CONTROL OF ROBOT MANIPULATORS IN JOINT SPACE*

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Part I

PRELIMINARIES

Introduction

Robots occupy a privileged place in the modernization of numerous industrial sectors.

- The word *robot* finds its origins in “robota” which means “work” in Slavic languages.
- Was introduced by the Czech science fiction writer Karel Čapek.
- The term “robot” is nowadays used to denote *autonomous* machines.

These machines may be roughly classified as follows:

- Robot manipulators
- Mobile robots $\begin{cases} \text{Ground robots} \begin{cases} \text{Wheeled robots} \\ \text{Legged robots} \end{cases} \\ \text{Submarine robots} \\ \text{Aerial robots.} \end{cases}$

This book is exclusively devoted to robot *manipulators*.

Robotics

- Term coined by the science fiction writer Isaac Asimov
- Science devoted to the study of robots
- Incorporates a variety of fields:
  - Electric engineering, mechanical engineering, industrial engineering, computer science and applied mathematics
    - Automatic control of robot manipulators (*spine* of robotics).

The International Federation of Robotics (norm ISO/TR 8373) defines:

> A manipulating industrial robot is an automatically controlled, reprogrammable, multipurpose manipulator programmable in three or more axes, which may be either fixed in place or mobile for use in industrial automation applications.

The number of joints of a manipulator determines as well, its number of *degrees of freedom* (*DOF*) —typically 6 DOF —.

- 3 determine the position of the end of the last link in the Cartesian space
- 3 more specify its orientation.

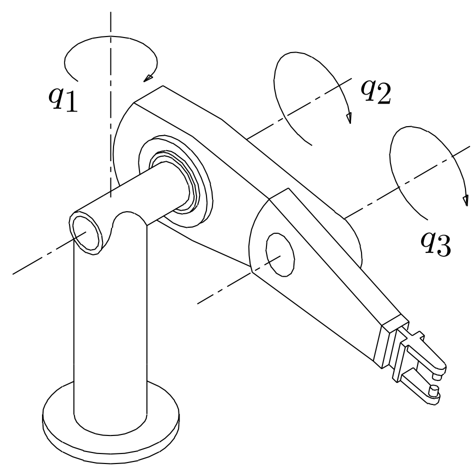

Figure 1: Robot manipulator.

In the present textbook:

> "A robot manipulator —or simply manipulator— is a mechanical articulated arm that is constituted of links interconnected through hinges or joints that allow a relative movement between two consecutive links".

- The joint positions of the robot are collected in the vector $\boldsymbol{q}$, i.e.,

$$\boldsymbol{q} := \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}.$$

- The *joint velocities* are: $\dot{\boldsymbol{q}} := \frac{d}{dt}\boldsymbol{q}$

- The torques and forces are collected in the vector: $\boldsymbol{\tau} := \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{bmatrix}.$

Ch. 1. What does ‘control of robots’ consists in?

Robot control consists in studying how to make a robot manipulator perform a task.

Control design may be divided roughly in the following steps:

- Familiarization with the physical system under consideration,
- Modeling.
- Control specifications.

Familiarization with the physical system under consideration

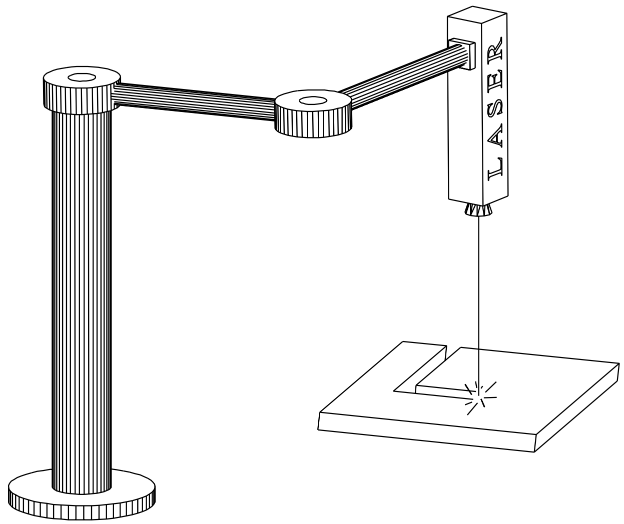

Figure 2: Freely moving robot.

- For this robot, the outputs $\boldsymbol{y}$, are the positions $\boldsymbol{q}$ and joint velocities $\dot{\boldsymbol{q}}$
- or the position and orientation of the end effector.

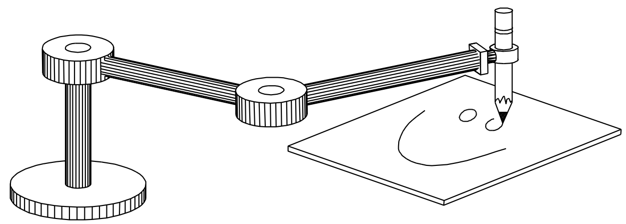

Figure 3: Robot interacting with its environment.

- In this case, the output $\boldsymbol{y}$ may include the torques and forces $\boldsymbol{f}$ exerted by the end tool over its environment.

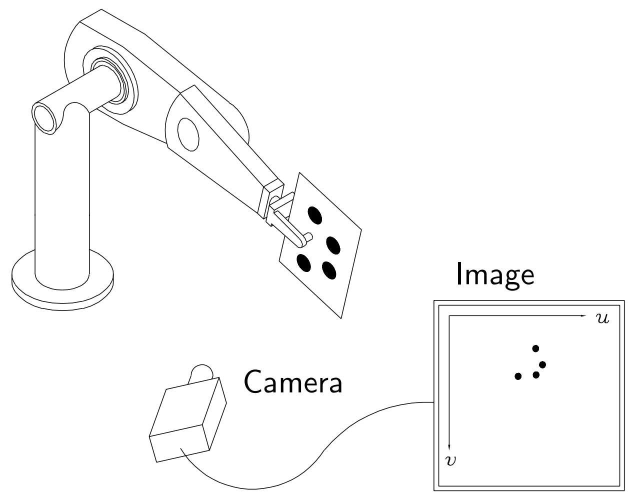

Figure 4: Robotic system: fixed camera.

- In this system, the output $\boldsymbol{y}$ may correspond to the coordinates associated to each of the marks with reference to a screen on a monitor.

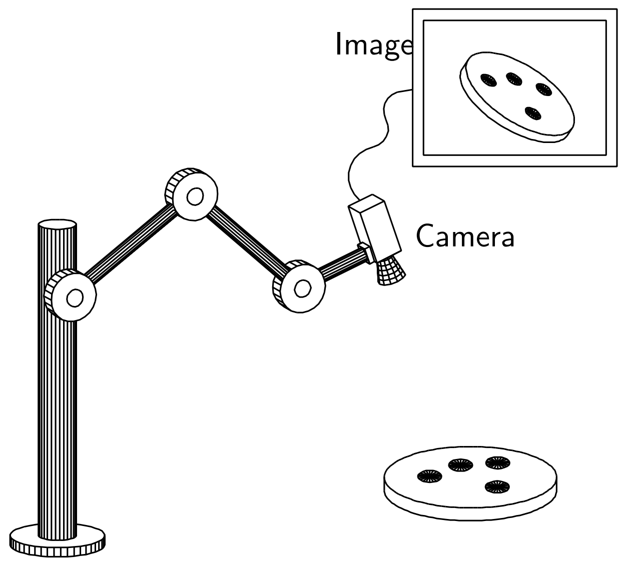

Figure 5: Robotic system: camera in hand.

- Output $\boldsymbol{y}$ may correspond to the coordinates of the screen.

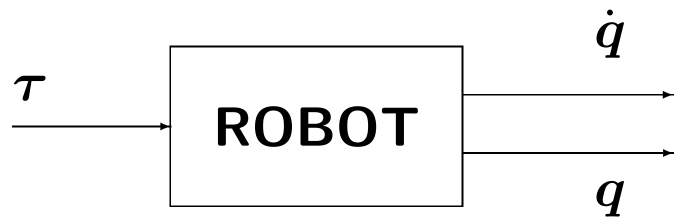

Figure 6: Input-output representation of a robot.

- The input variables, are basically the torques and forces $\boldsymbol{\tau}$,
- The outputs are the joint positions and velocities:

$$\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{f}) = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix}$$

Derivation of the dynamic model

The system's mathematical model is obtained typically via one of the two following techniques.

- *Analytic*. Physics laws of the system's motion.
- *Experimental*. Experimental data collected from the system itself.

Some interesting topics related with modelling are:

- *Robustness*. Faculty of a control system to cope with errors due to neglected dynamics.

- *Parametric identification*. The objective is to obtain the numeric values of different physical parameters.

The dynamic model of robot manipulators

- is derived in the analytic form using basically the laws of mechanics.

- an n DOF system (*multivariable nonlinear* system).

Control specifications

Definition of control objectives:

- Stability
- Regulation
- Trajectory tracking (motion control)
- Optimization.

- *Stability*. Consists in the property of a system by which it goes on working at certain regime or ‘closely’ to it ’for ever’.
  - *Lyapunov* stability theory.
  - *input-output* stability theory.

  In the case when the output $\boldsymbol{y}$ corresponds to the joint position $\boldsymbol{q}$ and velocity $\dot{\boldsymbol{q}}$.

- *Regulation* “Position control in joint coordinates”

- *Trajectory tracking* “Tracking control in joint coordinates”

Motion control of robot manipulators

- "Point-to-point". Determines a series of points in the manipulator's workspace, by which the end effector is required to pass.

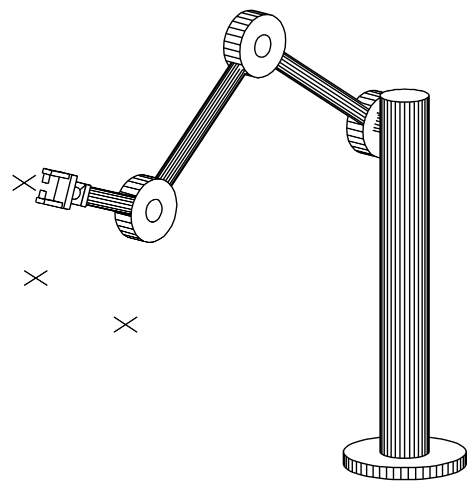

Figure 7: Point-to-point motion specification.

- (Continuous) trajectory. The control problem consists in making the end-effector follow a trajectory as closely as possible.

