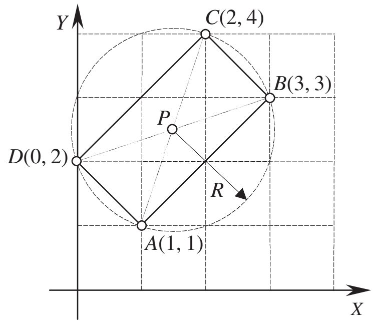
$$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\varepsilon}$$

$$\varepsilon = \frac{\alpha+\beta}{2} = \frac{71.57° + 108.43°}{2} = 90°$$

$$\text{Area} = \sqrt{4.1101 \times 2.3620 \times 3.5243 \times 1.0522 - 40\cos^2 90°} = 6$$

It just so happens that the quadrilateral is a cyclic quadrilateral.

Example: Calculate the center and radius of the circumscribed circle for a rectangle.



$$P_A = (1,1) \qquad P_B = (3,3) \qquad P_C = (2,4) \qquad P_D = (0,2)$$

The center of the circumscribed circle is

$$x_P = \tfrac{1}{2}(x_A + x_C) \qquad y_P = \tfrac{1}{2}(y_A + y_C)$$

$$x_P = \tfrac{1}{2}(1+2) = 1.5 \qquad y_P = \tfrac{1}{2}(1+4) = 2.5$$

The radius of the circumscribed circle is

$$R = \tfrac{1}{2}\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (x_B - x_C)^2 + (y_B - y_C)^2}$$

$$R = \tfrac{1}{2}\sqrt{(3-1)^2 + (3-1)^2 + (3-2)^2 + (3-4)^2} = \tfrac{1}{2}\sqrt{10}$$

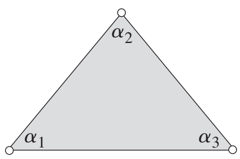
The circle has a radius of $\frac{1}{2}\sqrt{10}$ with a center at (1.5, 2.5).

2.5 Polygons

Example: Determine the internal angles of a polygon

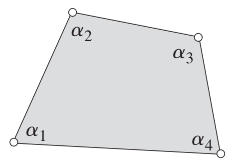
The internal angles of an n-sided polygon sum to $(n - 2) \times 180°$.

Triangle ($n = 3$)



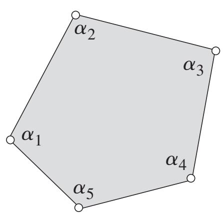
$$\sum_{i=1}^{3} \alpha_i = 180°$$

Quadrilateral ($n = 4$)



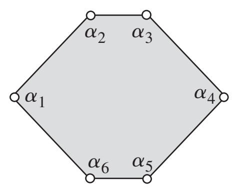
$$\sum_{i=1}^{4} \alpha_i = 360°$$

Pentagon ($n = 5$)



$$\sum_{i=1}^{5} \alpha_i = 540°$$

Hexagon ($n = 6$)

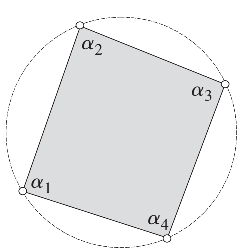


$$\sum_{i=1}^{6} \alpha_i = 720°$$

Example: Determine the alternate internal angles of a cyclic polygon

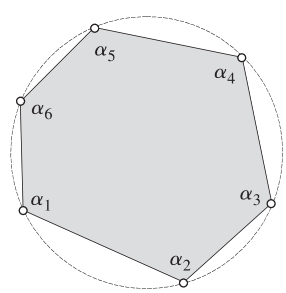
The alternate internal angles of an n-sided cyclic polygon sum to $(n - 2) \times 90°$
[$n \geqslant 4$ and is even].

Cyclic quadrilateral ($n = 4$)



$$\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = 180°$$

Cyclic hexagon ($n = 6$)



$$\alpha_1 + \alpha_3 + \alpha_5 = \alpha_2 + \alpha_4 + \alpha_6 = 360°$$

Example: Calculate the area of regular polygon

$$\text{Area} = \tfrac{1}{4} n s^2 \cot \frac{\pi}{n}$$

where n = number of sides
s = length of side

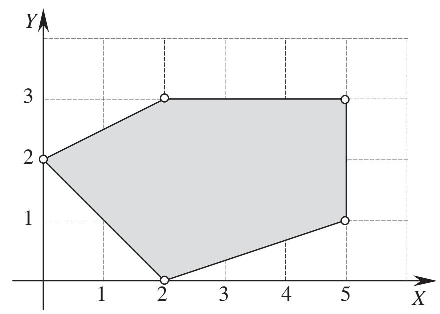
Let $s = 1$

n	Area
3	0.433
4	1
5	1.72
6	2.598
7	3.634
8	4.828

Example: Calculate the area of a polygon

The figure shows a polygon with the following vertices in counter-clockwise sequence

x	0	2	5	5	2
y	2	0	1	3	3



By inspection, the area is 10.5

The area of a polygon is given by

$$\text{Area} = \tfrac{1}{2} \sum_{i=0}^{n-1} \left(x_i y_{i+1(\text{mod } n)} - y_i x_{i+1(\text{mod } n)} \right)$$

$$\text{Area} = \tfrac{1}{2}(0 \times 0 + 2 \times 1 + 5 \times 3 + 5 \times 3 + 2 \times 2 - 2 \times 2 - 0 \times 5 - 1 \times 5 - 3 \times 2 - 3 \times 0)$$

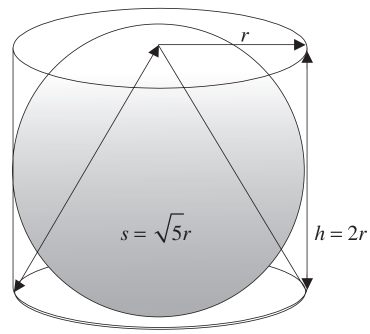
$$\text{Area} = \tfrac{1}{2}(36 - 15) = 10.5$$

2.6 Three-dimensional objects

2.6.1 Cone, cylinder and sphere

Example: Area and volume of a cone, cylinder and sphere

Area	$(h = 2r)\ (s = \sqrt{5}r)$	$(r = 1)$
Cone	$\pi r(r + s) = (1 + \sqrt{5})\pi r^2$	$(1 + \sqrt{5})\pi$
Sphere	$4\pi r^2$	4π
Cylinder	$2\pi r(r + h) = 6\pi r$	6π
Volume		
Cone	$\frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^3$	$\frac{2}{3}\pi$
Sphere	$\frac{4}{3}\pi r^3$	$\frac{4}{3}\pi$
Cylinder	$\pi r^2 h = 2\pi r^2$	2π



2.6.2 Conical frustum, spherical segment and torus

Example: Area and volume of a conical frustum, spherical segment and torus

Circular, conical frustum

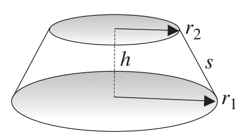
$$S = \pi(r_1^2 + r_2^2 + s(r_1 + r_2))$$

If $\quad r_1 = 2 \quad r_2 = 1 \quad h = 1 \quad s = \sqrt{2}$

$$S = \pi(4 + 1 + \sqrt{2}(2 + 1)) = 29.03$$

$$V = \tfrac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$$

$$V = \tfrac{1}{3}\pi(4 + 1 + 2) = 7.33$$



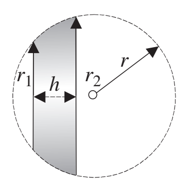
Spherical segment

$$S = 2\pi rh$$

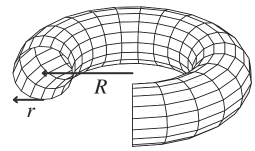
If $r = 1 \quad h = 1 \qquad S = 6.28$

$$V = \tfrac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2)$$

If $r_1 = 0 \quad r_2 = 1 \quad h = 1 \qquad V = 2.09$
(half the volume)



Torus



$$S = 4\pi^2 rR$$

If $r = 1 \quad R = 1 \qquad S = 39.48$

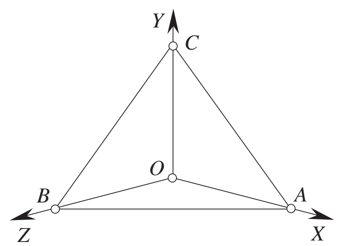
$$V = 2\pi^2 r^2 R$$

If $r = 1 \quad R = 1 \qquad V = 19.74$

2.6.3 Tetrahedron

Example: Volume of a tetrahedron

Tetrahedron



Let $A = (1, 0, 0) \quad B = (0, 0, 1) \quad C = (0, 1, 0)$

$$V = \frac{1}{6}\begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \frac{1}{6}$$

Note: If the vertices are reversed the volume is negative.

$$V = \frac{1}{6}\begin{vmatrix} x_b & y_b & z_b \\ x_a & y_a & z_a \\ x_c & y_c & z_c \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\frac{1}{6}$$

2.7 Coordinate systems

2.7.1 Cartesian coordinates in $\mathbb{R}^2$

Example: Distance in $\mathbb{R}^2$

Find the distance between the points (12, 16) and (9, 12).

Given $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

therefore $$d = \sqrt{(12-9)^2 + (16-12)^2} = \sqrt{9+16}$$
$$d = 5$$

2.7.2 Cartesian coordinates in $\mathbb{R}^3$

Example: Distance in $\mathbb{R}^3$

Find the distance between the points (12, 16, 22) and (9, 12, 20).

Given $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

therefore $$d = \sqrt{(12-9)^2 + (16-12)^2 + (22-20)^2}$$
$$= \sqrt{9+16+4} = \sqrt{29}$$
$$d = 5.39$$

2.7.3 Polar coordinates

Example: Conversion between Cartesian and polar coordinates

Find the polar coordinates (r, θ) for the points $(4, 3), (-4, 3), (-4, -3)$ and $(4, -3)$.

Given $$r = \sqrt{x^2 + y^2}$$

and $$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{(1st and 4th quadrants only)}$$

For (4, 3) $$r = \sqrt{16+9} = 5$$

and $$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.87°$$
$$(4, 3) \equiv (5, 36.87°)$$

For $(-4, 3)$ $$r = 5$$

and $$\theta = 180° - 36.87° = 143.13°$$
$$(-4, 3) \equiv (5, 143.13°)$$

For $(-4, -3)$ $r = 5$

and $\theta = 180° + 36.87° = 216.87°$

$(-4, -3) \equiv (5, 216.87°)$

For $(4, -3)$ $r = 5$

and $\theta = -36.87°$ or $323.13°$

$(4, -3) \equiv (5, 323.13°)$

Find the Cartesian coordinates (x, y) for the point $(5, 216.87°)$.

Given $x = r \cos \theta$

and $y = r \sin \theta$

For $(5, 216.87°)$ $x = 5 \cos 216.87° = -4$

and $y = 5 \sin 216.87° = -3$

$(5, 216.87°) \equiv (-4, -3)$

2.7.4 Cylindrical coordinates

Example: Conversion between Cartesian and cylindrical coordinates

Find the cylindrical coordinates (r, θ, z) for the points $(4, 3, 4)$, $(-4, 3, 4)$, $(-4, -3, 4)$ and $(4, -3, 4)$.

Given $r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$ (1st and 4th quadrants only)

and $z = z$

For $(4, 3, 4)$ $r = \sqrt{16 + 9} = 5$

$\theta = \tan^{-1}\left(\dfrac{3}{4}\right) = 36.87°$

and $z = 4$

$(4, 3, 4) \equiv (5, 36.87°, 4)$

For $(-4, 3, 4)$ $r = 5$

$\theta = 180° - 36.87° = 143.13°$

and $z = 4$

$(-4, 3, 4) \equiv (5, 143.13°, 4)$

For $(-4, -3, 4)$ $r = 5$

$\theta = 180° + 36.87° = 216.87°$

and $z = 4$

$(-4, -3, 4) \equiv (5, 216.87°, 4)$

For $(4, -3, 4)$ $\quad r = 5$

$$\theta = -36.87° \text{ or } 323.13°$$

and $\quad z = 4$

$$(4, -3, 4) \equiv (5, 216.87°, 4)$$

Find the Cartesian coordinates (x, y, z) for the point $(5, 216.87°, 4)$.

Given $\quad x = r\cos\theta$

$$y = r\sin\theta$$

and $\quad z = z$

For $(5, 216.87°, 4)$ $\quad x = 5\cos 216.87° = -4$

$$y = 5\sin 216.87° = -3$$

$$z = 4$$

$$(5, 216.87°, 4) \equiv (-4, -3, 4)$$

2.7.5 Spherical coordinates

Example: Conversion between Cartesian and spherical coordinates

Find the spherical coordinates (ρ, θ, ϕ) for the points $(4, 3, 4)$, $(-4, 3, 4)$, $(-4, -3, 4)$ and $(4, -3, 4)$.

Given $\quad \rho = \sqrt{x^2 + y^2 + z^2}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{(1st and 4th quadrants only)}$$

and $\quad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$

For $(4, 3, 4)$ $\quad \rho = \sqrt{16 + 9 + 16} = \sqrt{41} = 6.403$

$$\theta = \tan^{-1}\frac{3}{4} = 36.87°$$

and $\quad \phi = \cos^{-1}\frac{4}{6.403} = 51.34°$

$$(4, 3, 4) \equiv (6.403, 36.87°, 51.34°)$$

For $(-4, 3, 4)$ $\quad \rho = 6.403$

$$\theta = 180° - 36.87° = 143.13°$$

and $\quad \phi = 51.34°$

$$(-4, 3, 4) \equiv (6.403, 143.13°, 51.34°)$$

For $(-4, -3, 4)$ $\rho = 6.403$

$\theta = 180° + 36.87° = 216.87°$

and $\phi = 51.34°$

$(-4, -3, 4) \equiv (6.403, 216.87°, 51.34°)$

For $(4, -3, 4)$ $\rho = 6.403$

$$\theta = \tan^{-1}\left(\frac{-3}{4}\right) = -36.87° = 323.13°$$

and $\phi = 51.34°$

$(4, -3, 4) \equiv (6.403, 323.13°, 51.34°)$

2.8 Vectors

2.8.1 Vector between two points

Given $P_1(1,2,3)$ and $P_2(4,6,8)$

$$\overrightarrow{P_1P_2} = \mathbf{a} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$$

2.8.2 Scaling a vector

Given $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

scale by 3 $3\mathbf{a} = 9\mathbf{i} + 12\mathbf{j} + 15\mathbf{k}$

2.8.3 Reversing a vector

Given $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

$$-\mathbf{a} = -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$$

2.8.4 Magnitude of a vector

Given $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

$$\|\mathbf{a}\| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 7.071$$

2.8.5 Normalizing a vector to a unit length

Given $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

$$\hat{\mathbf{a}} = \frac{3}{\sqrt{50}}\mathbf{i} + \frac{4}{\sqrt{50}}\mathbf{j} + \frac{5}{\sqrt{50}}\mathbf{k} = 0.424\mathbf{i} + 0.566\mathbf{j} + 0.707\mathbf{k}$$

check $\|\hat{\mathbf{a}}\| = \sqrt{\frac{9}{50} + \frac{16}{50} + \frac{25}{50}} = 1$

2.8.6 Vector addition/subtraction

Given $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

$$\mathbf{a} + \mathbf{b} = 5\mathbf{i} + 8\mathbf{j} + 11\mathbf{k}$$

2.8.7 Position vector

Given a point $(3, 4, 5)$ its position vector is $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.

2.8.8 Scalar (dot) product

Given $\qquad \mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \quad \text{and} \quad \mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

$$\mathbf{a} \cdot \mathbf{b} = 3 \times 2 + 4 \times 4 + 5 \times 6 = 52$$

2.8.9 Angle between two vectors

Given $\qquad \mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \quad \text{and} \quad \mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

Let α be the angle between **a** and **b**.

$$\|\mathbf{a}\| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \quad \text{and} \quad \|\mathbf{b}\| = \sqrt{2^2 + 4^2 + 6^2} = \sqrt{56}$$

$$\alpha = \cos^{-1}\left(\frac{x_a x_b + y_a y_b + z_a z_b}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}\right)$$

$$\alpha = \cos^{-1}\left(\frac{3 \times 2 + 4 \times 4 + 5 \times 6}{\sqrt{50}\sqrt{56}}\right) = \cos^{-1}\left(\frac{52}{52.915}\right) = 10.67^\circ$$

2.8.10 Vector (cross) product

Given $\qquad \mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} \quad \text{and} \quad \mathbf{b} = \mathbf{i} + \mathbf{j} + 8\mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 5 \\ 1 & 1 & 8 \end{vmatrix} = 11\mathbf{i} - 19\mathbf{j} + \mathbf{k}$$

$11\mathbf{i} - 19\mathbf{j} + \mathbf{k}$ is orthogonal to **a** and **b**.

Remember that $\qquad \mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$

Proof $\qquad \mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 8 \\ 3 & 2 & 5 \end{vmatrix} = -11\mathbf{i} + 19\mathbf{j} - \mathbf{k}$

$-11\mathbf{i} + 19\mathbf{j} - \mathbf{k}$ is still orthogonal to **a** and **b** but is in the opposite direction to $11\mathbf{i} - 19\mathbf{j} + \mathbf{k}$.

2.8.11 Scalar triple product

Given $\mathbf{a} = 2\mathbf{j} + 2\mathbf{k} \qquad \mathbf{b} = 10\mathbf{k} \qquad \mathbf{c} = 5\mathbf{i}$

$$\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix}$$

$$\text{Volume} = \mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 0 & 2 & 2 \\ 0 & 0 & 10 \\ 5 & 0 & 0 \end{vmatrix} = 100$$

2.8.12 Vector normal to a triangle

Given $P_1(5,0,0) \qquad P_2(0,0,5) \qquad P_3(10,0,5)$

$$\mathbf{a} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix}$$

$$\mathbf{a} = -5\mathbf{i} + 5\mathbf{k} \qquad \mathbf{b} = 5\mathbf{i} + 5\mathbf{k}$$

$$\mathbf{n} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 0 & 5 \\ 5 & 0 & 5 \end{vmatrix} = 50\mathbf{j}$$

Surface normal $\mathbf{n} = 50\mathbf{j}$

2.8.13 Area of a triangle

Given $P_1(5,0,0) \qquad P_2(0,0,5) \qquad P_3(10,0,5)$

$$\mathbf{a} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix}$$

$$\mathbf{a} = -5\mathbf{i} + 5\mathbf{k} \qquad \mathbf{b} = 5\mathbf{i} + 5\mathbf{k}$$

$$\text{Area} = \tfrac{1}{2}\|\mathbf{a}\times\mathbf{b}\| = \tfrac{1}{2}\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 0 & 5 \\ 5 & 0 & 5 \end{vmatrix} \right\| = \tfrac{1}{2}\|50\mathbf{j}\|$$

$$\text{Area} = 25$$

2.9 Quaternions

2.9.1 Quaternion addition and subtraction

$$\mathbf{q}_1 \pm \mathbf{q}_2 = [(s_1 \pm s_2) + (x_1 \pm x_2)\mathbf{i} + (y_1 \pm y_2)\mathbf{j} + (z_1 \pm z_2)\mathbf{k}]$$

Given $\mathbf{q}_1 = [1 + 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}]$

and $\mathbf{q}_2 = [1 - \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}]$

then $\mathbf{q}_1 + \mathbf{q}_2 = [2 + \mathbf{i} + 5\mathbf{j} + 9\mathbf{k}]$

2.9.2 Quaternion multiplication

$$\mathbf{q}_1\mathbf{q}_2 = [(s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2), s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2]$$

Given $\mathbf{q}_1 = [1 + \mathbf{i}]$

and $\mathbf{q}_2 = [1 + \mathbf{j}]$

then $\mathbf{q}_1\mathbf{q}_2 = [1 + \mathbf{i} + \mathbf{j} + \mathbf{k}]$

2.9.3 Magnitude of a quaternion

$$\|\mathbf{q}_1\| = \sqrt{s^2 + x^2 + y^2 + z^2}$$

Given $\mathbf{q}_1 = [1 + 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}]$

then $\|\mathbf{q}_1\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$

2.9.4 The inverse quaternion

$$\mathbf{q}_1^{-1} = \frac{[s - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}]}{\|\mathbf{q}_1\|^2}$$

Given $\mathbf{q}_1 = [1 + 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}]$

then $\mathbf{q}_1^{-1} = \frac{1}{30}[1 - 2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}] = [\frac{1}{30} - \frac{1}{15}\mathbf{i} - \frac{1}{10}\mathbf{j} - \frac{2}{15}\mathbf{k}]$

2.9.5 Rotating a vector

Rotate $\mathbf{p}$ using $\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$ where $\mathbf{q} = [\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})\hat{\mathbf{v}}]$

Let $\mathbf{p}$ be the quaternion for $(1, 0, 0)$ i.e. $\mathbf{p} = [0 + \mathbf{i}]$

Let $\mathbf{q}$ be a unit quaternion aligned with the z-axis which rotates $\mathbf{p}$ 180°

i.e. $\quad \mathbf{q} = [\cos 90°, \sin 90°(\mathbf{k})] = [0 + \mathbf{k}]$

then $\quad \mathbf{q}^{-1} = [-\mathbf{k}]$

but $\quad \|\mathbf{q}\| = 1$

therefore $\quad \mathbf{p}' = [0 + \mathbf{k}] \cdot [0 + \mathbf{i}] \cdot [0 - \mathbf{k}] = [0 + \mathbf{j}] \cdot [0 - \mathbf{k}] = [0 - \mathbf{i}]$

$[0 - \mathbf{i}]$ points to the rotated point: $(-1, 0, 0)$, which is correct.

