2

Problems with Dirichlet Boundary Conditions

2.1 Function Spaces

In what follows, we use the same notation for norms and inner products on spaces of scalar and vector-valued functions, in accordance with the convention adopted in §1.1. We also point out that all these function spaces are complex.

We start by introducing spaces of functions that depend on a complex parameter p; their properties, studied in [2], are listed below without proof.

Let $m \in \mathbb{R}$, $p \in \mathbb{C}$, $k \in \mathbb{R}$, and $S \subset \mathbb{R}^2$.

$H_m(\mathbb{R}^2)$, $H_m(S)$, and $H_m(\partial S)$ are the standard Sobolev spaces whose elements are defined on $\mathbb{R}^2$, S, and the boundary ∂S of S, respectively.

$H_{m,p}(\mathbb{R}^2)$ is the space that coincides as a set with $H_m(\mathbb{R}^2)$ but is equipped with the norm

$$\|u\|_{m,p} = \left\{ \int\limits_{\mathbb{R}^2} (1 + |p|^2 + |\xi|^2)^m |\tilde{u}(\xi)|^2 \, d\xi \right\}^{1/2},$$

where $\tilde{u}$ is the (distributional) Fourier transform of the three-component distribution $u \in \mathcal{S}'(\mathbb{R}^2)$ (see the Appendix). Clearly, for any fixed $p \in \mathbb{C}$, the norms on $H_{m,p}(\mathbb{R}^2)$ and $H_m(\mathbb{R}^2)$ are equivalent.

$\mathring{H}_{m,p}(S)$ is the subspace of all $u \in H_{m,p}(\mathbb{R}^2)$ such that $\operatorname{supp} u \subset \bar{S}$.

$H_{m,p}(S)$ is the space of the restrictions to S of all $v \in H_{m,p}(\mathbb{R}^2)$. The norm of $u \in H_{m,p}(S)$ is defined by

$$\|u\|_{m,p;S} = \inf_{v \in H_{m,p}(\mathbb{R}^2):\, v|_S = u} \|v\|_{m,p}.$$

The inner products in $L^2(\mathbb{R}^2)$, $L^2(S)$, and $L^2(\partial S)$ are denoted by $(\cdot\,,\cdot)_0$, $(\cdot\,,\cdot)_{0;S}$, and $(\cdot\,,\cdot)_{0;\partial S}$, respectively.

$H_{-m,p}(\mathbb{R}^2)$ is the dual of $H_{m,p}(\mathbb{R}^2)$ with respect to the duality generated by $(\cdot\,,\cdot)_0$.

$H_{-m,p}(S)$ is the dual of $\mathring{H}_{m,p}(S)$.

$H_{1/2,p}(\partial S)$ is the space of the traces on ∂S of all the elements of $H_{1,p}(S)$. It coincides as a set with $H_{1/2}(\partial S)$ but is equipped with the norm

$$\|f\|_{1/2,p;\partial S} = \inf_{u \in H_{1,p}(S):\gamma u = f} \|u\|_{1,p;S}.$$

Here γ is the trace operator, which maps $H_{1,p}(S)$ continuously to $H_{1/2,p}(\partial S)$. We mention that the continuity of γ is uniform with respect to $p \in \mathbb{C}$; that is,

$$\|\gamma u\|_{1/2,p;\partial S} \leq c\|u\|_{1,p;S},$$

where c does not depend on $p \in \mathbb{C}$ [2]. We recall that the trace operators corresponding to the interior and exterior domains $S^{\pm}$ are denoted by $\gamma^{\pm}$.

$H_{-1/2,p}(\partial S)$ is the dual of $H_{1/2,p}(\partial S)$ with respect to the duality generated by $(\cdot\,,\cdot)_{0;\partial S}$.

Next, l^+ and l^- are extension operators that, in the context of our function spaces, map $H_{1/2,p}(\partial S)$ to $H_{1,p}(S^+)$ and $H_{1,p}(S^-)$ continuously and uniformly with respect to $p \in \mathbb{C}$.

$H_{m,k,\kappa}(S)$ is the space of all $\hat{u}(x,p)$, $x \in S$, $p \in \mathbb{C}_\kappa$, such that the mapping $U(p) = \hat{u}(\cdot\,,p)$ is holomorphic from $\mathbb{C}_\kappa$ to $H_m(S)$ and

$$\|\hat{u}\|^2_{m,k,\kappa;S} = \sup_{\sigma>\kappa} \int\limits_{-\infty}^{\infty} (1+|p|^2)^k \|U(p)\|^2_{m,p;S}\,d\tau < \infty, \quad p = \sigma + i\tau. \tag{2.1}$$

Formula (2.1) defines the norm on $H_{m,k,\kappa}(S)$. It is readily seen from its definition that $U(p) \in H_{m,p}(S)$ for any $p \in \mathbb{C}_\kappa$. In what follows, we write $\hat{u}(x,p)$ if we want to emphasize that this is an element of $H_{m,p}(S)$, and $U(p)$ when we want to regard it as a mapping from $\mathbb{C}_\kappa$ to $H_m(S)$.

$H_{\pm 1/2,k,\kappa}(\partial S)$ are introduced similarly; that is, these spaces consist of all $\hat{f}(x,p)$, $x \in \partial S$, $p \in \mathbb{C}_\kappa$, such that the corresponding mapping $F(p) = \hat{f}(\cdot\,,p)$ is holomorphic from $\mathbb{C}_\kappa$ to $H_{\pm 1/2}(\partial S)$ and

$$\|\hat{f}\|^2_{\pm 1/2,k,\kappa;\partial S} = \sup_{\sigma>\kappa} \int\limits_{-\infty}^{\infty} (1+|p|^2)^k \|F(p)\|^2_{\pm 1/2,p;\partial S}\,d\tau < \infty.$$

Once again, the above equality defines the norms on these spaces and, as above, we interpret $\hat{f}(x,p)$ as an element of $H_{\pm 1/2,p}(\partial S)$ and $F(p)$ as a mapping from $\mathbb{C}_\kappa$ to $H_{\pm 1/2}(\partial S)$.

