2. The Universal Generating Function in Reliability Analysis of Binary Systems

While the most effective applications of the UGF method lie in the field of the MSS reliability, it can also be used for evaluating the reliability of binary systems. The theory of binary systems is well developed. Many algorithms exist for evaluating the reliability of different types of binary system. However, no universal systematic approach has been suggested for the wide range of system types. This chapter demonstrates the ability of the UGF approach to handle the reliability assessment problem for different types of binary system.

Since very effective specialized algorithms were developed for each type of system, the UGF-based procedures may not appear to be very effective in comparison with the best known algorithms (these algorithms can be found in the comprehensive book of Kuo and Zuo [12]). The aim of this chapter is to demonstrate how the UGF technique can be adapted for solving a variety of reliability evaluation problems.

2.1 Basic Notions of Binary System Reliability

System reliability analysis considers the relationship between the functioning of the system's elements and the functioning of the system as a whole. An element is an entity in a system that is not further subdivided. This does not imply that an element cannot be made of parts; rather, it means that, in a given reliability study, it is regarded as a self-contained unit and is not analyzed in terms of the functioning of its constituents.

In binary system reliability analysis it is assumed that each system element, as well as the entire system, can be in one of two possible states, *i.e.* working or failed. Therefore, the state of each element or the system can be represented by a binary random variable such that X_j indicates the state of element j: $X_j = 1$ if element j is in working condition and $X_j = 0$ if element j is failed; X indicates the state of the entire system: $X = 1$ if the system works, $X = 0$ if the system is failed.

The states of all n elements composing the system are represented by the so-called element state vector $(X_1, \ldots, X_n)$. It is assumed that the states of the system elements (the realization of the element state vector) unambiguously determine the state of the system. Thus, the relationship between the element state vector and the system state variable X can be expressed by the deterministic function

$$X = \phi(X_1,..., X_n) \tag{2.1}$$

This function is called the system structure function.

Example 2.1

Consider an air conditioning system that consists of two air conditioners supplied from a single power source. The system fails if neither air conditioner works.

The two air conditioners constitute a subsystem that fails if and only if all of its elements are failed. Such subsystems are called parallel.

Assume that the random binary variables X_1 and X_2 represent the states of the air conditioners and the random binary variable X_c represents the state of the subsystem. The structure function of the subsystem can be expressed as

$$X_c = \phi_{par}(X_1, X_2) = \max(X_1, X_2) = 1-(1-X_1)(1-X_2)$$

The entire system fails either if the power source fails or if the subsystem of the conditioners fails. The system that works if and only if all of its elements work is called a series system. Assume that the random binary variable X_3 represents the state of the power source. The structure function of the entire system takes the form

$$X = \phi_{ser}(X_3, X_c) = \min(X_3, X_c) = X_3 X_c$$

Combining the two expressions one can obtain the structure function of the entire system:

$$\begin{aligned} X &= \phi_{ser}(X_3, X_c) = \phi_{ser}(X_3, \phi_{par}(X_1, X_2)) \\ &= \min(X_3, \max(X_1, X_2)) = X_3(1-(1-X_1)(1-X_2)) \end{aligned}$$

In order to represent the nature of the relationship among the elements in the system, reliability block diagrams are usually used. The reliability block diagram of the system considered is presented in Figure 2.1.

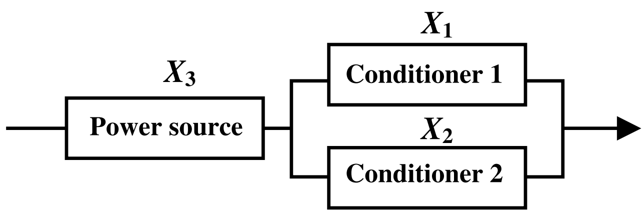

Figure 2.1. Reliability block diagram of an air conditioning system

The reliability is a property of any element or the entire system to be able to perform its intended task. Since we represent the system state by random binary variable X and $X = 1$ corresponds to the system state in which it performs its task, the measures of the system reliability should express its ability to be in the state $X = 1$. Different reliability measures can be defined in accordance with the conditions of the system's functioning.

When the system has a fixed mission time (for example, a satellite that should supply telemetric information during the entire preliminarily defined time of its mission), the reliability of such a system (and its elements) is defined to be the probability that it will perform its task during the mission time under specified working conditions. For any system element j its reliability p_j is

$$p_j = \Pr\{X_j = 1\} \tag{2.2}$$

and for the entire system its reliability R is

$$R = \Pr\{X = 1\} \tag{2.3}$$

Observe that the reliability can be expressed as the expected value of the state variable:

$$p_j = E(X_j), \quad R = E(X) \tag{2.4}$$

The reliabilities of the elements compose the element reliability vector $\boldsymbol{p} = (p_1, \ ..., \ p_n)$. Usually this vector is known and we are interested in obtaining the system reliability as a function of $\boldsymbol{p}$:

$$R = R(\boldsymbol{p}) = R(p_1, ..., p_n) \tag{2.5}$$

In systems with independent elements, such functions exist and depend on the system structure functions.

Example 2.2

Consider the system from Example 2.1 and assume that the reliabilities of the system elements p_1, p_2 and p_3 are known. Since the elements are independent, we can obtain the probability of each realization of the element state vector $(X_1, \ X_2, \ X_3) = (x_1, \ x_2, \ x_3)$ as

$$\begin{aligned}&\Pr\{X_1 = x_1 \bigcap X_2 = x_2 \bigcap X_3 = x_3\} \\ &= p_1{}^{x_1}(1-p_1)^{1-x_1} p_2{}^{x_2}(1-p_2)^{1-x_2} p_3{}^{x_3}(1-p_3)^{1-x_3}\end{aligned}$$

Having the system structure function

$$X = \min(X_3,\ \max(X_1,\ X_2))$$

and the probability of each realization of the element state vector, we can obtain the probabilities of each system state that defines the p.m.f. of the system state variable X. This p.m.f. is presented in Table 2.1.

Table 2.1. p.m.f. of the system structure function

Realization of (X_1, X_2, X_3)	Realization probability	Realization of X
0,0,0	$(1-p_1)(1-p_2)(1-p_3)$	0
0,0,1	$(1-p_1)(1-p_2)p_3$	0
0,1,0	$(1-p_1)p_2(1-p_3)$	0
0,1,1	$(1-p_1)p_2p_3$	1
1,0,0	$p_1(1-p_2)(1-p_3)$	0
1,0,1	$p_1(1-p_2)p_3$	1
1,1,0	$p_1p_2(1-p_3)$	0
1,1,1	$p_1p_2p_3$	1

The system reliability can now be defined as the expected value of the random variable X (which is equal to the sum of the probabilities of states corresponding to $X = 1$):

$$R = E(X) = (1-p_1)p_2p_3+p_1(1-p_2)p_3+p_1p_2p_3$$

$$= [(1-p_1)p_2+p_1(1-p_2)+p_1p_2]\ p_3 = (p_1+p_2-p_1p_2)p_3$$

When the system operates for a long time and no finite mission time is specified, we need to know how the system's ability to perform its task changes over time. In this case, a dynamic measure called the reliability function is used. The reliability function of element j $p_j(t)$ or the entire system $R(t)$ is defined as the probability that the element (system) will perform its task beyond time t, while assuming that at the beginning of the mission the element (system) is in working condition: $p_j(0) = R(0) = 1$.

Having the reliability functions of independent system elements $p_j(t)$ $(1\le j\le n)$ one can obtain the system reliability function $R(t)$ using the same relationship $R(\boldsymbol{p})$ that was defined for the fixed mission time and by substituting p_j with $p_j(t)$.

Example 2.3

Consider the system from Example 2.2 and assume that the reliability functions of the system elements are

$$p_1(t) = \mathrm{e}^{-\lambda_1 t},\quad p_2(t) = \mathrm{e}^{-\lambda_2 t},\quad p_3(t) = \mathrm{e}^{-\lambda_3 t}$$

The system reliability function takes the form

$$R(t)=E(X(t))=(p_1(t)+p_2(t)-p_1(t)p_2(t))p_3(t)$$

$$=(\mathrm{e}^{-\lambda_1 t}+\mathrm{e}^{-\lambda_2 t}-\mathrm{e}^{-(\lambda_1+\lambda_2)t})\mathrm{e}^{-\lambda_3 t}$$

In many practical cases the failed system elements can be repaired. While the failures bring the elements to a non-working state, repairs performed on them bring them back to a working state. Therefore, the state of each element and the state of the entire system can change between 0 and 1 several times during the system's mission. The probability that the element (system) is able to perform its task at a given time t is called the element (system) availability function:

$$a_j(t) = \Pr\{X_j = 1\} \tag{2.6}$$

$$A(t) = \Pr\{X = 1\}$$

For a repairable system, $X_j = 1$ ($X = 1$) indicates that the element (system) can perform its task at time t regardless of the states experienced before time t.

While the reliability reflects the internal properties of the element (system), the availability reflects both the ability of the element (system) to work without failures and the ability of the system's environment to bring the failed element (system) to a working condition. The same system working in a different maintenance environment has a different availability.

As a rule, the availability function is difficult to obtain. Instead, the steady-state system availability is usually used. It is assumed that enough time has passed since the beginning of the system operation so that the system's initial state has practically no influence on its availability and the availabilities of the system elements become constant

$$a_j = \lim_{t\to\infty} a_j(t) \tag{2.7}$$

Having the long-run average (steady-state) availabilities of the system elements one can obtain the steady-state availability A of the system by substituting in Equation (2.5) R with A and p_j with a_j.

One can see that since all of the reliability measures presented are probabilities, the same procedure of obtaining the system reliability measure from the reliability measures of its elements can be used in all cases. This procedure presumes:

- Obtaining the probabilities of each combination of element states from the element reliability vector.
- Obtaining the system state (the value of the system state variable) for each combination of element states (the realization of the element state vector) using the system structure function.

- Calculating the expected value of the system state variable from its p.m.f. defined by the element state combination probabilities and the corresponding values of the structure function.

This procedure can be formalized by using the UGF technique. In fact, the element reliability vector (p_1, …, p_n) determines the p.m.f. of each binary element that can be represented in the form of u-functions

$$u_j(z) = (1-p_j)\, z^0 + p_j\, z^1 \text{ for } 1 \le j \le n \tag{2.8}$$

Having the u-functions of system elements that represent the p.m.f. of discrete random variables X_1, ..., X_n and having the system structure function $X = \phi(X_1, \ ..., \ X_n)$ we can obtain the u-function representing the p.m.f. of the system state variable X using the composition operator over u-functions of individual system elements:

$$U(z) = \underset{\phi}{\otimes}(u_1(z),...,u_n(z)) \tag{2.9}$$

The system reliability measure can now be obtained as $E(X) = U'(1)$.

Note that the same procedure can be applied for any reliability measure considered. The system reliability measure (the fixed mission time reliability, the value of the reliability function at a specified time or availability) corresponds to the reliability measures used to express the state probabilities of elements. Therefore, we use the term reliability and presume that any reliability measure can be considered in its place (if some specific measure is not explicitly specified).