2.9.6 Quaternion as a matrix

$$R(\theta) = \begin{bmatrix} s^2 + x^2 - y^2 - z^2 & 2(xy - sz) & 2(xy + sz) \\ 2(xy + sz) & s^2 + y^2 - x^2 - z^2 & 2(yz - sx) \\ 2(xz - sy) & 2(yz + sx) & s^2 + z^2 - x^2 - y^2 \end{bmatrix}$$

Let's express the previous rotation quaternion as a matrix:

Given $\quad [0 + \mathbf{k}] \quad$ then $\quad s = 0, \quad x = 0, \quad y = 0, \quad z = 1$

therefore
$$R(\theta) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then
$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

which confirms the previous result.

2.10 Transformations

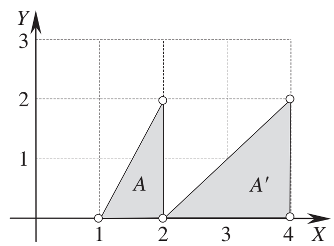
In the following examples the coordinates of the original shape A are shown on the right-hand side of the transform enclosed in brackets, whilst the coordinates of the transformed shape A' are shown on the left-hand side.

2.10.1 Scaling relative to the origin in $\mathbb{R}^2$

Scale shape A by a factor of 2 in the x-direction and 1 in the y-direction relative to the origin.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 2 & 4 & 4 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

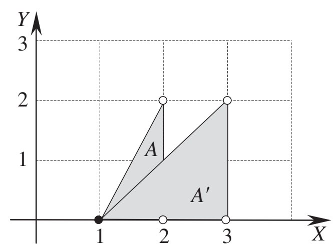


2.10.2 Scaling relative to a point in $\mathbb{R}^2$

Scale shape A by a factor of 2 in the x-direction and 1 in the y-direction relative to the point $(1, 0)$.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & x_P(1-S_x) \\ 0 & S_y & y_P(1-S_y) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

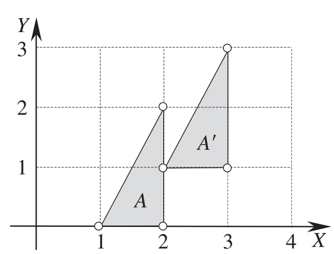


2.10.3 Translation in $\mathbb{R}^2$

Translate shape A by 1 in the x-direction and 1 in the y-direction.

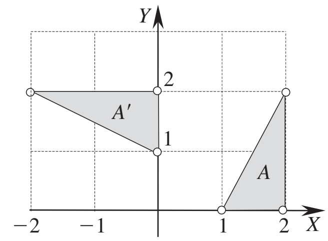
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 2 & 3 & 3 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$



2.10.4 Rotation about the origin in $\mathbb{R}^2$

Rotate shape A 90° about the origin.

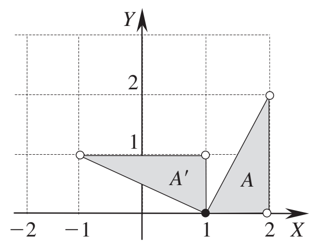


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.5 Rotation about a point in $\mathbb{R}^2$

Rotate shape A 90° about the point $(1, 0)$.

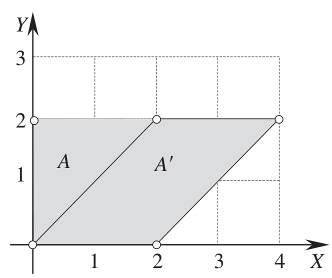


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & x_p(1-\cos\alpha) + y_p\sin\alpha \\ \sin\alpha & \cos\alpha & y_p(1-\cos\alpha) - x_p\sin\alpha \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.6 Shearing along the x-axis in $\mathbb{R}^2$

Shear shape A 45° along the x-axis.

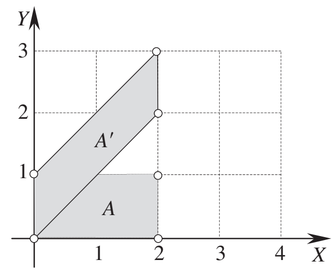


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 2 & 4 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

2.10.7 Shearing along the y-axis in $\mathbb{R}^2$

Shear shape A 45° along the y-axis.

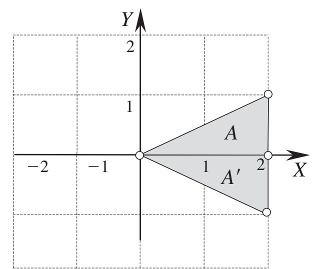


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \tan\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

2.10.8 Reflection about the x-axis in $\mathbb{R}^2$

Reflect shape A about the x-axis.

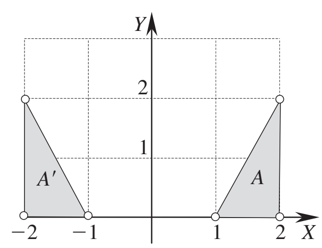


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.9 Reflection about the y-axis in $\mathbb{R}^2$

Reflect shape A about the y-axis.

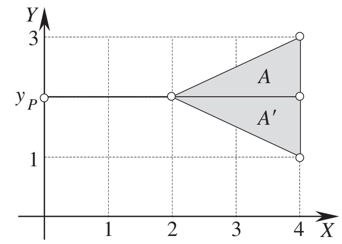


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} -1 & -2 & -2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.10 Reflection about a line parallel with the *x*-axis in $\mathbb{R}^2$

Reflect shape A about the line $y_p = 2$.

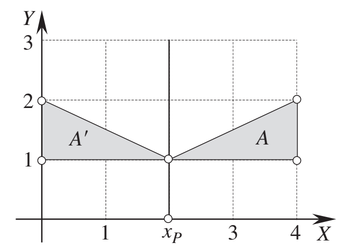


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2y_P \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.11 Reflection about a line parallel with the *y*-axis in $\mathbb{R}^2$

Reflect shape A about the line $x_p = 2$.

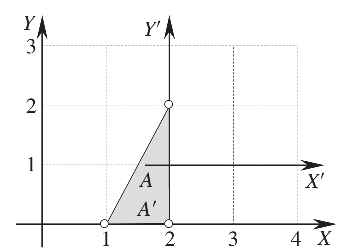


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2x_P \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} -1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 2 & 4 & 4 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.12 Translated change of axes in $\mathbb{R}^2$

The axes are subjected to a translation of $(2, 1)$.

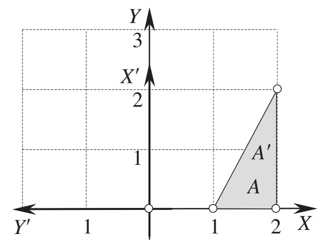


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -x_T \\ 0 & 1 & -y_T \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.13 Rotated change of axes in $\mathbb{R}^2$

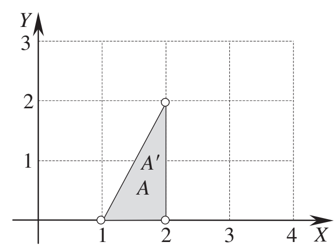
Rotate the axes 90°.



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 0 & 2 \\ -1 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.14 The identity matrix in $\mathbb{R}^2$

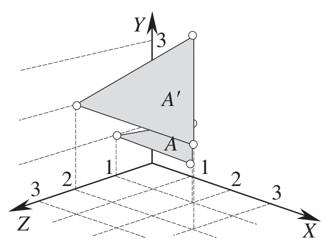


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.15 Scaling relative to the origin in $\mathbb{R}^3$

Scale shape A 1.5 in the x-direction, 2 in the y-direction and 2 in the z-direction.

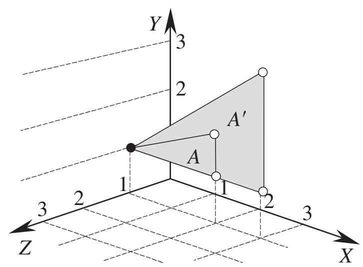


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 3 & 3 \\ 2 & 2 & 4 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.16 Scaling relative to a point in $\mathbb{R}^3$

Scale shape A 1.5 in the x-direction, 2 in the y-direction and 2 in the z-direction relative to the point $(0, 1, 1)$.

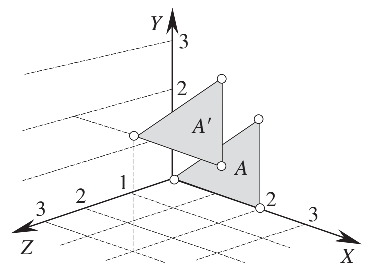


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & x_P(1-S_x) \\ 0 & S_y & 0 & y_P(1-S_y) \\ 0 & 0 & S_z & z_P(1-S_z) \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 0 & 3 & 3 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.17 Translation in $\mathbb{R}^3$

Translate shape A by $(2, 2, 3)$.

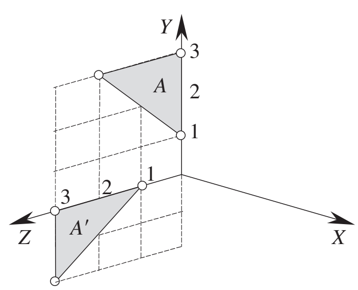


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 4 \\ 3 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.18 Rotation about the x-axis in $\mathbb{R}^3$

Rotate shape A about the x-axis 90°.

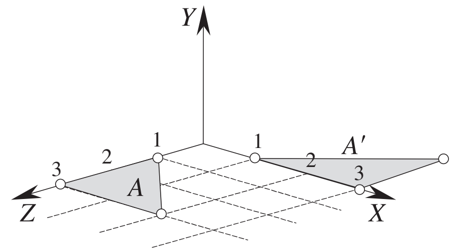


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 3 & 3 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.19 Rotation about the *y*-axis in $\mathbb{R}^3$

Rotate shape A about the y-axis 90°.

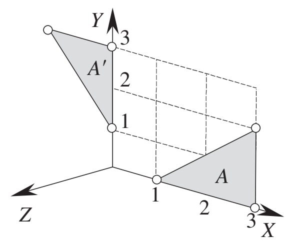


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.20 Rotation about the *z*-axis in $\mathbb{R}^3$

Rotate shape A about the z-axis 90°.



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 0 & -2 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.21 Rotation about an arbitrary axis in $\mathbb{R}^3$

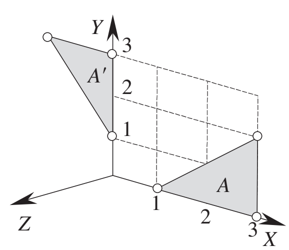
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a^2K + \cos\alpha & abK - c\sin\alpha & acK + b\sin\alpha & 0 \\ abK + c\sin\alpha & b^2K + \cos\alpha & bcK - a\sin\alpha & 0 \\ acK - b\sin\alpha & bcK + a\sin\alpha & c^2K + \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$K = 1 - \cos\alpha$

Axis $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\|\mathbf{v}\| = 1$

Given $\mathbf{v} = \mathbf{k}$ and $\alpha = 90°$

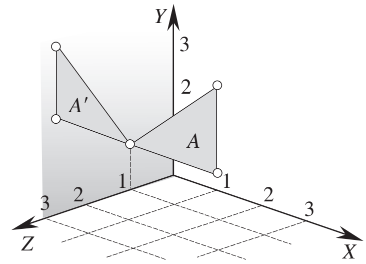
then $K = 1$



$$\underset{A'}{\begin{bmatrix} 0 & 0 & -2 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.22 Reflection about the *yz*-plane in $\mathbb{R}^3$

Reflect shape A in the yz-plane.

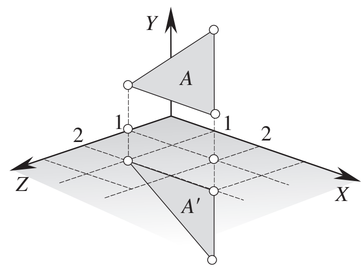


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & -2 & -2 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.23 Reflection about the *zx*-plane in $\mathbb{R}^3$

Reflect shape A in the zx-plane.

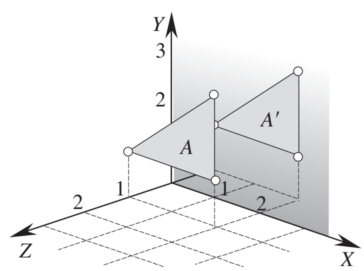


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 2 & 2 \\ -1 & -1 & -3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.24 Reflection about the *xy*-plane in $\mathbb{R}^3$

Reflect shape A in the xy-plane.

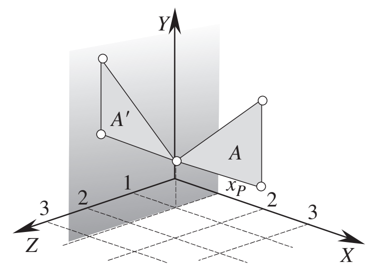


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underset{A'}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 3 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.25 Reflection about a plane parallel with the *yz*-plane in $\mathbb{R}^3$

Reflect shape A in the yz-plane $x_p = 1$.

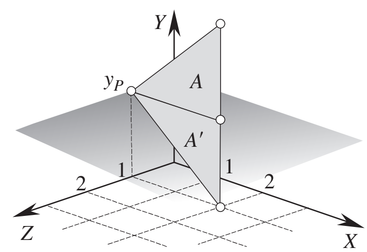


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 2x_P \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.26 Reflection about a plane parallel with the *zx*-plane in $\mathbb{R}^3$

Reflect shape A in the zx-plane $y_p = 2$.

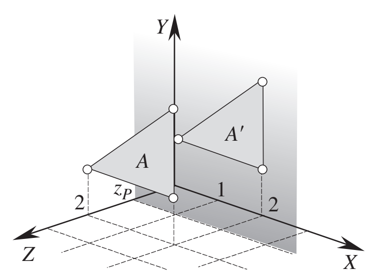


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2y_P \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 0 & 2 & 2 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 2 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.27 Reflection about a plane parallel with the *xy*-plane in $\mathbb{R}^3$

Reflect shape A in the xy-plane $z_p = 1$.

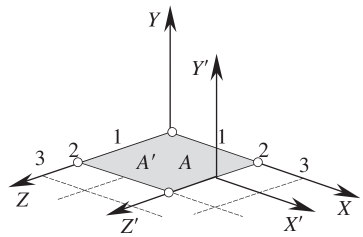


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2z_P \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

2.10.28 Translated axes in $\mathbb{R}^3$

The axes are subjected to a translation of $(2, 0, 1)$.

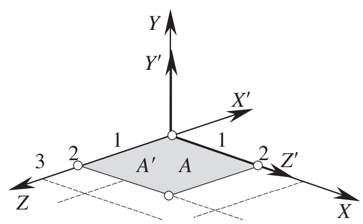


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -x_T \\ 0 & 1 & 0 & -y_T \\ 0 & 0 & 1 & -z_T \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

2.10.29 Rotated axes in $\mathbb{R}^3$

The axes are subjected to a rotation as illustrated.

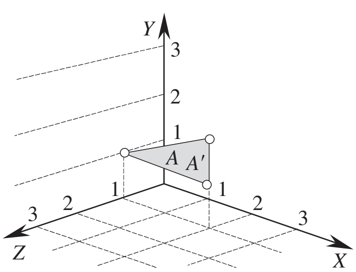


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\overset{A'}{\begin{bmatrix} 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

2.10.30 The identity matrix in $\mathbb{R}^3$

$$\overset{A'}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \overset{\text{Transform}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$



2.11 Two-dimensional straight lines

2.11.1 Convert the normal form of the line equation to its general form and the Hessian normal form

Given the normal form of the line equation

$$y = -\frac{3}{4}x + \frac{5}{4}$$

The general form of the line equation is obtained by rearranging the equation to

$$3x + 4y - 5 = 0$$

The Hessian normal form is obtained be dividing throughout by the magnitude of the line's normal vector:

$$\frac{3x + 4y - 5}{\sqrt{3^2 + 4^2}} = 0$$

$$\frac{3}{5}x + \frac{4}{5}y - 1 = 0$$

The line intersects the x-axis at $x = 1\frac{2}{3}$ and the y-axis at $y = 1\frac{1}{4}$. The unit normal vector to the line $\hat{\mathbf{n}} = 0.6\mathbf{i} + 0.8\mathbf{j}$ and the perpendicular from the origin to the line is 1.

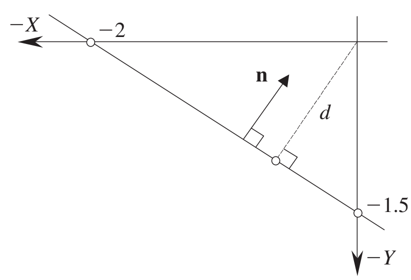
2.11.2 Derive the unit normal vector and perpendicular from the origin to the line for the line equation $3x + 4y + 6 = 0$

The normal vector is $\mathbf{n} = 3\mathbf{i} + 4\mathbf{j}$

The unit normal vector is $\hat{\mathbf{n}} = \frac{1}{\sqrt{3^2 + 4^2}}(3\mathbf{i} + 4\mathbf{j})$

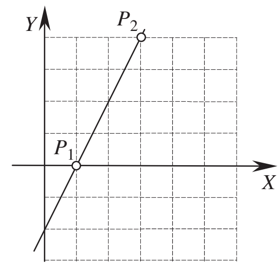
$= 0.6\mathbf{i} + 0.8\mathbf{j}$

The distance is $d = \frac{|c|}{\sqrt{3^2 + 4^2}} = \frac{6}{5} = 1.2$



2.11.3 Derive the straight-line equation from two points

Normal form of the line equation



Given $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

and $y = mx + c$

then $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

and $c = y_1 - x_1\left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)$

If the two points are $P_1(1, 0)$ and $P_2(3, 4)$

then $y = \left(\dfrac{4-0}{3-1}\right)x + 0 - 1\left(\dfrac{4-0}{3-1}\right)$

and $y = 2x - 2$

General form of the line equation

Given $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

and $Ax + By + C = 0$

then $A = y_2 - y_1 \qquad B = x_1 - x_2 \qquad C = -(x_1 y_2 - x_2 y_1)$

If the two points are $P_1(1, 0)$ and $P_2(3, 4)$

then $(4 - 0)x + (1 - 3)y - (1 \times 4 - 3 \times 0) = 0$

and $4x - 2y - 4 = 0$

or $2x - y - 2 = 0$

Determinant form of the line equation

Given $\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} x + \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} y = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$

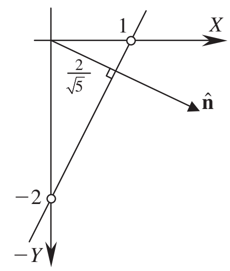
If the two points are $P_1(1, 0)$ and $P_2(3, 4)$

then $\begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} x + \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} y = \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix}$

and $4x - 2y - 4 = 0$

or $2x - y - 2 = 0$

Hessian normal form of the line equation



Given $4x - 2y - 4 = 0$

The normalizing factor is $\dfrac{1}{\sqrt{a^2+b^2}} = \dfrac{1}{\sqrt{16+4}} = \dfrac{1}{\sqrt{20}}$

then $\dfrac{4}{\sqrt{20}}x - \dfrac{2}{\sqrt{20}}y - \dfrac{4}{\sqrt{20}} = 0$

and $\dfrac{2}{\sqrt{5}}x - \dfrac{1}{\sqrt{5}}y - \dfrac{2}{\sqrt{5}} = 0$

The normal unit vector to the line is $\hat{\mathbf{n}} = \dfrac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$

The perpendicular from the origin to the line $= \dfrac{2}{\sqrt{5}}$

Parametric form of the line equation

Given $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

and $\mathbf{p} = \mathbf{p}_1 + \lambda\mathbf{v}$

and $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$

If the two points are $P_1(1, 0)$ and $P_2(3, 4)$

$$\mathbf{v} = 2\mathbf{i} + 4\mathbf{j}$$

Therefore $x = 1 + 2\lambda$

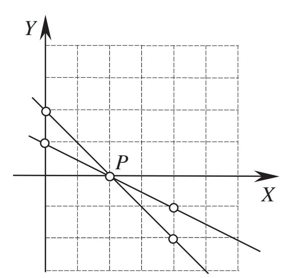
and $y = 4\lambda$

For example, when $\lambda = 0$ $x = 1$ $y = 0$

and when $\lambda = -0.5$ $x = 0$ $y = -2$

2.11.4 Point of intersection of two straight lines

General form of the line equation



Given $a_1x + b_1y + c_1 = 0$

and $a_2x + b_2y + c_2 = 0$

They intersect at $x_P = \dfrac{c_2b_1 - c_1b_2}{a_1b_2 - a_2b_1}$ $y_P = \dfrac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$

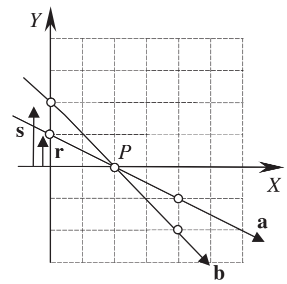
Let the straight lines be $2x + 2y - 4 = 0$ and $2x + 4y - 4 = 0$

Therefore $x_P = \dfrac{-4 \times 2 + 4 \times 4}{2 \times 4 - 2 \times 2} = \dfrac{8}{4} = 2$

and $y_P = \dfrac{2 \times -4 - 2 \times -4}{2 \times 4 - 2 \times 2} = \dfrac{0}{4} = 0$

The point of intersection is $(2, 0)$ as confirmed by the diagram.