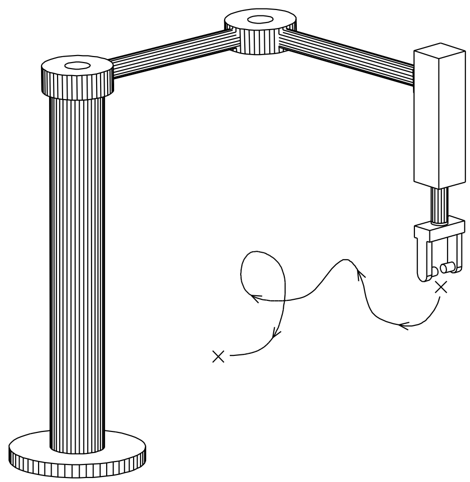

Figure 8: Trajectory motion specification.

- Set-point control problem. The specified trajectory is simply a point in the workspace.

Robot navigation problem consists in solving, in one single step, the following subproblems.

- Path planning. Determines a curve in the state space of the desired posture
- Trajectory generation. Parameterizes in time above curve
- Control design. Solves the control problem

Ch. 2. Introduction to Lyapunov stability theory

Basic notation

Throughout the text we employ the following symbols.

$\forall$	"for all"
$\exists$	"there exists"
$\in$	"belong(s) to"
$\Longrightarrow$	"implies"
$\Longleftrightarrow$	"is equivalent to" or "if and only if"
$\rightarrow$	"tends to"
$:=$ and $=:$	"is defined as" and "equals by definition to" .

Preliminaries: Linear algebra

Vectors

- IR^n denotes the real Euclidean space of dimension n
- Vector $\boldsymbol{x}$ of dimension n

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \;\; x_2 \;\; \cdots \;\; x_n]^T$$

- Denoted by **bold** small letters, either Latin or Greek.

Euclidean norm

The *Euclidean norm* $\|\boldsymbol{x}\|$ of a vector $\boldsymbol{x} \in \mathbb{R}^n$ is defined as

$$\|\boldsymbol{x}\| := \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\boldsymbol{x}^T\boldsymbol{x}}$$

Matrices

$\mathbb{R}^{n\times m}$ denotes the set of real *matrices* A of dimension $n \times m$

$$A = \{a_{ij}\} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

Eigenvalues

For each square matrix $A \in \mathrm{IR}^{n\times n}$ there exist n eigenvalues (in general, complex numbers)

- denoted by $\lambda_1\{A\}, \lambda_2\{A\}, \cdots, \lambda_n\{A\}$.
- satisfy: $\det\left[\lambda_i\{A\}I - A\right] = 0, \qquad \text{for } i = 1, 2, \cdots, n$

For the case of a symmetric matrix $A = A^T \in \mathrm{IR}^{n\times n}$,

- $\lambda_1\{A\}, \lambda_2\{A\}, \cdots, \lambda_n\{A\} \in \mathrm{IR}$.
- theorem of Rayleigh–Ritz establishes that for all $\boldsymbol{x} \in \mathrm{IR}^n$

$$\lambda_{\mathrm{Max}}\{A\}\left\|\boldsymbol{x}\right\|^2 \geq \boldsymbol{x}^T A \boldsymbol{x} \geq \lambda_{\min}\{A\}\left\|\boldsymbol{x}\right\|^2.$$

Spectral norm

The *spectral norm* $\|A\|$ of a matrix $A \in \mathrm{IR}^{n\times m}$ is defined as

$$\|A\| = \sqrt{\lambda_{\mathrm{Max}}\{A^T A\}},$$

Fixed points

Consider a continuous function $\boldsymbol{f} : \mathrm{IR}^n \rightarrow \mathrm{IR}^n$. The vector $\boldsymbol{x}^* \in \mathrm{IR}^n$ is a *fixed point* of $\boldsymbol{f}(\boldsymbol{x})$ if

$$\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{x}^*.$$

- If $\boldsymbol{x}^*$ is a fixed point of the function $\boldsymbol{f}(\boldsymbol{x})$, then $\boldsymbol{x}^*$ is a solution of $\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{x} = \mathbf{0}$.

- The Contraction Mapping Theorem provides a sufficient condition for the existence and unicity of fixed points.

Theorem 2.1 —Contraction Mapping. Consider $\Omega \subset \mathrm{IR}^m$ and the following continuous function

$$\begin{aligned} \boldsymbol{f} : \mathrm{IR}^n \times \Omega &\rightarrow \mathrm{IR}^n \\ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\theta} \end{bmatrix} &\mapsto \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\theta}) \end{aligned}$$

where $\boldsymbol{\theta} \in \Omega$ stands for a vector of parameters.

- Assume that there exists a non-negative constant k such that

$$\|\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\theta}) - \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{\theta})\| \leq k \|\boldsymbol{x} - \boldsymbol{y}\| \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathrm{IR}^n \text{ and all } \boldsymbol{\theta} \in \Omega.$$

  - If $k < 1$, then for each $\boldsymbol{\theta} \in \Omega$, the function $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\theta})$ possesses a unique fixed point $\boldsymbol{x}^* \in \mathrm{IR}^n$.

Lyapunov stability

Second method of Lyapunov or **direct method of Lyapunov**.

Dynamical systems described by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \qquad \boldsymbol{x} \in \mathrm{IR}^n, \quad t \in \mathrm{IR}_+ \,, \tag{1}$$

where

- $\boldsymbol{x}$ corresponds to the state.
- We will use $\boldsymbol{x}(t)$ to denote a solution to (1) in place of $\boldsymbol{x}(t, t_\circ, \boldsymbol{x}(t_\circ))$.

We assume that $\boldsymbol{f} : \mathrm{IR}_+ \times \mathrm{IR}^n \to \mathrm{IR}^n$ is continuous in t and $\boldsymbol{x}$ and is such that

- it has a unique solution corresponding to each initial condition $t_\circ$, $\boldsymbol{x}(t_\circ)$ and
- the solution $\boldsymbol{x}(t, t_\circ, \boldsymbol{x}(t_\circ))$ depends continuously on the initial conditions $t_\circ$, $\boldsymbol{x}(t_\circ)$.
- If $\boldsymbol{f}(t, \boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})$ then, is said to be autonomous.
  - we can safely consider that $t_\circ = 0$.
- $\boldsymbol{f}(t, \boldsymbol{x}) = A(t)\boldsymbol{x} + \boldsymbol{u}(t)$ is linear differential equation
  - In the opposite case it is nonlinear

The concept of equilibrium

Definition 2.1 A *constant* vector $\boldsymbol{x}_e \in \mathrm{IR}^n$ is an *equilibrium* or equilibrium state of the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$, if

$$\boldsymbol{f}(t, \boldsymbol{x}_e) = \mathbf{0} \qquad \forall\, t \geq 0\,.$$

Typically $\boldsymbol{x} = \mathbf{0} \in \mathrm{IR}^n$ is an equilibrium of $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$. If this is not the case, this may be translated to the origin.

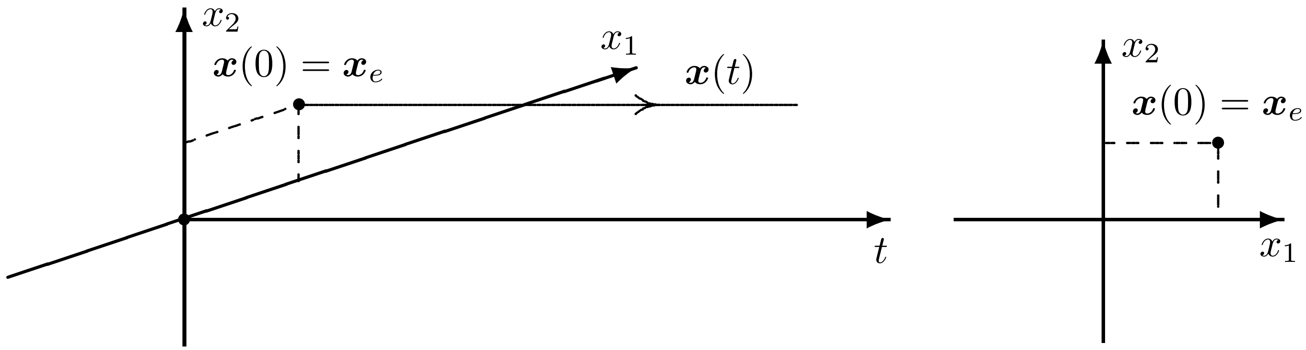

Figure 9: Equilibrium

If the initial state $\boldsymbol{x}(t_\circ) \in \mathrm{IR}^n$ is an equilibrium $(\boldsymbol{x}(t_\circ) = \boldsymbol{x}_e \in \mathrm{IR}^n)$ then,

- $\boldsymbol{x}(t) = \boldsymbol{x}_e \qquad \forall\, t \geq t_\circ \geq 0$

- $\dot{\boldsymbol{x}}(t) = \mathbf{0} \qquad \forall\, t \geq t_\circ \geq 0.$

Example 2.2

Consider a pendulum whose dynamic model is

$$J\ddot{q} + mgl\,\sin(q) = \tau(t)$$

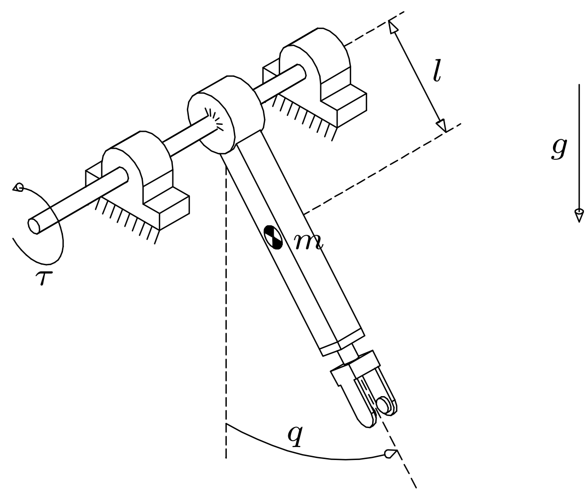

Figure 10: Pendulum.

where

- m is the mass of the pendulum
- J is the total moment of inertia about the joint axis
- l is the distance from its axis of rotation to the center of mass
- q is the angular position.

In terms of the state $[q \quad \dot{q}]^T$, the dynamic model is given by

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ J^{-1}\left[\tau(t) - mgl\,\sin(q)\right] \end{bmatrix}.$$

If $\tau(t) = 0$,

- the equilibrium states are given by

$$[q \quad \dot{q}]^T = [n\pi \quad 0]^T \quad \text{for} \quad n = \cdots, -2, -1, 0, 1, 2, \cdots$$

since $mgl\ \sin(n\pi) = 0$.

If $\tau(t) = \tau^*$ such that $|\tau^*| > mgl$,

- there does not exist any equilibrium since there is no $\boldsymbol{q}^* \in$ IR such that $\tau = mgl\ \sin(\boldsymbol{q}^*)$.