$H_{m,k,\kappa}(G)$ and $H_{\pm 1/2,k,\kappa}(\Gamma)$ consist, respectively, of the inverse Laplace transforms u and f of the elements $\hat{u}$ and $\hat{f}$ of $H_{m,k,\kappa}(S)$ and $H_{\pm 1/2,k,\kappa}(\partial S)$; these spaces are equipped with the norms

$$\begin{aligned} \|u\|_{m,k,\kappa;G} &= \|\hat{u}\|_{m,k,\kappa;S}, \\ \|f\|_{\pm 1/2,k,\kappa;\Gamma} &= \|\hat{f}\|_{\pm 1/2,k,\kappa;\partial S}. \end{aligned} \tag{2.2}$$

By the Paley–Wiener theorem and Parseval's equality [12], for a nonnegative integer k the spaces $H_{1,k,\kappa}(G)$ consist of all three-component distributions u defined on $S \times \mathbb{R}$ that vanish for $t < 0$ and are such that

$$\int\limits_G e^{-2\kappa t} \sum_{|\alpha|+\alpha_t \le 1} |(\partial_x^\alpha \partial_t^{\alpha_t + k} u)(x,t)|^2 \, dx \, dt < \infty, \tag{2.3}$$

where α is a two-component multi-index, α_t is a nonnegative integer, and ∂_x^α is the partial differentiation operator acting with respect to the space variables. The norm on $H_{1,k,\kappa}(G)$ defined by (2.3) is equivalent to (2.2). A similar remark is also valid for $H_{1/2,k,\kappa}(\Gamma)$.

In what follows, we relax the terminology and refer to the elements of all of the above spaces as "functions" instead of "distributions" or "generalized functions", since the former term, although technically incorrect, is more familiar to the nonspecialist reader.

Finally, $C_0^\infty(\bar{G}^\pm)$ are the spaces of infinitely differentiable functions with compact support in $\bar{G}^\pm$, respectively.

2.2 Solvability of the Transformed Problems

In this section, we discuss the problems ($\mathrm{D}_p^\pm$) obtained after applying the Laplace transformation with respect to the time variable in the original problems ($\mathrm{DD}^\pm$). Our aim is to establish the unique solvability of ($\mathrm{D}_p^\pm$) for every $p \in \mathbb{C}_0$ and derive certain estimates for their solutions. We use two different approaches to solve these problems. The first one is based on the Fredholm Alternative and works well in the case of interior domains. Unfortunately, the same cannot be done in exterior problems because here the operators occurring in the corresponding functional equations lose their compactness in the natural spaces where such problems are set. This makes it necessary for us to modify the method when we deal with exterior domains.

The transformed problems that we consider here are more general in that they include the contribution of the body forces and moments q occurring on the right-hand side in the equation of motion (1.8). Thus, the classical version of these more general problems ($\mathrm{D}_p^\pm$) consists in finding $\hat{u} \in C^2(S^\pm) \cap C(\bar{S}^\pm)$ such that

$$\begin{gathered} Bp^2 \hat{u}(x,p) + (A\hat{u})(x,p) = \hat{q}(x,p), \quad x \in S^\pm, \\ \hat{u}^\pm(x,p) = \hat{f}(x,p), \quad x \in \partial S, \end{gathered} \tag{2.4}$$

where

$$\hat{u}(x,p) = \mathcal{L}u(x,t), \quad \hat{q}(x,p) = \mathcal{L}q(x,t), \quad \hat{f}(x,p) = \mathcal{L}f(x,t).$$

In order to simplify the notation, and since there is no danger of ambiguity, throughout this section we omit the hat from the symbols of functions that depend on x and p, it being understood that, unless otherwise stipulated, we are carrying out our arguments in spaces of Laplace transforms.

As usual, to derive the variational version of $(\mathrm{D}_p^\pm)$, we multiply (2.4) termwise by $v^* \in C_0^\infty(S^\pm)$ and integrate the result over $S^\pm$; thus, we obtain

$$p^2(Bu,v)_{0;S^\pm} + a_\pm(u,v) = (q,v)_{0;S^\pm}, \tag{2.5}$$

where

$$(Bu,v)_{0;S^\pm} = (B^{1/2}u, B^{1/2}v)_{0;S^\pm},$$

$$a_\pm(u,v) = 2\int\limits_{S^\pm} E(u, v^{*\mathrm{T}})\,dx,$$

and $E(u,v)$ is the sesquilinear form defined by the internal energy density; that is [9],

$$\begin{aligned} 2E(u,v) &= h^2E_0(u,v) + h^2\mu(\partial_2u_1 + \partial_1u_2)(\partial_2\bar{v}_1 + \partial_1\bar{v}_2)\\ &\qquad + \mu[(u_1 + \partial_1u_3)(\bar{v}_1 + \partial_1\bar{v}_3) + (u_2 + \partial_2u_3)(\bar{v}_2 + \partial_2\bar{v}_3)],\\ E_0(u,v) &= (\lambda + 2\mu)\big[(\partial_1u_1)(\partial_1\bar{v}_1) + (\partial_2u_2)(\partial_2\bar{v}_2)\big]\\ &\qquad\qquad + \lambda\big[(\partial_1u_1)(\partial_2\bar{v}_2) + (\partial_2u_2)(\partial_1\bar{v}_1)\big]. \end{aligned}$$

It is obvious that

$$a_\pm(u,v) = \overline{a_\pm(v,u)}, \tag{2.6}$$

where the superposed bar denotes complex conjugation. Equation (2.5) indicates that the variational version of problems (2.4) should consist in finding $u \in H_{1,p}(S^\pm)$ such that

$$\begin{gathered} p^2(B^{1/2}u, B^{1/2}v)_{0;S^\pm} + a_\pm(u,v) = (q,v)_{0;S^\pm} \quad \forall v \in \mathring{H}_{1,p}(S^\pm),\\ \gamma^\pm u = f. \end{gathered} \tag{2.7}$$

Throughout what follows, we denote by the same symbol c all positive constants occurring in estimates, which are independent of the functions in these estimates and of $p \in \mathbb{C}_\kappa$, but may depend on κ.

2.1 Theorem. *For any $f \in H_{1/2,p}(\partial S)$ and $q \in H_{-1,p}(S^+)$, $p \in \bar{\mathbb{C}}_\kappa$, $\kappa > 0$, problem (D_p^+) has a unique weak solution $u \in H_{1,p}(S^+)$ and*

$$\|u\|_{1,p;S^+} \le c|p|(\|q\|_{-1,p;S^+} + \|f\|_{1/2,p;\partial S}). \tag{2.8}$$

Proof. First we consider (D_p^+) with homogeneous boundary conditions, which consists in finding $u_0 \in \mathring{H}_{1,p}(S^+)$ such that

$$p^2(B^{1/2}u_0, B^{1/2}v)_{0;S^+} + a_+(u_0,v) = (q,v)_{0;S^+} \quad \forall v \in \mathring{H}_{1,p}(S^+). \tag{2.9}$$