Example 2.4

The u-functions of the system elements from Example 2.2 are

$$u_1(z) = (1-p_1)z^0 + p_1 z^1 , \quad u_2(z) = (1-p_2)z^0 + p_2 z^1 , \quad u_3(z) = (1-p_3)z^0 + p_3 z^1$$

The system structure function is

$$X = \phi(X_1, X_2, X_3) = \min(X_3, \max(X_1, X_2))$$

Using the composition operator we obtain the system u-function representing the p.m.f. of the random variable X:

$$
\begin{aligned}
U(z) &= \underset{\phi}{\otimes}(u_1(z), u_2(z), u_3(z)) \\
&= \underset{\phi}{\otimes}\left(\sum_{i=0}^{1} p_1^i (1-p_1)^{1-i} z^i, \sum_{k=0}^{1} p_2^k (1-p_2)^{1-k} z^k, \sum_{m=0}^{1} p_3^m (1-p_3)^{1-m} z^m\right) \\
&= \sum_{i=0}^{1} \sum_{k=0}^{1} \sum_{m=0}^{1} p_1^i (1-p_1)^{1-i} p_2^k (1-p_2)^{1-k} p_3^m (1-p_3)^{1-m} z^{\min(\max(i,k),m)}
\end{aligned}
$$

The resulting u-function takes the form

$$
\begin{aligned}
U(z) &= (1-p_1)(1-p_2)(1-p_3) z^{\min(\max(0,0),0)} \\
&+ (1-p_1)(1-p_2) p_3 z^{\min(\max(0,0),1)} + (1-p_1) p_2 (1-p_3) z^{\min(\max(0,1),0)} \\
&+ (1-p_1) p_2 p_3 z^{\min(\max(0,1),1)} + p_1 (1-p_2)(1-p_3) z^{\min(\max(1,0),0)} \\
&+ p_1 (1-p_2) p_3 z^{\min(\max(1,0),1)} + p_1 p_2 (1-p_3) z^{\min(\max(1,1),0)} \\
&+ p_1 p_2 p_3 z^{\min(\max(1,1),1)} \\
&= (1-p_1)(1-p_2)(1-p_3) z^0 + (1-p_1)(1-p_2) p_3 z^0 \\
&+ (1-p_1) p_2 (1-p_3) z^0 + (1-p_1) p_2 p_3 z^1 + p_1 (1-p_2)(1-p_3) z^0 \\
&+ p_1 (1-p_2) p_3 z^1 + p_1 p_2 (1-p_3) z^0 + p_1 p_2 p_3 z^1
\end{aligned}
$$

After collecting the like terms we obtain

$$
\begin{aligned}
U(z) &= [(1-p_1)(1-p_2)(1-p_3) + (1-p_1)(1-p_2) p_3 \\
&+ (1-p_1) p_2 (1-p_3) + p_1 (1-p_2)(1-p_3) + p_1 p_2 (1-p_3)] z^0 \\
&+ [p_1 (1-p_2) p_3 + (1-p_1) p_2 p_3 + p_1 p_2 p_3] z^1
\end{aligned}
$$

The system reliability is equal to the expected value of variable X that has the p.m.f. represented by the u-function $U(z)$. As we know, this expected value can be obtained as the derivative of $U(z)$ at $z = 1$:

$$
\begin{aligned}
R = E(X) = U'(1) &= p_1 (1-p_2) p_3 + (1-p_1) p_2 p_3 + p_1 p_2 p_3 \\
&= (p_1 + p_2 - p_1 p_2) p_3
\end{aligned}
$$

It can easily be seen that the total number of combinations of states of the elements in the system with n elements is equal to 2^n. For systems with a great number of elements, the technique presented is associated with an enormous number of evaluations of the structure function value (the u-function of the system state variable X before the like term collection contains 2^n terms). Fortunately, the structure function can usually be defined recursively and the p.m.f. of intermediate variables corresponding to some subsystems can be obtained. These p.m.f. always consist of two terms. Substituting all the combinations of the elements composing the subsystem with its two-term p.m.f. (obtained by collecting the like terms in the u-function corresponding to the subsystem) allows one to achieve considerable reduction of the computational burden.

Example 2.5

Consider a series-parallel system consisting of five binary elements (Figure 2.2). The structure function of this system is

$$X = \phi(X_1, X_2, X_3, X_4, X_5) = \max(\max(X_1, X_2)\, X_3,\ X_4\, X_5)$$

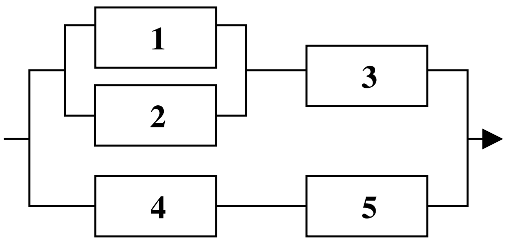

Figure 2.2. Reliability block diagram of series-parallel binary system

The u-functions of the elements take the form

$$u_j(z) = (1-p_j)\, z^0 + p_j\, z^1, \quad \text{for} \quad 1 \le j \le 5$$

Direct application of the operator $\underset{\phi}{\otimes}(u_1(z), u_2(z), u_3(z), u_4(z), u_5(z))$ requires $2^5 = 32$ evaluations of the system structure function.

The system structure function can be defined recursively:

$X_6 = \max(X_1, X_2)$

$X_7 = X_6\, X_3$

$X_8 = X_4\, X_5$

$X = \max(X_7, X_8)$

where X_6 is the state variable corresponding to the subsystem consisting of elements 1 and 2, X_7 is the state variable corresponding to the subsystem consisting of elements 1, 2 and 3, X_8 is the state variable corresponding to the subsystem consisting of elements 4 and 5.

The u-functions corresponding to variables X_6, X_7 and X_8 consist of two terms (after collecting the like terms) as well as u-functions corresponding to variables X_1, …, X_5. The number of evaluations of the structure functions representing the p.m.f. of variables X_6, X_7, X_8, and X is four. Therefore, the total number of such evaluations is 16. Note that the structure functions evaluated for the intermediate variables are much simpler than the structure function of the entire system that must be evaluated when applying the direct approach.

The process of obtaining the system reliability using the recursive approach is as follows:

$$U_6(z) = u_1(z) \underset{\max}{\otimes} u_2(z) = [p_1 z^1 + (1-p_1)z^0] \underset{\max}{\otimes} [p_2 z^1 + (1-p_2)z^0]$$
$$= p_1 p_2 z^{\max(1,1)} + p_1(1-p_2)z^{\max(1,0)} + (1-p_1)p_2 z^{\max(0,1)}$$
$$+ (1-p_1)(1-p_2)z^{\max(0,0)} = p_1 p_2 z^1 + p_1(1-p_2)z^1 + (1-p_1)p_2 z^1$$
$$+ (1-p_1)(1-p_2)z^0 = (p_1 + p_2 - p_1 p_2)z^1 + (1-p_1)(1-p_2)z^0$$

$$U_7(z) = U_6(z) \underset{\times}{\otimes} u_3(z) = [(p_1 + p_2 - p_1 p_2)z^1$$
$$+ (1-p_1)(1-p_2)z^0] \underset{\times}{\otimes} [p_3 z^1 + (1-p_3)z^0] = (p_1 + p_2 - p_1 p_2)p_3 z^{1\times 1}$$
$$+ (1-p_1)(1-p_2)p_3 z^{0\times 1} + (p_1 + p_2 - p_1 p_2)(1-p_3)z^{1\times 0}$$
$$+ (1-p_1)(1-p_2)(1-p_3)z^{0\times 0} = (p_1 + p_2 - p_1 p_2)p_3 z^1$$
$$+ [(1-p_1)(1-p_2) + (p_1 + p_2 - p_1 p_2)(1-p_3)]z^0$$

$$U_8(z) = u_4(z) \underset{\times}{\otimes} u_5(z) = [p_4 z^1 + (1-p_4)z^0] \underset{\times}{\otimes} [p_5 z^1 + (1-p_5)z^0]$$
$$= p_4 p_5 z^{1\times 1} + p_4(1-p_5)z^{1\times 0} + (1-p_4)p_5 z^{0\times 1} + (1-p_4)(1-p_5)z^{0\times 0}$$
$$= p_4 p_5 z^1 + (1 - p_4 p_5)z^0$$

$$U(z) = U_7(z) \underset{\max}{\otimes} U_8(z) = \{(p_1 + p_2 - p_1 p_2)p_3 z^1 + [(1-p_1)(1-p_2)$$
$$+ (p_1 + p_2 - p_1 p_2)(1-p_3)]z^0\} \underset{\max}{\otimes} [p_4 p_5 z^1 + (1 - p_4 p_5)z^0]$$

$$
\begin{aligned}
&= (p_1 + p_2 - p_1 p_2) p_3 p_4 p_5 z^{\max(1,1)} \\
&+ (p_1 + p_2 - p_1 p_2) p_3 (1 - p_4 p_5) z^{\max(1,0)} \\
&+ [(1 - p_1)(1 - p_2) + (p_1 + p_2 - p_1 p_2)(1 - p_3)] p_4 p_5 z^{\max(0,1)} \\
&+ [(1 - p_1)(1 - p_2) + (p_1 + p_2 - p_1 p_2)(1 - p_3)](1 - p_4 p_5) z^{\max(0,0)} \\
&= \{(p_1 + p_2 - p_1 p_2) p_3 + [(1 - p_1)(1 - p_2) \\
&+ (p_1 + p_2 - p_1 p_2)(1 - p_3)] p_4 p_5\} z^1 \\
&+ [(1 - p_1)(1 - p_2) + (p_1 + p_2 - p_1 p_2)(1 - p_3)](1 - p_4 p_5) z^0
\end{aligned}
$$

The system reliability (availability) can now be obtained as

$$
\begin{aligned}
&R = E(X) = U'(1) = (p_1 + p_2 - p_1 p_2) p_3 + [(1 - p_1)(1 - p_2) \\
&+ (p_1 + p_2 - p_1 p_2)(1 - p_3)] p_4 p_5
\end{aligned}
$$

In order to reduce the number of arithmetical operations in the term multiplication procedures performed when obtaining the u-functions of the system variable, the u-function of the binary elements that takes the form

$$u_j(z) = p_j z^1 + (1 - p_j) z^0 \tag{2.10}$$

can be represented in the form

$$u_j(z) = p_j (z^1 + q_j z^0) \tag{2.11}$$

where

$$q_j = p_j^{-1} - 1 \tag{2.12}$$

Factoring out the probability p_j from $u_j(z)$ results in fewer computations associated with performing the operators $U(z) \underset{\phi}{\otimes} u_j(z)$ for any $U(z)$ because the multiplications by 1 are implicit.

Example 2.6

In this example we obtain the reliability of the series-parallel system from Example 2.5 numerically for $p_1 = 0.8$, $p_2 = 0.9$, $p_3 = 0.7$, $p_4 = 0.9$, $p_5 = 0.7$.