Definitions of stability

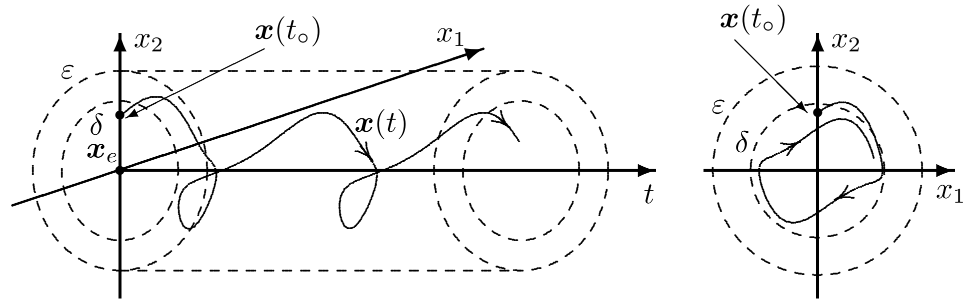

Figure 11: Stability

Definition 2.2 —Stability.

The origin is a *stable equilibrium* (in the sense of Lyapunov) of Equation (1) if, for each pair of numbers $\varepsilon > 0$ and $t_\circ \geq 0$, there exists $\delta = \delta(\varepsilon, t_\circ) > 0$ such that

$$\|\boldsymbol{x}(t_\circ)\| < \delta \implies \|\boldsymbol{x}(t)\| < \varepsilon \qquad \forall\ t \geq t_\circ \geq 0\,. \tag{2}$$

Example 2.3

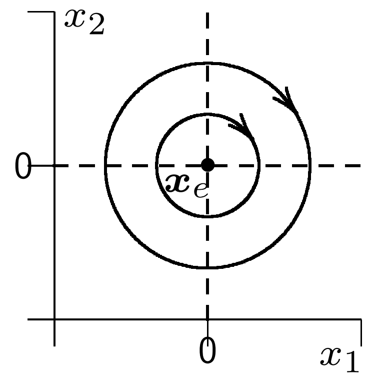

Figure 12: Harmonic oscillator: Phase plane

System described by the equations:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1\end{aligned}$$

The origin is the unique equilibrium point and is stable.

Definition 2.3 —Uniform stability.

The origin is a *uniformly stable equilibrium* (in the sense of Lyapunov) of Equation (1) if for each number $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that (2) holds.

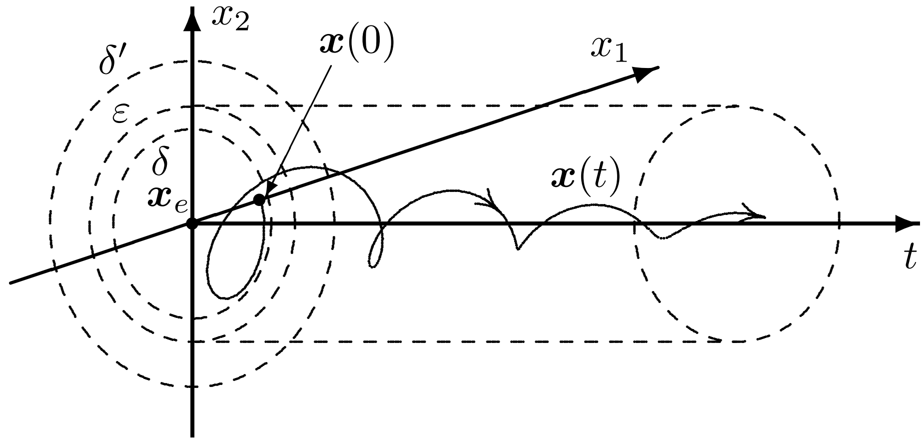

Figure 13: Asymptotic stability

Definition 2.4 —Asymptotic stability.

The origin is an *asymptotically stable equilibrium* of (1) if

1. the origin is stable and
2. the origin is attractive, i.e., for each $t_\circ$, there exists $\delta' = \delta'(t_\circ) > 0$:

$$\|\boldsymbol{x}(t_\circ)\| < \delta' \quad \Longrightarrow \quad \|\boldsymbol{x}(t)\| \to 0 \quad \text{as} \quad t \to \infty\,, \quad \forall\, t_\circ \geq 0. \tag{3}$$

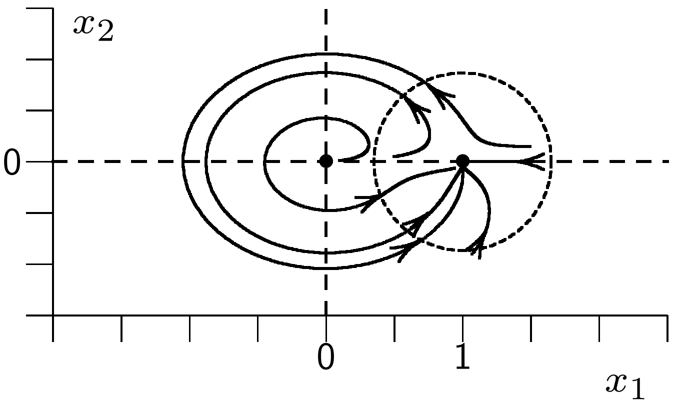

Figure 14: Attractive but unstable equilibrium ($x_1 = r\cos(\theta)$ and $x_2 = r\,\sin(\theta)$)

Example 2.4

$$\begin{aligned} \dot{r} &= \frac{5}{100} r(1-r) \\ \dot{\theta} &= \sin^2(\theta/2) \qquad \theta \in [0, 2\pi). \end{aligned}$$

Equilibria at the origin $[r \;\; \theta]^T = [0 \;\; 0]^T$ and at $[r \;\; \theta]^T = [1 \;\; 0]^T$.

Definition 2.5 —Uniform asymptotic stability.

The origin is an *uniformly asymptotically stable equilibrium* of (1) if

1. the origin is uniformly stable and
2. the origin is uniformly attractive, that is, there exists a number $\delta' > 0$ such that (3) holds with a rate of convergence independent of $t_\circ$.

Definition 2.6 —Global asymptotic stability.

The origin is a *globally asymptotically stable* equilibrium

1. the origin is stable and

2. the origin is globally attractive, that is,

$$\|\boldsymbol{x}(t)\| \to 0 \quad \text{as} \quad t \to \infty, \qquad \forall\ \boldsymbol{x}(t_\circ) \in \mathrm{IR}^n, t_\circ \geq 0.$$

Definition 2.7 —Global uniform asymptotic stability.

The origin is a *globally uniformly asymptotically stable* equilibrium of Equation (1) if:

1. the origin is uniformly stable with $\delta(\varepsilon)$ in (2) which satisfies $\delta(\varepsilon) \to \infty$ as $\varepsilon \to \infty$ (uniform boundedness) and

2. the origin is globally uniformly attractive, i.e., for all $\boldsymbol{x}(t_\circ) \in \mathrm{IR}^n$ and all $t_\circ \geq 0$,
$$\|\boldsymbol{x}(t)\| \to 0 \quad \text{as} \quad t \to \infty$$

with a convergence rate that is independent of $t_\circ$.

Definition 2.8 —Global exponential stability.

The origin is a *globally exponentially stable* equilibrium of (1) if there exist positive constants α and β, independent of $t_\circ$, such that

$$\|\boldsymbol{x}(t)\| < \alpha \,\|\boldsymbol{x}(t_\circ)\|\, e^{-\beta(t-t_\circ)}, \qquad \forall\, t \geq t_\circ \geq 0, \qquad \forall\, \boldsymbol{x}(t_\circ) \in \mathrm{I\!R}^n\,.$$

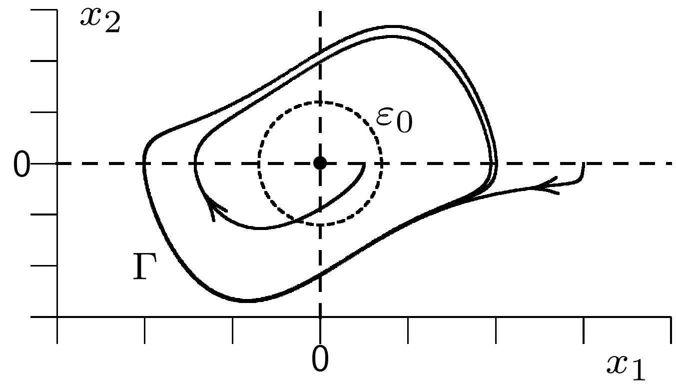

Figure 15: Van der Pol: Phase plane.

Definition 2.9 —Instability.

The origin of Equation (1) is *unstable* if it is *not stable*.

Example 2.6 Van der Pol system,

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 + (1 - x_1^2)x_2.
\end{aligned}
$$

Lyapunov functions

Definition 2.10 —Locally and globally positive definite function.

A continuous function $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is said to be *locally positive definite* if

1. $W(\mathbf{0}) = 0$,
2. $W(\boldsymbol{x}) > 0 \quad \text{for small} \quad \|\boldsymbol{x}\| \neq 0$.

It is said to be *globally positive definite* (or simply *positive definite*) if

1. $W(\mathbf{0}) = 0$,
2. $W(\boldsymbol{x}) > 0 \quad \forall\, \boldsymbol{x} \neq \mathbf{0}$.

- $W(\boldsymbol{x})$ is said to be (locally) negative definite if $-W(\boldsymbol{x})$ is (locally) positive definite.

- A function $V : \mathrm{IR}_+ \times \mathrm{IR}^n \to \mathrm{IR}_+$ is (resp. locally) positive definite if there exists a positive definite function $W : \mathrm{IR}^n \to \mathrm{IR}_+$ such that

  1. $V(t, \mathbf{0}) = 0 \;\forall\; t \;\geq 0$
  2. $V(t, \boldsymbol{x}) \geq W(\boldsymbol{x}), \quad \forall\; t \geq 0, \quad \forall\; \boldsymbol{x} \in \mathrm{IR}^n \quad \text{(resp. for small } \|\boldsymbol{x}\|\text{)}.$

Definition 2.11 —Radially unbounded function and decrescent function.

A continuous function $W : \mathrm{IR}^n \to \mathrm{IR}$ is said to be *radially unbounded* if

$$W(\boldsymbol{x}) \to \infty \qquad \text{as} \qquad \| \boldsymbol{x} \| \to \infty \,.$$

- $V(t, \boldsymbol{x})$ is radially unbounded if $V(t, \boldsymbol{x}) \geq W(\boldsymbol{x})$ for all $t \geq 0$.

A continuous function $V : \mathrm{IR}_+ \times \mathrm{IR}^n \to \mathrm{IR}$ is (resp. locally) *decrescent* if there exists a (resp. locally) positive definite function $W : \mathrm{IR}^n \to \mathrm{IR}_+$ such that

$$V(t, \boldsymbol{x}) \leq W(\boldsymbol{x}) \qquad \forall\, t \geq 0 \qquad \forall\, \boldsymbol{x} \in \mathrm{IR}^n \;\; (\text{resp. for small } \|\boldsymbol{x}\|).$$

- If $V(t, \boldsymbol{x}) = V(\boldsymbol{x})$ then $V(\boldsymbol{x})$ is decrescent.