Parametric form of the line equation



Given $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$ $\qquad \mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

where $\mathbf{r} = x_R\mathbf{i} + y_R\mathbf{j}$ $\qquad \mathbf{s} = x_S\mathbf{i} + y_S\mathbf{j}$

and $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j}$ $\qquad \mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j}$

then $\lambda = \dfrac{x_b(y_S - y_R) - y_b(x_S - x_R)}{x_b y_a - x_a y_b}$

Point of intersection $x_P = x_R + \lambda x_a$ $\qquad y_P = y_R + \lambda y_a$

Given $\mathbf{r} = \mathbf{j}$ $\quad \mathbf{a} = 2\mathbf{i} - \mathbf{j}$ $\quad \mathbf{s} = 2\mathbf{j}$ $\quad \mathbf{b} = 2\mathbf{i} - 2\mathbf{j}$

$$\lambda = \frac{2(2-1) + 2(0-0)}{2 \times (-1) - 2 \times (-2)} = \frac{2}{2} = 1$$

$$x_P = 0 + 2 = 2 \qquad y_P = 1 + 1 \times (-1) = 0$$

The point of intersection is $(2, 0)$ as confirmed by the diagram.

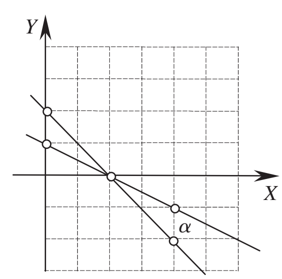
2.11.5 Calculate the angle between two straight lines

General form of the line equation

Given $a_1x + b_1y + c_1 = 0$ $\qquad a_2x + b_2y + c_2 = 0$

where $\mathbf{n} = a_1\mathbf{i} + b_1\mathbf{j}$ $\qquad \mathbf{m} = a_2\mathbf{i} + b_2\mathbf{j}$

Angle $\alpha = \cos^{-1}\left(\dfrac{\mathbf{n} \bullet \mathbf{m}}{\|\mathbf{n}\| \cdot \|\mathbf{m}\|}\right)$



Let the line equations be $2x + 2y - 4 = 0$

and $2x + 4y - 4 = 0$

Therefore $\alpha = \cos^{-1}\left(\dfrac{2 \times 2 + 2 \times 4}{\sqrt{2^2 + 2^2}\sqrt{2^2 + 4^2}}\right)$

$= 18.435°$

Normal form of the line equation

Given $y = m_1x + c_1 \qquad y = m_2x + c_2$

Angle $\alpha = \cos^{-1}\left(\dfrac{1+m_1m_2}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}\right)$

Let the line equations be $y = -x + 2$ and $y = -\dfrac{x}{2}+1$

where $m_1 = -1 \qquad m_2 = -\frac{1}{2}$

Therefore $\alpha = \cos^{-1}\left(\dfrac{1+(-1)(-\frac{1}{2})}{\sqrt{1+(-1)^2}\sqrt{1+(-\frac{1}{2})^2}}\right) = 18.435°$

Parametric form of the line equation

Given $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a} \qquad \mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

Angle $\alpha = \cos^{-1}\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}\right)$

Let the line equations be $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$ and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

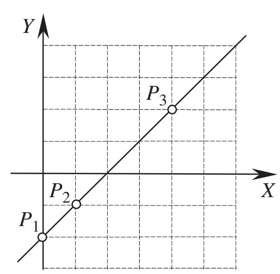
where $\mathbf{r} = \mathbf{j} \qquad \mathbf{a} = 2\mathbf{i} - \mathbf{j} \qquad \mathbf{s} = -2\mathbf{j} \qquad \mathbf{b} = 2\mathbf{i} - 2\mathbf{j}$

Therefore $\alpha = \cos^{-1}\left(\dfrac{2\times 2+(-1)(-2)}{\sqrt{5}\sqrt{8}}\right) = 18.435°$

2.11.6 Test if three points lie on a straight line

Given $P_1(x_1, y_1), P_2(x_2, y_2)$ and $P_3(x_3, y_3)$

and $\mathbf{r} = \overrightarrow{P_1P_2}$ and $\mathbf{s} = \overrightarrow{P_1P_3}$



The three points lie on a straight line when $\mathbf{s} = \lambda\mathbf{r}$.

Let the points be $P_1(0, -2) \qquad P_2(1, -1) \qquad P_3(4, 2)$

Therefore $\mathbf{r} = \mathbf{i} + \mathbf{j}$ and $\mathbf{s} = 4\mathbf{i} + 4\mathbf{j}$

and $\mathbf{s} = 4\mathbf{r}$

Therefore the points lie on a straight line as confirmed by the diagram.

2.11.7 Test for parallel and perpendicular lines

General form of the line equation

Given $a_1x + b_1y + c_1 = 0$ $a_2x + b_2y + c_2 = 0$

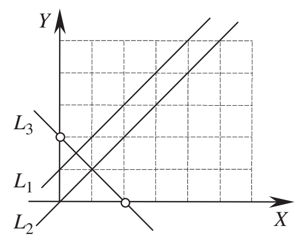
where $\mathbf{n} = a_1\mathbf{i} + b_1\mathbf{j}$ $\mathbf{m} = a_2\mathbf{i} + b_2\mathbf{j}$

The lines are parallel if $\mathbf{n} = \lambda\mathbf{m}$.

The lines are mutually perpendicular if $\mathbf{n} \cdot \mathbf{m} = 0$.

Given three lines

$$L_1: x - y + 1 = 0$$
$$L_2: x - y = 0$$
$$L_3: x + y - 2 = 0$$



L_1 and L_2 are parallel because the normal vectors to the lines are

$$\mathbf{n}_1 = \mathbf{i} - \mathbf{j} \quad \text{and} \quad \mathbf{n}_2 = \mathbf{i} - \mathbf{j}$$

and

$$\mathbf{n}_1 = \lambda\mathbf{n}_2 \quad (\lambda = 1)$$

L_1 and L_2 are perpendicular because

$$\mathbf{n} \cdot \mathbf{m} = 0 \qquad 1 \times 1 + (-1) \times 1 = 0$$

Normal form of the line equation

Given $y = m_1x + c_1$ $y = m_2x + c_2$

The lines are parallel if $m_1 = m_2$.

The lines are mutually perpendicular if $m_1m_2 = -1$

Given three lines

$$L_1 : y = x + 1$$
$$L_2 : y = x$$
$$L_3 : y = -x + 2$$

L_1 and L_2 are parallel because

$$m_1 = m_2 = 1$$

L_1 and L_3 are perpendicular because

$$m_1m_3 = -1 \qquad 1 \times (-1) = -1$$

Parametric form of the line equation

Given $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$ $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

where $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j}$ $\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j}$

The lines are parallel if $\mathbf{a} = k\mathbf{b}$.

The lines are mutually perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$.

Given three lines $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$ $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$ $\mathbf{u} = \mathbf{t} + \beta\mathbf{c}$

where $L_1 : \mathbf{a} = \mathbf{i} + \mathbf{j}$

and $L_2 : \mathbf{b} = \mathbf{i} + \mathbf{j}$

and $L_3 : \mathbf{c} = \mathbf{i} - \mathbf{j}$

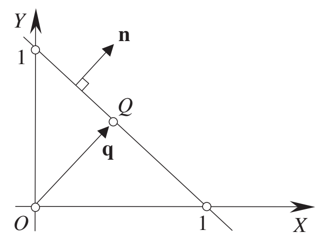
L_1 and L_2 are parallel because $\mathbf{a} = \mathbf{b}$

L_1 and L_3 are perpendicular because

$$x_a x_c + y_a y_c = 0 \qquad 1 \times 1 + 1 \times (-1) = 0$$

2.11.8 Find the position and distance of the nearest point on a line to the origin

General form of the line equation



Given $ax + by + c = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$

$\mathbf{q} = \lambda \mathbf{n}$

where $\lambda = \dfrac{-c}{\mathbf{n} \bullet \mathbf{n}}$

Distance $OQ = \|\mathbf{q}\| = \|\lambda \mathbf{n}\|$

Given the line equation $x + y - 1 = 0$

where $a = 1 \qquad b = 1 \qquad c = -1$

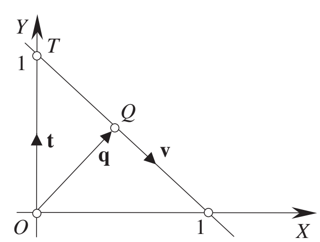
Therefore $\lambda = \frac{1}{2}$

and $x_Q = \lambda x_n = \frac{1}{2} \qquad y_Q = \lambda y_n = \frac{1}{2}$

The nearest point is $Q\left(\frac{1}{2}, \frac{1}{2}\right)$

Distance $OQ = \frac{1}{2}\|\mathbf{n}\| = \frac{1}{2}\sqrt{2} = 0.7071$

Parametric form of the line equation



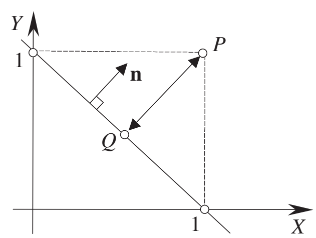
Given $\mathbf{q} = \mathbf{t} + \lambda \mathbf{v}$

where $\lambda = \dfrac{-\mathbf{v} \bullet \mathbf{t}}{\mathbf{v} \bullet \mathbf{v}}$

Distance	$OQ = \|\mathbf{q}\|$
Given the direction vectors	$\mathbf{t} = \mathbf{j} \qquad \mathbf{v} = \mathbf{i} - \mathbf{j}$
	$\lambda = \frac{1}{2}$
	$x_Q = x_T + \lambda x_v = 0 + \frac{1}{2} \times 1 = \frac{1}{2}$
	$y_Q = y_T + \lambda y_v = 1 + \frac{1}{2} \times (-1) = \frac{1}{2}$
The nearest point is	$Q\left(\frac{1}{2}, \frac{1}{2}\right)$
Distance	$OQ = \|\mathbf{t} + \lambda\mathbf{v}\| = \|\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\| = 0.7071$

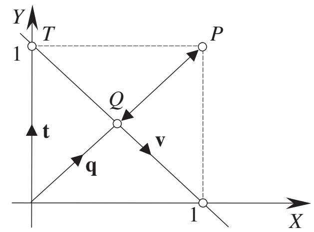
2.11.9 Find the position and distance of the nearest point on a line to a point

General form of the line equation



Given	$ax + by + c = 0$
where	$\mathbf{n} = a\mathbf{i} + b\mathbf{j}$
	$\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$
where	$\lambda = -\dfrac{\mathbf{n} \bullet \mathbf{p} + c}{\mathbf{n} \bullet \mathbf{n}}$
Distance	$PQ = \|\lambda\mathbf{n}\|$
Given	$P(1, 1) \quad \text{and} \quad x + y - 1 = 0$
then	$a = 1 \qquad b = 1 \qquad c = -1$
	$\lambda = -\dfrac{2-1}{2} = -\frac{1}{2}$
Therefore	$x_Q = x_P + \lambda x_n = 1 - \frac{1}{2} \times 1 = \frac{1}{2}$
	$y_Q = y_P + \lambda y_n = 1 - \frac{1}{2} \times 1 = \frac{1}{2}$
The nearest point is	$Q\left(\frac{1}{2}, \frac{1}{2}\right)$
Distance	$PQ = \|\lambda\mathbf{n}\| = \frac{1}{2}\|\mathbf{i} + \mathbf{j}\| = 0.7071$

Parametric form of the line equation



Given $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

where $\lambda = \dfrac{\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})}{\mathbf{v} \bullet \mathbf{v}}$

Distance $PQ = \|\mathbf{p} - \mathbf{t} - \lambda\mathbf{v}\|$

Given the direction vectors $\mathbf{t} = \mathbf{j}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j}$

and $\mathbf{p} = \mathbf{i} + \mathbf{j}$

$$\lambda = \tfrac{1}{2}$$

$$x_Q = x_T + \lambda x_v = 0 + \tfrac{1}{2} = \tfrac{1}{2}$$

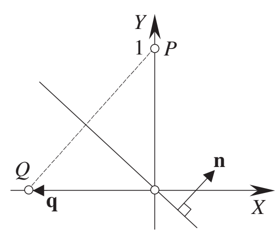
$$y_Q = y_T + \lambda y_v = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

The nearest point is $Q\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$

Distance $PQ = \|\mathbf{p} - \mathbf{t} - \lambda\mathbf{v}\| = \|\tfrac{1}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j}\| = 0.7071$

2.11.10 Find the reflection of a point in a line passing through the origin

General form of the line equation



Given $ax + by + c = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$

$$\mathbf{q} = \mathbf{p} - \lambda\mathbf{n}$$

$$\lambda = \frac{2(\mathbf{n} \bullet \mathbf{p} + c)}{\mathbf{n} \bullet \mathbf{n}}$$

Given the line equation $x + y = 0$

where $a = 1 \quad b = 1 \quad c = 0 \quad P(1, 1)$

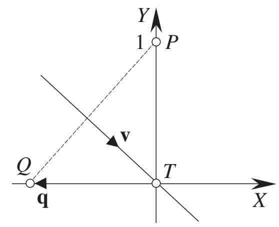
$$\lambda = \frac{2 \times 1}{2} = 1$$

Therefore $x_Q = x_P - \lambda x_n = 0 - 1 \times 1 = -1$

$y_Q = y_P - \lambda y_n = 1 - 1 \times 1 = 0$

The reflection point is $Q(-1, 0)$

Parametric form of the line equation



Given $\mathbf{s} = \mathbf{t} + \lambda\mathbf{v}$

$\mathbf{q} = 2\mathbf{t} + \varepsilon\mathbf{v} - \mathbf{p}$

where $\varepsilon = \dfrac{2\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})}{\mathbf{v} \bullet \mathbf{v}}$

Given $x_P = 0 \quad y_P = 1 \quad \mathbf{t} = 0 \quad \mathbf{v} = \mathbf{i} - \mathbf{j}$

$$\varepsilon = \frac{2 \times (-1)}{2} = -1$$

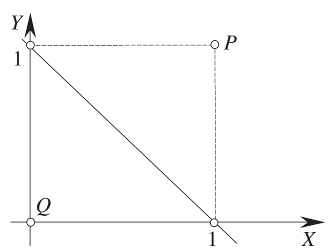
Therefore $x_Q = 2x_T + \varepsilon x_v - x_P = 2 \times 0 - 1 \times 1 - 0 = -1$

$y_Q = 2y_T + \varepsilon y_v - y_P = 2 \times 0 - 1 \times (-1) - 1 = 0$

The reflection point is $Q(-1, 0)$

2.11.11 Find the reflection of a point in a line

General form of the line equation



Given $ax + by + c = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$

$\mathbf{q} = \mathbf{p} - \lambda\mathbf{n}$

$$\lambda = \frac{2(\mathbf{n} \bullet \mathbf{p} + c)}{\mathbf{n} \bullet \mathbf{n}}$$

Given the line equation $x + y - 1 = 0$

where $a = 1 \quad b = 1 \quad c = -1 \quad x_P = 1 \quad y_P = 1$

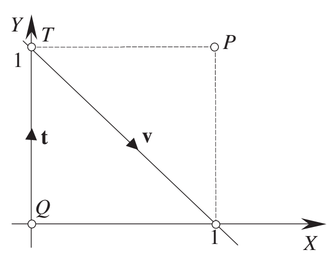
$$\lambda = \frac{2 \times (2 - 1)}{2} = 1$$

Therefore $x_Q = x_P - \lambda x_n = 1 - 1 \times 1 = 0$

$y_Q = y_P - \lambda y_n = 1 - 1 \times 1 = 0$

The reflection point is $Q(0, 0)$

Parametric form of the line equation



Given $\mathbf{s} = \mathbf{t} + \lambda\mathbf{v}$

$\mathbf{q} = 2\mathbf{t} + \varepsilon\mathbf{v} - \mathbf{p}$

where $\varepsilon = \dfrac{2\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})}{\mathbf{v} \bullet \mathbf{v}}$

Given $x_P = 1 \quad y_P = 1 \quad \mathbf{t} = \mathbf{j} \quad \mathbf{v} = \mathbf{i} - \mathbf{j}$

$$\varepsilon = \frac{2 \times 1}{2} = 1$$

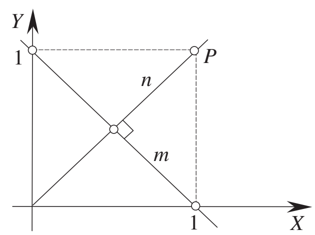
Therefore $x_Q = 2x_T + \varepsilon x_v - x_P = 2 \times 0 + 1 \times 1 - 1 = 0$

$$y_Q = 2y_T + \varepsilon y_v - y_P = 2 \times 1 + 1 \times (-1) - 1 = 0$$

The reflection point is $Q(0, 0)$

2.11.12 Find the normal to a line through a point

General form of the line equation



If line m is $ax + by + c = 0$

and line n is perpendicular to m passing through the point $P(x_P, y_P)$

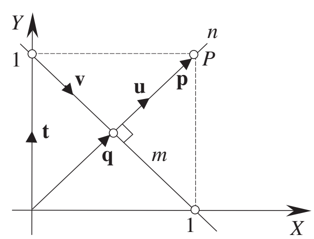
The line equation for n is $-bx + ay + bx_P - ay_P = 0$

Given m is $x + y - 1 = 0$

then $a = 1 \quad b = 1 \quad x_P = 1 \quad y_P = 1$

Line n is $-x + y = 0$

Parametric form of the line equation



Given line m $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$ and a point P

$$\mathbf{u} = \mathbf{p} - (\mathbf{t} + \lambda\mathbf{v})$$

where $$\lambda = \frac{\mathbf{v} \bullet (\mathbf{p} - \mathbf{t})}{\mathbf{v} \bullet \mathbf{v}}$$

line n is $\mathbf{p} + \varepsilon\mathbf{u}$ where ε is a scalar.

Given $\mathbf{v} = \mathbf{i} - \mathbf{j} \quad \mathbf{p} = \mathbf{i} + \mathbf{j} \quad \mathbf{t} = \mathbf{j}$

$$\lambda = \frac{(\mathbf{i} - \mathbf{j}) \bullet \mathbf{i}}{(\mathbf{i} - \mathbf{j}) \bullet (\mathbf{i} - \mathbf{j})} = \frac{1}{2}$$

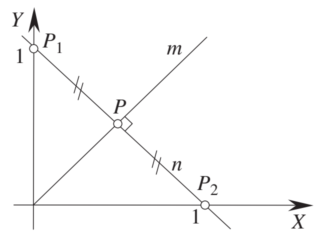
$$\mathbf{u} = (\mathbf{i} + \mathbf{j}) - (\mathbf{j} + \tfrac{1}{2}(\mathbf{i} - \mathbf{j})) = \tfrac{1}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j}$$

Line n is $$\mathbf{n} = (\mathbf{i} + \mathbf{j}) + \varepsilon(\tfrac{1}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j}) = (1 + \tfrac{1}{2}\varepsilon)\mathbf{i} + (1 + \tfrac{1}{2}\varepsilon)\mathbf{j}$$

where ε is a scalar, which is equivalent to $-x + y = 0$.