The u-functions of the elements take the form

$$u_1(z) = 0.8z^1 + 0.2z^0,\ u_2(z) = u_4(z) = 0.9z^1 + 0.1z^0,\ u_3(z) = u_5(z) = 0.7z^1 + 0.3z^0$$

Following the procedure presented in Example 2.5 we obtain:

$$U_6(z) = u_1(z) \underset{\max}{\otimes} u_2(z) = (0.8z^1 + 0.2z^0) \underset{\max}{\otimes} (0.9z^1 + 0.1z^0)$$

$$= 0.8 \cdot 0.9z^1 + 0.8 \cdot 0.1z^1 + 0.2 \cdot 0.9z^1 + 0.2 \cdot 0.1z^0 = 0.98z^1 + 0.02z^0$$

$$U_7(z) = U_6(z) \underset{\times}{\otimes} u_3(z) = (0.98z^1 + 0.02z^0) \underset{\times}{\otimes} (0.7z^1 + 0.3z^0)$$

$$= 0.98 \times 0.7z^1 + 0.98 \times 0.3z^0 + 0.02 \times 0.7z^0 + 0.02 \times 0.3z^0$$
$$= 0.686z^1 + 0.314z^0$$

$$U_8(z) = u_4(z) \underset{\times}{\otimes} u_5(z) = (0.9z^1 + 0.1z^0) \underset{\times}{\otimes} (0.7z^1 + 0.3z^0)$$

$$= 0.9 \times 0.7z^1 + 0.9 \times 0.3z^0 + 0.1 \times 0.7z^0 + 0.1 \times 0.3z^0 = 0.63z^1 + 0.37z^0$$

$$U(z) = U_7(z) \underset{\max}{\otimes} U_8(z) = (0.686z^1 + 0.314z^0) \underset{\max}{\otimes} (0.63z^1 + 0.37z^0)$$

$$= 0.686 \times 0.63z^1 + 0.686 \times 0.37z^1 + 0.314 \times 0.63z^1 + 0.314 \times 0.37z^0$$

$$= 0.88382z^1 + 0.11618z^0$$

And, finally:

$$R = U'(1) = 0.88382 \approx 0.884$$

Representing the u-functions of the elements in the form

$$u_1(z) = 0.8(z^1+0.25)z^0,\ u_2(z) = u_4(z) = 0.9(z^1+0.111)z^0$$

$$u_3(z) = u_5(z) = 0.7(z^1+0.429)z^0$$

we can obtain the same result by fewer calculations:

$$U_6(z) = u_1(z) \underset{\max}{\otimes} u_2(z) = 0.8(z^1 + 0.25z^0) \underset{\max}{\otimes} 0.9(z^1 + 0.111z^0)$$

$$= 0.8 \cdot 0.9(z^1 + 0.111z^1 + 0.25z^1 + 0.25 \times 0.111z^0)$$
$$= 0.72(1.361z^1 + 0.0278)z^0$$

$$U_7(z) = U_6(z) \underset{\times}{\otimes} u_3(z) = 0.72(1.361z^1 + 0.028z^0) \underset{\times}{\otimes} 0.7(z^1 + 0.429z^0)$$

$$= 0.72 \times 0.7(1.361z^1 + 0.028z^0 + 1.361 \times 0.429z^0 + 0.028 \times 0.429z^0)$$
$$= 0.504(1.361z^1 + 0.623z^0)$$

$$U_8(z) = u_4(z) \underset{\times}{\otimes} u_5(z) = 0.9(z^1 + 0.111z^0) \underset{\times}{\otimes} 0.7(z^1 + 0.429z^0)$$

$$= 0.9 \times 0.7(z^1 + 0.111z^0 + 0.429z^0 + 0.111 \times 0.429z^0) = 0.63(z^1 + 0.588z^0)$$

$$U(z) = U_7(z) \underset{\max}{\otimes} U_8(z) = 0.504(1.361z^1 + 0.623z^0) \underset{\max}{\otimes} 0.63(z^1 + 0.588z^0)$$

$$= 0.504 \times 0.63(1.361z^1 + 0.623z^1 + 1.361 \times 0.588z^1 + 0.623 \times 0.588z^0)$$

$$= 0.3175(2.784z^1 + 0.366z^0)$$

$$R = U'(1) = 0.3175 \cdot 2.784 \approx 0.884$$

The simplification method presented is efficient in numerical procedures. In future examples we do not use it in order to preserve their clarity.

There are many cases where estimating the structure function of the binary system is a very complicated task. In some of these cases the structure function and the system reliability can be obtained recursively, as in the case of the complex series-parallel systems. The following sections of this chapter are devoted to such cases.

2.2 *k*-out-of-*n* Systems

Consider a system consisting of *n* independent binary elements that can perform its task (is "good") if and only if at least *k* of its elements are in working condition. This type of system is called a *k*-out-of-*n:G* system. The system that fails to perform its task if and only if at least *k* of its elements fail is called a *k*-out-of-*n:F* system. It can be seen that a *k*-out-of-*n:G* system is equivalent to an (*n*–*k*+1)-out-of-*n:F* system. Therefore, we consider only *k*-out-of-*n:G* systems and omit *G* from their denomination.

The pure series and pure parallel systems can be considered to be special cases of *k*-out-of-*n* systems. Indeed, the series system works if and only if all of its elements work. This corresponds to an *n*-out-of-*n* system. The parallel system works if and only if at least one of its elements works, which corresponds to a 1-out-of-*n* system.

The *k*-out-of-*n* systems are widely used in different technical applications. For example, an airplane survives if no more than two of its four engines are destroyed. The power generation system can meet its demand when at least three out of five of its generators function.

Consider the *k*-out-of-*n* system consisting of identical elements with reliability *p*. It can be seen that the number of working elements in the system follows the binomial distribution: the probability R_j that exactly *j* out of *n* elements work ($1 \le j \le n$) takes the following form:

$$R_j = \binom{n}{j} p^j (1-p)^{n-j} \tag{2.13}$$

Since the system reliability is equal to the probability that the number of working elements is not less than k, the overall system reliability can be found as

$$R = \sum_{j=k}^{n} R_j = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j} \tag{2.14}$$

Using this equation one can readily obtain the reliability of the k-out-of-n system with independent identical binary elements. When the elements are not identical (have different reliabilities) the evaluation of the system reliability is a more complicated problem. The structure function of the system takes the form

$$\phi(X_1, \ ..., \ X_n) = 1(\sum_{i=1}^{n} X_i \geq k) \tag{2.15}$$

In order to obtain the probability R_j that exactly j out of n elements work ($1 \leq j \leq n$), one has to sum up the probabilities of all of the possible realizations of the element state vector $(X_1, \ ..., \ X_n)$ in which j state variables exactly take on the value of 1. Observe that, in such realizations, number i_1 of the first variable X_{i_1} from the vector that should be equal to 1 can vary from 1 to n–j+1. Indeed, if $X_{i_1} = 0$ for $1 \leq i_1 \leq n-j+1$, then the maximal number of variables taking a value of 1 is not greater than j–1. Using the same consideration, we can see that if the number of the first variable that is equal to 1 is i_1, the number of the second variable taking this value can vary from i_1+1 to n–j+2 and so on. Taking into account that $\Pr\{X_i = 1\} = p_i$ and $\Pr\{X_i = 0\} = 1 - p_i$ for any i: $1 \leq i \leq n$, we can obtain

$$R_j = \left[\prod_{i=1}^{n}(1-p_i)\right]\left[\sum_{i_1=1}^{n-j+1} \frac{p_{i_1}}{1-p_{i_1}} \sum_{i_2=i_1+1}^{n-j+2} \frac{p_{i_2}}{1-p_{i_2}} ... \sum_{i_j=i_{j-1}+1}^{n} \frac{p_{i_j}}{1-p_{i_j}}\right] \tag{2.16}$$

The reliability of the system is equal to the probability that j is greater than or equal to k. Therefore:

$$\begin{aligned} R &= \sum_{j=k}^{n} R_j \\ &= \left[\prod_{i=1}^{n}(1-p_i)\right]\sum_{j=k}^{n}\left[\sum_{i_1=1}^{n-j+1} \frac{p_{i_1}}{1-p_{i_1}} \sum_{i_2=i_1+1}^{n-j+2} \frac{p_{i_2}}{1-p_{i_2}} ... \sum_{i_j=i_{j-1}+1}^{n} \frac{p_{i_j}}{1-p_{i_j}}\right] \end{aligned} \tag{2.17}$$

The computation of the system reliability based on this equation is very complicated. The UGF approach provides for a straightforward method of k-out-of-n system reliability computation that considerably reduces the computational complexity. The basics of this method were mentioned in the early Reliability Handbook by Kozlov and Ushakov [13]; the efficient algorithm was suggested by Barlow and Heidtmann [14].

Since the p.m.f. of each element state variable X_j can be represented by the u-function

$$u_j(z) = p_j z^1 + (1 - p_j) z^0 \tag{2.18}$$

the operator

$$U(z) = \underset{+}{\otimes}(u_1(z), \ldots, u_n(z)) \tag{2.19}$$

gives the distribution of the random variable X:

$$X = \sum_{i=1}^{n} X_i \tag{2.20}$$

which is equal to the total number of working elements in the system.

The resulting u-function representing the p.m.f. of the variable X takes the form

$$U(z) = \sum_{j=0}^{n} R_j z^j \tag{2.21}$$

where $R_j = \Pr\{X = j\}$ is the probability that exactly j elements work. By summing the coefficients of the u-function $U(z)$ corresponding to $k \le j \le n$, we obtain the system reliability.

Taking into account that the operator $\otimes_+$ possesses the associative property (Equation (1.27)) and using the structure function formalism we can define the following procedure that obtains the reliability of a k-out-of-n system:

1. Determine u-functions of each element in the form (2.18).
2. Assign $U_1(z) = u_1(z)$.
3. For j = 2, …, n obtain $U_j(z) = U_{j-1}(z) \otimes_+ u_j(z)$ (the final u-function $U_n(z)$ represents the p.m.f. of random variable X).
4. Obtain u-function $U(z)$ representing the p.m.f. of structure function (2.15) as $U(z) = U_n(z) \underset{\varphi}{\otimes} k$, where $\varphi(X, k) = 1(X \ge k)$.
5. Obtain the system reliability as $E(\varphi(X, k)) = U'(1)$.

Example 2.7

Consider a 2-out-of-4 system consisting of elements with reliabilities $p_1 = 0.8$, $p_2 = 0.6$, $p_3 = 0.9$, and $p_4 = 0.7$.

First, determine the u-functions of the elements:

$$u_1(z) = 0.8z^1 + 0.2z^0$$
$$u_2(z) = 0.6z^1 + 0.4z^0$$
$$u_3(z) = 0.9z^1 + 0.1z^0$$
$$u_4(z) = 0.7z^1 + 0.3z^0$$

Follow step 2 and assign

$$U_1(z) = u_1(z) = 0.8z^1 + 0.2z^0$$

Using the recursive equation (step 3 of the procedure) obtain

$$U_2(z) = (0.8z^1 + 0.2z^0)\underset{+}{\otimes}(0.6z^1 + 0.4z^0)$$
$$= (0.8z^1 + 0.2z^0)(0.6z^1 + 0.4z^0) = 10^{-2}(48z^2 + 44z^1 + 8z^0)$$

$$U_3(z) = 10^{-2}(48z^2 + 44z^1 + 8z^0)\underset{+}{\otimes}(0.9z^1 + 0.1z^0)$$
$$= 10^{-3}(48z^2 + 44z^1 + 8z^0)(9z^1 + 1z^0)$$
$$= 10^{-3}(432z^3 + 444z^2 + 116z^1 + 8z^0)$$

$$U_4(z) = 10^{-3}(432z^3 + 444z^2 + 116z^1 + 8z^0)\underset{+}{\otimes}(0.7z^1 + 0.3z^0)$$
$$= 10^{-4}(432z^3 + 444z^2 + 116z^1 + 8z^0)(7z^1 + 3z^0)$$
$$= 10^{-4}(3024z^4 + 4404z^3 + 2144z^2 + 404z^1 + 24z^0)$$

Following step 4 obtain

$$U(z) = U_4(z)\underset{\varphi}{\otimes}2 = 10^{-4}(3024z^1 + 4404z^1 + 2144z^1 + 404z^0 + 24z^0)$$
$$= 0.9572z^1 + 0.0428z^0$$

The system reliability can now be obtained as

$$U'(1) = 0.9572$$

Note that the UGF method requires less computational effort than simple enumeration of possible combinations of states of the elements. In order to obtain $U_2(z)$ we used four term multiplication operations. In order to obtain $U_3(z)$, six operations were used (because $U_2(z)$ has only three different terms after collecting the like terms). In order to obtain $U_4(z)$, eight operations were used (because $U_3(z)$ has only four different terms after collecting the like terms). The total number of the term multiplication operations used in the example is 18.