Repeating the proof of Lemma 2.3 in [7], we can show that $a_+(u,v)$ is coercive on $[\mathring{H}_{1,p}(S^+)]^2$. Since the form is also continuous on this space, we conclude that for any $q \in H_{-1,p}(S^+)$, the variational equation

$$a_+(u_0, v) = (q, v)_{0;S^+} \quad \forall v \in \mathring{H}_{1,p}(S^+) \tag{2.10}$$

has a unique solution $u_0 \in \mathring{H}_{1,p}(S^+)$, which satisfies the estimate

$$\|u_0\|_1 \le c\|q\|_{-1;S^+}.$$

On the other hand, since $a_+(u_0, v)$ defines a bounded antilinear (conjugate linear) functional on $\mathring{H}_{1,p}(S^+)$ for any $u_0 \in \mathring{H}_{1,p}(S^+)$, it can be written in the form (2.10) with some $q \in H_{-1,p}(S^+)$. Let $\mathcal{A}$ be the operator defined by $a_+(u_0, v)$, which associates $u_0 \in \mathring{H}_{1,p}(S^+)$ with $q \in H_{-1,p}(S^+)$ as described above; thus,

$$a_+(u_0, v) = (\mathcal{A}u_0, v)_{0;S^+} \quad \forall u_0, v \in \mathring{H}_1(S^+).$$

$\mathcal{A}$ is a homeomorphism from $\mathring{H}_1(S^+)$ to $H_{-1}(S^+)$. Additionally, from (2.6) it follows that $\mathcal{A}$ is self-adjoint in the sense that

$$(\mathcal{A}u_0, v)_{0;S^+} = (u_0, \mathcal{A}v)_{0;S^+} \quad \forall u_0, v \in \mathring{H}_{1,p}(S^+).$$

This is easily verified, since for any u_0, $v \in \mathring{H}_{1,p}(S^+)$,

$$\begin{aligned}(\mathcal{A}u_0, v)_{0;S^+} &= a_+(u_0, v) = \overline{a_+(v, u_0)} \\ &= \overline{(\mathcal{A}v, u_0)}_{0;S^+} = (u_0, \mathcal{A}v)_{0;S^+}.\end{aligned}$$

Equation (2.9) can now be written in the form

$$p^2 Bu_0 + \mathcal{A}u_0 = q. \tag{2.11}$$

Applying $\mathcal{A}^{-1}$ on both sides in (2.11), we arrive at the equivalent equation

$$p^2 \mathcal{A}^{-1} Bu_0 + u_0 = \mathcal{A}^{-1} q \tag{2.12}$$

in the Banach space $\mathring{H}_{1,p}(S^+)$. We denote by $\mathcal{B}_0$ the restriction of $\mathcal{A}^{-1}B$ from $H_{-1}(S^+)$ to $\mathring{H}_1(S^+)$ and claim that $\mathcal{B}_0$ is compact on $\mathring{H}_1(S^+)$. Let $\{u_n\}_{n=1}^\infty$ be a weakly convergent sequence in $\mathring{H}_1(S^+)$. Since S^+ is a bounded domain, Rellich's lemma implies that the set $\{u_n\}_{n=1}^\infty$ is strongly compact in $L^2(S^+)$. Therefore, there is a subsequence $\{u_{n_j}\}_{j=1}^\infty$ that converges strongly in $L^2(S^+)$, consequently, also in $H_{-1}(S^+)$. Because $\mathcal{A}^{-1}B$ is continuous from $H_{-1}(S^+)$ to $\mathring{H}_1(S^+)$, the sequence $\{\mathcal{B}_0 u_{n_j}\}_{j=1}^\infty$ is strongly convergent in the space $\mathring{H}_1(S^+)$, which proves that $\mathcal{B}_0$ is a compact operator.

While studying the solvability of (2.12), we are under the conditions of the Fredholm Alternative, so (2.12)—hence, also (2.11)—has a unique solution $u_0 \in \mathring{H}_1(S^+)$ if and only if the homogeneous equation

$$p^2 \mathcal{A}^{-1} B u_0 + u_0 = 0 \tag{2.13}$$

has only the trivial solution. In turn, (2.13) can be rewritten in the equivalent form

$$p^2 B u_0 + \mathcal{A} u_0 = 0,$$

or

$$p^2 (B^{1/2} u_0, B^{1/2} v)_{0;S^+} + a_+(u_0, v) = 0 \quad \forall v \in \mathring{H}_1(S^+). \tag{2.14}$$

Taking $v = u_0$ in (2.14), writing $p = \sigma + i\tau$, and separating the real and imaginary parts, we arrive at

$$(\sigma^2 - \tau^2) \|B^{1/2} u_0\|^2_{0;S^+} + a_+(u_0, u_0) = 0, \tag{2.15}$$

$$2\sigma\tau \|B^{1/2} u_0\|^2_{0;S^+} = 0. \tag{2.16}$$

If $\tau \neq 0$, then from (2.16) it follows that $B^{1/2} u_0 = 0$; hence, $u_0 = 0$. If $\tau = 0$, then the equality $u_0 = 0$ follows from (2.15). So (2.11) is uniquely solvable in $\mathring{H}_1(S^+)$ for any $q \in H_{-1}(S^+)$.

We now establish estimate (2.8). Taking $v = u_0$ in (2.9) and separating the real and imaginary parts, we obtain

$$(\sigma^2 - \tau^2) \|B^{1/2} u_0\|^2_{0;S^+} + a_+(u_0, u_0) = \operatorname{Re}(q, u_0)_{0;S^+}, \tag{2.17}$$

$$2\sigma\tau \|B^{1/2} u_0\|^2_{0;S^+} = \operatorname{Im}(q, u_0)_{0;S^+}. \tag{2.18}$$

Multiplying (2.18) by $\sigma^{-1}\tau$ and adding the new equality to (2.17), we find that

$$\begin{aligned} |p|^2 \|B^{1/2} u_0\|^2_{0;S^+} &+ a_+(u_0, u_0) \\ &= \operatorname{Re}(q, u_0)_{0;S^+} + \sigma^{-1}\tau \operatorname{Im}(q, u_0)_{0;S^+} \\ &= \sigma^{-1} \operatorname{Re}\{\bar{p}(q, u_0)_{0;S^+}\}. \end{aligned}$$

Because $p \in \bar{\mathbb{C}}_\kappa$, it follows that $\sigma \geq \kappa$. Taking into account the inequality $a_+(u_0, u_0) \geq c\|u_0\|_1^2$, we obtain

$$\|u_0\|^2_{1,p;S^+} \leq c|p| |(q, u_0)_{0;S^+}|,$$

from which

$$\|u_0\|_{1,p;S^+} \leq c|p| \|q\|_{-1,p;S^+}.$$

We now return to the full problem (2.7). Let $w = l^+ f \in H_1(S^+)$. We recall that since the extension operator l^+ is continuous (uniformly with respect to $p \in \mathbb{C}$), we have

$$\|w\|_{1,p;S^+} \le c\|f\|_{1/2,p;\partial S}. \tag{2.19}$$

Representing u in the form $u = w + u_0$, we see that $u_0 \in \mathring{H}_{1,p}(S^+)$ satisfies the equation

$$\begin{aligned} p^2(B^{1/2}u_0, B^{1/2}v) &+ a_+(u_0, v) \\ &= (q, v)_{0;S^+} - p^2(B^{1/2}w, B^{1/2}v)_{0;S^+} - a_+(w, v) \\ &\qquad \forall v \in \mathring{H}_{1,p}(S^+). \end{aligned} \tag{2.20}$$