2.11.13 Find the line equidistant from two points

General form of the line equation



Given $ax + by + c = 0$

Line n is $x + y - 1 = 0$

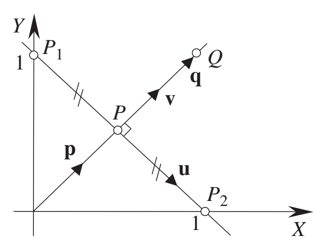
Line m is given by $(x_2 - x_1)\,x + (y_2 - y_1)\,y - \frac{1}{2}(x_2^2 - x_1^2 + y_2^2 - y_1^2) = 0$

with $P_1(0, 1)$ and $P_2(1, 0)$

Line m is $(1 - 0)\,x + (0 - 1)\,y - \frac{1}{2}(1 - 0 + 0 - 1) = 0$

$$x - y = 0$$

Parametric form of the line equation



Given $\mathbf{q} = \mathbf{p} + \lambda\mathbf{v}$

where $\mathbf{q} = (\frac{1}{2}(x_1 + x_2) - \lambda(y_2 - y_1))\mathbf{i} + (\frac{1}{2}(y_1 + y_2) + \lambda(x_2 - x_1))\mathbf{j}$

with $P_1(0, 1)$ and $P_2(1, 0)$

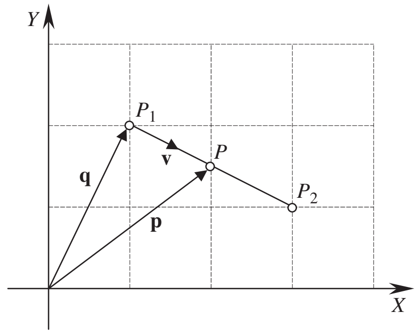
Therefore $\mathbf{q} = (\frac{1}{2}(0 + 1) - \lambda(0 - 1))\mathbf{i} + (\frac{1}{2}(1 + 0) + \lambda(1 - 0))\mathbf{j}$

$$\mathbf{q} = (\tfrac{1}{2} + \lambda)\mathbf{i} + (\tfrac{1}{2} + \lambda)\mathbf{j}$$

e.g. when $\lambda = 0$ we have $P\left(\frac{1}{2}, \frac{1}{2}\right)$ and when $\lambda = \frac{1}{2}$ the point is $Q(1, 1)$

This is equivalent to $y = x$ or $-x + y = 0$

2.11.14 Creating the parametric line equation for a line segment



P_1 (x_1, y_1) and P_2 (x_2, y_2) delimit the line segment and the parametric line equation is given by

$$\mathbf{p} = \mathbf{q} + \lambda\,\mathbf{v}$$

where $\quad \mathbf{q} = x_1\mathbf{i} + y_1\mathbf{j} \quad \text{and} \quad \mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$

Therefore $\quad x_P = x_1 + \lambda(x_2 - x_1)$

$$y_P = y_1 + \lambda(y_2 - y_1)$$

Given P_1 $(1, 2)$ and $P_2(3, 1)$. P is between P_1 and P_2 for $\lambda \in [0, 1]$

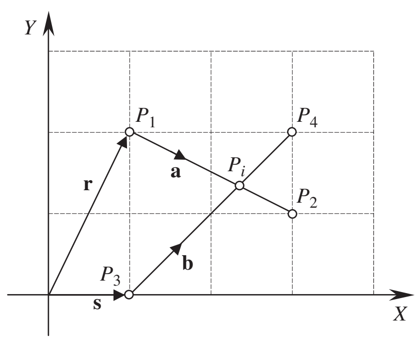
i.e. $\quad x_P = 1 + \lambda(3 - 1) = 1 + 2\lambda$

$$y_P = 2 + \lambda(1 - 2) = 2 - \lambda$$

For example, when $\lambda = 0.5$

$$x_{\frac{1}{2}} = 2 \quad \text{and} \quad y_{\frac{1}{2}} = 1.5$$

2.11.15 Intersecting two line segments



Given two line segments with equations $\mathbf{r} + \lambda\mathbf{a}$ and $\mathbf{s} + \varepsilon\mathbf{b}$

where $\quad \mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} \quad \text{and} \quad \mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j}$

The point of intersection is $x_i = x_r + \lambda x_a \quad y_i = y_r + \lambda y_a$

where $$\lambda = \frac{x_b(y_3 - y_1) - y_b(x_3 - x_1)}{x_b y_a - x_a y_b}$$

Let the two line segments be $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$ with $P_1(1,2), P_2(3,1), P_3(1,0), P_4(3,1)$

Therefore $\mathbf{r} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$

$\mathbf{s} = \mathbf{i}$ and $\mathrm{b} = 2\mathbf{i} + 2\mathbf{j}$

Therefore $$\lambda = \frac{2(0-2) - 2(1-1)}{2 \times (-1) - 2 \times 2} = \tfrac{2}{3}$$

As $0 < \lambda < 1$ there is a point of intersection

$$x_i = 1 + \tfrac{2}{3} \times 2 = 2\tfrac{1}{3}$$

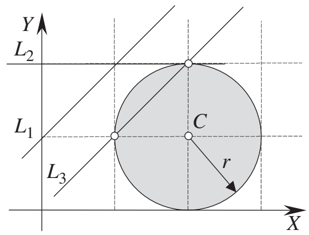
$$y_i = 2 + \tfrac{2}{3}(-1) = 1\tfrac{1}{3}$$

The point of intersection is $(2\tfrac{1}{3}, 1\tfrac{1}{3})$, which is correct.

2.12 Lines and circles

2.12.1 Line intersecting a circle

General form of the line equation



The diagram shows a circle radius $r = 1$ centered at $C(x_C, y_C) = (2, 1)$ and three lines: L_1, L_2 and L_3 that miss, touch and intersect the circle respectively.

The line equation is $\quad ax + by + c = 0$

Point(s) of intersection

$$x = x_C - ac_T \pm \sqrt{c_T^2(a^2 - 1) + b^2r^2} \tag{1}$$

$$y = y_C - bc_T \pm \sqrt{c_T^2(b^2 - 1) + a^2r^2}$$

where $\quad c_T = ax_C + by_C + c$

Miss condition

Line L_1 is

$$-x + y - 1 = 0 \tag{2}$$

L_1 normalized is

$$-\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - \frac{1}{\sqrt{2}} = 0$$

where

$$a = -\frac{1}{\sqrt{2}} \qquad b = \frac{1}{\sqrt{2}} \qquad c = -\frac{1}{\sqrt{2}}$$

then

$$c_T = -\frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$$

$$c_T^2(b^2 - 1) + a^2r^2 = 2\left(\tfrac{1}{2} - 1\right) + \tfrac{1}{2} = -\tfrac{1}{2}$$

The negative discriminant confirms the non-intersection.

Touch condition

Line L_2 is $\quad y - 2 = 0$ (which is already normalized) (3)

therefore $\quad a = 0 \qquad b = 1 \qquad c = -2$

and $\quad c_T = 0 + 1 - 2 = -1$

$$c_T^2(b^2 - 1) + a^2r^2 = 1(1 - 1) = 0$$

The zero discriminant confirms the touch condition:

using (1) $\quad x = 2$

and (2) $\quad y = 2$

Therefore the touching point is (2, 2) which is correct.

Intersect condition

Line L_3 is $\quad x - y = 0$ (4)

L_3 normalized is $\quad \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y = 0$

where $\quad a = \frac{1}{\sqrt{2}} \qquad b = -\frac{1}{\sqrt{2}} \qquad c = 0$

$$c_T = \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$c_T^2(b^2 - 1) + a^2r^2 = \tfrac{1}{4}$$

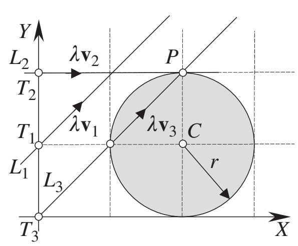
The positive discriminant confirms the intersect condition.

Using (1) $\quad x = 2 - \frac{1}{2} \pm \sqrt{\frac{1}{4}} = 2$ and 1

and (4) $\quad y = 2$ and 1

The intersection points are (2, 2) and (1, 1) which are correct.

Parametric form of the line equation



The diagram shows a circle radius $r = 1$ centered at $C(x_C, y_C) = (2, 1)$ and three lines: L_1, L_2 and L_3 that miss, touch, and intersect the circle respectively.

The lines are $\quad \mathbf{p}_1 = \mathbf{t}_1 + \lambda \mathbf{v}_1 \quad \mathbf{p}_2 = \mathbf{t}_2 + \lambda \mathbf{v}_2 \quad \mathbf{p}_3 = \mathbf{t}_3 + \lambda \mathbf{v}_3$

where $\quad \mathbf{t}_1 = \mathbf{j} \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

$\mathbf{t}_2 = 2\mathbf{j} \quad \mathbf{v}_2 = \mathbf{i}$

$\mathbf{t}_3 = 0 \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

and $\quad \mathbf{c} = 2\mathbf{i} + \mathbf{j}$

Let us substitute the lines into the following equations:

Point(s) of intersection $\quad x_p = x_T + \lambda x_v$

$y_P = y_T + \lambda y_v$

where $\quad \lambda = \mathbf{s} \bullet \mathbf{v} \pm \sqrt{(\mathbf{s} \bullet \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2}$

$\mathbf{s} = \mathbf{c} - \mathbf{t}$

L_1: $\quad \mathbf{s} = 2\mathbf{i}$

$(\mathbf{s} \bullet \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 2 - 4 + 1 = -1$

The negative discriminant confirms a miss condition.

L_2: $\quad \mathbf{s} = 2\mathbf{i} - \mathbf{j}$

$(\mathbf{s} \bullet \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 4 - 5 + 1 = 0$

The zero discriminant confirms a touch condition.

Therefore $\quad \lambda = 2$

The touch point is $\quad x_P = 2 \quad y_P = 2 \quad$ which is correct.

L_3: $\quad \mathbf{s} = 2\mathbf{i} + \mathbf{j}$

$(\mathbf{s} \bullet \mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 4.5 - 5 + 1 = \frac{1}{2}$

The positive discriminant confirms an intersect condition.

Therefore $\quad \lambda = \frac{3}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} = 2\sqrt{2} \quad$ and $\quad \sqrt{2}$

The intersection points are

$\lambda = 2\sqrt{2} \qquad x_P = 0 + 2\sqrt{2}\frac{1}{\sqrt{2}} = 2$

$$y_P = 0 + 2\sqrt{2}\frac{1}{\sqrt{2}} = 2$$

$\lambda = \sqrt{2}$

$$x_P = 0 + \sqrt{2}\frac{1}{\sqrt{2}} = 1$$

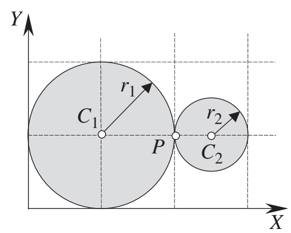
$$y_P = 0 + \sqrt{2}\frac{1}{\sqrt{2}} = 1$$

The intersection points are (1, 1) and (2, 2) which are correct.

2.12.2 Touching and intersecting circles

Touching circles

The diagram shows two circles touching one another at a point $P(x_P, y_P)$.



One circle with radius $r_1 = 1$ is centered at $C_1(1, 1)$, the other with radius $r_2 = 0.5$ is centered at C_2 (2.5, 1).

Given $$d = \sqrt{(x_{C2} - x_{C1})^2 + (y_{C2} - y_{C1})^2}$$

The touch condition is $$d = r_1 + r_2$$

The touch point is $$x_P = x_{C1} + \frac{r_1}{d}(x_{C2} - x_{C1}) \quad \text{and} \quad y_P = y_{C1} + \frac{r_1}{d}(y_{C2} - y_{C1})$$

then $$d = \sqrt{(2.5 - 1)^2 + (1 - 1)^2} = 1.5$$

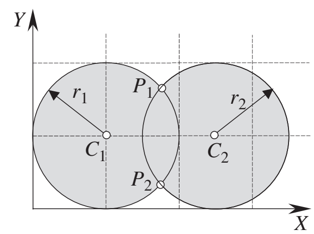
The touch condition is satisfied.

$$x_P = 1 + \frac{1}{1.5}(2.5 - 1) = 2$$

$$y_P = 1 + \frac{1}{1.5}(1 - 1) = 1$$

Therefore the touch point is $P(2, 1)$ which is correct.

Intersecting circles



The diagram shows two circles intersecting one another at points $P_1(x_{P1}, y_{P1})$ and $P_2(x_{P2}, y_{P2})$.

One circle with radius $r_1 = 1$ is centered at $C_1(1, 1)$, the other with radius $r_2 = 1$ is centered at $C_2(2.5, 1)$.

The intersect condition $\quad d < r_1 + r_2$

The points of intersection are

$$x_{P1} = x_{C1} + \lambda x_d - \varepsilon y_d$$

$$y_{P1} = y_{C1} + \lambda y_d + \varepsilon x_d$$

$$x_{P2} = x_{C1} + \lambda x_d + \varepsilon y_d$$

$$y_{P2} = y_{C1} + \lambda y_d - \varepsilon x_d$$

where

$$\lambda = \frac{r_1^2 - r_2^2 + d^2}{2d^2}$$

and

$$\varepsilon = \left|\sqrt{\frac{r_1^2}{d^2} - \lambda^2}\right|$$

$$d = 1$$

Therefore the intersect condition is satisfied.

$$\lambda = \frac{1 - 1 + 2.25}{2 \times 2.25} = \frac{1}{2}$$

$$\varepsilon = \left|\sqrt{\frac{4}{9} - \frac{1}{4}}\right| = \frac{\sqrt{7}}{6}$$

therefore

$$x_{P1} = 1 + \frac{1}{2}\frac{3}{2} = 1\tfrac{3}{4} \qquad y_{P1} = \frac{\sqrt{7}}{6}\frac{3}{2} = \tfrac{\sqrt{7}}{4}$$

and

$$x_{P2} = 1 + \frac{1}{2}\frac{3}{2} = 1\tfrac{3}{4} \qquad y_{P2} = -\frac{\sqrt{7}}{6}\frac{3}{2} = -\tfrac{\sqrt{7}}{4}$$

The intersection points are $\left(1\tfrac{3}{4}, \tfrac{\sqrt{7}}{4}\right)$ and $\left(1\tfrac{3}{4}, -\tfrac{\sqrt{7}}{4}\right)$ which are correct.

2.13 Second degree curves

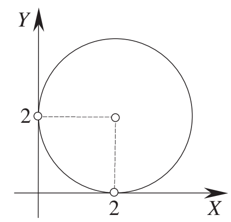
2.13.1 Circle

General equation

Center (x_C, y_C) $\quad (x - x_C)^2 + (y - y_C)^2 = r^2$

Given a radius $r = 2$ and center $(2, 2)$

then $\quad (x - 2)^2 + (y - 2)^2 = 4$



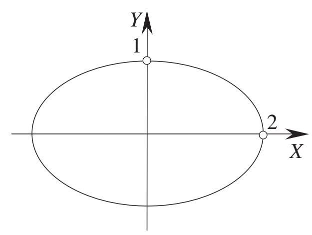
2.13.2 Ellipse

General equation

Center origin $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

with $a = 2, b = 1$

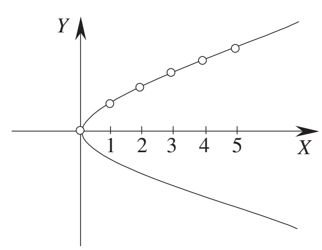
then $\quad \dfrac{x^2}{4} + y^2 = 1$



2.13.3 Parabola

Parametric equation

Vertex origin $\quad \left.\begin{aligned} x &= t^2 \\ y &= 2t \end{aligned}\right\} \quad t \in [-5, 5]$



t	0	± 1	± 2	± 3	± 4	± 5
x	0	1	4	9	16	25
y	0	± 2	± 4	± 6	± 8	± 10

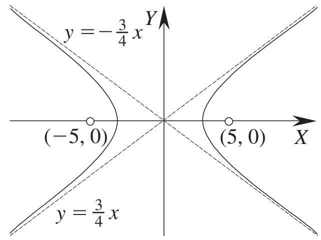
2.13.4 Hyperbola

General equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Foci at $(\pm c, 0)$ $\quad c = \sqrt{a^2 + b^2}$

then $\quad \frac{x^2}{16} - \frac{y^2}{9} = 1 \quad$ with $\quad c = 5$

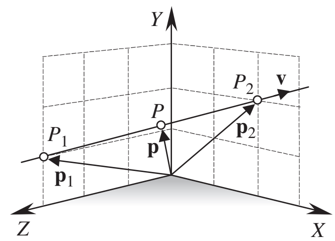


2.14 Three-dimensional straight lines

2.14.1 Derive the straight-line equation from two points

Given P_1 and P_2 $\quad \mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$

$\mathbf{p} = \mathbf{p}_1 + \lambda\mathbf{v}$



Given $\quad P_1(0, 1, 3)$ and $P_2(2, 2, 0)$

$P_1 = \mathbf{j} + 3\mathbf{k}$

$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

and $\quad \mathbf{p} = \mathbf{p}_1 + \lambda\mathbf{v}$

2.14.2 Intersection of two straight lines

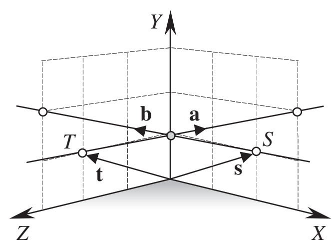
Given two lines $\quad \mathbf{p} = \mathbf{t} + \lambda\mathbf{a}$ and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

where $\quad \mathbf{t} = x_t\mathbf{i} + y_t\mathbf{j} + z_t\mathbf{k}$ and $\mathbf{s} = x_s\mathbf{i} + y_s\mathbf{j} + z_s\mathbf{k}$

$\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$ and $\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}$

Step 1: If $\mathbf{a} \times \mathbf{b} = 0$ the lines are parallel and do not intersect.

Step 2: If $(\mathbf{t} - \mathbf{s}) \cdot (\mathbf{a} \times \mathbf{b}) \neq 0$ the lines do not touch.



Step 3: Solving

$$\lambda x_a - \varepsilon x_b = x_s - x_t$$
$$\lambda y_a - \varepsilon y_b = y_s - y_t$$
$$\lambda z_a - \varepsilon z_b = z_s - z_t$$

provides values for λ and ε which, when substituted in the original line equations, reveal the intersection point.

Given $\quad \mathbf{t} = \mathbf{j} + 2\mathbf{k}$ and $\mathbf{s} = 2\mathbf{i} + \mathbf{j}$

$\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

Step 1: Prove that the lines are not parallel.
Although it is obvious that **a** and **b** are not parallel, let's prove it by ensuring that $\mathbf{a} \times \mathbf{b} \neq 0$.

	i	**j**	**k**
a	3	1	−2
b	−2	1	3
a × b	5	5	5

Therefore the lines are not parallel.

Step 2: Prove that the lines are touching.
If $(\mathbf{t} - \mathbf{s}) \cdot (\mathbf{a} \times \mathbf{b}) = 0$ the lines touch.
Therefore $(2\mathbf{i} + 2\mathbf{k}) \cdot (5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}) = 0$ so the lines touch.

Step 3: Compute the intersection point.
Create the three equations:

$$3\lambda + 2\varepsilon = 2 \quad (1)$$
$$\lambda - \varepsilon = 0 \quad (2)$$
$$-2\lambda - 3\varepsilon = -2 \quad (3)$$

From (2) $\lambda = \varepsilon$

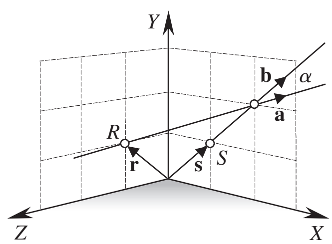
Substituting $\lambda = \varepsilon$ in (1) $\lambda = \frac{2}{5}$ and $\varepsilon = \frac{2}{5}$

Substitute λ and ε in the original line equations

$$\mathbf{p} = (\mathbf{j} + 2\mathbf{k}) + \tfrac{2}{5}(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \tfrac{6}{5}\mathbf{i} + \tfrac{7}{5}\mathbf{j} + \tfrac{6}{5}\mathbf{k}$$

The intersection point is $(\frac{6}{5}, \frac{7}{5}, \frac{6}{5})$

2.14.3 Calculate the angle between two straight lines



Given $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$

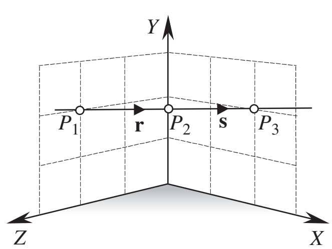
and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

angle $\alpha = \cos^{-1}\left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}\right)$

Given $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$

$$\alpha = \cos^{-1}\left(\frac{(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (\mathbf{i} + \mathbf{j})}{\sqrt{6}\sqrt{2}}\right)$$
$$= \cos^{-1}\left(\frac{3}{\sqrt{12}}\right) = 30°$$

2.14.4 Test if three points lie on a straight line



Given three points P_1, P_2, P_3.