Example 2.7

Consider the graphs of the functions $V_i(x)$ with $i = 1, \ldots, 4$ as depicted in Figure 16.

- $V_1(x)$ is locally positive definite but is not (globally) positive definite.
- $V_2(x)$ is locally positive definite and (globally) positive definite. Also it is radially unbounded.
- $V_3(x)$ is locally positive definite and (globally) positive definite but it is not radially unbounded.
- $V_4(x)$ is positive definite and radially unbounded.

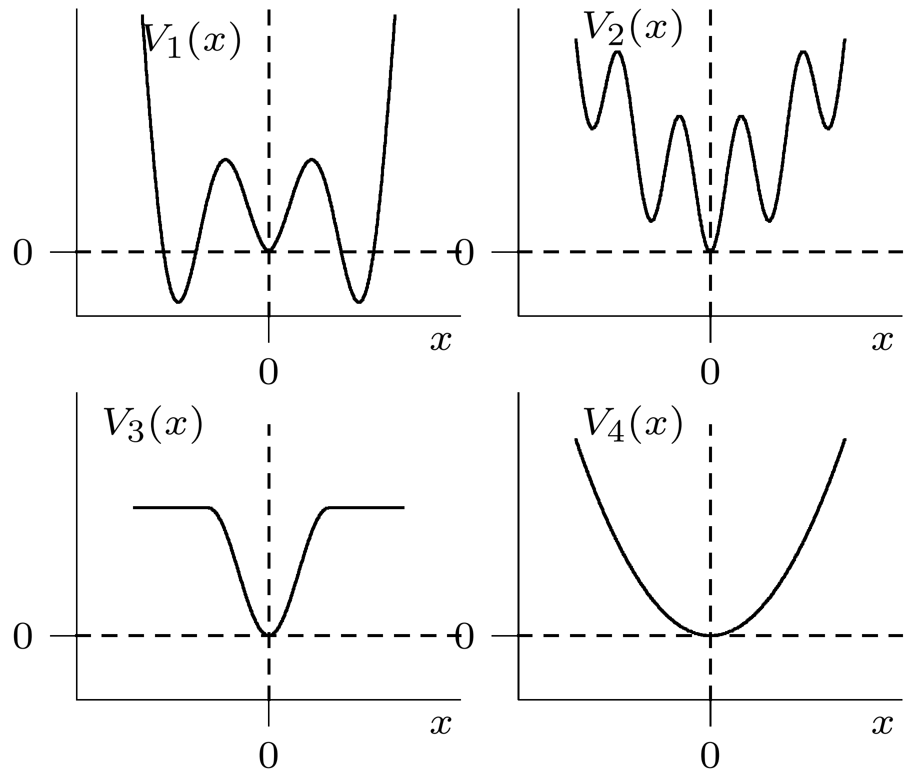

Figure 16: Examples

Definition 2.12 —Lyapunov function candidate.

A function $V : \mathrm{IR}_+ \times \mathrm{IR}^n \rightarrow \mathrm{IR}_+$ is said to be a *Lyapunov function candidate* for the equilibrium $\boldsymbol{x} = \boldsymbol{0} \in \mathrm{IR}^n$ of the equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ if

1. $V(t, \boldsymbol{x})$ is locally positive definite,
2. $\frac{\partial V(t,\boldsymbol{x})}{\partial t}$ is continuous with respect to t and $\boldsymbol{x}$,
3. $\frac{\partial V(t,\boldsymbol{x})}{\partial \boldsymbol{x}}$ is continuous with respect to t and $\boldsymbol{x}$.

Definition 2.13 —Time derivative of a Lyapunov function candidate.

Let $V(t, \boldsymbol{x})$ be a Lyapunov function candidate for the equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$. The total time derivative of $V(t, \boldsymbol{x})$ along the trajectories of (1), denoted by $\dot{V}(t, \boldsymbol{x})$, is given by

$$\dot{V}(t, \boldsymbol{x}) := \frac{d}{dt}\{V(t, \boldsymbol{x})\} = \frac{\partial V(t, \boldsymbol{x})}{\partial t} + \frac{\partial V(t, \boldsymbol{x})}{\partial \boldsymbol{x}}^T f(t, \boldsymbol{x}).$$

Definition 2.14 —Lyapunov function.

A Lyapunov function candidate $V(t, \boldsymbol{x})$ for Equation (1) is a *Lyapunov function* for (1) if its total time derivative along the trajectories of (1) satisfies

$$\dot{V}(t, \boldsymbol{x}) \leq 0 \qquad \forall\, t \geq 0$$

and for small $\|\boldsymbol{x}\|$.

Lyapunov's direct method

Theorem 2.2 —Stability and uniform stability.

The origin is a *stable* equilibrium of Equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$, if there exists

- $V(t, \mathbf{0}) = 0 \ \forall \ t \ \geq 0$
- $V(t, \boldsymbol{x}) \geq W_1(\boldsymbol{x}) > 0, \qquad \forall \ t \geq 0$ locally
- $\dot{V}(t, \boldsymbol{x}) \leq 0, \qquad \forall \ t \geq 0 \qquad \text{for small } \|\boldsymbol{x}\|.$
- If moreover $V(t, x) \leq W_2(\boldsymbol{x}), \quad \forall \ t \geq 0$ (decrescent) for small $\|\boldsymbol{x}\|$, then the origin is uniformly stable.

$W_1(\boldsymbol{x})$ and $W_2(\boldsymbol{x})$ are positive definite functions.

Theorem 2.3 —(Uniform) boundedness of solutions plus uniform stability.

The origin is a uniformly stable equilibrium of $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ and the solutions $\boldsymbol{x}(t)$ are uniformly bounded for all initial conditions $(t_\circ, \boldsymbol{x}(t_\circ)) \in \mathrm{IR}_+ \times \mathrm{IR}^n$ if there exists

- $V(t, \boldsymbol{x}) \geq W_1(\boldsymbol{x}) > 0, \qquad \forall\, t \geq 0 \quad \forall\, \boldsymbol{x} \in \mathrm{IR}^n$
  - and $V(t, \mathbf{0}) = 0\ \forall\ t\ \geq 0$ (positive definite condition)
  - with $W_1(\boldsymbol{x}) \to \infty \qquad \text{as} \qquad \|\, \boldsymbol{x}\, \| \to \infty$ (radially unbounded)
- $V(t, x) \leq W_2(\boldsymbol{x}) \qquad \forall\, t \geq 0 \quad \forall\, \boldsymbol{x} \in \mathrm{IR}^n$ (decrescent)
- $\dot{V}(t, \boldsymbol{x}) \leq 0 \qquad \forall\, t \geq t_\circ \geq 0 \qquad \forall\, \boldsymbol{x} \in \mathrm{IR}^n$.

$W_1(\boldsymbol{x})$ and $W_2(\boldsymbol{x})$ are positive definite functions.

Example 2.11

Consider the dynamic model of an ideal pendulum without friction

$$J\ddot{q} + mgl\,\sin(q) = 0 \quad \text{with} \quad q(0), \dot{q}(0) \in \mathbb{R}$$

or, in the state-space form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{mgl}{J}\sin(x_1), \end{aligned}$$

where $x_1 = q$ and $x_2 = \dot{q}$.

- Autonomous nonlinear equation

- It has multiple equilibria,
  - at $[q \;\; \dot{q}]^T = [n\pi \;\; 0]^T$ for $n = \cdots, -2, -1, 0, 1, 2, \cdots$
  - the origin is an equilibrium

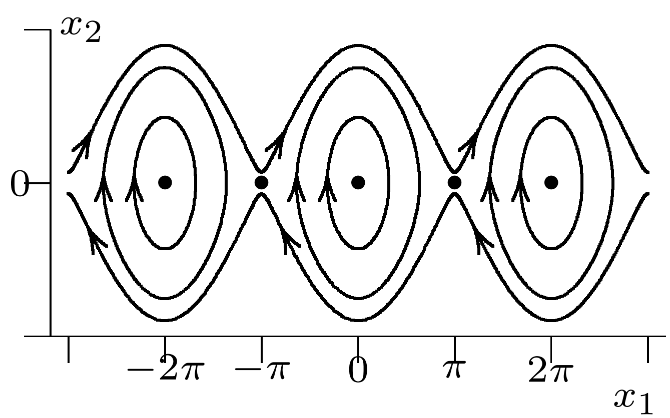

Figure 17: Pendulum: Phase plane

- Lyapunov function candidate

$$V(x_1, x_2) = mgl\ \ [1 - \cos(x_1)] + J\ \frac{x_2^2}{2}.$$

which is locally positive definite

- Total time derivative of $V(x_1, x_2)$

$$\begin{aligned} \dot{V}(x_1, x_2) &= mgl\ \sin(x_1)\dot{x}_1 + Jx_2\dot{x}_2 \\ &= 0\,. \end{aligned}$$

- According with Theorem 2.2 the origin is a stable equilibrium.

Theorem 2.4 —Global (uniform) asymptotic stability

The origin of $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ is *globally asymptotically stable* if there exists

- $V(t, \boldsymbol{x}) \geq W_1(\boldsymbol{x}) > 0, \qquad \forall\, t \geq 0 \quad \forall\, \boldsymbol{x} \in \mathrm{IR}^n$
  - and $V(t, \mathbf{0}) = 0\ \forall\ t\ \geq 0$ (positive definite condition)
  - with $W_1(\boldsymbol{x}) \to \infty \qquad \text{as} \qquad \| \boldsymbol{x} \| \to \infty$ (radially unbounded)
- $\dot{V}(t, \boldsymbol{x}) \leq W_3(\boldsymbol{x}) < 0 \qquad \forall\, t \geq 0 \quad \forall\, \boldsymbol{x} \in \mathrm{IR}^n$.
  - and $\dot{V}(t, \mathbf{0}) = 0\ \forall\ t\ \geq 0$ (negative definite condition)
- If moreover $V(t, x) \leq W_2(\boldsymbol{x}) \qquad \forall\, t \geq 0 \quad \forall\, \boldsymbol{x} \in \mathrm{IR}^n$ (decrescent), the origin is globally uniformly asymptotically stable.

$W_1(\boldsymbol{x})$, $W_2(\boldsymbol{x})$ and $W_3(\boldsymbol{x})$ are positive definite functions.

Theorem 2.5 —Global exponential stability.