We claim that the form $p^2(B^{1/2}w, B^{1/2}v)_{0;S^+} + a_+(w, v)$ defines a bounded antilinear (conjugate linear) functional on $\mathring{H}_{1,p}(S^+)$; true,

$$\begin{aligned} |p^2(B^{1/2}w, B^{1/2}v)_{0;S^+} &+ a_+(w, v)| \\ &\le c(|p|^2\|w\|_{0;S^+}\|v\|_{0;S^+} + \|w\|_{1;S^+}\|v\|_1) \\ &\le c\|w\|_{1,p;S^+}\|v\|_{1,p} \quad \forall v \in \mathring{H}_{1,p}(S^+). \end{aligned}$$

Consequently, there is $\tilde{q} \in H_{-1,p}(S^+)$ such that

$$p^2(B^{1/2}w, B^{1/2}v)_{0;S^+} + a_+(w, v) = (\tilde{q}, v)_{0;S^+} \quad \forall v \in \mathring{H}_{1,p}(S^+)$$

and

$$\|\tilde{q}\|_{-1,p;S^+} \le c\|w\|_{1,p;S^+} \le c\|f\|_{1/2,p;\partial S}.$$

Equation (2.20) takes the form

$$p^2(B^{1/2}u_0, B^{1/2}v)_{0;S^+} + a_+(u_0, v) = (q - \tilde{q}, v)_{0;S^+} \quad \forall v \in \mathring{H}_{1,p}(S^+).$$

As we already know, the latter problem is uniquely solvable and its solution satisfies the estimate

$$\begin{aligned} \|u_0\|_{1,p} &\le c|p|(\|q\|_{-1,p;S^+} + \|\tilde{q}\|_{-1,p;S^+}) \\ &\le c|p|(\|q\|_{-1,p;S^+} + \|f\|_{1/2,p;\partial S}). \end{aligned} \tag{2.21}$$

Combining (2.21) and (2.19), we arrive at (2.8).

If u_1 and u_2 are two solutions of (2.7), then $u_0 = u_1 - u_2 \in \mathring{H}_{1,p}(S^+)$ is a solution of (2.14); hence, $u_0 = 0$, and the theorem is proved. □

As was mentioned above, we cannot repeat this proof in the case of S^- because Rellich's lemma is not valid for an unbounded domain. Consequently, we need to modify our approach.

2.2 Theorem. *For any* $f \in H_{1/2,p}(\partial S)$ *and* $q \in H_{-1,p}(S^-)$, $p \in \bar{\mathbb{C}}_\kappa$, $\kappa > 0$, *problem* (D_p^-) *has a unique solution* $u \in H_{1,p}(S^-)$ *and*

$$\|u\|_{1,p;S^-} \le c|p|(\|q\|_{-1,p;S^-} + \|f\|_{1/2,p;\partial S}).$$

Proof. Again, first we assume that $f = 0$. In this case we seek $u_0 \in H_{1,p}(S^-)$ such that

$$p^2(B^{1/2}u_0, B^{1/2}v)_{0;S^-} + a_-(u_0, v) = (q, v)_{0;S^-} \quad \forall v \in \mathring{H}_{1,p}(S^-). \tag{2.22}$$

To prove the unique solvability of (2.22), we consider an auxiliary variational problem that consists in finding $u_0 \in \mathring{H}_1(S^-)$ such that

$$\tfrac{1}{2}\,\kappa^2(B^{1/2}u_0, B^{1/2}v)_{0;S^-} + a_-(u_0, v) = (q, v)_{0;S^-} \quad \forall v \in \mathring{H}_1(S^-), \tag{2.23}$$

where $q \in H_{-1}(S^-)$ is prescribed. Repeating the proof of Lemma 2.2 in [7], we find that there is a constant $c > 0$ such that

$$a_-(u, u) + \|u\|^2_{0;S^-} \ge c\|u\|^2_{1;S^-} \quad \forall u \in H_1(S^-). \tag{2.24}$$

From (2.24) it follows that the form

$$a_{-,\kappa}(u, v) = \tfrac{1}{2}\,\kappa^2(B^{1/2}u, B^{1/2}v)_{0;S^-} + a_-(u, v),$$

which is continuous on $\big[\mathring{H}_1(S^-)\big]^2$, is coercive on this space. The Lax–Milgram lemma then implies that (2.23) has a unique solution $u_0 \in \mathring{H}_1(S^-)$ for any $q \in H_{-1}(S^-)$. On the other hand, for any $u_0 \in \mathring{H}_1(S^-)$, the form $a_{-,\kappa}(u_0, v)$ generates a bounded antilinear (conjugate linear) functional on $\mathring{H}_1(S^-)$; therefore, it can be written in the form (2.23). This enables us to define an operator $\mathcal{A}_\kappa$ through the equality

$$(\mathcal{A}_\kappa u_0, v)_{0;S^-} = a_{-,\kappa}(u_0, v) \quad \forall v \in \mathring{H}_1(S^-),$$

which is a homeomorphism from $\mathring{H}_1(S^-)$ to $H_{-1}(S^-)$. Equation (2.23) can be rewritten as $\mathcal{A}_\kappa u_0 = q$. In turn, (2.22) can be written in the form

$$\mathcal{A}_\kappa u_0 + \big(p^2 - \tfrac{1}{2}\,\kappa^2\big)Bu_0 = q. \tag{2.25}$$

Applying $\mathcal{A}_\kappa^{-1}$ on both sides in (2.25), we arrive at the equivalent equation

$$u_0 + \big(p^2 - \tfrac{1}{2}\,\kappa^2\big)\mathcal{A}_\kappa^{-1}Bu_0 = \mathcal{A}_\kappa^{-1}q. \tag{2.26}$$

Setting $B^{1/2}u_0 = u_b$, we again rewrite (2.26) in the equivalent form

$$u_b + \big(p^2 - \tfrac{1}{2}\,\kappa^2\big)B^{1/2}\mathcal{A}_\kappa^{-1}B^{1/2}u_b = B^{1/2}\mathcal{A}_\kappa^{-1}q. \tag{2.27}$$

Equation (2.27) is solvable in $\mathring{H}_1(S^-)$. If $u_b \in \mathring{H}_1(S^-)$ is its solution, then u_b is, at the same time, the solution of (2.27) in $L^2(S^-)$. Conversely, let $u_b \in L^2(S^-)$ be a solution of (2.27). Since

$$B^{1/2}\mathcal{A}_\kappa^{-1}B^{1/2}u_b \in \mathring{H}_1(S^-), \quad B^{1/2}\mathcal{A}_\kappa^{-1}q \in \mathring{H}_1(S^-),$$

it follows that $u_b \in \mathring{H}_1(S^-)$. This means that problem (2.27) in $\mathring{H}_1(S^-)$ is equivalent to itself in $L^2(S^-)$. We now study the properties of the restriction $\mathcal{B}_\kappa$ of $B^{1/2}\mathcal{A}_\kappa^{-1}B^{1/2}$ from $H_1(S^-)$ to $L^2(S^-)$.