Let $\mathbf{r} = \overrightarrow{P_1P_2}$ and $\mathbf{s} = \overrightarrow{P_1P_3}$

The points lie on a straight line when $\mathbf{s} = \lambda\mathbf{r}$ where λ is a scalar.

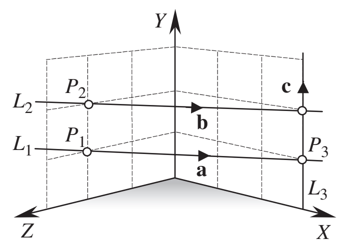
Given $P_1(0, 2, 2) \quad P_2(1, 2, 1) \quad P_3(2, 2, 0)$

therefore $\mathbf{r} = \mathbf{i} - \mathbf{k}$ and $\mathbf{s} = 2\mathbf{i} - 2\mathbf{k}$

and $\mathbf{s} = 2\mathbf{r}$

Therefore the points lie on a straight line.

2.14.5 Test for parallel and perpendicular straight lines



Given $\mathbf{p} = \mathbf{r} + \mu\mathbf{a}$

and $\mathbf{q} = \mathbf{s} + \varepsilon\mathbf{b}$

The lines are parallel if $\mathbf{a} = \lambda\mathbf{b}$ where λ is a scalar.
The lines are perpendicular if $\mathbf{a}\bullet\mathbf{b} = 0$.

Given three lines L_1: $\mathbf{p}_1 + \mu\mathbf{a}$

L_2: $\mathbf{p}_2 + \varepsilon\mathbf{b}$

L_3: $\mathbf{p}_3 + \lambda\mathbf{c}$

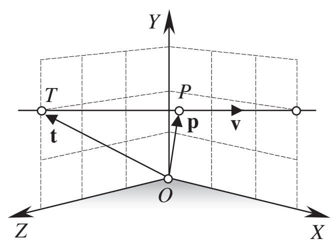
where $\mathbf{a} = 3\mathbf{i} - 2\mathbf{k}$

$\mathbf{b} = 3\mathbf{i} - 2\mathbf{k}$

$\mathbf{c} = \mathbf{j}$

L_1 and L_2 are parallel because $\mathbf{a} = \mathbf{b}$.
L_1 and L_3 are perpendicular because $\mathbf{a}\bullet\mathbf{c} = (3\mathbf{i} - 2\mathbf{k})\bullet(\mathbf{j}) = 0$.

2.14.6 Find the position and distance of the nearest point on a line to the origin



Given $\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

where $\lambda = \dfrac{-\mathbf{v}\bullet\mathbf{t}}{\mathbf{v}\bullet\mathbf{v}}$

Distance $OP = \|\mathbf{p}\|$

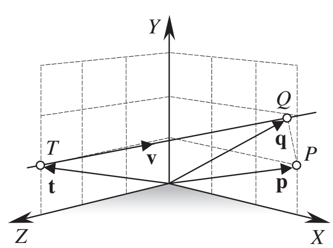
Given $\mathbf{t} = 2\mathbf{j} + 3\mathbf{k} \quad \mathbf{v} = 3\mathbf{i} - 3\mathbf{k}$

$$\lambda = \frac{-(3\mathbf{i} - 3\mathbf{k})\bullet(2\mathbf{j} + 3\mathbf{k})}{(3\mathbf{i} - 3\mathbf{k})\bullet(3\mathbf{i} - 3\mathbf{k})} = \frac{9}{18} = \frac{1}{2}$$

therefore
$$x_P = x_T + \lambda x_v = 0 + \tfrac{1}{2}\times 3 = 1\tfrac{1}{2}$$
$$y_P = y_T + \lambda y_v = 2 + \tfrac{1}{2}\times 0 = 2$$
$$z_P = z_T + \lambda z_v = 3 + \tfrac{1}{2}\times(-3) = 1\tfrac{1}{2}$$

Distance $OP = \|\mathbf{p}\| = \|\tfrac{3}{2}\mathbf{i} + 2\mathbf{j} + \tfrac{3}{2}\mathbf{k}\| = 2.92$

2.14.7 Find the position and distance of the nearest point on a line to a point



Given $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

where $\lambda = \dfrac{\mathbf{v}\bullet(\mathbf{p} - \mathbf{t})}{\mathbf{v}\bullet\mathbf{v}}$

Distance $PQ = \|\mathbf{p} - (\mathbf{t} + \lambda\mathbf{v})\|$

Given $\mathbf{t} = \mathbf{j} + 3\mathbf{k}$

$\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

and $\mathbf{p} = 3\mathbf{i} + \mathbf{j}$

$$\lambda = \frac{(3\mathbf{i}+\mathbf{j}-3\mathbf{k})\bullet(3\mathbf{i}-3\mathbf{k})}{(3\mathbf{i}+\mathbf{j}-3\mathbf{k})\bullet(3\mathbf{i}+\mathbf{j}-3\mathbf{k})} = \frac{18}{19}$$

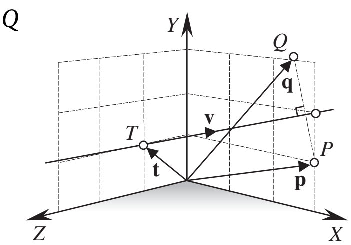
then $x_Q = x_T + \lambda x_v = 0 + \frac{18}{19}\times 3 = 2.842$

$y_Q = y_T + \lambda y_v = 1 + \frac{18}{19}\times 1 = 1.947$

$z_Q = z_T + \lambda z_v = 3 + \frac{18}{19}\times(-3) = 0.1579$

Distance $PQ = \|(3\mathbf{i}+\mathbf{j}) - ((\mathbf{j}+3\mathbf{k}) + \frac{18}{19}(3\mathbf{i}+\mathbf{j}-3\mathbf{k}))\| = 0.9733$

2.14.8 Find the reflection of a point in a line



Given $\mathbf{s} = \mathbf{t} + \lambda\mathbf{v}$ and a point P with reflection Q

$\mathbf{q} = 2\mathbf{t} + \varepsilon\mathbf{v} - \mathbf{p}$

where $\varepsilon = \dfrac{2\mathbf{v}\bullet(\mathbf{p}-\mathbf{t})}{\mathbf{v}\bullet\mathbf{v}}$

Given $\mathbf{t} = \mathbf{j} + \mathbf{k}$

$\mathbf{v} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

and $\mathbf{p} = 3\mathbf{i} + \mathbf{j}$

$$\varepsilon = \frac{2(3\mathbf{i}+\mathbf{j}-\mathbf{k})\bullet(3\mathbf{i}-\mathbf{k})}{(3\mathbf{i}+\mathbf{j}-\mathbf{k})\bullet(3\mathbf{i}+\mathbf{j}-\mathbf{k})} = \frac{20}{11}$$

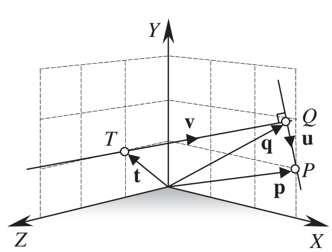
then $x_Q = 2x_T + \varepsilon x_v - x_P = 2\times 0 + \frac{20}{11}\times 3 - 3 = 2.4545$

$y_Q = 2y_T + \varepsilon y_v - y_P = 2\times 1 + \frac{20}{11}\times 1 - 1 = 2.8181$

$z_Q = 2z_T + \varepsilon z_v - z_P = 2\times 1 + \frac{20}{11}\times(-1) - 0 = 0.1818$

The reflection point is $Q(2.45, 2.82, 0.18)$

2.14.9 Find the normal to a line through a point



Given $\mathbf{q} = \mathbf{t} + \lambda\mathbf{v}$

the normal is $\mathbf{u} = \mathbf{p} - (\mathbf{t} + \lambda\mathbf{v})$

where $\lambda = \dfrac{\mathbf{v}\bullet(\mathbf{p}-\mathbf{t})}{\mathbf{v}\bullet\mathbf{v}}$

Given $\mathbf{t} = \mathbf{j} + \mathbf{k}$

$\mathbf{v} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

and $\mathbf{p} = 3\mathbf{i} + \mathbf{j}$

therefore $$\lambda = \frac{(3\mathbf{i}+\mathbf{j}-\mathbf{k})\bullet(3\mathbf{i}-\mathbf{k})}{(3\mathbf{i}+\mathbf{j}-\mathbf{k})\bullet(3\mathbf{i}+\mathbf{j}-\mathbf{k})} = \frac{10}{11}$$

and $$x_u = x_P - (x_T + \lambda x_v) = 3 - (0 + \tfrac{10}{11} \times 3) = 0.2727$$

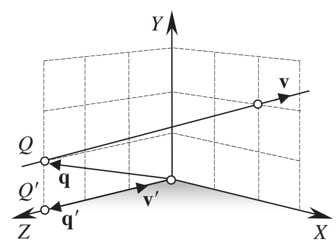
$$y_u = y_P - (y_T + \lambda y_v) = 1 - (1 + \tfrac{10}{11} \times 1) = -0.909$$

$$z_u = z_P - (z_T + \lambda z_v) = 0 - (1 + \tfrac{10}{11} \times (-1)) = -0.0909$$

therefore $$\mathbf{u} = 0.273\mathbf{i} - 0.909\mathbf{j} - 0.091\mathbf{k}$$

The line equation for the normal is $$\mathbf{n} = \mathbf{p} + \varepsilon\mathbf{u}$$

2.14.10 Find the shortest distance between two skew lines



Given $$\mathbf{p} = \mathbf{q} + t\mathbf{v}$$
and $$\mathbf{p}' = \mathbf{q}' + \tau\mathbf{v}'$$

Shortest distance $$d = \frac{\|(\mathbf{q} - \mathbf{q}') \cdot (\mathbf{v} \times \mathbf{v}')\|}{\|\mathbf{v} \times \mathbf{v}'\|}$$

Given $$\mathbf{q} = \mathbf{j} + 3\mathbf{k}$$
$$\mathbf{q}' = 3\mathbf{k}$$
$$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$
$$\mathbf{v}' = -\mathbf{k}$$

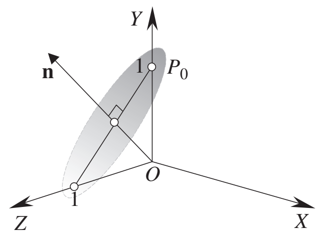
Calculate $\mathbf{v} \times \mathbf{v}'$

	i	**j**	**k**
$\mathbf{v}$	2	1	−3
$\mathbf{v}'$	0	0	−1
$\mathbf{v} \times \mathbf{v}'$	−1	2	0

$$d = \frac{\|\mathbf{j} \cdot (-\mathbf{i} + 2\mathbf{j})\|}{\|-\mathbf{i} + 2\mathbf{j}\|} = \frac{2}{\sqrt{5}} = 0.8944$$

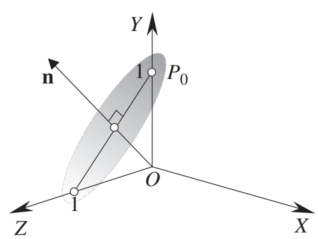
2.15 Planes

2.15.1 Cartesian form of the plane equation



Given	$ax + by + cz = d$
where the normal is	$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$
	$\mathbf{p}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$
and	$d = \mathbf{n} \bullet \mathbf{p}_0$
If the normal is	$\mathbf{n} = \mathbf{j} + \mathbf{k}$
and the point	$P_0(0, 1, 0)$
then	$0x + 1y + 1z = 0 \times 0 + 1 \times 1 + 1 \times 0 = 1$
The plane equation is	$y + z = 1$

2.15.2 General form of the plane equation



Given	$ax + by + cz - (ax_0 + by_0 + cz_0) = 0$
where the normal is	$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$
and a point is	$\mathbf{p}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$
If the normal is	$\mathbf{n} = \mathbf{j} + \mathbf{k}$
and the point	$P_0(0, 1, 0)$
then	$0x + 1y + 1z - (0 \times 0 + 1 \times 1 + 1 \times 0) = 0$
The plane equation is	$y + z - 1 = 0$

2.15.3 Hessian normal form of the plane equation

To convert the previous equation into Hessian normal form, rearrange the formula and divide throughout by $||\mathbf{n}||$.

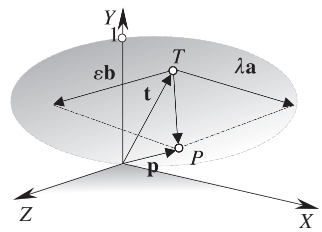
Given	$y + z - 1 = 0$
where the normal is	$\mathbf{n} = \mathbf{j} + \mathbf{k}$
	$||\mathbf{n}|| = \sqrt{2}$

therefore $$\frac{1}{\sqrt{2}}y+\frac{1}{\sqrt{2}}z-\frac{1}{\sqrt{2}}=0$$

or $$\tfrac{1}{2}\sqrt{2}y+\tfrac{1}{2}\sqrt{2}z-\tfrac{1}{2}\sqrt{2}=0$$

2.15.4 Parametric form of the plane equation

Given vectors **a** and **b** that are parallel to the plane and point T is on the plane



where $\mathbf{c}=\lambda\mathbf{a}+\varepsilon\mathbf{b}$

and $\mathbf{p}=\mathbf{t}+\mathbf{c}$

then
$$x_P=x_T+\lambda x_a+\varepsilon x_b$$
$$y_P=y_T+\lambda y_a+\varepsilon y_b$$
$$z_P=z_T+\lambda z_a+\varepsilon z_b$$

The plane is parallel with the xz-plane and intersects the y-axis at $y=1$. Let **a** and **b** be unit vectors parallel with the plane

i.e. $\mathbf{a}=\mathbf{i}\qquad\mathbf{b}=\mathbf{k}$

and $T(1,1,1)$ is a point on the plane

therefore $\mathbf{t}=\mathbf{i}+\mathbf{j}+\mathbf{k}$

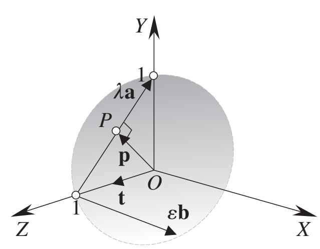
and $\mathbf{p}=\mathbf{t}+\lambda\mathbf{a}+\varepsilon\mathbf{b}$

As **a** and **b** are unit vectors, λ and ε measure Euclidean distances. Therefore if $\lambda=2$ and $\varepsilon=1$

$$x_P=1+2\times1+1\times0=3$$
$$y_P=1+2\times0+1\times0=1$$
$$z_P=1+2\times0+1\times1=2$$

2.15.5 Converting a plane equation from parametric form to general form

Given $\mathbf{p}=\mathbf{t}+\lambda\mathbf{a}+\varepsilon\mathbf{b}$



for P to be perpendicular to O

$$\lambda=\frac{(\mathbf{a}\bullet\mathbf{b})(\mathbf{b}\bullet\mathbf{t})-(\mathbf{a}\bullet\mathbf{t})\|\mathbf{b}\|^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2-(\mathbf{a}\bullet\mathbf{b})^2}$$

and $$\varepsilon=\frac{(\mathbf{a}\bullet\mathbf{b})(\mathbf{a}\bullet\mathbf{t})-(\mathbf{b}\bullet\mathbf{t})\|\mathbf{a}\|^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2-(\mathbf{a}\bullet\mathbf{b})^2}$$

then $$\frac{x_P}{\|\mathbf{p}\|}x + \frac{y_P}{\|\mathbf{p}\|}y + \frac{z_P}{\|\mathbf{p}\|}z - \|\mathbf{p}\| = 0$$

We know in advance that the general equation of this plane is

$$\tfrac{1}{2}\sqrt{2}y + \tfrac{1}{2}\sqrt{2}z - \tfrac{1}{2}\sqrt{2} = 0$$

and intersects the y-axis and z-axis at $y = 1$ and $z = 1$ respectively.
The vectors for the parametric equation are

$$\mathbf{a} = \mathbf{j} - \mathbf{k}$$
$$\mathbf{b} = \mathbf{i}$$
$$\mathbf{t} = \mathbf{k}$$

therefore $$\lambda = \frac{(0)(0) - (-1) \times 1}{2 \times 1 - (0)} = \frac{1}{2}$$

and $$\varepsilon = \frac{(0)(-1) - (0) \times 2}{2 \times 1 - (0)} = 0$$

therefore $$x_P = 0 + \tfrac{1}{2} \times 0 + 0 \times 1 = 0$$

$$y_P = 0 + \tfrac{1}{2} \times 1 + 0 \times 0 = \tfrac{1}{2}$$

$$z_P = 1 + \tfrac{1}{2}(-1) + 0 \times 0 = \tfrac{1}{2}$$

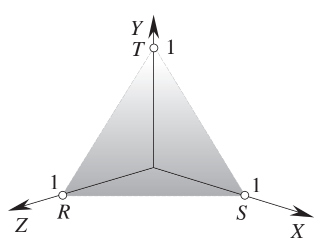
$$\|\mathbf{p}\| = \sqrt{0^2 + \tfrac{1}{2}^2 + \tfrac{1}{2}^2} = \tfrac{1}{2}\sqrt{2}$$

The plane equation is $$0x + \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{2}}y + \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{2}}z - \tfrac{1}{2}\sqrt{2} = 0$$

and $$\tfrac{1}{2}\sqrt{2}y + \tfrac{1}{2}\sqrt{2}z - \tfrac{1}{2}\sqrt{2} = 0$$

or $$y + z - 1 = 0$$

2.15.6 Plane equation from three points



Given three points $R(x_R, y_R, z_R), S(x_S, y_S, z_S), T(x_T, y_T, z_T,)$
the plane equation is $ax + by + cz + d = 0$
where

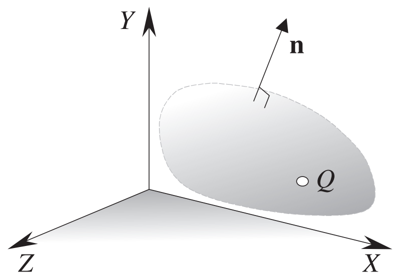
$$a = \begin{vmatrix} y_R & z_R & 1 \\ y_S & z_S & 1 \\ y_T & z_T & 1 \end{vmatrix} \qquad b = \begin{vmatrix} z_R & x_R & 1 \\ z_S & x_S & 1 \\ z_T & x_T & 1 \end{vmatrix} \qquad c = \begin{vmatrix} x_R & y_R & 1 \\ x_S & y_S & 1 \\ x_T & y_T & 1 \end{vmatrix} \qquad d = -(ax_R + by_R + cz_R)$$

If the three points are $R(0,0,1), S(1,0,0), T(0,1,0)$

$$a = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1 \qquad b = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \qquad c = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \qquad d = -(1\times 0 + 1\times 0 + 1\times 1) = -1$$

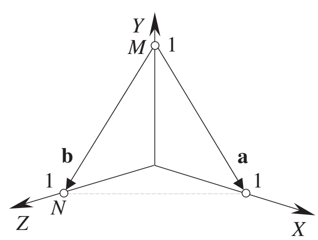
then the plane equation is $x + y + z - 1 = 0$

2.15.7 Plane through a point and normal to a line



Given $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $Q(x_Q, y_Q, z_Q)$
the plane equation is $ax + by + cz - (ax_Q + by_Q + cz_Q) = 0$
If the line is $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $Q(0,1,0)$
the plane is $x + y + z - 1 = 0$

2.15.8 Plane through two points and parallel to a line



Given a line's direction vector **a** and two points $M(x_M, y_M, z_M)$ and $N(x_N, y_N, z_N)$
where $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$

and	$\mathbf{b} = (x_N - x_M)\mathbf{i} + (y_N - y_M)\mathbf{j} + (z_N - z_M)\mathbf{k}$
the plane equation is	$ax + by + cz - (ax_M + by_M + cz_M) = 0$
where	$a = y_a z_b - y_b z_a \quad b = z_a x_b - z_b x_a \quad c = x_a y_b - x_b y_a$
Given	$M = (0, 1, 0)$ and $N = (0, 0, 1)$
and	$\mathbf{a} = \mathbf{i} - \mathbf{j}$
therefore	$\mathbf{a} \times \mathbf{b} = \mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$
and	$-x - y - z - (0 - 1 + 0) = 0$
The plane equation is	$-x - y - z + 1 = 0$
or	$x + y + z - 1 = 0$

2.15.9 Intersection of two planes

Given two planes	$a_1x + b_1y + c_1z + d_1 = 0$	$a_2x + b_2y + c_2z + d_2 = 0$
where	$\mathbf{n}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$	$\mathbf{n}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$

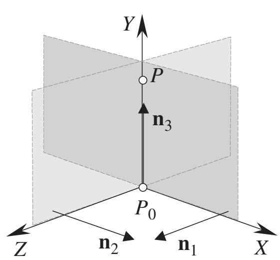
The direction vector of the intersection line is given by $\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2$
and the point P_0 on the intersection line is given by

$$DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad x_0 = \frac{d_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - d_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{DET}$$

$$y_0 = \frac{d_2\begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} - d_1\begin{vmatrix} a_3 & c_3 \\ a_2 & c_2 \end{vmatrix}}{DET} \qquad z_0 = \frac{d_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}{DET}$$

Example 1

Let the two intersecting planes be the xy-plane and the xz-plane, which means that the line of intersection will be the y-axis.