When the enumerative approach is used, one has to evaluate the probabilities of $2^4 = 16$ combinations of the states of the elements. For each combination the product of four element state probabilities should be obtained. This requires three multiplication operations. The total number of the multiplication operations is $16\times 3 = 48$. The difference in the computational burden increases with the growth of n.

The computational complexity of this algorithm can be further reduced in its modification that avoids calculating the probabilities R_j. Note that it does not matter for the k-out-of-n system how many elements work if the number of the working elements is not less than k. Therefore, we can introduce the intermediate variable X^*:

$$X^* = \min\{k, \sum_{i=1}^{n} X_i\} \tag{2.22}$$

and define the system structure function as

$$\phi(X_1,..., X_n) = \eta(X^*, k) = 1(X^* = k) \tag{2.23}$$

In order to obtain the u-function of the variable X^* we introduce the following composition operator:

$$U(z) = \underset{\theta_k}{\otimes}(u_1(z),..., u_n(z)) \tag{2.24}$$

where

$$\theta_k(x_1,..., x_n) = \min\{k, \sum_{i=1}^{n} x_i\} \tag{2.25}$$

It can be easily seen that this operator possesses the associative and commutative properties and, therefore, the u-function of X^* can be obtained recursively:

$$U_1(z) = u_1(z) \tag{2.26}$$

$$U_j(z) = U_{j-1}(z) \underset{\theta_k}{\otimes} u_j(z) \text{ for } j = 2, \ldots, n \tag{2.27}$$

The *u*-functions $U_j(z)$ for $j<k$ do not contain terms with exponents equal to k. The first *u*-function that contains such a term is $U_k(z)$. This *u*-function can be represented as

$$U_k(z) = \sum_{i=0}^{k-1} \alpha_i z^i + \varepsilon_k z^k \tag{2.28}$$

Applying the operator $\underset{\theta_k}{\otimes}$ over $U_k(z)$ and $u_{k+1}(z)$ we obtain

$$\begin{aligned} U_{k+1}(z) &= (\sum_{i=0}^{k-1} \alpha_i z^i + \varepsilon_k z^k) \underset{\theta_k}{\otimes} [p_{k+1} z^1 + (1-p_{k+1}) z^0] \\ &= p_{k+1} \sum_{i=0}^{k-1} \alpha_i z^{\theta_k(i,1)} + (1-p_{k+1}) \sum_{i=0}^{k-1} \alpha_i z^{\theta_k(i,0)} \\ &+ \varepsilon_k p_{k+1} z^{\theta_k(k,1)} + \varepsilon_k (1-p_{k+1}) z^{\theta_k(k,0)} \\ &= p_{k+1} \sum_{i=1}^{k} \alpha_{i-1} z^i + (1-p_{k+1}) \sum_{i=0}^{k-1} \alpha_i z^i + \varepsilon_k [p_{k+1} + (1-p_{k+1})] z^k \\ &= \sum_{i=0}^{k-1} \beta_i z^i + \varepsilon_{k+1} z^k + \varepsilon_k z^k \end{aligned} \tag{2.29}$$

where

$$\varepsilon_{k+1} = p_{k+1} \alpha_{k-1} \tag{2.30}$$

Here, we used the important property of the function θ_k that for any $x \ge 0$

$$\theta_k(k, x) = k \tag{2.31}$$

It can be proven by induction that

$$U_n(z) = \sum_{i=0}^{k-1} \upsilon_i z^i + (\varepsilon_n + \varepsilon_{n-1} + ... + \varepsilon_{k+1} + \varepsilon_k) z^k \tag{2.32}$$

where ε_i (for $k \le i \le n$) is a product of the coefficient of the term with z^{k-1} from $U_{i-1}(z)$ and p_i. Observe that the coefficient of the term with z^k from U_{i-1}(z) does not participate in calculating ε_i, …, ε_n.

The *u*-function of the system structure function can now be determined as

$$U(z) = U_n(z) \underset{\eta}{\otimes} k = (\sum_{i=0}^{k-1} \upsilon_i) z^0 + (\varepsilon_n + \varepsilon_{n-1} + ... + \varepsilon_{k+1} + \varepsilon_k) z^1 \tag{2.33}$$

and the system reliability can be calculated as

$$R = U'(1) = \varepsilon_n + \varepsilon_{n-1} + ... + \varepsilon_{k+1} + \varepsilon_k \tag{2.34}$$

The considerations presented lie at the base of the following simplified algorithm of the system reliability determination:

1. Determine u-functions of each element in the form (2.18).
2. Assign $R = 0$, $U_1(z) = u_1(z)$.
3. For $j = 2, \ldots, n$:
   3.1. Obtain $U_j(z) = U_{j-1}(z) \underset{\theta_k}{\otimes} u_j(z)$.
   3.2. If the u-function $U_j(z)$ contains a term with z^k, remove this term from $U_j(z)$ and add its coefficient to R.

After termination of the algorithm, R is equal to the system reliability.

Note that in this algorithm the operator $\underset{\theta_k}{\otimes}$ can be replaced by the operator $\underset{+}{\otimes}$. Indeed, after removing the term with z^k from $U_j(z)$ the operators $U_j(z) \underset{+}{\otimes} u_{j+1}(z)$ and $U_j(z) \underset{\theta_k}{\otimes} u_{j+1}(z)$ become equivalent (since the function $\theta_k(x,1)$ for $x<k$ is equivalent to function $x+1$).

In the simplified algorithm, obtaining $U_j(z)$ for $j \le k$ requires $2j$ term multiplication operations and obtaining $U_j(z)$ for $k < j \le n$ requires $2k$ term multiplication operations. The total number of these operations is

$$\begin{aligned} &2 \sum_{j=2}^{k} j + \sum_{j=k+1}^{n} 2k(n-k) = 2[0.5k(k+1) - 1] + 2k(n-k) \\ &= 2nk + k - k^2 - 2 \end{aligned} \tag{2.35}$$

Example 2.8

Consider a 2-out-of-4 system from Example 2.7 and apply the technique described for the recursive derivation of the system reliability:

Assign $R = 0$;

$$U_1(z) = u_1(z) = 0.8z^1 + 0.2z^0$$

$$U_2(z) = (0.8z^1 + 0.2z^0) \underset{+}{\otimes} (0.6z^1 + 0.4z^0)$$

$$= (0.8z^1 + 0.2z^0)(0.6z^1 + 0.4z^0) = 10^{-2}(48z^2 + 44z^1 + 8z^0)$$

Remove the term with z^2 from $U_2(z)$ and add its coefficient to R:

$$R = 0.48$$

$$U_2(z) = 10^{-2}(44z^1 + 8z^0)$$

Further:

$$U_3(z) = 10^{-2}(44z^1 + 8z^0)\underset{+}{\otimes}(0.9z^1 + 0.1z^0)$$

$$= 10^{-3}(44z^1 + 8z^0)(9z^1 + 1z^0) = 10^{-3}(396z^2 + 116z^1 + 8z^0)$$

Remove the term with z^2 from $U_3(z)$ and add its coefficient to R:

$$R = 0.48 + 0.396 = 0.876$$

$$U_3(z) = 10^{-3}(116z^1 + 8z^0)$$

In the next step

$$U_4(z) = 10^{-3}(116z^1 + 8z^0)\underset{+}{\otimes}(0.7z^1 + 0.3z^0)$$

$$= 10^{-4}(116z^1 + 8z^0)(7z^1 + 3z^0) = 10^{-4}(812z^2 + 404z^1 + 24z^0)$$

Finally, after adding the coefficient of the term with z^2 from $U_4(z)$ to R we obtain:

$$R = 0.876 + 0.0812 = 0.9572$$

Observe that the total number of the term multiplication operations used in this example is 12.

2.3 Consecutive *k*-out-of-*n* Systems

Consider a system consisting of n independent binary elements that are linearly connected in such a way that the system fails if and only if at least k consecutive elements fail. Such a model named linear consecutive k-out-of-n:F system is used for evaluating system reliability in telecommunications, oil pipeline systems, spacecraft relay stations, *etc*. [15-20].

Example 2.9

Consider a set of n+2 radio relay stations with a transmitter allocated at the first station and a receiver allocated at the last station. Each one of n intermediate stations has retransmitters generating signals that cover the distance including the next k stations. The aim of the system is to provide propagation of a signal from transmitter to receiver. It is evident that the system fails if at least k consecutive retransmitters fail. An example of the radio relay system with $k = 3$ and $n = 8$ is shown in Figure 2.3. Whenever the number of consecutive failures is less than three, the signal flow is not interrupted and the signal reaches the receiver.

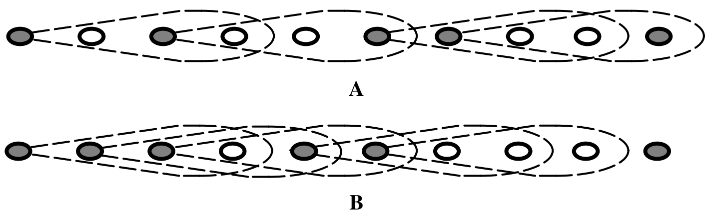

Figure 2.3. Linear consecutive 3-out-of-8 system in working state (A) and in failed state (B)

Example 2.10

In the pipeline systems transporting oil from a source point to a destination point the pressure is provided by n equally spaced pump stations. Each pump station provides pressure sufficient to transport oil to a distance that includes the k next stations. If m out of k stations following the working one fail ($m<k$), then the flow of oil is not interrupted because the remaining $k-m$ stations still carry the load. When the k adjacent stations fail, no working pumps remain in the part of the pipeline reached by the oil transported by the last working station. The oil flow is interrupted and the system fails.

The system in which n independent binary elements are linearly connected in such a way that the system works if and only if at least k consecutive elements are working is named the k-out-of-n:G system. The k-out-of-n:F and k-out-of-n:G systems are duals of each other [19]. This means that if the reliability of any element j in a k-out-of-n:F system is equal to the unreliability of element j a in k-out-of-n:G system (with the same k and n), then the reliability of the entire k-out-of-n:F system is equal to the unreliability of the entire k-out-of-n:G system. Therefore, the same algorithms for reliability evaluation can be applied to both types of system. In this chapter we consider only k-out-of-n:F systems and omit F from their denomination.

The linear consecutive k-out-of-n system was formally introduced by Chiang and Niu [15] but had been previously mentioned by Kontoleon [21]. The methods for evaluating the reliability of a linear consecutive k-out-of-n system with identical elements were suggested in [15, 16, 22-26]. The more complex case of systems with different elements was studied in [20, 21, 27].

The model in which the elements are circularly connected so that the first and the nth elements become adjacent to each other is named a circular consecutive k-out-of-n system. Examples of such a system can be found in monitoring, nuclear accelerators, *etc*. [12].

Example 2.11

For taking pictures of high-energy particles in a nuclear accelerator, n high-speed cameras are installed around the accelerator. If more then k adjacent cameras fail to take pictures, the particle behaviour cannot be analyzed.

Example 2.12

The vacuum system of an electronic accelerator consists of a large number of vacuum bulbs placed evenly along a ring. The vacuum system fails if at least k adjacent vacuum bulbs fail.

The circular consecutive k-out-of-n system was introduced by Derman *et al.* [22]. The algorithms for evaluating the system reliability were suggested in [22, 24, 25, 28–30] for a system with identical elements and in [20, 27, 31, 32] for a system with different elements.

The series and parallel systems can be considered as special cases of the consecutive k-out-of-n system. Indeed, when $k = 1$ the failure of any element causes the failure of the entire system, and the system becomes series one. When $k = n$ the entire system fails only if all of its elements fail, which corresponds to the parallel system.