Let q, $\psi \in L^2(S^-)$ be arbitrary, and let

$$u_0 = \mathcal{A}_\kappa^{-1}B^{1/2}q, \quad v = \mathcal{A}_\kappa^{-1}B^{1/2}\psi.$$

From the definition of $\mathcal{A}_\kappa$ it follows that

$$a_{-,\kappa}(u_0, v) = (B^{1/2}q, v)_{0;S^-},$$
$$a_{-,\kappa}(v, u_0) = (B^{1/2}\psi, u_0)_{0;S^-};$$

hence,

$$(B^{1/2}q, v)_{0;S^-} = \overline{(B^{1/2}\psi, u_0)}_{0;S^-} = (u_0, B^{1/2}\psi)_{0;S^-}. \tag{2.28}$$

In turn, (2.28) can be rewritten as

$$(\mathcal{A}_\kappa^{-1}B^{1/2}q, B^{1/2}\psi)_{0;S_-} = (B^{1/2}q, \mathcal{A}_\kappa^{-1}B^{1/2}\psi)_{0;S^-},$$

or

$$(\mathcal{B}_\kappa q, \psi)_{0;S^-} = (q, \mathcal{B}_\kappa\psi)_{0;S^-} \quad \forall q,\ \psi \in L^2(S^-). \tag{2.29}$$

By (2.29), $\mathcal{B}_\kappa$ is a symmetric operator on $L^2(S^-)$; therefore, it is self-adjoint (as a symmetric operator defined on the whole of a Hilbert space [3]).

Once again, let $q \in L^2(S^-)$ and $u_0 = \mathcal{A}_\kappa^{-1}B^{1/2}q$; then

$$\begin{aligned}(\mathcal{B}_\kappa q, q)_{0;S^-} &= (\mathcal{A}_\kappa^{-1}B^{1/2}q, B^{1/2}q)_{0;S^-}\\ &= (u_0, \mathcal{A}_\kappa u_0)_{0;S^-} = a_{-,\kappa}(u_0, u_0) \geq 0,\end{aligned}$$

which means that $\mathcal{B}_\kappa$ is nonnegative. Since the spectrum of a self-adjoint nonnegative operator lies on the half-line $[0,\infty)$ in the complex plane, every point of $\mathbb{C} \setminus [0,\infty)$ is a regular point for $\mathcal{B}_\kappa$ [3]. All that remains to do now is to remark that for any $p \in \mathbb{C}_\kappa$,

$$\left(\tfrac{1}{2}\kappa^2 - p^2\right)^{-1} \notin [0,\infty).$$

This implies that (2.27) is uniquely solvable for any $q \in H_{-1}(S^-)$. Consequently, equation (2.22) has a unique solution $u_0 \in \mathring{H}_{1,p}(S^-)$ for any $q \in H_{-1,p}(S^-)$. To complete the proof, we repeat the last part of the proof of Theorem 2.1, replacing the extension operator l^+ by l^-. $\square$

2.3 Solvability of the Time-dependent Problems

We start with the variational version of problems (DD$^{\pm}$) for the nonhomogeneous equation of motion. The classical formulation of (DD^{+}) asks for a function $u \in C^2(G^+) \cap C^1(\bar{G}^+)$ such that

$$\begin{gathered} B(\partial_t^2 u)(X) + (Au)(X) = q(X), \quad X \in G^+, \\ u(x, 0+) = (\partial_t u)(x, 0+) = 0, \quad x \in S^+, \\ u^+(X) = f(X), \quad X \in \Gamma. \end{gathered} \tag{2.30}$$

Multiplying the first equation (2.30) by v^*, where $v \in C_0^\infty(\bar{G}^+)$ is such that $v^+ = 0$, integrating the new equality over S^+ with respect to x and over $[0, \infty)$ with respect to t, and taking into account the initial data for u and the boundary value of v, we arrive at

$$\int_0^\infty \big[a_+(u, v) - (B^{1/2}\partial_t u, B^{1/2}\partial_t v)_{0;S^+}\big]\, dt = \int_0^\infty (q, v)_{0;S^+}\, dt. \tag{2.31}$$

Conversely, if $u \in C^2(G^+) \cap C^1(\bar{G}^+)$ satisfies (2.31) for any $v \in C_0^\infty(\bar{G}^+)$ such that $v^+ = 0$, $u(x, 0+) = 0$, $x \in S^+$, and $u^+ = f$, then, integrating by parts in (2.31), we find that u is the solution of (2.30). Equation (2.31) suggests that the variational problem (DD^{+}) should consist in finding $u \in H_{1,0,\kappa}(G^+)$ that satisfies

$$\begin{gathered} \int_0^\infty \big[a_+(u, v) - (B^{1/2}\partial_t u, B^{1/2}\partial_t v)_{0;S^+}\big]\, dt \\ = \int_0^\infty (q, v)_{0;S^+}\, dt \quad \forall v \in C_0^\infty(\bar{G}^+),\ v^+ = 0, \\ \gamma^+ u = f, \end{gathered} \tag{2.32}$$

where q and f are prescribed. Similarly, the variational problem (DD^{-}) consists in finding $u \in H_{1,0,\kappa}(G^-)$ that satisfies

$$\begin{gathered} \int_0^\infty \big[a_-(u, v) - (B^{1/2}\partial_t u, B^{1/2}\partial_t v)_{0;S^-}\big]\, dt \\ = \int_0^\infty (q, v)_{0;S^-}\, dt, \quad \forall v \in C_0^\infty(\bar{G}^-),\ v^- = 0, \\ \gamma^- u = f. \end{gathered} \tag{2.33}$$

2.3 Theorem. *For any* $q \in H_{-1,1,\kappa}(G^+)$ *and* $f \in H_{1/2,1,\kappa}(\Gamma)$, $\kappa > 0$, *problems* (2.32) *and* (2.33) *have unique solutions* $u \in H_{1,0,\kappa}(G^\pm)$. *If* $q \in H_{-1,k,\kappa}(G^\pm)$ *and* $f \in H_{1/2,k,\kappa}(\Gamma)$, $k \in \mathbb{R}$, *then* $u \in H_{1,k-1,\kappa}(G^\pm)$ *and*

$$\|u\|_{1,k-1,\kappa;G^\pm} \le c(\|q\|_{-1,k,\kappa;G^\pm} + \|f\|_{1/2,k,\kappa;\Gamma}). \tag{2.34}$$

Proof. We prove the assertion for (DD$^+$); the case of (DD$^-$) is treated similarly.