The plane equations are $z = 0$ and $x = 0$

where	$\mathbf{n}_1 = \mathbf{k} \quad \mathbf{n}_2 = \mathbf{i} \quad d_1 = 0 \quad d_2 = 0$
and	$\mathbf{n}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$

Therefore $DET = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1$ $\quad x_0 = \dfrac{0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$

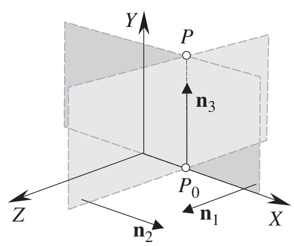
$$y_0 = \frac{0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0 \qquad z_0 = \frac{0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}{1} = 0$$

therefore the line equation is $\mathbf{p} = \lambda\mathbf{n}_3$

where $\mathbf{n}_3 = \mathbf{j}$

Example 2

Let the two intersecting planes be the xy-plane and the plane $x = 1$, which means that the line of intersection will be parallel with the y-axis passing through the point $(1, 0, 0)$



The plane equations are $z = 0$ and $x - 1 = 0$

where $\mathbf{n}_1 = \mathbf{k} \qquad \mathbf{n}_2 = \mathbf{i} \qquad d_1 = 0 \qquad d_2 = -1$

and $\mathbf{n}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$

and $DET = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1$ $\quad x_0 = \dfrac{-1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}{1} = 1$

$$y_0 = \frac{-1\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0 \qquad z_0 = \frac{-1\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}{1} = 0$$

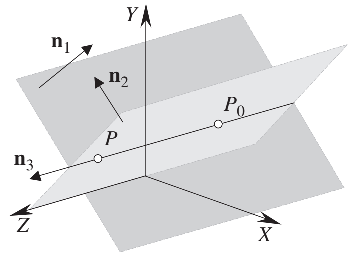
Therefore the line equation is $\mathbf{p} = \mathbf{p}_0 + \lambda\mathbf{n}_3$

where $\mathbf{p}_0 = \mathbf{i}$

and $\mathbf{n}_3 = \mathbf{j}$

Example 3

Let the two intersecting planes be $x + y - 1 = 0$ and $-x + y = 0$.



Therefore $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} \qquad \mathbf{n}_2 = -\mathbf{i} + \mathbf{j} \qquad d_1 = -1 \qquad d_2 = 0$

and
$$\mathbf{n}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 2\mathbf{k}$$

$$DET = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 4 \qquad x_0 = \frac{0\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}}{4} = \frac{1}{2}$$

$$y_0 = \frac{0\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}}{4} = \frac{1}{2} \qquad z_0 = \frac{0\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix}}{4} = 0$$

Therefore the line equation is $\mathbf{p} = \mathbf{p}_0 + \lambda\mathbf{n}_3$

where $\mathbf{p}_0 = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$

and $\mathbf{n}_3 = 2\mathbf{k}$

2.15.10 Intersection of three planes

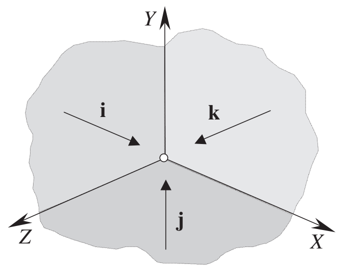
Given three planes
$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

the intersection point (x, y, z) is

$$x = -\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{DET} \qquad y = -\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{DET} \qquad z = -\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{DET}$$

where
$$DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example 1



Given the planes $x = 0 \quad y = 0 \quad z = 0$

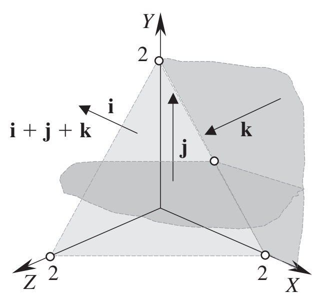
which are the three orthogonal planes intersecting at the origin.

$$DET = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$x = -\begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 \qquad y = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 \qquad z = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

The intersection point is the origin, which is correct.

Example 2



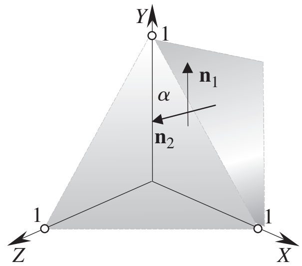
Given the planes $x + y + z - 2 = 0 \quad z = 0 \quad y - 1 = 0$

$$DET = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1$$

$$x = -\frac{\begin{vmatrix} -2 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix}}{-1} = 1 \qquad y = -\frac{\begin{vmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}}{-1} = 1 \qquad z = -\frac{\begin{vmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}}{-1} = 0$$

The intersection point is $(1, 1, 0)$ which is correct.

2.15.11 Angle between two planes



Given two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

where $\mathbf{n}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{n}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$

the angle between the normals is $\alpha = \cos^{-1}\left(\dfrac{\mathbf{n}_1 \bullet \mathbf{n}_2}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_2\|}\right)$

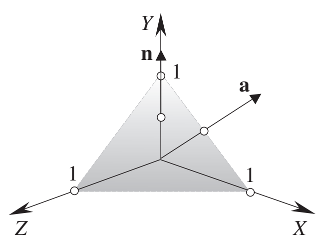
Given the planes $x + y + z - 1 = 0$ and $z = 0$

where $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{k}$

$$\|\mathbf{n}_1\| = \sqrt{3} \quad \text{and} \quad \|\mathbf{n}_2\| = 1$$

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 54.74°$$

2.15.12 Angle between a line and a plane



Given the plane $ax + by + cz + d = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and the line $\mathbf{p} = \mathbf{r} + \lambda\mathbf{a}$

the angle between the line and the plane's normal is

$$\alpha = \cos^{-1}\left(\frac{\mathbf{n} \bullet \mathbf{a}}{\|\mathbf{n}\| \cdot \|\mathbf{a}\|}\right)$$

Given the plane $x + y + z - 1 = 0$

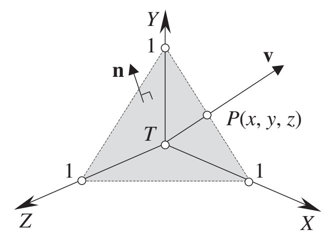
then $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

and $\mathbf{a} = \mathbf{i} + \mathbf{j}$

$$\|\mathbf{n}\| = \sqrt{3} \quad \text{and} \quad \|\mathbf{a}\| = \sqrt{2}$$

$$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) = 35.26°$$

2.15.13 Intersection of a line and a plane



Given a plane $ax + by + cz + d = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and a line $\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

for the intersection point P $\lambda = \dfrac{-(\mathbf{n} \cdot \mathbf{t} + d)}{\mathbf{n} \cdot \mathbf{v}}$

Example 1

Given the plane $x + y + z - 1 = 0$

and the line $\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$

where $\mathbf{t} = 0$

and $\mathbf{v} = \mathbf{i} + \mathbf{j}$

then $\lambda = \dfrac{-(1\times 0 + 1\times 0 + 1\times 0 - 1)}{1\times 1 + 1\times 1 + 1\times 0} = \dfrac{1}{2}$

The point of intersection is $P\left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

Example 2

With the same plane $x + y + z - 1 = 0$

but $\mathbf{t} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

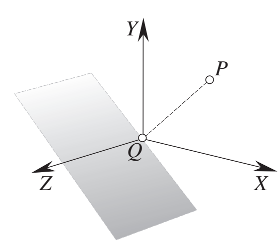
and $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\lambda = \frac{-(1\times 1 + 1\times 1 + 1\times 1 - 1)}{1\times 1 + 1\times 1 + 1\times 1} = -\frac{2}{3}$$

$$\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$$

The point of intersection is $P\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

2.15.14 Position and distance of the nearest point on a plane to a point



Given the plane $ax + by + cz + d = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and a point P with position vector $\mathbf{p}$.

The position vector of the nearest point Q is given by $\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$

where $\lambda = \dfrac{-(\mathbf{n} \bullet \mathbf{p} + d)}{\mathbf{n} \bullet \mathbf{n}}$

The distance PQ is $PQ = \|\lambda\mathbf{n}\|$

Given the plane $x + y = 0$

where $\mathbf{n} = \mathbf{i} + \mathbf{j}$

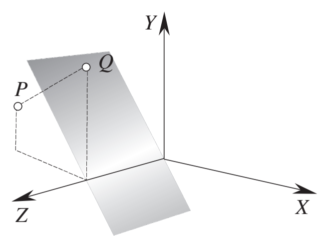
and a point $P(1, 1, 0)$ where $\mathbf{p} = \mathbf{i} + \mathbf{j}$

$$\lambda = \frac{-(2)}{2} = -1$$

The nearest point is $Q(0, 0, 0)$ the origin.

The distance is $PQ = \|-1(\mathbf{i} + \mathbf{j})\| = \sqrt{2}$

2.15.15 Reflection of a point in a plane



Given the plane $ax + by + cz + d = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

and P is a point with position vector $\mathbf{p}$

P's reflection Q is given by $\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$

where $\lambda = \dfrac{-2(\mathbf{n} \bullet \mathbf{p} + d)}{\mathbf{n} \bullet \mathbf{n}}$

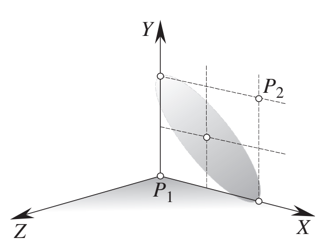
Given the plane $x + y = 0$ and $P(-1, 0, 1)$

$$\mathbf{n} = \mathbf{i} + \mathbf{j}$$

$$\lambda = \frac{-2(-1)}{2} = 1$$

The reflection point is $(0, 1, 1)$.

2.15.16 Plane equidistant from two points



Given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ the plane equation is

$$(x_2 - x_1)x + (y_2 - y_1)y + (z_2 - z_1)z - \tfrac{1}{2}(x_2^2 - x_1^2 + y_2^2 - y_1^2 + z_2^2 - z_1^2) = 0$$

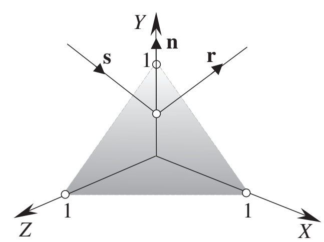
Given $P_1(0, 0, 0)$ and $P_2(2, 2, 0)$

the plane equation is $2x + 2y - \tfrac{1}{2}(4 + 4) = 0$

or $x + y - 2 = 0$

2.15.17 Reflected ray on a surface

Given the surface normal $\mathbf{n}$
the incident ray $\mathbf{s}$
the reflected ray $\mathbf{r}$



then $\mathbf{r} = \mathbf{s} + \lambda\mathbf{n}$

where $$\lambda = \frac{-2\mathbf{n}\bullet\mathbf{s}}{\mathbf{n}\bullet\mathbf{n}}$$

Given $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

and $$\mathbf{s} = \mathbf{i} - \frac{1}{4}\mathbf{j} - \frac{1}{4}\mathbf{k}$$

then $$\lambda = -\frac{1}{3}$$

and $$x_r = 1 - \frac{1}{3} = \frac{2}{3}$$

$$y_r = -\frac{1}{4} - \frac{1}{3} = -\frac{7}{12}$$

$$z_r = -\frac{1}{4} - \frac{1}{3} = -\frac{7}{12}$$

with $$\mathbf{r} = \frac{2}{3}\mathbf{i} - \frac{7}{12}\mathbf{j} - \frac{7}{12}\mathbf{k}$$

Let's check this vector out. Its magnitude should equal the magnitude of the incident vector **s**, and the reflection angle should equal the incident angle.

$$\|\mathbf{s}\| = \sqrt{1^2 + \left(\frac{-1}{4}\right)^2 + \left(\frac{-1}{4}\right)^2} = \frac{\sqrt{18}}{4}$$

$$\|\mathbf{r}\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{-7}{12}\right)^2 + \left(\frac{-7}{12}\right)^2} = \frac{\sqrt{18}}{4}$$

The reflection angle equals $\theta = \cos^{-1}\left(\dfrac{\mathbf{n}\bullet\mathbf{r}}{\|\mathbf{n}\|\cdot\|\mathbf{r}\|}\right)$

The incident angle equals $\alpha = \cos^{-1}\left(\dfrac{\mathbf{n}\bullet-\mathbf{s}}{\|\mathbf{n}\|\cdot\|-\mathbf{s}\|}\right)$

For $\theta = \alpha$ $\dfrac{\mathbf{n}\bullet\mathbf{r}}{\|\mathbf{n}\|\cdot\|\mathbf{r}\|} = \dfrac{\mathbf{n}\bullet-\mathbf{s}}{\|\mathbf{n}\|\cdot\|-\mathbf{s}\|}$

but $\|\mathbf{s}\| = \|\mathbf{r}\|$

therefore $\mathbf{n}\bullet\mathbf{r} = \mathbf{n}\bullet-\mathbf{s}$

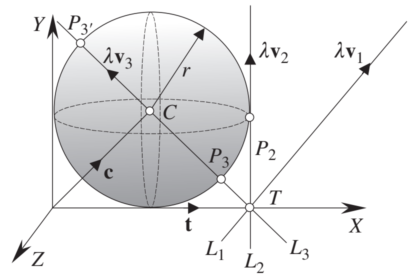
$$\mathbf{n}\bullet\mathbf{r} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\bullet\left(\frac{2}{3}\mathbf{i}-\frac{7}{12}\mathbf{j}-\frac{7}{12}\mathbf{k}\right) = -\frac{1}{2}$$

$$\mathbf{n}\bullet-\mathbf{s} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\bullet\left(-\mathbf{i}+\frac{1}{4}\mathbf{j}+\frac{1}{4}\mathbf{k}\right) = -\frac{1}{2}$$

which confirms that the angle of reflection equals the angle of incidence.

2.16 Lines, planes and spheres

2.16.1 Line intersecting a sphere



Given a sphere with radius r located at C with position vector $\mathbf{c}$

and a line equation $\mathbf{p} = \mathbf{t} + \lambda\mathbf{v}$ where $\|\mathbf{v}\| = 1$

a touch, miss or intersect condition is determined by λ

where $\lambda = \mathbf{s}\cdot\mathbf{v} \pm \sqrt{(\mathbf{s}\cdot\mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2}$

and $\mathbf{s} = \mathbf{c} - \mathbf{t}$

The diagram shows a sphere with radius $r = 1$ centered at C with position vector $\mathbf{c} = \mathbf{i} + \mathbf{j}$ and three lines L_1, L_2 and L_3 that miss, touch and intersect the sphere respectively.

The lines are of the form $\mathbf{p} = \mathbf{t} + \lambda\,\mathbf{v}$

therefore $\mathbf{p}_1 = \mathbf{t}_1 + \lambda\mathbf{v}_1 \qquad \mathbf{p}_2 = \mathbf{t}_2 + \lambda\mathbf{v}_2 \qquad \mathbf{p}_3 = \mathbf{t}_3 + \lambda\mathbf{v}_3$

where $\mathbf{t}_1 = 2\mathbf{i} \qquad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

$\mathbf{t}_2 = 2\mathbf{i} \qquad \mathbf{v}_2 = \mathbf{j}$

$\mathbf{t}_3 = 2\mathbf{i} \qquad \mathbf{v}_3 = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

and $\mathbf{c} = \mathbf{i} + \mathbf{j}$

Let us substitute the lines in the original equations:

L_1: $\mathbf{s} = -\mathbf{i} + \mathbf{j}$

$$(\mathbf{s}\cdot\mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 0 - 2 + 1 = -1$$

The negative discriminant confirms a miss condition.

L_2: $\mathbf{s} = -\mathbf{i} + \mathbf{j}$

$$(\mathbf{s}\cdot\mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 1 - 2 + 1 = 0$$

The zero discriminant confirms a touch condition, therefore $\lambda = 1$.
The touch point is $P_2(2, 1, 0)$ which is correct.

L_3: $\mathbf{s} = -\mathbf{i} + \mathbf{j}$

$$(\mathbf{s}\cdot\mathbf{v})^2 - \|\mathbf{s}\|^2 + r^2 = 2 - 2 + 1 = 1$$

The positive discriminant confirms an intersect condition therefore $\lambda = \frac{2}{\sqrt{2}} \pm 1 = 1 + \sqrt{2} \quad \text{or} \quad \sqrt{2} - 1$

The intersection points are:

if $\lambda = 1 + \sqrt{2}$

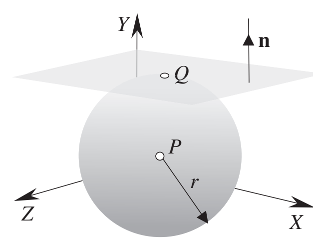
$$x_P = 2 + (1 + \sqrt{2})\left(-\frac{1}{\sqrt{2}}\right) = 1 - \frac{1}{\sqrt{2}}$$
$$y_P = 0 + (1 + \sqrt{2})\frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}}$$
$$z_P = 0$$

if $\lambda = \sqrt{2} - 1$

$$x_P = 1 + (\sqrt{2} - 1)\left(-\frac{1}{\sqrt{2}}\right) = 1 + \frac{1}{\sqrt{2}}$$
$$y_P = 0 + (\sqrt{2} - 1)\frac{1}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}$$
$$z_P = 0$$

The intersection points are $P_{3'}\left(1 - \frac{1}{\sqrt{2}},\ 1 + \frac{1}{\sqrt{2}}, 0\right)$ and $P_3\left(1 + \frac{1}{\sqrt{2}},\ 1 - \frac{1}{\sqrt{2}}, 0\right)$ which are correct.

2.16.2 Sphere touching a plane



Given a plane $ax + by + cz + d = 0$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

the nearest point Q on the plane to a point P is given by

$$\mathbf{q} = \mathbf{p} + \lambda\mathbf{n}$$

where $\lambda = -\frac{\mathbf{n} \bullet \mathbf{p} + d}{\mathbf{n} \bullet \mathbf{n}}$

The distance is given by $\|\lambda\mathbf{n}\|$

for a plane and a sphere $\|\lambda\mathbf{n}\| = r$

The diagram shows a sphere radius $r = 1$ centered at $P(1, 1, 1)$

The plane equation is $y - 2 = 0$

therefore $\mathbf{n} = \mathbf{j}$

and $\mathbf{p} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

therefore $\lambda = -(1 - 2) = 1$

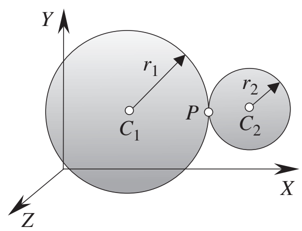
which equals the sphere's radius and therefore the sphere and the plane touch.

The touch point is

$$x_Q = 1 + 1 \times 0 = 1$$
$$y_Q = 1 + 1 \times 1 = 2$$
$$z_Q = 1 + 1 \times 0 = 1$$

therefore the touch point is $Q(1, 2, 1)$ which is correct.