The origin of $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ is *globally exponentially stable* if there exists a Lyapunov function candidate $V(t, \boldsymbol{x})$ and positive constants α, β, γ and $p \geq 1$ such that:

- $\alpha \|\boldsymbol{x}\|^p \leq V(t, \boldsymbol{x}) \leq \beta \|\boldsymbol{x}\|^p$,
- $\dot{V}(t, \boldsymbol{x}) \leq -\gamma \|\boldsymbol{x}\|^p \quad \forall\, t \geq t_\circ \geq 0 \quad \forall\, \boldsymbol{x} \in \mathrm{IR}^n$.

If all the above conditions hold only for small $\|\boldsymbol{x}\|$ then we say that the origin is an exponentially stable equilibrium.

Theorem 2.6

Consider the differential equation

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n, \quad t \in \mathbb{R}_+ \\
\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n.
\end{aligned}
$$

The unicity of an existing equilibrium point is necessary for the following properties (or, in other words, the following properties imply the unicity of an equilibrium)

- Global asymptotic stability,
- Global exponential stability.

Theorem 2.7 —La Salle

Consider the autonomous differential equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, whose origin $\boldsymbol{x} = \mathbf{0} \in \mathrm{IR}^n$ is an equilibrium. Assume that there exists

- a globally positive definite and radially unbounded Lyapunov function candidate $V(\boldsymbol{x})$, such that
- $\dot{V}(\boldsymbol{x}) \leq 0 \quad \forall\ \boldsymbol{x} \in \mathrm{IR}^n$.

Define the set $\Omega = \left\{ \boldsymbol{x} \in \mathrm{IR}^n : \dot{V}(\boldsymbol{x}) = 0 \right\}$. If

- $\boldsymbol{x}(0) = \mathbf{0}$ is the only initial state in Ω, such that $\boldsymbol{x}(t) \in \Omega$ for all $t \geq 0$,
- then the origin $\boldsymbol{x} = \mathbf{0} \in \mathrm{IR}^n$ is globally asymptotically stable.

Corollary 2.1 —Simplified La Salle.
Consider the set of autonomous differential equations

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}_x(\boldsymbol{x}, \boldsymbol{z}), \qquad & \boldsymbol{x} \in \mathbb{R}^n \\ \dot{\boldsymbol{z}} &= \boldsymbol{f}_z(\boldsymbol{x}, \boldsymbol{z}), \qquad & \boldsymbol{z} \in \mathbb{R}^m . \end{aligned}$$

where $\boldsymbol{f}_x(\mathbf{0}, \mathbf{0}) = \boldsymbol{f}_z(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. That is, the origin is an equilibrium point. Let $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ be globally positive definite and radially unbounded in both arguments. Assume that there exists a globally positive definite function $W : \mathbb{R}^m \to \mathbb{R}_+$ such that

$$\dot{V}(\boldsymbol{x}, \boldsymbol{z}) = -W(\boldsymbol{z}) .$$

If $\boldsymbol{x} = \mathbf{0}$ is the only solution of $\boldsymbol{f}_z(\boldsymbol{x}, \mathbf{0}) = \mathbf{0}$ then the origin $[\boldsymbol{x}^T \, \boldsymbol{z}^T]^T = \mathbf{0}$ is globally asymptotically stable.

Lemma 2.2 Consider the continuously differentiable functions $\boldsymbol{x} : \mathrm{IR}_+ \to \mathrm{IR}^n$, $\boldsymbol{z} : \mathrm{IR}_+ \to \mathrm{IR}^m$, $h : \mathrm{IR}_+ \to \mathrm{IR}_+$ and $P : \mathrm{IR}_+ \to \mathrm{IR}^{(n+m)\times(n+m)}$. Assume $P(t) = P(t)^T > 0$ for each $t \in \mathrm{IR}_+$.

- Define: $V(t, \boldsymbol{x}, \boldsymbol{z}, h) = \begin{bmatrix} \boldsymbol{x} \\ \\ \boldsymbol{z} \end{bmatrix}^T P(t) \begin{bmatrix} \boldsymbol{x} \\ \\ \boldsymbol{z} \end{bmatrix} + h(t) \geq 0\,.$

- If $\dot{V}(t, \boldsymbol{x}, \boldsymbol{z}, h) = - \begin{bmatrix} \boldsymbol{x} \\ \\ \boldsymbol{z} \end{bmatrix}^T \begin{bmatrix} Q(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \\ \boldsymbol{z} \end{bmatrix} \leq 0$

  for all $t \in \mathrm{IR}_+$, $\boldsymbol{x} \in \mathrm{IR}^n$, $\boldsymbol{z} \in \mathrm{IR}^m$ and $h \in \mathrm{IR}_+$, where $Q(t) = Q(t)^T > 0$ for all $t \geq 0$, then,

1. $\boldsymbol{x}(t)$, $\boldsymbol{z}(t)$ and $h(t)$ are bounded for all $t \geq 0$ and
2. $\boldsymbol{x}(t)$ is square-integrable, i.e.,

$$\int_0^\infty \|\boldsymbol{x}(t)\|^2 \, dt < \infty .$$

If moreover $\dot{\boldsymbol{x}}$ is also bounded then we have that

$$\lim_{t \to \infty} \boldsymbol{x}(t) = \mathbf{0} .$$

Ch. 3. Robot dynamics

We consider robot manipulators formed by an open *kinematic* chain.

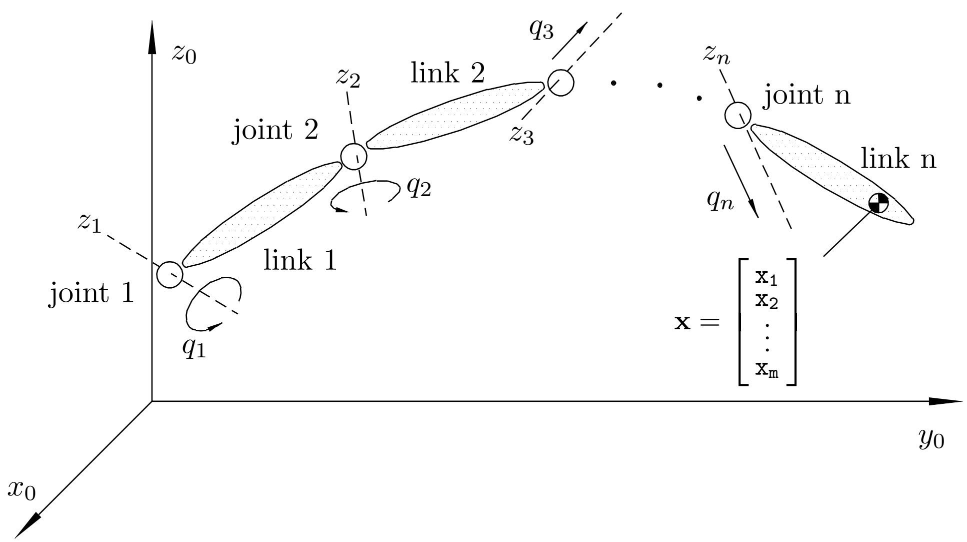

Figure 18: Abstract diagram of an n-DOF robot manipulator.

The *generalized* joint coordinate q_i is the angular displacement around z_i (revolute) or linear displacement along z_i (prismatic).

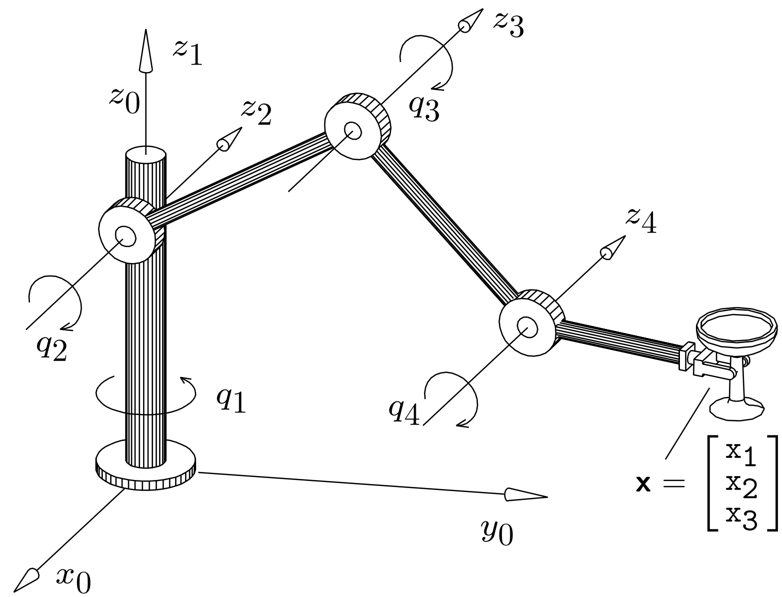

Figure 19: Example of a 3-DOF robot.

The vector of joint positions $\boldsymbol{q}$ has n elements

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \in \mathrm{IR}^n.$$

The position and orientation of the robot's end-effector, are collected in the vector $\mathbf{x}$ of *operational* positions .

$$\mathbf{x} = \begin{bmatrix} \mathtt{x}_1 \\ \mathtt{x}_2 \\ \vdots \\ \mathtt{x}_\mathtt{m} \end{bmatrix}$$

where $m \leq n$.

The *direct kinematic model* describes the relation betwwen $\boldsymbol{q}$ and $\mathbf{x}$. It is a function $\boldsymbol{\varphi} : \mathbb{R}^n \to \mathbb{R}^m$ such that

- $\mathbf{x} = \boldsymbol{\varphi}(\boldsymbol{q})$.

The *inverse kinematic model* consists on the inverse relation of the direct kinematic model

- $\boldsymbol{q} = \boldsymbol{\varphi}^{-1}(\mathbf{x})$.
- The computation may be highly complex.

The *dynamic model* of a robot consists of an ordinary differential equation.

- In general, these are second order nonlinear models

$$\boldsymbol{f}_{EL}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},\boldsymbol{\tau}) = \mathbf{0}\,, \tag{4}$$

$$\boldsymbol{f}_{C}(\mathsf{x},\dot{\mathsf{x}},\ddot{\mathsf{x}},\boldsymbol{\tau}) = \mathbf{0}\,. \tag{5}$$

- $\boldsymbol{\tau}$ stands for the forces and torques applied.
- The dynamic model (4) is called joint dynamic model,
- while (5) corresponds to the operational dynamic model.
- We focus on the joint dynamic model.

Lagrange's equations of motion

Now we describe how to derive the dynamic model of a robot.

Consider the robot manipulator with n links depicted in Figure 20.