Let $\hat{u} \in H_{1,p}(S^+)$ be the (weak) solution of the problem

$$\begin{aligned} p^2 B\hat{u}(x,p) + (A\hat{u})(x,p) &= \hat{q}(x,p), \quad x \in S^+, \\ \gamma^+ \hat{u}(x,p) &= \hat{f}(x,p), \quad x \in \partial S, \end{aligned} \tag{2.35}$$

obtained by applying the Laplace transformation in (DD$^+$) with a nonhomogeneous equation of motion. The existence of $\hat{u}$ was proved in the previous section. For simplicity, in what follows we write

$$\hat{u}(\cdot\,,p) = U(p), \quad \hat{q}(\cdot\,,p) = Q(p), \quad \hat{f}(\cdot\,,p) = F(p),$$

and regard U, Q, and F as functions from $\mathbb{C}_\kappa$ to $H_1(S^+)$, $H_{-1}(S^+)$, and $H_{1/2}(\partial S)$, respectively.

We claim that the inverse Laplace transform u of $\hat{u}$ belongs to the space $H_{1,k-1,\kappa}(G^+)$ if $q \in H_{-1,k,\kappa}(G^+)$ and $f \in H_{1/2,k,\kappa}(\Gamma)$. To show this, first we verify that U is holomorphic from $\mathbb{C}_\kappa$ to $H_1(S^+)$. Let $p_0 \in \mathbb{C}_\kappa$, and let $K_R(p_0)$ be a circle with center at p_0 and radius R (to be specified later), and such that $\bar{K}_R(p_0) \subset \mathbb{C}_\kappa$. We recall that the solution $U(p_0)$ of the problem

$$\begin{gathered} p_0^2 BU(p_0) + (AU)(p_0) = Q(p_0), \\ \gamma^+ U(p_0) = F(p_0) \end{gathered}$$

satisfies the estimates

$$\begin{aligned} \|U(p_0)\|_{1;S^+} &\le c|p_0|(\|Q(p_0)\|_{-1,p_0;S^+} + \|F(p_0)\|_{1/2,p_0;\partial S}) \\ &\le c(\|Q(p_0)\|_{-1;S^+} + \|F(p_0)\|_{1/2;\partial S}). \end{aligned}$$

Rewriting (2.35) in the form

$$\begin{gathered} p_0^2 BU(p) + (AU)(p) = Q(p) - (p^2 - p_0^2)BU(p), \\ \gamma^+ U(p) = F(p), \end{gathered} \tag{2.36}$$

we see that

$$\begin{aligned} \|U(p)\|_{1;S^+} \le c(\|Q(p)\|_{-1;S^+} &+ \|F(p)\|_{1/2;\partial S} \\ &+ |p^2 - p_0^2| \|U(p)\|_{-1;S^+}). \end{aligned}$$

Since $\|U(p)\|_{-1;S^+} \le \|U(p)\|_{1;S^+}$, it follows that for p satisfying

$$c|p^2 - p_0^2| \le \tfrac{1}{2} \tag{2.37}$$

we have

$$\|U(p)\|_{1;S^+} \le c(\|Q(p)\|_{-1;S^+} + \|F(p)\|_{1/2;\partial S}). \tag{2.38}$$

We choose $R > 0$ so that $\bar{K}_R(p_0) \subset \mathbb{C}_\kappa$ and (2.37) holds for $p \in K_R(p_0)$. Since Q and F are holomorphic from $\mathbb{C}_\kappa$ to $H_{-1}(S^+)$ and $H_{1/2}(\partial S)$, respectively, they are bounded in these spaces for $p \in \bar{K}_R(p_0)$. Estimate (2.38) shows that U is also bounded in $H_1(S^+)$ for $p \in \bar{K}_R(p_0)$. By (2.36),

$$\begin{aligned} p_0^2 B\big[U(p) - U(p_0)\big] &+ A\big[U(p) - U(p_0)\big] \\ &= Q(p) - Q(p_0) - (p^2 - p_0^2)BU(p), \\ \gamma^+ U(p) - \gamma^+ U(p_0) &= F(p) - F(p_0); \end{aligned}$$

hence, for $p \in K_R(p_0)$,

$$\begin{aligned} \|U(p) - U(p_0)\|_{1;S^+} & \\ \le c\big(\|Q(p) - Q(p_0)\|_{-1;S^+} &+ \|F(p) - F(p_0)\|_{1/2;\partial S} \\ &+ |p^2 - p_0^2|\|BU(p)\|_{-1;S^+}\big). \end{aligned}$$

As $U(p)$ is bounded in $H_1(S^+)$—hence, also in $H_{-1}(S^+)$—for $p \in K_R(p_0)$, it follows that

$$\lim_{p \to p_0} \|U(p) - U(p_0)\|_{1;S^+} = 0,$$

which means that U is continuous from $\mathbb{C}_\kappa$ to $H_1(S^+)$ at p_0. Finally, let $V \in H_1(S^+)$ be the solution of the problem

$$\begin{aligned} p_0^2 BV - AV &= Q'(p_0) - 2p_0 U(p_0), \\ \gamma^+ V &= F'(p_0); \end{aligned}$$

then the function

$$W(p) = (p - p_0)^{-1}\big[U(p) - U(p_0)\big] - V \in H_1(S^+)$$

satisfies

$$\begin{aligned} p_0^2 BW(p) + (AW)(p) & \\ &= (p - p_0)^{-1}\big[Q(p) - Q(p_0)\big] - Q'(p_0) \\ &\qquad - B\big[(p + p_0)U(p) - 2p_0 U(p_0)\big], \\ \gamma^+ W &= (p - p_0)^{-1}\big[F(p) - F(p_0)\big] - F'(p_0). \end{aligned}$$

Next,

$$\|W(p)\|_{1;S^+} \le c\left\{\left\|\frac{Q(p)-Q(p_0)}{p-p_0} - Q'(p_0)\right\|_{-1;S^+} + \left\|\frac{F(p)-F(p_0)}{p-p_0} - F'(p_0)\right\|_{1/2;\partial S} + \|(p+p_0)U(p) - 2p_0U(p_0)\|_{-1;S^+}\right\}.$$

Since U is continuous at p_0,

$$\lim_{p\to p_0} \|W(p)\|_{1;S^+} = 0,$$

which means that $U'(p_0)$ exists and $U'(p_0) = V$. The arbitrariness of p_0 in $\mathbb{C}_\kappa$ implies that the mapping U is holomorphic from $\mathbb{C}_\kappa$ to $H_1(S^+)$.

