Preface

The notion of an order plays an important rôle not only throughout mathematics but also in adjacent disciplines such as logic and computer science. The purpose of the present text is to provide a basic introduction to the theory of ordered structures. Taken as a whole, the material is mainly designed for a postgraduate course. However, since prerequisites are minimal, selected parts of it may easily be considered suitable to broaden the horizon of the advanced undergraduate. Indeed, this has been the author's practice over many years.

A basic tool in analysis is the notion of a continuous function, namely a mapping which has the property that the inverse image of an open set is an open set. In the theory of ordered sets there is the corresponding concept of a residuated mapping, this being a mapping which has the property that the inverse image of a principal down-set is a principal down-set. It comes therefore as no surprise that residuated mappings are important as far as ordered structures are concerned. Indeed, albeit beyond the scope of the present exposition, the naturality of residuated mappings can perhaps best be exhibited using categorical concepts. If we regard an ordered set as a small category then an order-preserving mapping $f : A \to B$ becomes a functor. Then f is residuated if and only if there exists a functor $f^+ : B \to A$ such that (f, f^+) is an adjoint pair.

Residuated mappings play a central rôle throughout this exposition, with fundamental concepts being introduced whenever possible in terms of natural properties of them. For example, an order isomorphism is precisely a bijection that is residuated; an ordered set E is a meet semilattice if and only if, for every principal down-set $x^{\downarrow}$, the canonical embedding of $x^{\downarrow}$ into E is residuated; and a Heyting algebra can be characterised as a lattice-based algebra in which every translation $\lambda_x : y \mapsto x \wedge y$ is residuated. The important notion of a closure operator, which arises in many situations that concern ordered sets, is intimately related to that of a residuated mapping. Likewise, Galois connections can be described in terms of residuated mappings, and vice versa. Residuated mappings have the added advantage that they can be composed to form new residuated mappings. In particular, the set $\mathrm{Res}\, E$ of residuated mappings on an ordered set E forms a semigroup, and here we include descriptions of the types of semigroup that arise.

A glance at the list of contents will reveal how the material is marshalled. Roughly speaking, the text may be divided into two parts though it should be stressed that these are not mutually independent. In Chapters 1 to 8 we deal with the essentials of ordered sets and lattices, including boolean algebras, p-algebras, Heyting algebras, and their subdirectly irreducible algebras. In Chapters 9 to 14 we provide an introduction to ordered algebraic structures, including ordered groups, rings, fields, and semigroups. In particular, we include a characterisation of the real numbers as, to within isomorphism, the only Dedekind complete totally ordered field, something that is rarely seen by mathematics graduates nowadays. As far as ordered groups are concerned, we develop the theory as far as proving that every archimedean lattice-ordered group is commutative. In dealing with ordered semigroups we concentrate mainly on naturally ordered regular and inverse semigroups and provide a unified account which highlights those that admit an ordered group as an image under a residuated epimorphism, culminating in structure theorems for various types of Dubreil-Jacotin semigroups.

Throughout the text we give many examples of the structures arising, and interspersed with the theorems there are bundles of exercises to whet the reader's appetite. These are of varying degrees of difficulty, some being designed to help the student gain intuition and some serving to provide further examples to supplement the text material. Since this is primarily designed as a non-encyclopaedic introduction to the vast area of ordered structures we also include relevant references.

We are deeply indebted to Professora Doutora Maria Helena Santos and Professor Doutor Herberto Silva for their assistance in the proof-reading. The traditional free copy is small recompense for their labour. Finally, our thanks go to the editorial team at Springer for unparalleled courtesy and efficiency.

St Andrews, Scotland T.S. Blyth

http://www.springer.com/978-1-85233-905-0

Lattices and Ordered Algebraic Structures
Blyth, T.S.
2005, X, 304 p., Hardcover
ISBN: 978-1-85233-905-0