2.16.3 Touching spheres



Given

$$d = \sqrt{(x_{C2} - x_{C1})^2 + (y_{C2} - y_{C1})^2 + (z_{C2} - z_{C1})^2}$$

the touch condition is

$$d = r_1 + r_2$$

the touch point is

$$x_P = x_{C1} + \frac{r_1}{d}(x_{C2} - x_{C1})$$
$$y_P = y_{C1} + \frac{r_1}{d}(y_{C2} - y_{C1})$$
$$z_P = z_{C1} + \frac{r_1}{d}(z_{C2} - z_{C1})$$

Given that one sphere with radius $r_1 = 1$ is centered at $C_1(1, 1, 1)$ and the other with radius $r_2 = 0.5$ is centered at $C_2(2.5, 1, 1)$

then

$$d = \sqrt{(2.5 - 1)^2 + (1 - 1)^2 + (1 - 1)^2} = 1.5$$

The touch condition is satisfied

and

$$x_P = 1 + \frac{1}{1.5}(2.5 - 1) = 2$$
$$y_P = 1 + \frac{1}{1.5}(1 - 1) = 1$$
$$z_P = 1 + \frac{1}{1.5}(1 - 1) = 1$$

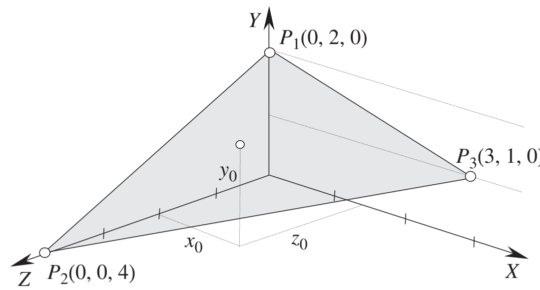
therefore the touch point is $P(2, 1, 1)$ which is correct.

2.17 Three-dimensional triangles

2.17.1 Coordinates of a point inside a triangle

To locate points inside and outside the triangle P_1, P_2, P_3 using barycentric coordinates.

For any point $P_0(x_0, y_0, z_0)$ we can state



$$x_0 = \varepsilon x_1 + \lambda x_2 + \beta x_3$$
$$y_0 = \varepsilon y_1 + \lambda y_2 + \beta y_3$$
$$z_0 = \varepsilon z_1 + \lambda z_2 + \beta z_3$$

where $\quad \varepsilon + \lambda + \beta = 1$

The table below shows values of P_0 for various values of ε, λ and β. Let us check that the positions of P_0 reside on the plane of the triangle.

The vertices of the triangle are $P_1(0, 2, 0), P_2(0, 0, 4), P_3(3, 1, 0)$ therefore the Cartesian plane equation is

$ax + by + cz = d$ (see plane equation from three points)

where

$$a = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \quad b = \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix} \quad c = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad d = ax_1 + by_1 + cz_1$$

$$a = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 4 \quad b = \begin{vmatrix} 0 & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix} = 12 \quad c = \begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 6 \quad \begin{aligned} d &= 4 \times 0 + 12 \times 2 + 6 \times 0 \\ &= 24 \end{aligned}$$

therefore the plane equation is $4x + 12y + 6z = 24$

The table also confirms that the values of P_0 satisfy the plane equation.

ε	λ	β	x_0	y_0	z_0	$4x_0 + 12y_0 + 6z_0$
1	0	0	0	2	0	24
0	1	0	0	0	4	24
0	0	1	3	1	0	24
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$1\frac{1}{2}$	1	1	24
0	$\frac{1}{2}$	$\frac{1}{2}$	$1\frac{1}{2}$	$\frac{1}{2}$	2	24
$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	2	24
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	1	$\frac{4}{3}$	24

2.17.2 Unknown coordinate value inside a triangle

The x and z-coordinates of a point P_0 are known and it is required to determine its y-coordinate inside the triangle P_1, P_2, P_3.

Using barycentric coordinates we have

$$y_0 = \varepsilon y_1 + \lambda y_2 + (1 - \varepsilon - \lambda) y_3$$

where
$$\frac{\varepsilon}{\begin{vmatrix} x_0 & z_0 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} x_0 & z_0 & 1 \\ x_3 & z_3 & 1 \\ x_1 & z_1 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}}$$

For P_0 to be inside the triangle $(\varepsilon, \lambda) \in [0, 1]$.

If P_0 is positioned at P_1 i.e. $x_0 = z_0 = 0$, y_0 should be 2.

Therefore
$$\frac{\varepsilon}{\begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 1 \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 1 \end{vmatrix}}$$

and
$$\frac{\varepsilon}{-12} = \frac{\lambda}{0} = \frac{1}{-12}$$

which makes $\varepsilon = 1$ and $\lambda = 0$

therefore $y_0 = 1 \times 2 + 0 \times 0 + (1 - 1 - 0)1 = 2$ which is correct.

The table below shows the values of $\varepsilon, \lambda, 1 - \varepsilon - \lambda$ and y_0 for different values of x_0 and z_0.
Let us check that the interpolated values of P_0 reside on the plane of the triangle.

The vertices of the triangle are $P_1(0, 2, 0), P_2(0, 0, 4), P_3(3, 1, 0)$ therefore the Cartesian plane equation is $ax + by + cz = d$ (see plane equation from three points)

where

$$a = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \quad b = \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix} \quad c = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad d = ax_1 + by_1 + cz_1$$

$$a = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 4 \quad b = \begin{vmatrix} 0 & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix} = 12 \quad c = \begin{vmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 6 \quad \begin{aligned} d &= 4 \times 0 + 12 \times 2 + 6 \times 0 \\ &= 24 \end{aligned}$$

therefore the plane equation is $4x + 12y + 6z = 24$

x_0	y_0	z_0	ε	λ	$1-\varepsilon-\lambda$	$4x+12y+6z$
0	2	0	1	0	0	24
3	1	0	0	0	1	24
0	0	4	0	1	0	24
1	$\frac{2}{3}$	2	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	24
2	$\frac{5}{6}$	1	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{2}{3}$	24
1	$\frac{7}{6}$	1	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	24

The table below also confirms that the above values of P_0 satisfy the plane equation. Let us test a point outside the triangle's boundary, e.g. $P_0(4,0,0)$

$$\frac{\varepsilon}{\begin{vmatrix} 4 & 0 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 1 \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} 4 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 1 \end{vmatrix}}$$

$$\frac{\varepsilon}{4} = \frac{\lambda}{0} = \frac{1}{-12}$$

therefore $\varepsilon = -\frac{1}{3}$

which confirms that P_0 is outside the triangle's boundary.

Similarly, for $P_0(0,0,5)$

$$\frac{\varepsilon}{\begin{vmatrix} 0 & 5 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 1 \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} 0 & 5 & 1 \\ 3 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & 1 \\ 3 & 0 & 1 \end{vmatrix}}$$

$$\frac{\varepsilon}{3} = \frac{\lambda}{-15} = \frac{1}{-12}$$

therefore $\varepsilon = -\frac{1}{4}$ and $\lambda = 1\frac{1}{4}$

which confirms that P_0 is also outside the triangle's boundary.

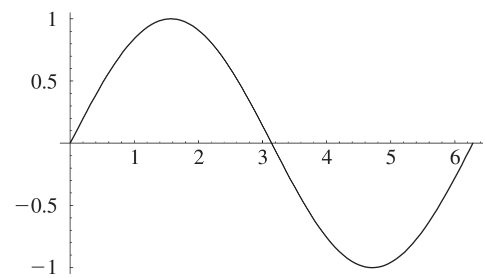
2.18 Parametric curves and patches

The following examples illustrate how various curves can be created by mixing together different parametric functions.

2.18.1 Parametric curves in $\mathbb{R}^2$

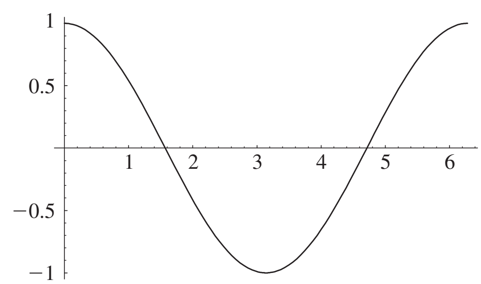
Sine curve

$$t_{\max} = 2\pi$$
$$a = 1$$
$$\left.\begin{array}{l} x = t \\ y = a\sin t \end{array}\right\} \quad t \in [0,\ t_{\max}]$$



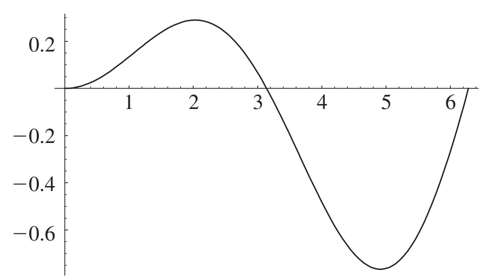
Cosine curve

$$t_{\max} = 2\pi$$
$$a = 1$$
$$\left.\begin{array}{l} x = t \\ y = a\cos t \end{array}\right\} \quad t \in [0,\ t_{\max}]$$



Sine curve with growing amplitude

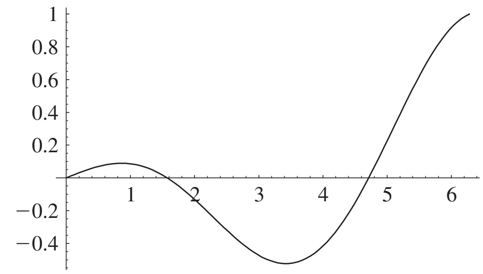
$$t_{\max} = 2\pi$$
$$\left.\begin{array}{l} a = \dfrac{t}{t_{\max}} \\ x = t \\ y = a\sin t \end{array}\right\} \quad t \in [0,\ t_{\max}]$$



Cosine curve with growing amplitude

$$t_{\max} = 2\pi$$

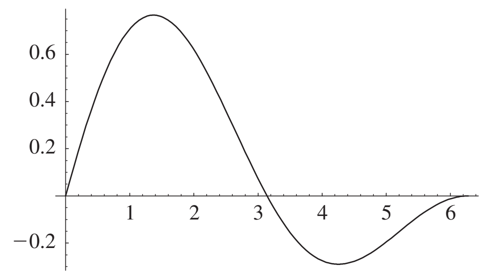
$$\left.\begin{aligned} a &= \frac{t}{t_{\max}} \\ x &= t \\ y &= a\cos t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



Sine curve with decaying amplitude

$$t_{\max} = 2\pi$$

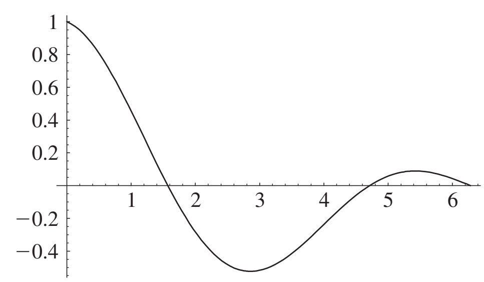
$$\left.\begin{aligned} a &= 1 - \frac{t}{t_{\max}} \\ x &= t \\ y &= a\sin t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



Cosine curve with decaying amplitude

$$t_{\max} = 2\pi$$

$$\left.\begin{aligned} a &= 1 - \frac{t}{t_{\max}} \\ x &= t \\ y &= a\cos t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$

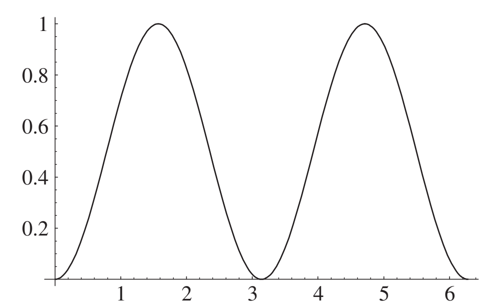


Sine-squared curve

$$t_{\max} = 2\pi$$

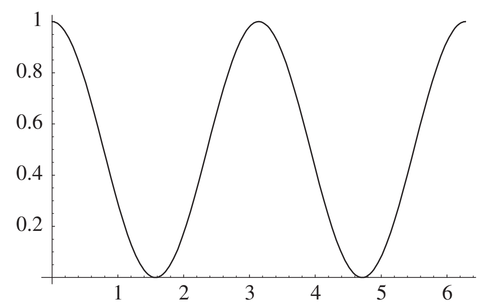
$$a = 1$$

$$\left.\begin{aligned} x &= t \\ y &= a\sin^2 t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



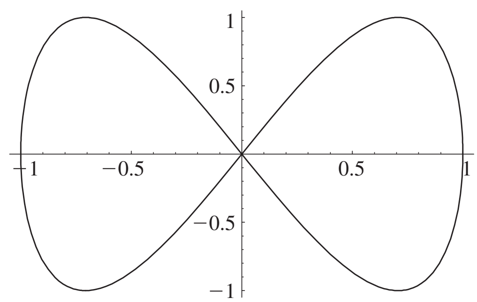
Cosine-squared curve

$$t_{max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= t \\ y &= a\cos^2 t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



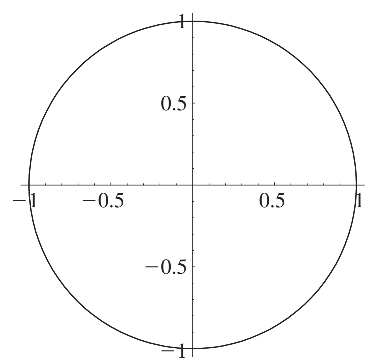
Lissajous curve

$$t_{max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a\sin t \\ y &= a\sin 2t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



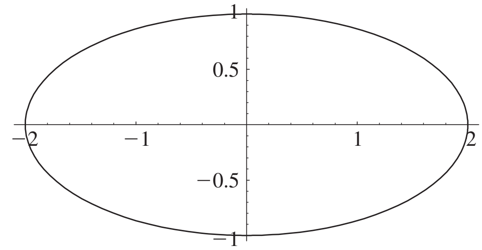
Circle

$$t_{max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



Ellipse

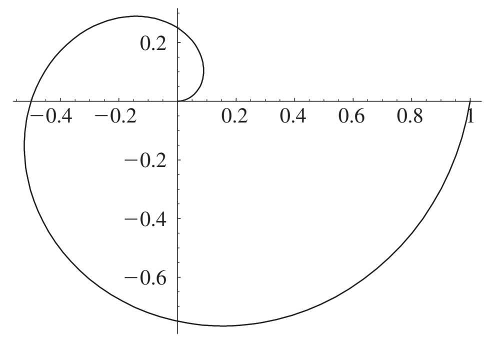
$$t_{max} = 2\pi$$
$$a = 2$$
$$b = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



Spiral

$$t_{\max} = 2\pi$$

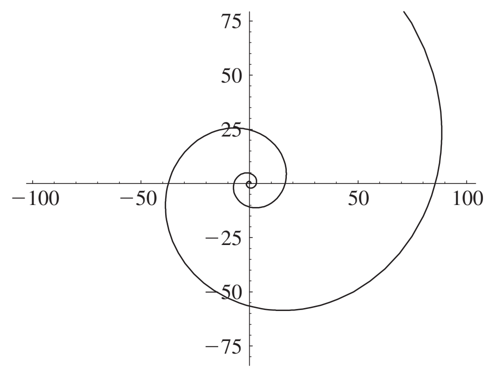
$$\left.\begin{aligned} r &= \frac{t}{t_{\max}} \\ x &= r\cos t \\ y &= r\sin t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



Logarithmic spiral

$$t_{\max} = 2\pi$$
$$a = 0.6$$
$$b = 3.8$$

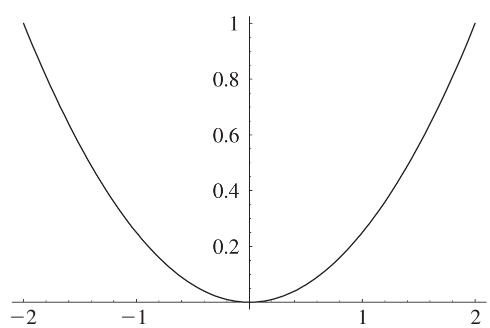
$$\left.\begin{aligned} x &= ae^{t}\cos bt \\ y &= ae^{t}\sin bt \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



Parabola

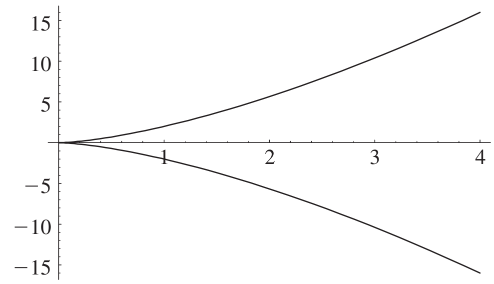
$$t_{\max} = 4$$
$$p = 2$$

$$\left.\begin{aligned} x &= t \\ y &= \frac{1}{2p}t^2 \end{aligned}\right\} \quad t \in \left[-\frac{t_{\max}}{2},\ \frac{t_{\max}}{2}\right]$$



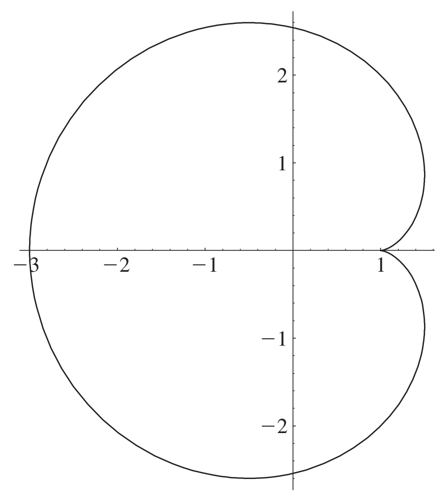
Neil's parabola

$$t_{\max} = 4$$
$$a = 2$$
$$\left.\begin{aligned} x &= t^2 \\ y &= at^3 \end{aligned}\right\} \quad t \in \left[-\frac{t_{\max}}{2},\ \frac{t_{\max}}{2}\right]$$



Cardioid

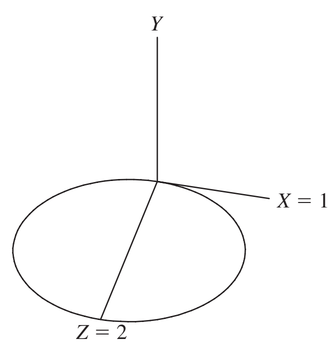
$$t_{\max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a(2\cos t - \cos 2t) \\ y &= a(2\sin t - \sin 2t) \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



2.18.2 Parametric curves in $\mathbb{R}^3$

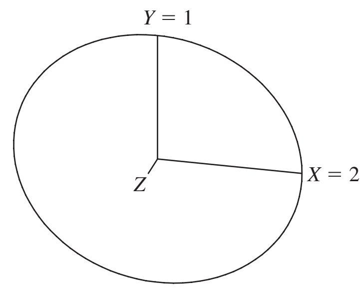
Circle

$$t_{\max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= 0 \\ z &= a + a\sin t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



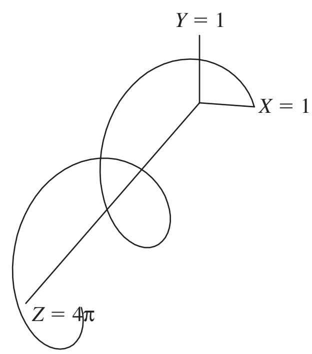
Ellipse

$$t_{\max} = 2\pi$$
$$a = 2$$
$$b = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin t \\ z &= 0 \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



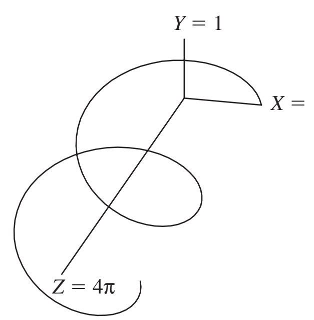
Spiral 1

$$t_{\max} = 4\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= a\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



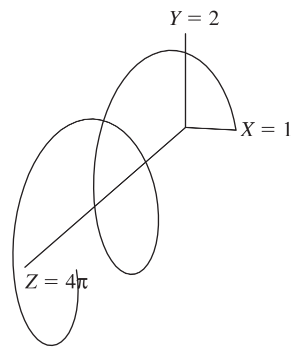
Spiral 2

$$t_{\max} = 4\pi$$
$$a = 2$$
$$b = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



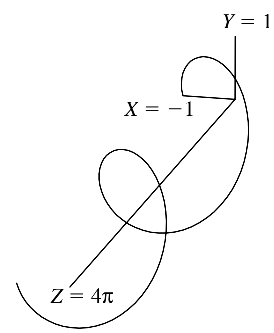
Spiral 3

$$t_{max} = 4\pi$$
$$a = 1$$
$$b = 2$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



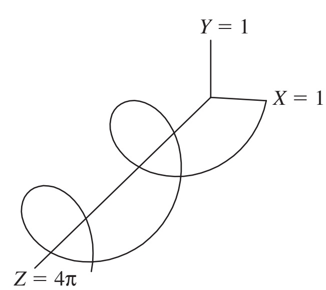
Spiral 4

$$t_{max} = 4\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= -a\cos t \\ y &= a\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