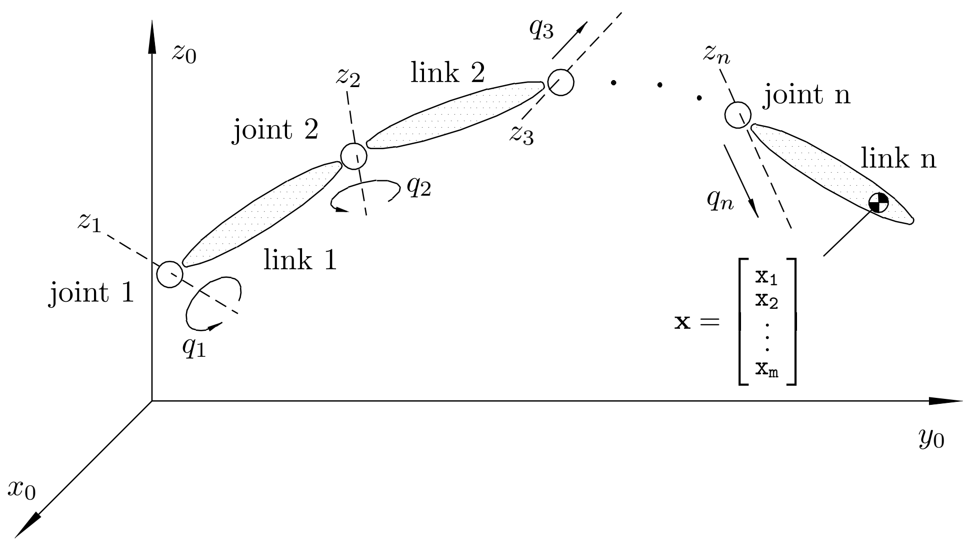

Figure 20: Abstract diagram of an n-DOF robot manipulator.

The *Lagrangian* $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is defined as

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{K}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q}). \tag{6}$$

where

- $\mathcal{K}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is the kinetic energy function
- $\mathcal{U}(\boldsymbol{q})$ is the potential energy function (we assume only conservative forces)

The total energy $\mathcal{E}$ of a robot manipulator of n DOF is

$$\mathcal{E}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{K}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathcal{U}(\boldsymbol{q})$$

where $\boldsymbol{q} = [q_1, \cdots, q_n]^T$.

The equations of motion of Lagrange for a manipulator of n DOF, are given by

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}\right] - \frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{q}} = \boldsymbol{\tau},$$

or in the equivalent form by

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{q}_i}\right] - \frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial q_i} = \tau_i, \qquad i = 1, \cdots, n \tag{7}$$

where

- τ_i correspond to the external as well as to other non conservative forces and torques at each joint.

The use of Lagrange's equations in the derivation of the robot dynamics can be reduced to four main stages:

1. Computation of the kinetic energy function $\mathcal{K}(\boldsymbol{q}, \dot{\boldsymbol{q}})$.

2. Computation of the potential energy function $\mathcal{U}(\boldsymbol{q})$.

3. Computation of the Lagrangian (6) $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})$.

4. Development of Lagrange's equations (7).

Example 3.3

Consider the robot manipulator with 2 DOF shown in Figure 21.

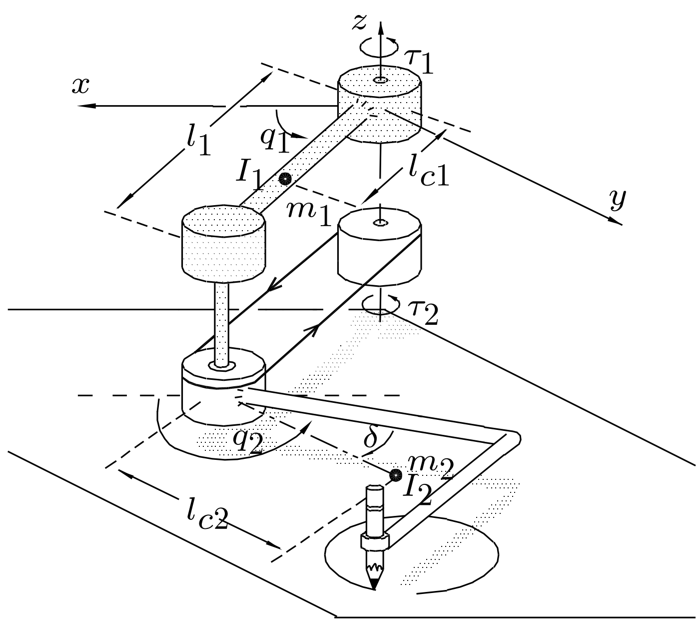

Figure 21: 2-DOF robot.

where,

- l_1 is the length the first link
- m_1 and m_2 are the masses of the links
- l_{c1} and l_{c2} are the distances of the rotating axes to the centers of their respective mass
- I_1 and I_2 denote the moments of inertia of the links
- $\boldsymbol{q} = [q_1 \;\; q_2]^T$ defines the vector of joint positions

Notice that the center of mass of link 2 may be physically placed "out" of the link itself! This is determined by the value of the constant angle δ

The kinetic energy function $\mathcal{K}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ may be decomposed by

- $\mathcal{K}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{K}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathcal{K}_2(\boldsymbol{q}, \dot{\boldsymbol{q}})$,

  where

  - $\mathcal{K}_1(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is the kinetic energy associated to the mass m_1
  - $\mathcal{K}_2(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is the kinetic energy associated to the mass m_2
    with
    * $\mathcal{K}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2} m_1 l_{c1}^2 \dot{q}_1^2 + \frac{1}{2} I_1 \dot{q}_1^2$
    * $$\begin{aligned} \mathcal{K}_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \frac{m_2}{2} l_1^2 \dot{q}_1^2 + \frac{m_2}{2} l_{c2}^2 \dot{q}_2^2 + m_2 l_1 l_{c2} \cos(q_1 - q_2 + \delta) \dot{q}_1 \dot{q}_2 \\ &\quad + \frac{1}{2} I_2 \dot{q}_2^2 \,. \end{aligned}$$

The Lagrangian $\mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \mathcal{K}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q})$, takes the form

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \mathcal{K}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) + \mathcal{K}_2(\boldsymbol{q},\dot{\boldsymbol{q}}) \\
&= \frac{1}{2}(m_1 l_{c1}^2 + m_2 l_1^2)\dot{q}_1^2 + \frac{1}{2} m_2 l_{c2}^2 \dot{q}_2^2 \\
&\quad + m_2 l_1 l_{c2} \cos(q_1 - q_2 + \delta)\dot{q}_1 \dot{q}_2 \\
&\quad + \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2} I_2 \dot{q}_2^2 .
\end{aligned}
$$

By using Lagrange's equations $\frac{d}{dt}\left[\frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{q}_i}\right] - \frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial q_i} = \tau_i, \qquad i = 1, \cdots, n$
we have

$$\begin{aligned}
\tau_1 &= \left[m_1 l_{c1}^2 + m_2 l_1^2 + I_1\right] \ddot{q}_1 \\
&\quad + m_2 l_1 l_{c2} \cos(q_1 - q_2 + \delta)\ddot{q}_2 \\
&\quad + m_2 l_1 l_{c2} \sin(q_1 - q_2 + \delta)\dot{q}_2^2
\end{aligned}$$

and,

$$\begin{aligned}
\tau_2 &= m_2 l_1 l_{c2} \cos(q_1 - q_2 + \delta)\ddot{q}_1 \\
&\quad + [m_2 l_{c2}^2 + I_2]\ddot{q}_2 \\
&\quad - m_2 l_1 l_{c2} \sin(q_1 - q_2 + \delta)\dot{q}_1^2 \,.
\end{aligned}$$

Denoting $C_{21} = \cos(q_2 - q_1)$, $S_{21} = \sin(q_2 - q_1)$, one obtains

$$\begin{aligned} \tau_1 &= \left[m_1 l_{c1}^2 + m_2 l_1^2 + I_1\right] \ddot{q}_1 \\ &\quad + [m_2 l_1 l_{c2} \cos(\delta) C_{21} + m_2 l_1 l_{c2} \sin(\delta) S_{21}] \ddot{q}_2 \\ &\quad + [-m_2 l_1 l_{c2} \cos(\delta) S_{21} + m_2 l_1 l_{c2} \sin(\delta) C_{21}] \dot{q}_2^2 \end{aligned} \tag{8}$$

and

$$\begin{aligned} \tau_2 &= [m_2 l_1 l_{c2} \cos(\delta) C_{21} + m_2 l_1 l_{c2} \sin(\delta) S_{21}] \ddot{q}_1 \\ &\quad + [m_2 l_{c2}^2 + I_2] \ddot{q}_2 \\ &\quad + [m_2 l_1 l_{c2} \cos(\delta) S_{21} - m_2 l_1 l_{c2} \sin(\delta) C_{21}] \dot{q}_1^2 \end{aligned} \tag{9}$$

which are of the form (4) with

$$\boldsymbol{f}_{EL}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \boldsymbol{\tau}) = \begin{bmatrix} \tau_1 - \text{RHS(8)} \\ \tau_2 - \text{RHS(9)} \end{bmatrix}$$

where RHS(8) and RHS(9) denote the terms on the right hand side of (8) and (9) respectively.

Example 3.4

Consider the 3-DOF Cartesian robot manipulator shown in Figure 22.

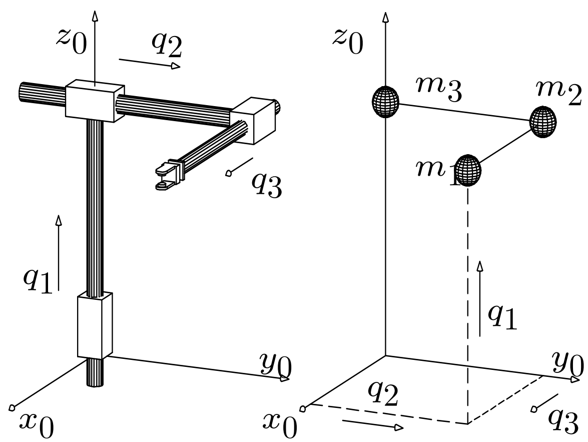

Figure 22: 3-DOF robot.

- $\mathcal{K}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\left[m_1\dot{q}_3^2 + [m_1+m_2]\dot{q}_2^2 + [m_1+m_2+m_3]\dot{q}_1^2\right]$.
- $\mathcal{U}(\boldsymbol{q}) = [m_1+m_2+m_3]gq_1$.