Given that

$$\|U(p)\|_{1,p;S^+} \le c|p|(\|Q(p)\|_{-1,p;S^+} + \|F(p)\|_{1/2,p;\partial S}),$$

we have

$$\begin{aligned}\|u\|^2_{1,k-1,\kappa;G^+} &= \sup_{\sigma>\kappa} \int\limits_{-\infty}^{\infty} (1+|p|^2)^{k-1}\|U(p)\|^2_{1,p;S^+}\,d\tau \\ &\le c\sup_{\sigma>\kappa} \int\limits_{-\infty}^{\infty} (1+|p|^2)^{k}(\|Q(p)\|^2_{-1,p;S^+} + \|F(p)\|^2_{1/2,p;\partial S})\,d\tau \\ &\le c(\|q\|^2_{-1,k,\kappa;G^+} + \|f\|^2_{1/2,k,\kappa;\Gamma}), \qquad (2.39)\end{aligned}$$

where $p = \sigma + i\tau$. This confirms (2.34).

To complete the proof of the theorem, we need to check that u is the only solution of (2.32). We recall [11] that any two functions $f_1(t)$ and $f_2(t)$ such that

$$\int\limits_0^{\infty} e^{-2\kappa_\nu t}|f_\nu(t)|^2\,dt < \infty, \quad \nu = 1,2,$$

satisfy Parseval's equality

$$\int\limits_0^{\infty} e^{-(\kappa_1+\kappa_2)t} f_1(t)\overline{f_2(t)}\,dt = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F_1(\kappa_1+i\tau)\overline{F_2(\kappa_2+i\tau)}\,d\tau, \qquad (2.40)$$

where $F_\nu(p)$ are the Laplace transforms of $f_\nu(t)$. Let $v \in C_0^\infty(\bar{G}^+)$ be such that $\gamma^+ v = 0$. We make the notation

$$v(x,0) = v_0(x), \;\; v_0 \in \mathring{H}_1(S^+), \quad v(\,\cdot\,,p) = V(p), \quad v_0(\cdot) = V_0,$$
$$p = \sigma + i\tau, \quad p^* = -\sigma + i\tau,$$

choose any $\sigma > \kappa$ and fix it, then take $\kappa_1 = \sigma$ and $\kappa_2 = -\sigma$ in (2.40) and find that the Laplace transform of $\partial_t v$ at the point p^* is $p^*V(p^*) - V_0$, and that

$$\int_0^\infty \bigl[a_+(u,v) - (B^{1/2}\partial_t u, B^{1/2}\partial_t v)_{0;S^+} - (q,v)_{0;S^+}\bigr]\,dt$$
$$= (2\pi)^{-1}\int_{-\infty}^{\infty} \bigl[a_+\bigl(U(p),V(p^*)\bigr) - \bigl(B^{1/2}pU(p), B^{1/2}(p^*V(p^*) - V_0)\bigr)_{0;S^+}$$
$$- \bigl(Q(p),V(p^*)\bigr)_{0;S^+}\bigr]\,d\tau. \qquad (2.41)$$

Since $U(p)$ is a weak solution of (2.4), it follows that

$$a_+\bigl(U(p),W\bigr) + p^2\bigl(B^{1/2}U(p), B^{1/2}W\bigr)_{0;S^+}$$
$$= \bigl(Q(p),W\bigr)_{0;S^+} \quad \forall W \in \mathring{H}_1(S^+). \qquad (2.42)$$

Taking $W = V(p^*) - (p^*)^{-1}V_0$ in (2.42), we obtain

$$a_+\bigl(U(p),V(p^*)\bigr) - \bigl(B^{1/2}pU(p), B^{1/2}(p^*V(p^*) - V_0)\bigr)_{0;S^+}$$
$$- \bigl(Q(p),V(p^*)\bigr)_{0;S^+}$$
$$= a_+\bigl(U(p),(p^*)^{-1}V_0\bigr) - \bigl(Q(p),(p^*)^{-1}V_0\bigr)_{0;S^+}$$
$$= p^{-1}\bigl(Q(p),V_0\bigr)_{0;S^+} - p^{-1}a_+\bigl(U(p),V_0\bigr).$$

Therefore, (2.41) takes the form

$$\int_0^\infty \bigl[a_+(u,v) - (B^{1/2}\partial_t u, B^{1/2}\partial_t v)_{0;S^+} - (q,v)_{0;S^+}\bigr]\,dt$$
$$= (2\pi)^{-1}\int_{-\infty}^{\infty} p^{-1}\bigl[\bigl(Q(p),V_0\bigr)_{0;S^+} - a_+\bigl(U(p),V_0\bigr)\bigr]\,d\tau. \quad (2.43)$$

We claim that the right-hand side in (2.43) vanishes. First, we remark that

$$\|\hat{q}\|^2_{-1;S^+} \le (1+|p|^2)\|\hat{q}\|^2_{-1,p;S^+} \quad \forall \hat{q} \in H_{-1}(S^+). \qquad (2.44)$$

For if $\hat{q} \in H_{-1}(\mathbb{R}^2)$ and $\tilde{q}$ is its Fourier transform, then

$$\begin{aligned}
\|\hat{q}\|_{-1}^2 &= \int\limits_{\mathbb{R}^2} (1+|\xi|^2)^{-1}|\tilde{q}(\xi)|^2\, d\xi \\
&\le (1+|p|^2) \int\limits_{\mathbb{R}^2} (1+|p|^2)^{-1}(1+|\xi|^2)^{-1}|\tilde{q}(\xi)|^2\, d\xi \\
&\le (1+|p|^2) \int\limits_{\mathbb{R}^2} (1+|p|^2+|\xi|^2)^{-1}|\tilde{q}(\xi)|^2\, d\xi \\
&= (1+|p|^2)\|\hat{q}\|_{-1,p}^2. \qquad (2.45)
\end{aligned}$$

Inequality (2.44) follows from (2.45) and the definition of the norms on the spaces $H_{-1}(S^+)$ and $H_{-1,p}(S^+)$. By (2.44),

$$\begin{aligned}
\int\limits_{-\infty}^{\infty} |(Q(p),V_0)_{0;S^+}|^2\, d\tau &\le \|V_0\|_1^2 \int\limits_{-\infty}^{\infty} \|Q(p)\|_{-1;S^+}^2\, d\tau \\
&\le \|V_0\|_1^2 \int\limits_{-\infty}^{\infty} (1+|p|^2)\|Q(p)\|_{-1,p;S^+}^2\, d\tau \\
&\le \|V_0\|_1^2\, \|q\|_{-1,1,\kappa;G^+}^2 < \infty.
\end{aligned}$$