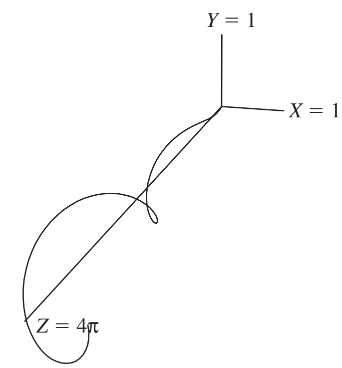
Spiral 5

$$t_{max} = 4\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= -a\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



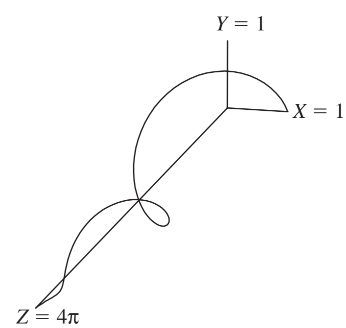
Spiral 6

$$t_{\max} = 4\pi$$
$$\left.\begin{aligned} r &= \frac{t}{t_{\max}} \\ x &= r\cos t \\ y &= r\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



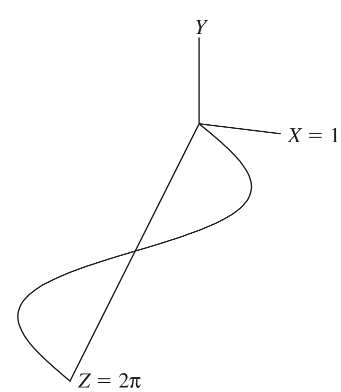
Spiral 7

$$t_{\max} = 4\pi$$
$$\left.\begin{aligned} r &= 1 - \frac{t}{t_{\max}} \\ x &= r\cos t \\ y &= r\sin t \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



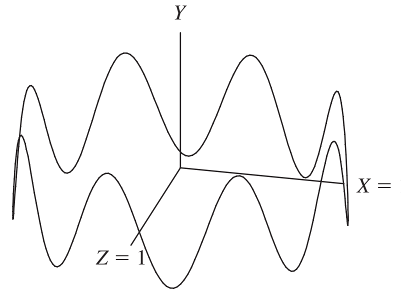
Sinusoid

$$t_{\max} = 2\pi$$
$$a = 1$$
$$\left.\begin{aligned} x &= a\sin t \\ y &= 0 \\ z &= t \end{aligned}\right\} \quad t \in [0,\ t_{\max}]$$



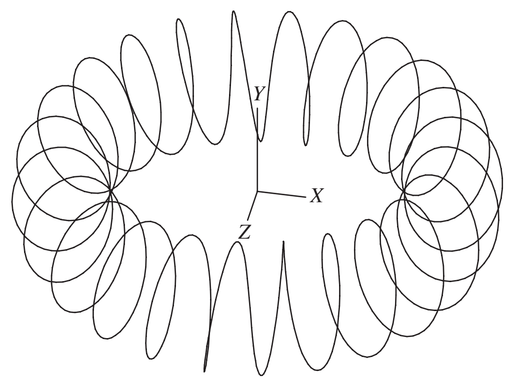
Sinusoidal ring

$$t_{max} = 2\pi$$
$$a = 1$$
$$b = 0.2$$
$$n = 8$$
$$\left.\begin{aligned} x &= a\cos t \\ y &= b\sin nt \\ z &= a\sin t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



Coiled ring

$$t_{max} = 2\pi$$
$$R = 2 \quad \text{(major radius)}$$
$$r = 0.5 \quad \text{(minor radius)}$$
$$n = 24$$
$$\left.\begin{aligned} x &= (R + r\cos nt)\cos t \\ y &= r\sin nt \\ z &= -(R + r\cos nt)\sin t \end{aligned}\right\} \quad t \in [0,\ t_{max}]$$



2.18.3 Planar patch

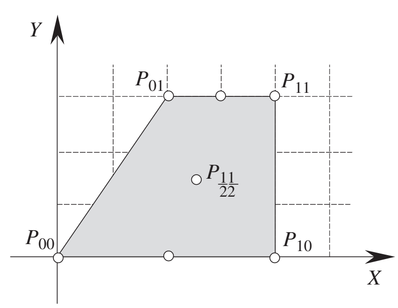
Given $P_{00}, P_{10}, P_{11}, P_{01}$ in $\mathbb{R}^2$

$$P_{uv} = \begin{bmatrix} u & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ 1 \end{bmatrix}$$

Given $P_{00}(0,0), P_{01}(2,3), P_{11}(4,3), P_{10}(4,0)$

$$x_{\frac{1}{2}\frac{1}{2}} = \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = 2\tfrac{1}{2}$$

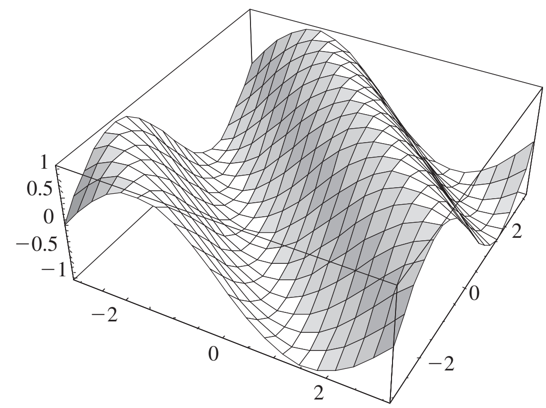
$$y_{\frac{1}{2}\frac{1}{2}} = \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = 1\tfrac{1}{2}$$



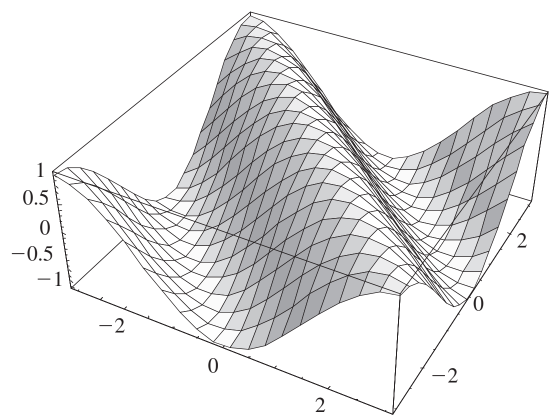
2.18.4 Parametric surfaces in $\mathbb{R}^3$

Modulated surface

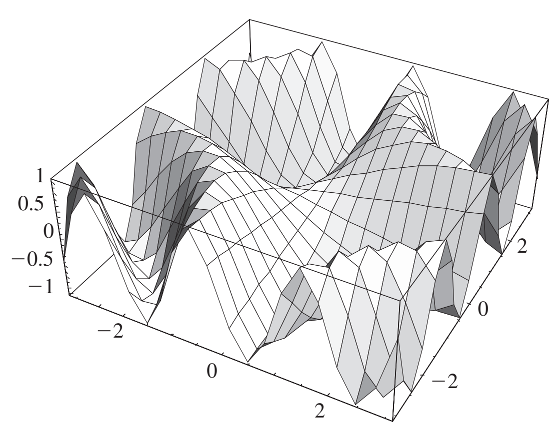
$y = \sin(x + z)$
$T = \pi$
$a = 1$
$y = a\sin(x + z)\}\quad (x, z) \in [-T,\ T]$



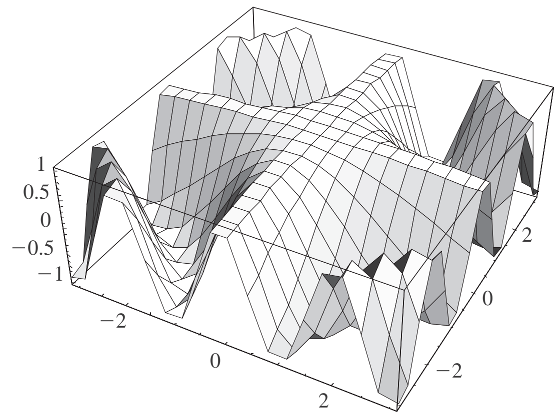
$y = \cos(x + z)$
$T = \pi$
$a = 1$
$y = a\cos(x + z)\}\quad (x, z) \in [-T,\ T]$



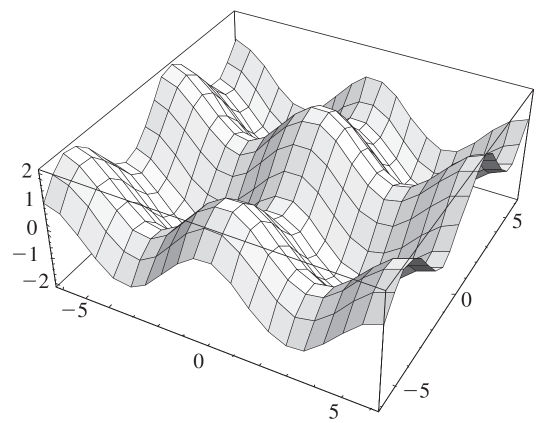
$y = \sin(xz)$
$T = \pi$
$a = 1$
$y = a\sin(xz)\}\quad (x, z) \in [-T,\ T]$



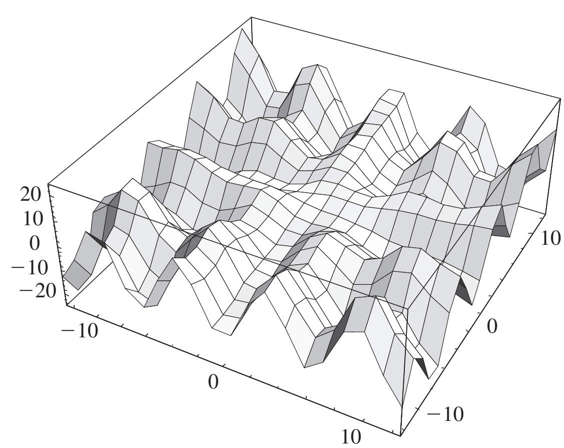
$y = \cos(xz)$
$T = \pi$
$a = 1$
$y = a\cos(xz)\}\quad (x, z) \in [-T,\ T]$



$y = \cos x + \sin z$
$T = 2\pi$
$a = 1$
$y = a\cos x + a\sin z\}\quad (x, z) \in [-T,\ T]$



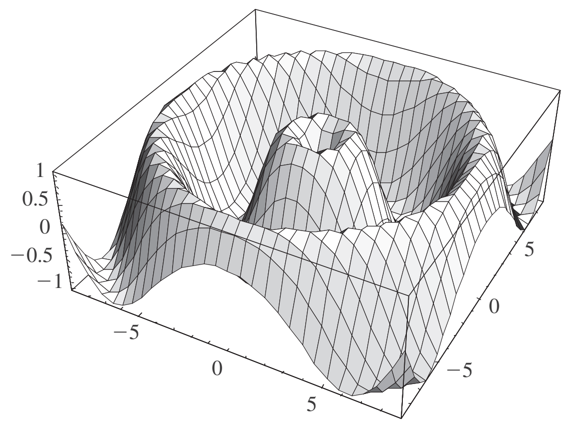
$y = z\cos x + x\sin z$
$T = 4\pi$
$y = z\cos x + x\sin z\}\quad (x, z) \in [-T,\ T]$



$\sin\left(\sqrt{x^2+y^2}\right)$

$T = 9$

$\sin\left(\sqrt{x^2+y^2}\right)\Big\}\quad (x, y) \in [-T, T]$



2.18.5 Quadratic Bézier curve

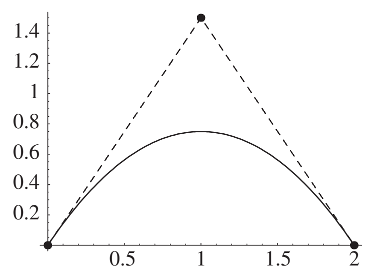
Quadratic Bézier curve in $\mathbb{R}^2$

A quadratic Bézier curve is given by

$\mathbf{p}(t) = (1-t)^2\,\mathbf{p}_1 + 2t(1-t)\,\mathbf{p}_C + t^2\mathbf{p}_2$

Given the points $P_1(0, 0), P_C(1, 1.5), P_2(2, 0)$

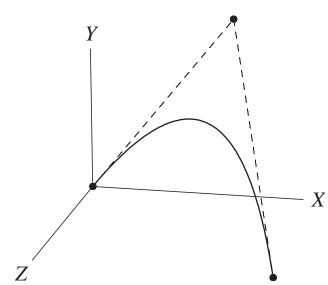
the quadratic Bézier curve is shown with its control points.



Quadratic Bézier curve in $\mathbb{R}^3$

Given the points $P_1(0, 0, 0), P_C(2, 2.5, 0), P_2(3, 0, 3)$

the quadratic Bézier curve is shown with its control points.



2.18.6 Cubic Bézier curve

Cubic Bézier curve in $\mathbb{R}^2$

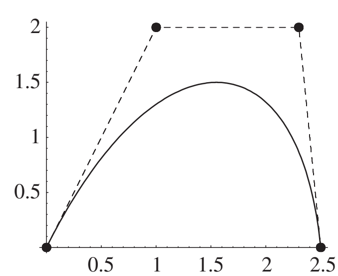
A cubic Bézier curve is given by

$\mathbf{p}(t) = (1-t)^3\,\mathbf{p}_1 + 3t(1-t)^2\,\mathbf{p}_{C1} + 3t^2\,(1-t)\,\mathbf{p}_{C2} + t^3\mathbf{p}_2$

Given the points

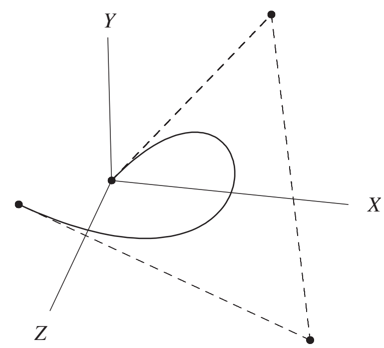
$P_1(0, 0), P_{C1}(1, 2), P_{C2}(2.3, 2), P_2(2.5, 0)$

the cubic Bézier curve is shown with its control points.



Cubic Bézier curve in $\mathbb{R}^3$

Given the points $P_1(0,0,0)$, $P_{C1}(2,2.5,0)$, $P_{C2}(3,0,3)$, $P_2(0,2,4)$ the cubic Bézier curve is shown with its control points.



2.18.7 Quadratic Bézier patch

A quadratic surface patch is described by

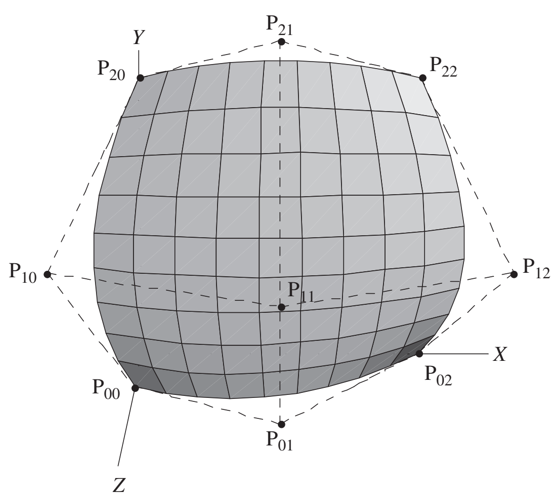
$$\mathbf{p}(u,v) = \begin{bmatrix}(1-u)^2 & 2u(1-u) & u^2\end{bmatrix}\begin{bmatrix}\mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02}\\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12}\\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22}\end{bmatrix}\begin{bmatrix}(1-v)^2\\ 2v(1-v)\\ v^2\end{bmatrix}$$

Given $\quad \mathbf{p}_{00} = (0,0,1) \quad \mathbf{p}_{01} = (1,0,2) \quad \mathbf{p}_{02}\,(2,0,0)$

$$\mathbf{p}_{10} = (-0.5,\ 1,2) \quad \mathbf{p}_{11} = (1,\ 1,3) \quad \mathbf{p}_{12}(2\tfrac{1}{2},\ 1,2)$$

$$\mathbf{p}_{20} = (0,2\tfrac{1}{2},0) \quad \mathbf{p}_{21} = (1,2\tfrac{1}{2},2) \quad \mathbf{p}_{22}(2,2,0)$$

The surface patch is shown in the diagram



2.18.8 Cubic Bézier patch

A cubic surface patch is described by

$$\mathbf{p}(u,v) = [(1-u)^3 \quad 3u(1-u)^2 \quad 3u^2(1-u) \quad u^3] \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} & \mathbf{p}_{03} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23} \\ \mathbf{p}_{30} & \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33} \end{bmatrix} \begin{bmatrix} (1-v)^3 \\ 3v(1-v)^2 \\ 3v^2(1-v) \\ v^3 \end{bmatrix}$$

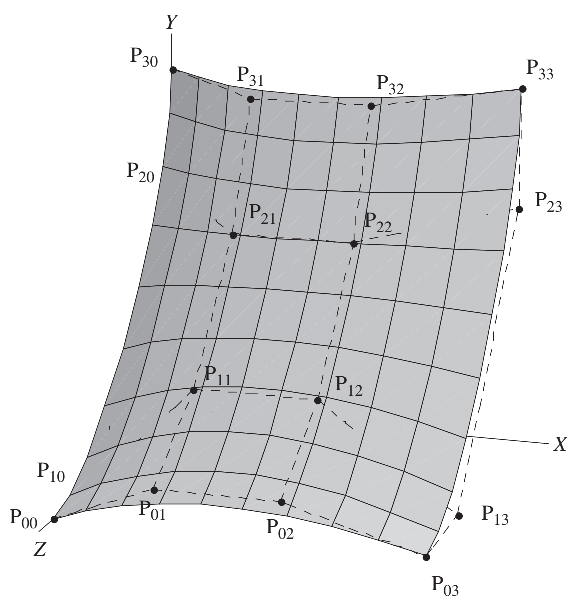
Given $\mathbf{p}_{00} = (0,0,3) \quad \mathbf{p}_{01} = (1, \frac{1}{2}, 3\frac{1}{2}) \quad \mathbf{p}_{02}(2, \frac{1}{2}, 3\frac{1}{2}) \quad \mathbf{p}_{03}(3,0,3)$

$$\mathbf{p}_{10} = (0,0,2) \quad \mathbf{p}_{11} = (1,\ 1, 2\tfrac{1}{2}) \quad \mathbf{p}_{12}(2,\ 1, 2\tfrac{1}{2}) \quad \mathbf{p}_{13}(3,0,2)$$

$$\mathbf{p}_{20} = (0,2,0) \quad \mathbf{p}_{21} = (1,2,1\tfrac{1}{2}) \quad \mathbf{p}_{22}(2,2,1\tfrac{1}{2}) \quad \mathbf{p}_{23}(3,2,0)$$

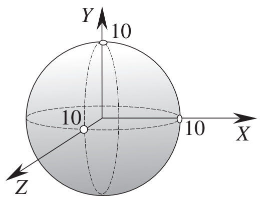
$$\mathbf{p}_{30} = (0,3,0) \quad \mathbf{p}_{31} = (1,3,\ 1) \quad \mathbf{p}_{32}(2,3,\ 1) \quad \mathbf{p}_{33}(3,3,0)$$

The surface patch is shown in the diagram



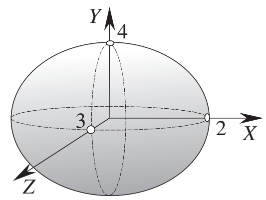
2.19 Second degree surfaces in standard form

Sphere



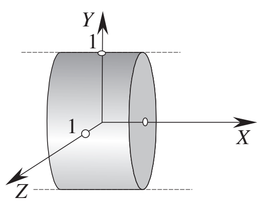
$x^2 + y^2 + z^2 = 100$

Ellipsoid



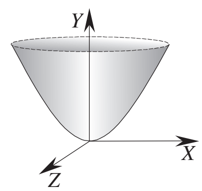
$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

Elliptic cylinder



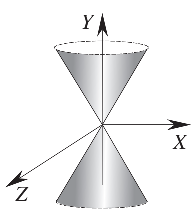
$y^2 + z^2 = 1$

Elliptic paraboloid



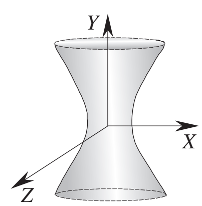
$x^2 + z^2 = y$

Elliptic cone



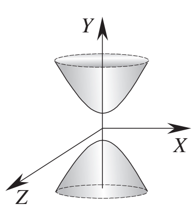
$x^2 + z^2 = y^2$

Elliptic hyperboloid of one sheet

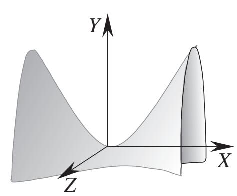


$x^2 + z^2 = 1 + y^2$

Elliptic hyperboloid of two sheets



$x^2 + z^2 = y^2 - 1$



$y = x^2 - z^2$

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