2.3.1 Consecutive *k*-out-of-*n* Systems with Identical Elements

In this section we consider the consecutive k-out-of-n system in which all of its elements are identical, *i.e.* each individual element has the same reliability p. In the algorithms suggested by Lambiris and Papastavridis [24] and Goulden [25] for evaluating the reliability of such systems, the generating function approach is used in order to determine the number of ways to arrange j failed elements in a line with $n-j$ working elements such that no k or more failed elements are consecutive. Having this number $N(j, k, n)$ for any j we can obtain the reliability of linear consecutive k-out-of-n system $R_L(k, n)$ as

$$R_{\mathrm{L}}(k,n) = \sum_{j=0}^{n}(1-p)^j p^{n-j} N(j,k,n) \tag{2.36}$$

If the system contains exactly j failed elements, then the remaining $n-j$ working elements divide the system into $n-j+1$ segments (the first segment is to the left of the first working element, $n-j-1$ segments are between any two working elements that are close to each other, and the last segment is to the right of the last working element). Each segment may contain from 0 to j failed elements. The allocations in which no one segment contains k or more failed elements correspond to the system's success.

Let u-function $u_i(z)$ represent the distribution of the number of ways the failed elements can be allocated in section i such that the system does not fail:

$$u_i(z) = z^0 + z^1 + \ldots + z^{k-1} \tag{2.37}$$

This representation corresponds to the fact that from 0 to k–1 elements can be allocated in the section (which is expressed by exponents of the u-function) and only one way exists to allocate any number of failed elements in this single section (all the coefficients are equal to 1). The distributions of the number of ways the failed elements can be allocated in several sections can be obtained using the $\otimes_+$ operator.

Indeed, in the u-function

$$u_i(z) \underset{+}{\otimes} u_m(z) = (z^0 + z^1 + \ldots + z^{k-1})(z^0 + z^1 + \ldots + z^{k-1})$$
$$= \sum_{h=0}^{2k-2} a_h z^h \tag{2.38}$$

a_h is equal to the number of ways h elements can be distributed between sections i and m. Applying the $\otimes_+$ operator over n–j+1 identical u-functions we obtain the resulting u-function

$$U(z) = \underset{+}{\otimes}(u_1(z), u_2(z), \ldots, u_{n-j+1}(z)) = (z^0 + z^1 + \ldots + z^{k-1})^{n-j+1}$$
$$= \sum_{h=1}^{(k-1)(n-j+1)} \alpha_h z^h \tag{2.39}$$

that represents the number of ways different numbers of failed elements can be allocated in n–j+1 sections. The coefficient α_j of term $\alpha_j z^j$ represents the number of ways exactly j failed elements can be allocated in n–j+1 segments. Therefore, $N(j,k,n) = \alpha_j$.

When j is close to n, the j failed elements cannot be allocated among $n-j+1$ segments in a manner where any segment contains less than k failed elements. There exists a maximum number of element failures $j_{\max}$ that the system may experience without failing independently on the location of the failed elements:

$$N(j_{\max}, k, n) > 0$$

$$N(j_{\max} + 1, k, n) = 0 \tag{2.40}$$

Indeed, when j failed elements are distributed among n–j+1 segments, the minimal number of elements in each segment is achieved when the elements are distributed as evenly as possible. In this case, some segments contain

$\lceil j/(n-j+1) \rceil$ failed elements and some segments contain $\lfloor j/(n-j+1) \rfloor$ failed elements. The system succeeds if

$$\lceil j/(n-j+1) \rceil \le k-1 \tag{2.41}$$

This inequality holds when

$$j \le (k-1)(n-j+1) \tag{2.42}$$

From this expression we obtain

$$j \le n+1-\frac{n+1}{k} \tag{2.43}$$

which means that the maximum possible value of j is

$$j_{\max} = \left\lfloor n+1-\frac{n+1}{k} \right\rfloor \tag{2.44}$$

Observe also that $N(0,k,n)=1$ for any k and n. Indeed, only one way exists for allocating zero failed elements among n+1 segments. Therefore, expression (2.36) can be rewritten as

$$R_{\mathrm{L}}(k,n) = p^n + \sum_{j=1}^{j_{\max}} (1-p)^j p^{n-j} N(j,k,n) \tag{2.45}$$

Example 2.13

Consider a linear consecutive 2-out-of-5 system with identical elements. This system should not contain more than k–1 = 2–1 = 1 failed elements in each segment between the working elements. Therefore, the u-function corresponding to a single segment is $u(z)=z^0+z^1$. According to (2.44)

$$j_{\max} = \left\lfloor 5+1-\frac{5+1}{2} \right\rfloor = 3$$

For j = 1, according to (2.39), we obtain

$$U(z) = (z^0+z^1)^{5-1+1} = (z^0+z^1)^5 = z^0+5z^1+10z^2+10z^3+5z^4+z^5$$

$N(1,k,n)$ is equal to the coefficient of the term with exponent 1: $N(1,k,n)$ = 5.

For $j = 2$:

$$U(z) = (z^0 + z^1)^{5-2+1} = (z^0 + z^1)^4 = z^0 + 4z^1 + 6z^2 + 4z^3 + z^4$$

$N(2,k,n)$ is equal to the coefficient of the term with exponent 2: $N(2,k,n) = 6$.
For $j = 3$:

$$U(z) = (z^0 + z^1)^{5-3+1} = (z^0 + z^1)^3 = z^0 + 3z^1 + 3z^2 + z^3$$

$N(3,k,n)$ is equal to the coefficient of the term with exponent 3: $N(3,k,n) = 1$.
According to (2.45) the system reliability is

$$R_\mathrm{L}(2,5) = p^5 + 5p^4(1-p) + 6p^3(1-p)^2 + p^2(1-p)^3$$

The problem of evaluating the reliability of a circular system with identical elements can be reduced to the problem of evaluating the reliability of a linear system (Derman *et al.* [22]). Indeed, consider a point between two arbitrary adjacent elements in a circular consecutive k-out-of-n system and find two working elements clockwise and counter clockwise to this point (Figure 2.4). These two working elements divide the circle into two fragments. The system reliability is equal to the probability that the fragment, including the marked point, contains less than k elements and the fragment not including the marked point forms a working linear consecutive k-out-of-n system.

Let n_1 and n_2 indicate the number of failed elements between the marked point and the first working elements. It can be easily seen that for any $i < n-1$

$$\Pr\{n_1 = i\} = \Pr\{n_2 = i\} = p(1-p)^i \tag{2.46}$$

The probability that the fragment, including the marked point, contains exactly i failed elements is

$$\begin{aligned}\Pr\{n_1 + n_2 = i\} &= \sum_{j=0}^{i} \Pr\{n_1 = j\}\Pr\{n_2 = i-j\} \\ &= \sum_{j=0}^{i} p(1-p)^j p(1-p)^{i-j} = \sum_{j=0}^{i} p^2(1-p)^i = (i+1)p^2(1-p)^i\end{aligned} \tag{2.47}$$

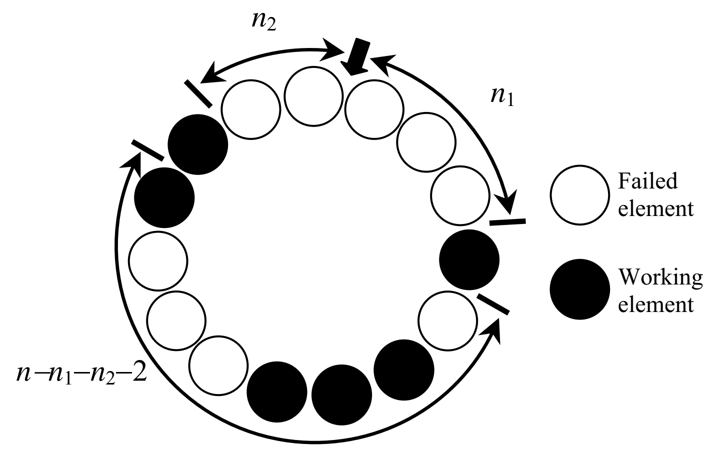

Figure 2.4. Circular consecutive *k*-out-of-*n* system with identical elements

The probability that the fragment contains less than the *k* failed elements is

$$\Pr\{n_1 + n_2 < k\} = \sum_{i=0}^{k-1} \Pr\{n_1 + n_2 = i\} = p^2 \sum_{i=0}^{k-1} (i+1)(1-p)^i \tag{2.48}$$

If the fragment, including the marked point, consists of i failed elements, then the second fragment consists of $n-i-2$ remaining elements. The reliability of the second fragment is $R_L(k, n-i-2)$. Therefore, the reliability of the entire circular system is

$$R_C(k,n) = p^2 \sum_{i=0}^{k-1} (i+1)(1-p)^i R_L(k, n-i-2) \tag{2.49}$$

Equation (2.49) is obtained on the assumption that $n>1$ and $k<n$. When $k=n$ we have a parallel system for which $R_C(n,n) = p^n$; when $n=1$ we have a trivial case: $R_C(1,1) = p$.

2.3.2 Consecutive *k*-out-of-*n* Systems with Different Elements

This section considers consecutive *k*-out-of-*n* systems consisting of elements with different reliabilities. In order to describe the UGF-based algorithm for evaluating the reliability of this type of system, we represent the linear consecutive *k*-out-of-*n* system as a set of n+2 consecutively ordered nodes: 0, 1, 2, …, n+1 (see Figure 2.5). Each node j $(1 \le j \le n)$ has two states. In state $X_j = 1$ the arcs from the node j

to nodes $\alpha(j+1)$, $\alpha(j+2), \ldots$, $\alpha(j+k)$ exist, where $\alpha(x) = \min\{x, n+1\}$. In state $X_j = 0$ no arcs from node j exist. The probabilities of states 1 and 0 of the node j are respectively p_j and $1-p_j$. Node 0 is fully reliable ($X_0 \equiv 1$) and provides arcs to nodes 1, 2, …, k. Node n+1 is a dummy node and its state does not matter. The system reliability is the probability that a path exists between the nodes 0 and n+1.

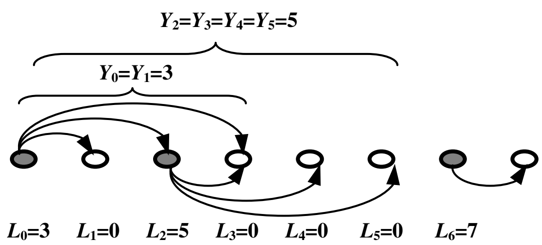

Figure 2.5. Example of state of linear consecutive 3-out-of-7 system

Let random value L_j be the number of the most remote node to which an arc from node j exists. It can be seen that, for $0 \le j \le n$, $L_j = X_j \alpha(j+k)$. The u-function $u_j(z)$ representing the p.m.f. of L_j takes the form

$$u_0(z) = z^k \text{ and } u_j(z) = p_j z^{\alpha(j+k)} + (1-p_j)z^0 \text{ for } 1 \le j \le n \tag{2.50}$$

Let random value Y_m be the number of the most remote node to which the path from node 0 provided by nodes 1, 2, …, m exists. It can be seen that if the path to the node m+1 provided by the nodes 1, 2, …, m exists $(Y_m \ge m+1)$, then the path to node max(m+1, L_{m+1}) also exists and the number of the most remote node connected with node 0 is equal to max(Y_m, L_{m+1}). If the path to the node m+1 does not exist ($Y_m < m+1$), then this node does not participate in prolonging the path and $Y_{m+1} = Y_m$. This consideration gives the recursive expression

$$Y_0 = L_0$$

$$Y_{m+1} = f(Y_m, L_{m+1}) = \begin{cases} \max\{Y_m, L_{m+1}\} & \text{if } Y_m > m \\ Y_m, & \text{if } Y_m < m+1 \end{cases} \quad \text{for } 0 \le m < n \tag{2.51}$$

The system structure function can be expressed as

$$\phi(X_1,...,X_n) = 1[Y_n(L_1,...,L_n) = n+1] \tag{2.52}$$

The u-function $U_m(z)$ representing the p.m.f. of each random variable Y_m can now be obtained as

$$U_0(z)=u_0(z)$$

$$U_{m+1}(z) = U_m(z) \underset{f}{\otimes} u_{m+1}(z),\ \ 0 \le m < n \tag{2.53}$$

The term of the u-function $U_n(z)$ that has the exponent n+1 corresponds to the system state in which the path from node 0 to node n+1 exists. The system reliability $R = E(1(Y_n = n+1))$ is equal to the coefficient of this term.