We obtain the Lagrangian as:

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \mathcal{K}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q}) \\
&= \frac{1}{2}\left[m_1\dot{q}_3^2 + [m_1+m_2]\dot{q}_2^2 + [m_1+m_2+m_3]\dot{q}_1^2\right] \\
&\quad - [m_1+m_2+m_3]gq_1\,,
\end{aligned}
$$

hence, the dynamic equations result

$$
\begin{aligned}
[m_1+m_2+m_3]\ddot{q}_1 + [m_1+m_2+m_3]g &= \tau_1 \\
[m_1+m_2]\ddot{q}_2 &= \tau_2 \\
m_1\ddot{q}_3 &= \tau_3
\end{aligned}
$$

In terms of the state vector $[q_1 \quad q_2 \quad q_3 \quad \dot{q}_1 \quad \dot{q}_2 \quad \dot{q}_3]$, above equations may be expressed as

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dfrac{1}{m_1+m_2+m_3}[\tau_1 - [m_1+m_2+m_3]g] \\ \dfrac{1}{m_1+m_2}\tau_2 \\ \dfrac{1}{m_1}\tau_3 \end{bmatrix}.$$

The necessary and sufficient condition for the existence of equilibria is

- $\tau_1 = [m_1 + m_2 + m_3]g$,
- $\tau_2 = 0$ and
- $\tau_3 = 0$

Actually we have an infinite number of them:

$$[q_1 \;\; q_2 \;\; q_3 \;\; \dot{q}_1 \;\; \dot{q}_2 \;\; \dot{q}_3]^T = [q_1^* \;\; q_2^* \;\; q_3^* \;\; 0 \;\; 0 \;\; 0]^T$$

with q_1^*, q_2^*, $q_3^* \in \mathrm{IR}$.

Dynamic model in compact form

Consider a robot manipulator of n DOF composed of rigid links interconnected by frictionless joints.

- $\mathcal{K}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}}$ expresses the kinetic energy

  where

  ⋆ $M(\boldsymbol{q}) = M(\boldsymbol{q})^T > 0$ for all $\boldsymbol{q} \in \mathbb{R}^n$ is the $n \times n$ **inertia matrix**.

- $\mathcal{U}(\boldsymbol{q})$ denotes the potential energy

- The Lagrangian results

$$\mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}} - \mathcal{U}(\boldsymbol{q})\,.$$

The Lagrange's equations of motion

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}\right] - \frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{q}} = \boldsymbol{\tau},$$

may be written as

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{\boldsymbol{q}}}\left[\frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}}\right]\right] - \frac{\partial}{\partial \boldsymbol{q}}\left[\frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}}\right] + \frac{\partial \mathcal{U}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \boldsymbol{\tau}\,.$$

which takes the compact form,

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

where

$$
\begin{aligned}
C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} &= \dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}\left[\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}}\right] \\
\boldsymbol{g}(\boldsymbol{q}) &= \frac{\partial \mathcal{U}(\boldsymbol{q})}{\partial \boldsymbol{q}}.
\end{aligned}
$$

- $C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ is a vector of dimension n called **vector of centrifugal and Coriolis forces**,
- $\boldsymbol{g}(\boldsymbol{q})$ is a vector of dimension n of **gravitational forces or torques** and
- $\boldsymbol{\tau}$ is a vector of dimension n called the **vector of external forces**

$C(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathrm{IR}^{n \times n}$ may be not unique, but $C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ is indeed unique.

- One way to obtain $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is through the Christoffel symbols

$$c_{ijk}(\boldsymbol{q}) = \frac{1}{2}\left[\frac{\partial M_{kj}(\boldsymbol{q})}{\partial q_i} + \frac{\partial M_{ki}(\boldsymbol{q})}{\partial q_j} - \frac{\partial M_{ij}(\boldsymbol{q})}{\partial q_k}\right] .$$

  – The kjth element of the matrix $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$, $C_{kj}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, is given by

$$C_{kj}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} c_{1jk}(\boldsymbol{q}) \\ c_{2jk}(\boldsymbol{q}) \\ \vdots \\ c_{njk}(\boldsymbol{q}) \end{bmatrix}^T \dot{\boldsymbol{q}} \, .$$

The model $M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$ may be viewed as

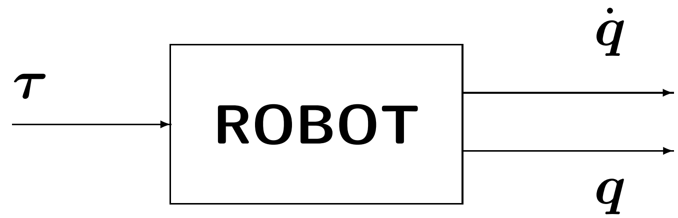

Figure 23: Input-output representation of a robot.

Example 3.6

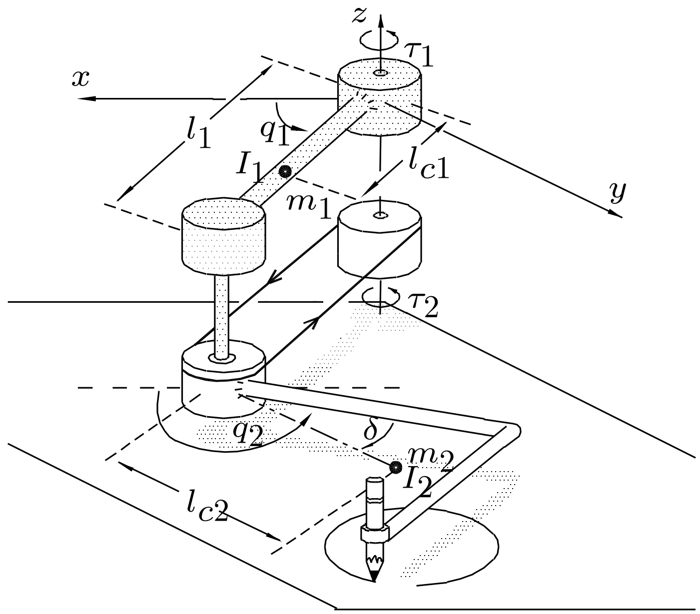

Figure 24: 2-DOF robot.

The dynamic model, for this robot, in compact form is

$$\underbrace{\begin{bmatrix} M_{11}(\boldsymbol{q}) & M_{12}(\boldsymbol{q}) \\ M_{21}(\boldsymbol{q}) & M_{22}(\boldsymbol{q}) \end{bmatrix}}_{M(\boldsymbol{q})} \ddot{\boldsymbol{q}} + \underbrace{\begin{bmatrix} C_{11}(\boldsymbol{q},\dot{\boldsymbol{q}}) & C_{12}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ C_{21}(\boldsymbol{q},\dot{\boldsymbol{q}}) & C_{22}(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{bmatrix}}_{C(\boldsymbol{q},\dot{\boldsymbol{q}})} \dot{\boldsymbol{q}} \quad = \quad \boldsymbol{\tau}(t)$$

where

$$\begin{aligned} M_{11}(\boldsymbol{q}) &= \left[m_1 l_{c1}^2 + m_2 l_1^2 + I_1\right] \\ M_{12}(\boldsymbol{q}) &= \left[m_2 l_1 l_{c2} \cos(\delta) C_{21} + m_2 l_1 l_{c2} \sin(\delta) S_{21}\right] \\ M_{21}(\boldsymbol{q}) &= \left[m_2 l_1 l_{c2} \cos(\delta) C_{21} + m_2 l_1 l_{c2} \sin(\delta) S_{21}\right] \\ M_{22}(\boldsymbol{q}) &= \left[m_2 l_{c2}^2 + I_2\right] \end{aligned}$$

$$\begin{aligned} C_{11}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= 0 \\ C_{12}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= [-m_2 l_1 l_{c2}\cos(\delta)S_{21} + m_2 l_1 l_{c2}\sin(\delta)C_{21}]\dot{q}_2 \\ C_{21}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= [m_2 l_1 l_{c2}\cos(\delta)S_{21} - m_2 l_1 l_{c2}\sin(\delta)C_{21}]\dot{q}_1 \\ C_{22}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= 0 \end{aligned}$$

That is, the gravitational forces vector is zero.

The dynamic model $M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$ in terms of the *state vector* $[\boldsymbol{q}^T \;\; \dot{\boldsymbol{q}}^T]^T$ can be expressed as

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{q}\\ \\ \dot{\boldsymbol{q}}\end{bmatrix} = \begin{bmatrix}\dot{\boldsymbol{q}}\\ \\ M(\boldsymbol{q})^{-1}\left[\boldsymbol{\tau}(t) - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right]\end{bmatrix}. \tag{10}$$

- The necessary and sufficient condition for the existence of equilibria is
  - ⋆ $\boldsymbol{\tau}(t)$ be constant (say, τ^*)
  - ⋆ and there exist a solution $\boldsymbol{q}^* \in \mathrm{IR}^n$ to $\boldsymbol{g}(\boldsymbol{q}^*) = \boldsymbol{\tau}$.

- The equilibria are given by $[\boldsymbol{q}^T \;\; \dot{\boldsymbol{q}}^T]^T = [\boldsymbol{q}^{*T} \;\; \boldsymbol{0}^T]^T \in \mathrm{IR}^{2n}$.

- When $\boldsymbol{\tau} \equiv 0$, the possible equilibria of (10) are given by $[\boldsymbol{q}^T \quad \dot{\boldsymbol{q}}^T]^T = [\boldsymbol{q}^{*T} \quad \boldsymbol{0}^T]^T$

  where

  ⋆ $\boldsymbol{q}^*$ is a solution of $\boldsymbol{g}(\boldsymbol{q}^*) = \boldsymbol{0}$

  ⋆ As $\boldsymbol{g}(\boldsymbol{q}) = \frac{\partial \mathcal{U}(\boldsymbol{q})}{\partial \boldsymbol{q}}$, we have that $\boldsymbol{q}^*$ corresponds to the vectors where the potential energy possesses extrema.

Dynamic model of robots with actuators

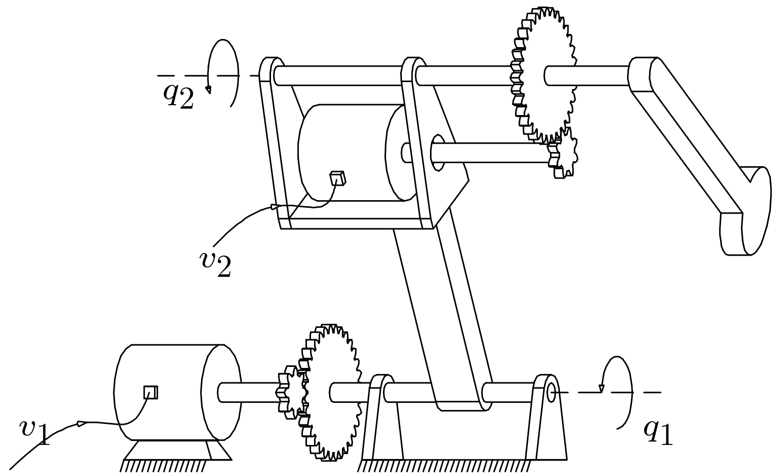

Figure 25: 2-DOF robot.

On a real robot manipulator the torques vector $\boldsymbol{\tau}$, is delivered by actuators

- electromechanical,
- pneumatic or
- hydraulic.