Consequently, the function $\varphi(t) = (q,v_0)_{0;S^+}$ satisfies

$$\int\limits_0^{\infty} e^{-2\sigma t}|\varphi(t)|^2\, dt < \infty,$$

so $\varphi \in L^1_{\mathrm{loc}}(0,\infty)$, which implies that

$$\psi(t) = \int\limits_0^t \varphi(\lambda)\, d\lambda$$

is continuous for $t \in [0,\infty)$; in particular, $\psi(0) = 0$. We have

$$0 = \psi(0) = (2\pi)^{-1} \int\limits_{-\infty}^{\infty} p^{-1}\big(Q(p),V_0\big)_{0;S^+}\, d\tau.$$

Next,

$$\int_{-\infty}^{\infty} |a_+(U(p),V_0)|^2\,d\tau \le c\|V_0\|_1^2 \int_{-\infty}^{\infty} \|U(p)\|_{1,p;S^+}^2\,d\tau$$

$$\le c\|V_0\|_1^2\|u\|_{1,0,\kappa;G^+}^2,$$

and the above arguments yield

$$\int_{-\infty}^{\infty} p^{-1}a_+(U(p),V_0)\,d\tau = 0.$$

Equality (2.43) now leads to

$$\int_0^{\infty} \big[a_+(u,v) - (B^{1/2}\partial_t u, B^{1/2}\partial_t v)_{0;S^+}\big]\,dt = \int_0^{\infty} (q,v)_{0;S^+}\,dt,$$

which means that u is a weak solution of (2.32). To prove that this solution is unique, suppose that $u \in H_{1,0,\kappa}(G^+)$, $\gamma^+ u = 0$, satisfies

$$\int_0^{\infty} \big[a_+(u,v) - (B^{1/2}\partial_t u, B^{1/2}\partial_t v)_{0;S^+}\big]\,dt = 0$$

$$\forall v \in C_0^{\infty}(\bar G^+),\ \gamma^+ v = 0. \qquad (2.46)$$

We fix an arbitrary $T > 0$. It is obvious that

$$u \in H_1((0,T);L^2(S^+)) \cap L^2((0,T);\mathring{H}_1(S^+));$$

that is, $U(t) = u(\cdot\,,t)$, regarded as a vector-valued function from $(0,T)$ to $L^2(S^+)$, belongs to $H_1(0,T)$ and

$$\int_0^{T} \big\{\|U(t)\|_{0;S^+}^2 + \|U'(t)\|_{0;S^+}^2\big\}\,dt < \infty. \qquad (2.47)$$

The same function, regarded as a mapping from $(0,T)$ to $\mathring{H}_1(S^+)$, belongs to $L^2(0,T)$ and

$$\int_0^{T} \|U(t)\|_1^2\,dt < \infty. \qquad (2.48)$$

From (2.47) and (2.48) we see that $H_1((0,T);L^2(S^+)) \cap L^2((0,T);\mathring{H}_1(S^+))$ can be equipped with the norm

$$\|u\|_{1;G_T^+}^2 = \int_0^T \int_{S^+} \sum_{|\alpha|+\alpha_t \le 1} |\partial^{\alpha+\alpha_t} u(x,t)|^2 \, dx \, dt,$$

where $G_T^+ = S^+ \times (0,T)$.

We now construct the function $Z(t) = z(\cdot\,,t)$, where

$$z(x,t) = \begin{cases} -\int_t^T u(x,\tau)\,d\tau, & t \le T, \\ 0, & t > T. \end{cases} \tag{2.49}$$

Clearly, the restriction of Z to $(0,T)$ belongs to $H_1((0,T);L^2(S^+)) \cap L^2((0,T);\mathring{H}_1(S^+))$ and that z can be approximated with any accuracy in the norm $\|\cdot\|_{1;G_T^+}$ by means of elements $v \in C_0^\infty(\bar{G}^+)$ such that $\gamma^+ v = 0$. Hence, we may set $v = z$ in (2.46) and obtain

$$\int_0^T \left[a_+(u,z) - (B^{1/2}\partial_t u, B^{1/2}\partial_t z)_{0;S^+}\right] dt = 0.$$

We remark that

$$Z'(t) = (\partial_t z)(\cdot\,,t) = \begin{cases} u(\cdot\,,t), & t < T, \\ 0, & t > T, \end{cases} \tag{2.50}$$

and rewrite (2.46) in the form

$$\int_0^T \left[a_+(\partial_t z, z) - (B^{1/2}\partial_t u, B^{1/2} u)_{0;S^+}\right] dt = 0,$$

or

$$\int_0^T \frac{d}{dt}\left[a_+(z,z) - \|B^{1/2}u\|_{0;S^+}^2\right] dt = 0. \tag{2.51}$$

Since U, regarded as a mapping from $(0,T)$ to $L^2(S^+)$, belongs to $H_1(0,T)$, it is absolutely continuous on $[0,T]$; hence,

$$\begin{aligned} \int_0^T \frac{d}{dt}\|B^{1/2}u\|_{0;S^+}^2 \, dt &= \|B^{1/2}U(T)\|_{0;S^+}^2 - \|B^{1/2}U(0)\|_{0;S^+}^2 \\ &= \|B^{1/2}U(T)\|_{0;S^+}^2. \end{aligned} \tag{2.52}$$

From (2.49) and (2.50), it follows that Z, as a mapping from $(0,T)$ to $\mathring{H}_1(S^+)$, belongs to $H_1(0,T)$; hence, Z is absolutely continuous on $[0,T]$ and

$$\begin{aligned}\int\limits_0^T \frac{d}{dt}\, a_+(z,z)\, dt &= a_+(Z(T),Z(T)) - a_+(Z(0),Z(0))\\ &= -a_+(Z(0),Z(0)).\end{aligned} \tag{2.53}$$

Formulas (2.51)–(2.53) now imply that

$$a_+(Z(0),Z(0)) + \|B^{1/2}U(T)\|^2_{0;S^+} = 0;$$

therefore, $u(T) = u(\cdot\,,T) = 0$ for any $T > 0$, which completes the proof. $\square$

http://www.springer.com/978-1-85233-888-6

Variational and Potential Methods for a Class of Linear
Hyperbolic Evolutionary Processes
Chudinovich, I.
2005, XII, 148 p., Hardcover
ISBN: 978-1-85233-888-6