When $Y_m < m+1$, the path from node 0 to nodes with numbers greater than m does not exist. Therefore, the states corresponding to $Y_m < m+1$ do not participate in the combinations of the node states that provide the success of the entire system. The terms corresponding to these states can be removed from the u-function $U_m(z)$. After removing the terms corresponding to $Y_m < m+1$, the function $f(Y_m, L_{m+1})$ takes the form $\max(Y_m, L_{m+1})$ and the simple operator $\underset{\max}{\otimes}$ can be used in Equation (2.53).

The following recursive procedure determines the reliability of the linear consecutive k-out-of-n system with different elements:

1. Define $u_0(z) = z^k$, $u_j(z) = p_j z^{\alpha(j+k)} + (1-p_j)z^0$ for $1 \le j \le n$.
2. Assign $U_0(z) = u_0(z)$.
3. For $0 \le j < n$: remove terms with z^s where $s \le j$ from $U_j(z)$ and obtain $U_{j+1}(z) = U_j(z) \underset{\max}{\otimes} u_{j+1}(z)$.
4. Obtain the system availability as the coefficient of the term with z^{n+1} in $U_n(z)$.

Example 2.14

Consider a linear consecutive 3-out-of-5 system with different elements. The u-functions of the individual elements are:

$$u_0(z) = z^3,\ u_1(z) = p_1 z^4 + q_1 z^0,\ u_2(z) = p_2 z^5 + q_2 z^0$$

$$u_3(z) = p_3 z^6 + q_3 z^0,\ u_4(z) = p_4 z^6 + q_4 z^0,\ u_5(z) = p_5 z^6 + q_5 z^0$$

where $q_j = 1-p_j$. Following the recursive procedure we obtain

$$U_0(z) = u_0(z) = z^3$$

$$U_1(z) = U_0(z) \underset{\max}{\otimes} u_1(z) = z^3 \underset{\max}{\otimes} (p_1 z^4 + q_1 z^0) = p_1 z^4 + q_1 z^3$$

$$U_2(z) = U_1(z) \underset{\max}{\otimes} u_2(z) = (p_1 z^4 + q_1 z^3) \underset{\max}{\otimes} (p_2 z^5 + q_2 z^0)$$
$$= p_2 z^5 + p_1 q_2 z^4 + q_1 q_2 z^3$$

$$U_3(z) = U_2(z) \underset{\max}{\otimes} u_3(z) = (p_2 z^5 + p_1 q_2 z^4 + q_1 q_2 z^3) \underset{\max}{\otimes} (p_3 z^6 + q_3 z^0)$$
$$= p_3 z^6 + p_2 q_3 z^5 + p_1 q_2 q_3 z^4 + q_1 q_2 q_3 z^3$$

After removing the term with z^3

$$U_3(z) = p_3 z^6 + p_2 q_3 z^5 + p_1 q_2 q_3 z^4$$

$$U_4(z) = U_3(z) \underset{\max}{\otimes} u_4(z) = (p_3 z^6 + p_2 q_3 z^5 + p_1 q_2 q_3 z^4) \underset{\max}{\otimes} (p_4 z^6 + q_4 z^0)$$
$$= (p_3 + p_2 q_3 + p_1 q_2 q_3) p_4 z^6 + p_3 q_4 z^6 + p_2 q_3 q_4 z^5 + p_1 q_2 q_3 q_4 z^4$$
$$= [p_4(1 - q_1 q_2 q_3) + p_3 q_4] z^6 + p_2 q_3 q_4 z^5 + p_1 q_2 q_3 q_4 z^4$$

After removing the term with z^4

$$U_4(z) = [p_4(1 - q_1 q_2 q_3) + p_3 q_4] z^6 + p_2 q_3 q_4 z^5$$

$$U_5(z) = U_4(z) \underset{\max}{\otimes} u_5(z)$$
$$= \{[p_4(1 - q_1 q_2 q_3) + p_3 q_4] z^6 + p_2 q_3 q_4 z^5\} \underset{\max}{\otimes} (p_5 z^6 + q_5 z^0)$$
$$= [p_4(1 - q_1 q_2 q_3) + p_3 q_4 + p_2 q_3 q_4 p_5] z^6 + p_1 q_2 q_3 q_4 q_5 z^5$$

The system reliability is equal to the coefficient of the term with z^6:

$$R = [p_4(1 - q_1 q_2 q_3) + p_3 q_4 + p_2 q_3 q_4 p_5]$$

The reduction of the problem of evaluating the reliability of the circular system with different independent elements into the problem of evaluating the reliability of the linear system was proposed by Hwang [27].

Let $R_{\mathrm{L}}(k, \langle i, j \rangle)$ be the reliability of the linear consecutively connected k-out-of-(j–i+1) subsystem consisting of elements i, i+1, …, j–1, j.

Consider the circular system presented in Figure 2.6. Let in this system f and l be the numbers of the first and last working elements respectively in the sequence from 1 to n. The reliability of the circular system is equal to the probability that the fragment between elements l and f through element n contains less than k elements and the fragment between elements f and l, not including element n, forms a working linear consecutive k-out-of-n system. There are $n-l+f-1$ elements in the first fragment (including element n). The probability that all of the elements belonging to this fragment fail while elements f and l work is

$$p_f \prod_{i=1}^{f-1}(1-p_i)p_l \prod_{j=l+1}^{n}(1-p_j) \tag{2.54}$$

The reliability of the second fragment (not including element n) is $R_{\mathrm{L}}(k,\langle f+1,\ l-1\rangle)$. The reliability of the circular consecutive k-out-of-n system $R_{\mathrm{C}}(k,n)$ is equal to the sum of the probabilities that the system works for all possible combinations of f and l meeting the constraint $n-l+f-1<k$:

$$\begin{aligned} R_{\mathrm{C}}(k,n) &= \sum_{n-l+f-1<k} p_f \prod_{i=1}^{f-1}(1-p_i)p_l \\ &\times \prod_{j=l+1}^{n}(1-p_j)R_{\mathrm{L}}(k,\langle f+1,l-1\rangle) \\ &= \sum_{a=0}^{k-1}\sum_{b=0}^{a} p_{b+1}\prod_{i=1}^{b}(1-p_i)p_{n-a+b} \\ &\times \prod_{j=n-a+b+1}^{n}(1-p_j)R_{\mathrm{L}}(k,\langle b+2,n-a+b-1\rangle) \end{aligned} \tag{2.55}$$

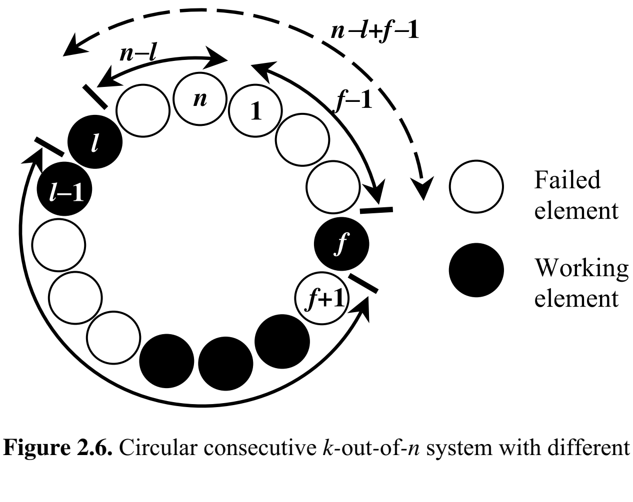

Figure 2.6. Circular consecutive k-out-of-n system with different elements

As in the case of a system with identical elements, Equation (2.55) is obtained on the assumption that $n>1$ and $k<n$. When $k=n$ we have a parallel system, for which $R_C(n,n)=\prod_{j=1}^{n} p_j$; when $n=1$, we have a trivial case: $R_C(1,1)=p_1$.

It should be mentioned that other recursive formulae for evaluating $R_C(k,n)$ were later suggested by Antonopoulou and Papastavridis [33], Korczak [34] and Chang *et al.* [35]. The last reference presents the most effective algorithm of the system reliability evaluation.

2.4 Consecutive *k*-out-of-*r*-from-*n* Systems

The linear consecutive *k*-out-of-*r*-from-*n*:*F* system has *n* ordered elements and fails if at least *k* out of any *r* consecutive elements fail. The system that works if at least *k* out of any *r* consecutive elements are working is called the consecutive *k*-out-of-*r*-from-*n*:*G* system. It can be seen that a *k*-out-of-*r*-from-*n*:*F* system is equivalent to an (*r*-*k*+1)-out-of-*r*-from-*n*:*G* system. Therefore, we consider only *k*-out-of-*n*:*F* systems and omit *F* from their denomination.

The linear consecutive *k*-out-of-*r*-from-*n* system was formally introduced by Griffith [36], but had been previously mentioned by Tong [37], Saperstein [38, 39], Naus [40] and Nelson [41] in connection with tests for non-random clustering, quality control and inspection procedures, service systems, and radar detection problems.

The models presented in the two previous sections can be considered as special cases of the linear consecutive *k*-out-of-*r*-from-*n* system. When $r=n$ one has the simple k-out-of-*n* system. When $k=r$ one has the consecutive *k*-out-of-*n* system.

Example 2.15

Consider a quality control system that randomly selects for a quality check *r* items produced consecutively by a manufacturing process. If within the selected sample at least *k* items are defective, then the system concludes that the process needs to be adjusted. If the process produces *n* items in a certain period of time, then we are interested in the probability that such a random quality check is able to detect a problem in the process.

Example 2.16

An outdoor industrial conveyor transports identical sealed containers (Figure 2.7). The containers are loaded onto pallets placed on the conveyor belt. The conveyor carries *r* pallets simultaneously. If the container sealing fails, its weight becomes greater due to humidity penetration. The maximum allowable load of the conveyor corresponds to $k-1$ containers with failed sealing. The system fails if more than *k* such containers are loaded on *r* consecutive pallets. Having the probability that each sealing fails, we obtain the system reliability as a probability that the system does not fail during the time when n containers are transported.

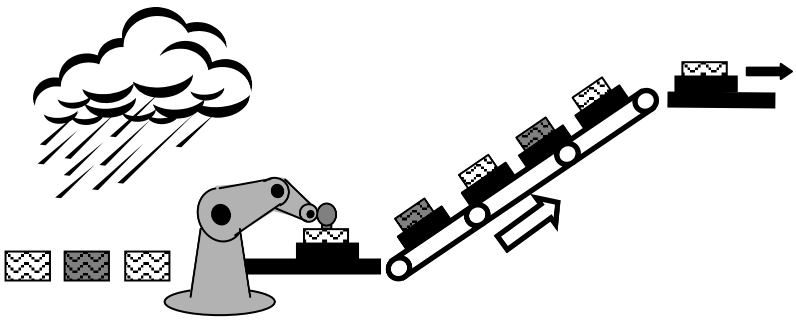

Figure 2.7. Industrial conveyor as an example of k-out-of-r-from-n system

The algorithms suggested for evaluating the reliability of linear consecutive k-out-of-r-from-n systems either consider the case of identical elements (elements with equal reliability) and a limited set of parameters [36, 40, 42] or provide bounds for system reliability [42-44] that are good enough only for element reliabilities very close to 1. Because of the difficulty in estimating the exact value of the system reliability, Psillakis [45] proposed a simulation approach and provided the error analysis. Malinowski and Preuss [46] suggested an enumerative algorithm for the exact evaluation of the system reliability based on recursive computation of conditional probabilities.

Let X_j be the binary state variable of element j. The u-function $u_j(z)$ that takes the form (2.8) represents the p.m.f. of X_j. The system succeeds if any group of r consecutive elements contains at least r–k+1 working elements. Therefore, the system reliability can be defined as

$$R = \Pr\{ \bigcup_{h=1}^{n-r+1} (\sum_{j=h}^{h+r-1} X_j > r\text{-}k)\} \tag{2.56}$$

Let V_h be a group of r consecutive elements numbered from h to h+r–1. The state of this group can be represented by a random binary state vector $\boldsymbol{Y}_h = \{X_h, \ldots, X_{h+r-1}\}$.

Each realization $\boldsymbol{y}_{h,m}$ of vector $\boldsymbol{Y}_h$ constitutes a state m of V_h. Since the elements are independent, the probability of any state of the group V_h is equal to the product of the probabilities of the corresponding states of the individual elements. The p.m.f. of $\boldsymbol{Y}_h$ can be represented by the u-function $U_h(z)$. The total number of different states of the group of r elements is equal to 2^r. Therefore, the u-function $U_h(z)$ consists of 2^r different terms.

The u-function corresponding to the hth group of r consecutive elements V_h takes the form

$$U_h(z) = \sum_{x_h=0}^{1} \sum_{x_{h+1}=0}^{1} \dots \sum_{x_{h+r-1}=0}^{1} [\prod_{j=h}^{h+r-1} p_j^{x_j} (1-p_j)^{1-x_j}] z^{(x_h,\dots,x_{h+r-1})} \quad (2.57)$$

Simplifying this representation one obtains

$$U_h(z) = \sum_{m=1}^{2^r} Q_{h,m} z^{\boldsymbol{y}_{h,m}} \quad (2.58)$$

where $Q_{h,m}$ is the probability that the hth group is in state m and r-length binary vector $\boldsymbol{y}_{h,m}$ represents the states of the elements when the group is in state m. The u-function obtained defines all of the possible states of the group V_h.

Let random variable S_h represent the sum of random binary variables X_h, X_{h+1}, …, X_{h+r-1} (which corresponds to the sum of the state variables of the elements belonging to the group V_h). According to definition (2.56) the system structure function takes the form

$$\phi(X_1,\dots,X_n) = \prod_{h=1}^{n-r+1} 1(S_h > r-k) \quad (2.59)$$

Having the vectors $\boldsymbol{y}_{h,m}$ representing states of elements belonging to V_h in any state m, one can obtain the realization of S_h in this state by summing the vector elements. Therefore, the p.m.f. of S_h can be represented by the u-function $\hat{U}_h(z)$, which is obtained from $U_h(z)$ by replacing the vectors $\boldsymbol{y}_{h,m}$ with sums of their elements. The u-function $\widetilde{U}_h(z)$, representing p.m.f. of the binary function $1(S_h \le r-k)$, can be obtained by applying the operator $\widetilde{U}_h(z) = \hat{U}_h(z) \underset{\varphi}{\otimes} (r-k)$, where $\varphi(Y, r-k) = 1(Y \le r-k)$. Calculating the expected value of the function $1(S_h \le r-k)$ one obtains the probability of failure of the hth group of r consecutive elements V_h:

$$\Pr\{S_h \le r-k\} = E(1(S_h \le r-k)) = \widetilde{U}'_h(1) \quad (2.60)$$

Observe that this operation is equivalent to summing the coefficients of terms containing in their state vectors k or more zeros in the u-function $U_h(z)$. Therefore, the failure probability can also be obtained by applying the following operator Θ_k directly over the u-function $U_h(z)$:

$$\Pr\{S_h \le r-k\} = \Theta_k(U_h(z)) = \sum_{m=1}^{2^r} Q_{h,m} \times 1(\theta(\boldsymbol{y}_{h,m}) \ge k) \quad (2.61)$$

where $\theta(\boldsymbol{y})$ is a sum of zeros in vector $\boldsymbol{y}$.

Example 2.17

Consider a system with $k = 2$, $n = 4$ and $r = 3$. The binary state variables of the system elements are X_1, X_2, X_3 and X_4. The state of the second group of three variables V_2 is represented by the vector $\boldsymbol{Y}_2 = (X_2, X_3, X_4)$. If the reliability of each element is $p_i = \Pr\{X_i = 1\}$, then the probability of each possible realization of the vector $\boldsymbol{Y}_2$ is

$$\Pr\{\boldsymbol{Y}_2 = (x_2, x_3, x_4)\} = p_2^{x_2}(1-p_2)^{1-x_2}\, p_3^{x_3}(1-p_3)^{1-x_3}\, p_4^{x_4}(1-p_4)^{1-x_4}$$

The condition of failure of the group V_2 is $S_2 \le r-k$, where $S_2 = X_2+X_3+X_4$ and $r-k = 3-2 = 1$. The u-function that represents the distribution of $\boldsymbol{Y}_2$ takes the form

$$U_2(z) = p_1p_2p_3z^{(1,1,1)} + p_1p_2q_3z^{(1,1,0)} + p_1q_2p_3z^{(1,0,1)} + p_1q_2q_3z^{(1,0,0)}$$
$$+ q_1p_2p_3z^{(0,1,1)} + q_1p_2q_3z^{(0,1,0)} + q_1q_2p_3z^{(0,0,1)} + q_1q_2q_3z^{(0,0,0)}$$

where $q_j = 1-p_j$. The u-functions $\hat{U}_2(z)$ and $\tilde{U}_2(z)$ that represent the distributions of the functions S_2 and $1(S_2 \le 1)$ respectively are

$$\hat{U}_2(z) = p_1p_2p_3z^3 + p_1p_2q_3z^2 + p_1q_2p_3z^2 + p_1q_2q_3z^1 + q_1p_2p_3z^2$$
$$+ q_1p_2q_3z^1 + q_1q_2p_3z^1 + q_1q_2q_3z^0 = p_1p_2p_3z^3 + (p_1p_2q_3 + p_1q_2p_3$$
$$+ q_1p_2p_3)z^2 + (p_1q_2q_3 + q_1p_2q_3 + q_1q_2p_3)z^1 + q_1q_2q_3z^0$$

and

$$\tilde{U}_2(z) = p_1p_2p_3z^0 + (p_1p_2q_3 + p_1q_2p_3 + q_1p_2p_3)z^0$$
$$+ (p_1q_2q_3 + q_1p_2q_3 + q_1q_2p_3)z^1 + q_1q_2q_3z^1$$

The failure probability is

$$\Pr\{S_2 \le 1\} = E(1(S_2 \le 1)) = \tilde{U}_2'(1) = p_1q_2q_3 + q_1p_2q_3 + q_1q_2p_3 + q_1q_2q_3$$

Note that the linear consecutive k-out-of-r-from-n system contains exactly $n-r+1$ groups of r consecutive elements and each element can belong to no more than r such groups. To obtain the u-functions corresponding to all the groups of r consecutive elements, the following definitions are introduced:

1. Define u-function $U_{1-r}(z)$ as follows:

$$U_{1-r}(\mathrm{z}) = z^{y_0} \tag{2.62}$$

where the vector $\boldsymbol{y}_0$ consists of r zeros.

2. Define the following shift operator over u-function $U_h(z)$:

$$\begin{aligned} U_h(z) \underset{\leftarrow}{\otimes} u_{h+r}(z) &= \left(\sum_{m=1}^{2^r} Q_{h,m} z^{\boldsymbol{y}_{h,m}}\right) \underset{\leftarrow}{\otimes} [p_{h+r} z^1 + (1-p_{h+r}) z^0] \\ &= p_{h+r} \sum_{m=1}^{2^r} Q_{h,m} z^{\boldsymbol{y}_{h,m} \leftarrow 1} + (1-p_{h+r}) \sum_{m=1}^{2^r} Q_{h,m} z^{\boldsymbol{y}_{h,m} \leftarrow 0} \end{aligned} \tag{2.63}$$

where operator $\boldsymbol{y} \leftarrow x$ over arbitrary vector $\boldsymbol{y}$ and value x shifts all the vector elements one position left: $y(j-1) = y(j)$ for $1 < j \le r$ and assigns the value x to the last element of $\boldsymbol{y}$: $y(r) = x$ (the first element of vector $\boldsymbol{y}$ disappears after applying the operator). The operator $\boldsymbol{y} \leftarrow x$ removes the state of the first element of the group and adds the state of the next (not yet considered) element to the group, preserving the order of the elements belonging to the group. Therefore, applying this operator over the u-function $U_h(z)$ that represents the state distribution of the group V_h, one obtains the u-function $U_{h+1}(z)$ representing the state distribution of the group V_{h+1}.

Using the operator $\underset{\leftarrow}{\otimes}$ in sequence as follows:

$$U_{j+1-r}(z) = U_{j-r}(z) \underset{\leftarrow}{\otimes} u_j(z) \tag{2.64}$$

for $j = 1, \ldots, n$ one obtains u-functions for all of the possible groups V_h: $U_1(z), \ldots, U_{n-r+1}(z)$. Note that the u-function $U_1(z)$ for the first group V_1 is obtained after applying the operator $\underset{\leftarrow}{\otimes}$ r times.

Consider a u-function $U_h(z)$ representing the distribution of the random vector $\boldsymbol{Y}_h = \{X_h, \ldots, X_{h+r-1}\}$. For each combination of values $X_{h+1}, \ldots, X_{h+r-1}$ it contains two terms corresponding to values 0 and 1 of X_h (states 0 and 1 of element h). After applying the operator $\underset{\leftarrow}{\otimes}$, X_h disappears from the vector $\boldsymbol{Y}_h$, being replaced with X_{h+1}. This produces two terms with the same state vector $\boldsymbol{y}_{h+1,m}$ for each state m of the group V_{h+1} in the u-function $U_{h+1}(z)$. The coefficients of the two terms with the same state vector $\boldsymbol{y}_{h+1,m}$ are equal to the probabilities that the group V_{h+1} is in state m while element h is in states 0 and 1 respectively. By summing these two coefficients (collecting the like terms in $U_{h+1}(z)$), one obtains a single term for each vector $\boldsymbol{y}_{h+1,m}$ with a coefficient equal to the overall probability that the group V_{h+1} is in state m. Therefore, the number of different terms in each u-function $U_h(z)$ is always equal to 2^r and

$$U_h(z) \underset{\leftarrow}{\otimes} u_{h+r}(z) = \sum_{m=1}^{2^r} Q_{h+1,m} z^{\boldsymbol{y}_{h+1,m}} \tag{2.65}$$

Applying the operator Θ_k (2.61) over the u-functions $U_1(z)$, …, $U_{n-r+1}(z)$ one can obtain the failure probability for each group of r consecutive elements.

Example 2.18

Consider the system from Example 2.17 and obtain the u-functions for all of the possible groups of three consecutive elements using the recursive procedure described above. There are two such groups in the system: V_1 with element state vector (X_1, X_2, X_3) and V_2 with element state vector (X_2, X_3, X_4).
First define

$$U_{-2}(z) = z^{(0,0,0)}$$

Following (2.63) obtain

$$U_{-1}(z) = U_{-2}(z) \underset{\leftarrow}{\otimes} u_1(z) = z^{(0,0,0)} \underset{\leftarrow}{\otimes} (p_1 z^1 + q_1 z^0)$$

$$= p_1 z^{(0,0,1)} + q_1 z^{(0,0,0)}$$

$$U_0(z) = U_{-1}(z) \underset{\leftarrow}{\otimes} u_2(z) = (p_1 z^{(0,0,1)} + q_1 z^{(0,0,0)}) \underset{\leftarrow}{\otimes} (p_2 z^1 + q_2 z^0)$$

$$= p_1 p_2 z^{(0,1,1)} + q_1 p_2 z^{(0,0,1)} + p_1 q_2 z^{(0,1,0)} + q_1 q_2 z^{(0,0,0)}$$

$$U_1(z) = U_0(z) \underset{\leftarrow}{\otimes} u_3(z) = (p_1 p_2 z^{(0,1,1)} + q_1 p_2 z^{(0,0,1)} + p_1 q_2 z^{(0,1,0)}$$

$$+ q_1 q_2 z^{(0,0,0)}) \underset{\leftarrow}{\otimes} (p_3 z^1 + q_3 z^0) = p_1 p_2 p_3 z^{(1,1,1)} + q_1 p_2 p_3 z^{(0,1,1)}$$

$$+ p_1 q_2 p_3 z^{(1,0,1)} + q_1 q_2 p_3 z^{(0,0,1)} + p_1 p_2 q_3 z^{(1,1,0)} + q_1 p_2 q_3 z^{(0,1,0)}$$
$$+ p_1 q_2 q_3 z^{(1,0,0)} + q_1 q_2 q_3 z^{(0,0,0)}$$

The u-function $U_1(z)$ represents the distribution of the random vector (X_1, X_2, X_3) and contains $2^3 = 8$ terms. In order to obtain the failure probability of group V_1 for $k = 2$ we apply the operator Θ_2 $(U_1(z))$:

$$\Pr\{S_1 \le 1\} = \Theta_2(U_1(z)) = q_1 q_2 p_3 + p_1 q_2 q_3 + q_1 p_2 q_3 + q_1 q_2 q_3$$

In order to obtain the u-function $U_2(z)$ representing the distribution of the random vector (X_2, X_3, X_4) we apply the operator $\underset{\leftarrow}{\otimes}$ once more:

$$U_2(z) = U_1(z) \underset{\leftarrow}{\otimes} u_4(z) = (p_1 p_2 p_3 z^{(1,1,1)} + q_1 p_2 p_3 z^{(0,1,1)}$$
$$+ p_1 q_2 p_3 z^{(1,0,1)} + q_1 q_2 p_3 z^{(0,0,1)} + p_1 p_2 q_3 z^{(1,1,0)} + q_1 p_2 q_3 z^{(0,1,0)}$$
$$+ p_1 q_2 q_3 z^{(1,0,0)} + q_1 q_2 q_3 z^{(0,0,0)}) \underset{\leftarrow}{\otimes} (p_4 z^1 + q_4 z^0) = p_1 p_2 p_3 p_4 z^{(1,1,1)}$$
$$+ q_1 p_2 p_3 p_4 z^{(1,1,1)} + p_1 q_2 p_3 p_4 z^{(0,1,1)} + q_1 q_2 p_3 p_4 z^{(0,1,1)}$$
$$+ p_1 p_2 q_3 p_4 z^{(1,0,1)} + q_1 p_2 q_3 p_4 z^{(1,0,1)} + p_1 q_2 q_3 p_4 z^{(0,0,1)}$$
$$+ q_1 q_2 q_3 p_4 z^{(0,0,1)} + p_1 p_2 p_3 q_4 z^{(1,1,0)} + q_1 p_2 p_3 q_4 z^{(1,1,0)}$$
$$+ p_1 q_2 p_3 q_4 z^{(0,1,0)} + q_1 q_2 p_3 q_4 z^{(0,1,0)} + p_1 p_2 q_3 q_4 z^{(1,0,0)}$$
$$+ q_1 p_2 q_3 q_4 z^{(1,0,0)} + p_1 q_2 q_3 q_4 z^{(0,0,0)} + q_1 q_2 q_3 q_4 z^{(0,0,0)}$$

This u-function contains pairs of terms with the same state vectors. For example, both terms $p_1 p_2 p_3 p_4 z^{(1,1,1)}$ and $q_1 p_2 p_3 p_4 z^{(1,1,1)}$ correspond to the cases when $X_2 = X_3 = X_4 = 1$, but the first term corresponds to probability $\Pr\{X_2 = X_3 = X_4 = 1,\ X_1 = 1\}$ whereas the second term corresponds to probability $\Pr\{X_2 = X_3 = X_4 = 1,\ X_1 = 0\}$. The overall probability $\Pr\{X_2 = X_3 = X_4 = 1\}$ is equal to the sum of the probabilities. Therefore:

$$\Pr\{X_2 = X_3 = X_4 = 1\} = p_1 p_2 p_3 p_4 + q_1 p_2 p_3 p_4 = p_2 p_3 p_4$$

By summing the coefficients of the terms with the same state vectors (collecting the like terms) we obtain the probabilities of the state combinations of the group V_2:

$$U_2(z) = p_1 p_2 p_3 z^{(1,1,1)} + p_1 p_2 q_3 z^{(1,1,0)} + p_1 q_2 p_3 z^{(1,0,1)}$$
$$+ p_1 q_2 q_3 z^{(1,0,0)} + q_1 p_2 p_3 z^{(0,1,1)} + q_1 p_2 q_3 z^{(0,1,0)} + q_1 q_2 p_3 z^{(0,0,1)}$$
$$+ q_1 q_2 q_3 z^{(0,0,0)}$$

This u-function also contains $2^3 = 8$ terms. The failure probability of the group V_2 represented by the u-function $U_2(z)$ is

$$\Pr\{S_2 \le 1\} = \Theta_2(U_2(z)) = p_1 q_2 q_3 + q_1 p_2 q_3 + q_1 q_2 p_3 + q_1 q_2 q_3$$

The variables S_h are mutually dependent because different groups V_h contain the same elements. Therefore, the failure probability of the entire system cannot be obtained as a sum of the probabilities $\Pr\{S_h \le r-k\}$ for $1 \le h \le n-r+1$. However, excluding the terms corresponding to the failure states from the u-functions we can obtain the system failure probability. Indeed, if for some combination of element states a certain group fails, then the entire system fails independently of the states of the elements that do not belong to this group. Therefore, the terms corresponding to the group failure can be removed from the u-function since they should not participate in determining further state combinations that cause system failures. This consideration lies at the base of the following algorithm, which evaluates the system reliability using the enumerative technique in order to obtain all of the possible element state combinations leading to the system's failure.

1. Initialization.

   $F = 0$; $U_{1-r}(z) = z^{\boldsymbol{y}_0}$ ($\boldsymbol{y}_0$ consists of r zeros).

2. Main loop. Repeat the following for $j = 1, \ldots, n$:

   Obtain $U_{j+1-r}(z) = U_{j-r}(z) \underset{\leftarrow}{\otimes} u_j(z)$ and collect like terms in the u-function obtained.

   If $j \geq r$, then add value $\Theta_k(U_{j+1-r}(z))$ to F and remove all of the terms with state vectors containing more than k zeros from $U_{j+1-r}(z)$.

Obtain the system reliability as $R = 1-F$. Alternatively, the system reliability can be obtained as the sum of the coefficients of the last u-function $U_{n+1-r}(z)$.

Here, we omit the proof that this algorithm obtains the system reliability. The proof can be found in [47].

Example 2.19

Consider the system from Example 2.17 and obtain the system reliability applying the algorithm presented above. The u-functions $U_{-2}(z)$, …, $U_1(z)$ are obtained in the same way as in Example 2.18. After obtaining $U_1(z)$ in the form

$$U_1(z) = p_1p_2p_3z^{(1,1,1)} + q_1p_2p_3z^{(0,1,1)} + p_1q_2p_3z^{(1,0,1)} + q_1q_2p_3z^{(0,0,1)}$$
$$+ p_1p_2q_3z^{(1,1,0)} + q_1p_2q_3z^{(0,1,0)} + p_1q_2q_3z^{(1,0,0)} + q_1q_2q_3z^{(0,0,0)}$$

we add the value of $\Theta_2(U_1(z)) = q_1q_2p_3 + p_1q_2q_3 + q_1p_2q_3 + q_1q_2q_3$ to F and remove the terms with two or more zeros from this u-function. The remaining u-function $U_1(z)$ takes the form

$$U_1(z) = p_1p_2p_3z^{(1,1,1)} + q_1p_2p_3z^{(0,1,1)} + p_1q_2p_3z^{(1,0,1)} + p_1p_2q_3z^{(1,1,0)}$$

Now we obtain $U_2(z)$ as

$$U_2(z) = U_1(z) \underset{\leftarrow}{\otimes} u_4(z) = [p_1p_2p_3z^{(1,1,1)} + q_1p_2p_3z^{(0,1,1)}$$
$$+ p_1q_2p_3z^{(1,0,1)} + p_1p_2q_3z^{(1,1,0)}] \underset{\leftarrow}{\otimes} [p_4z^1 + q_4z^0]$$
$$= p_2p_3p_4z^{(1,1,1)} + p_1q_2p_3p_4z^{(0,1,1)} + p_1p_2q_3p_4z^{(1,0,1)}$$
$$+ p_2p_3q_4z^{(1,1,0)} + p_1q_2p_3q_4z^{(0,1,0)} + p_1p_2q_3q_4z^{(1,0,0)}$$

Having $U_2(z)$ we obtain $\Theta_2(U_2(z)) = p_1(q_2 p_3 + p_2 q_3)q_4$. This probability is added to F. Now F takes the form

$$F = q_1 q_2 p_3 + p_1 q_2 q_3 + q_1 p_2 q_3 + q_1 q_2 q_3 + p_1(q_2 p_3 + p_2 q_3)q_4$$

After removing the terms with two or more zeros from $U_2(z)$ it takes the form

$$U_2(z) = p_2 p_3 p_4 z^{(1,1,1)} + p_1 q_2 p_3 p_4 z^{(0,1,1)}$$
$$+ p_1 p_2 q_3 p_4 z^{(1,0,1)} + p_2 p_3 q_4 z^{(1,1,0)}$$

The system reliability is equal to the sum of the coefficients of $U_2(z)$:

$$R = p_2 p_3 p_4 + p_1(q_2 p_3 + p_2 q_3)p_4 + p_2 p_3 q_4$$

The same result can be obtained as $R = 1-F$. This can be verified by summing R and F:

$$R + F = p_2 p_3 p_4 + p_1(q_2 p_3 + p_2 q_3)p_4 + p_2 p_3 q_4$$
$$+ q_1 q_2 p_3 + p_1 q_2 q_3 + q_1 p_2 q_3 + q_1 q_2 q_3 + p_1(q_2 p_3 + p_2 q_3)q_4$$
$$= p_2 p_3 p_4 + p_1(q_2 p_3 + p_2 q_3) + p_2 p_3 q_4 + q_1 q_2 p_3 + p_1 q_2 q_3$$
$$+ q_1 p_2 q_3 + q_1 q_2 q_3 = p_2 p_3 + p_2 q_3 + q_2 p_3 + q_2 q_3 = 1$$

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