They have their own dynamics,

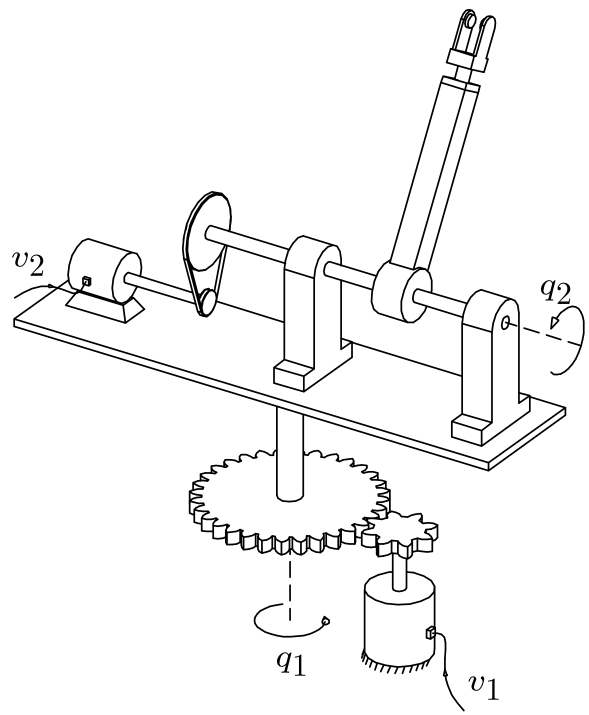

Figure 26: 2-DOF robot.

Actuators with linear dynamics

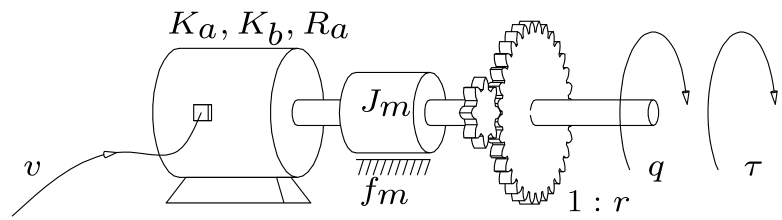

Figure 27: Diagram of a DC motor.

Simplified linear dynamic model of a DC motor

$$J_m\ddot{q} + f_m\dot{q} + \frac{K_aK_b}{R_a}\dot{q} + \frac{\tau}{r^2} = \frac{K_a}{rR_a}v$$

where:

- J_m : rotor's inertia [kg m^2],
- K_a : motor-torque constant [N m/A],
- R_a : armature resistance [Ω],
- K_b : back emf [V s/rad],
- f_m : rotor's friction coefficient with respect to its hinges [N m],
- τ : net torque applied after the set of gears at the load's axis [N m],
- q : angular position of the load's axis [rad],
- r : gear reduction ratio (in general $r \gg 1$),
- v : armature voltage [V].

We can obtain

$$J\ddot{\boldsymbol{q}} + B\dot{\boldsymbol{q}} + R\boldsymbol{\tau} = K\boldsymbol{v}$$

with

$$J = \mathrm{diag}\left\{J_{m_i}\right\}$$

$$B = \mathrm{diag}\left\{f_{m_i} + \left(\frac{K_a K_b}{R_a}\right)_i\right\}$$

$$R = \mathrm{diag}\left\{\frac{1}{r_i^2}\right\}$$

$$K = \mathrm{diag}\left\{\left(\frac{K_a}{R_a}\right)_i \frac{1}{r_i}\right\}$$

The complete dynamic model of a manipulator (considering friction in the joints) including its actuators is

$$(R\ M(q) + J)\,\ddot{\boldsymbol{q}} + R\ C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + R\ \boldsymbol{g}(\boldsymbol{q}) + R\ \boldsymbol{f}(\dot{\boldsymbol{q}}) + B\ \dot{\boldsymbol{q}} = K\boldsymbol{v}\,. \quad (11)$$

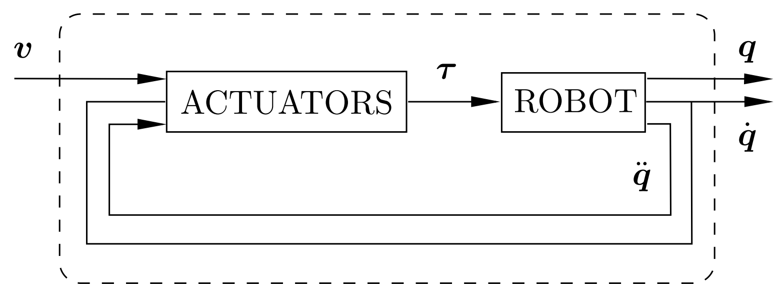

Figure 28: Block-diagram of a robot with its actuators.

Example 3.8

Consider the pendulum depicted in Figure 29.

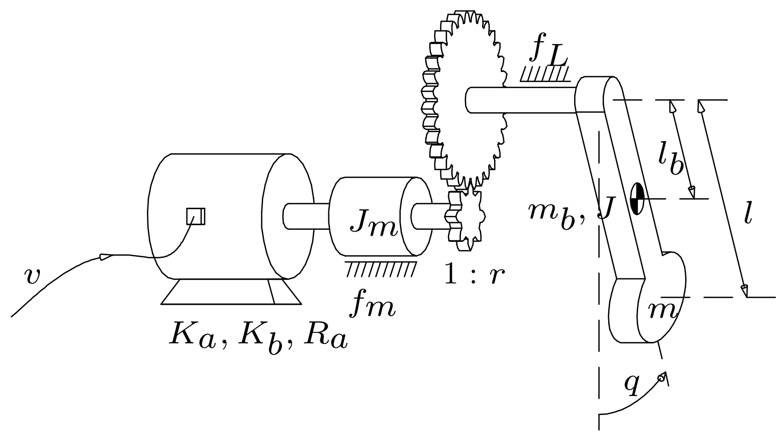

Figure 29: Pendular device with a DC motor.

The equation of motion for this device including its load is given by

$$J\ddot{q} + f_L\dot{q} + [m_b l_b + ml]\, g \sin(q) = \tau$$

where

- J : arm's inertia without load (i.e., with m=0),
- m_b : arm's mass (without load),
- l_b : distance from the rotating axis to the arm's center of mass (without load),
- m : load's mass (assumed to be punctual),
- l : distance from the rotating axis to the load m,
- g : gravity acceleration,
- τ : applied torque at the rotating axis,
- f_L : friction coefficient of the arm and its load.

The equation above may also be written in the compact form

$$J_L \ddot{q} + f_L \dot{q} + k_L \sin(q) = \tau$$

where

- $J_L = J + ml^2$
- $k_L = [m_b l_b + ml] g \, .$

The complete dynamic model of the pendular device may be obtained as:

$$\left[J_m + \frac{J_L}{r^2}\right]\ddot{q} + \left[f_m + \frac{f_L}{r^2} + \frac{K_a K_b}{R_a}\right]\dot{q} + \frac{k_L}{r^2}\sin(q) = \frac{K_a}{rR_a}v\,,$$

where, we identify

$$\begin{aligned} M(q) &= J_L & J &= J_m \\ R &= \frac{1}{r^2} & K &= \frac{K_a}{rR_a} \\ B &= f_m + \frac{K_a K_b}{R_a} & f(\dot{q}) &= f_L\dot{q} \\ C(q,\dot{q}) &= 0 & g(q) &= k_L\sin(q)\,. \end{aligned}$$

When the gear ratios r_i are sufficiently large, the robot-with-actuators equation (11) may be approximated by

$$J\ddot{\boldsymbol{q}} + B\dot{\boldsymbol{q}} \approx K\boldsymbol{v}\,.$$

In the opposite case, the equation (11) may be rewritten as

$$M'(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{f}(\dot{\boldsymbol{q}}) + R^{-1}B\dot{\boldsymbol{q}} = R^{-1}K\boldsymbol{v}$$

where $M'(\boldsymbol{q}) = M(\boldsymbol{q}) + R^{-1}J\,.$

In terms of the state vector $[\boldsymbol{q} \quad \dot{\boldsymbol{q}}]$ we have

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ M'(\boldsymbol{q})^{-1}\left[R^{-1}\ K\boldsymbol{v} - R^{-1}\ B\dot{\boldsymbol{q}} - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{f}(\dot{\boldsymbol{q}})\right] \end{bmatrix}.$$

- The problem of motion control is:
  - given $\boldsymbol{q}_d$, $\dot{\boldsymbol{q}}_d$ and $\ddot{\boldsymbol{q}}_d$,
  - determine $\boldsymbol{v}$, to be applied to the motors,
  - in such a manner that $\boldsymbol{q}$ follow precisely $\boldsymbol{q}_d$.

Ch. 4. Properties of the dynamic model

Dynamic model for n-DOF robots

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}\,.$$

These properties, may be classified as follows.

- Properties of the inertia matrix $M(\boldsymbol{q})$,
- Properties of the centrifugal and Coriolis forces matrix $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$,
- Properties of the gravitational forces and torques vector $\boldsymbol{g}(\boldsymbol{q})$,
- Residual dynamics.

Properties of the inertia matrix

Property 4.1 —Inertia matrix $M(\boldsymbol{q})$

The inertia matrix $M(\boldsymbol{q}) = M(\boldsymbol{q})^T > 0 \ \in \mathbb{R}^{n\times n}$, satisfies the following properties:

1. There exists a real positive number α such that

$$M(\boldsymbol{q}) \geq \alpha I \qquad \forall\ \boldsymbol{q} \in \mathbb{R}^n$$

where I denotes the identity matrix of dimension $n \times n$. The matrix $M(\boldsymbol{q})^{-1}$ exists and is positive definite.

2.

For robots having only revolute joints there exists a constant $\beta > 0$ such that

$$\lambda_{\mathrm{Max}}\{M(\boldsymbol{q})\} \leq \beta \qquad \forall\, \boldsymbol{q} \in \mathrm{IR}^n\,.$$

One way of computing β is

$$\beta \geq n \left(\max_{i,j,q} |M_{ij}(\boldsymbol{q})| \right)$$

where $M_{ij}(\boldsymbol{q})$ stands for the ijth element of the matrix $M(\boldsymbol{q})$.

3. For robots having only revolute joints there exists a constant $k_M > 0$ such that

$$\|M(\boldsymbol{x})\boldsymbol{z} - M(\boldsymbol{y})\boldsymbol{z}\| \leq k_M \; \|\boldsymbol{x} - \boldsymbol{y}\| \, \|\boldsymbol{z}\|$$

for all vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathrm{IR}^n$. One simple way to determine k_M is as follows.

$$k_M \geq n^2 \; \left(\max_{i,j,k,q} \left| \frac{\partial M_{ij}(\boldsymbol{q})}{\partial q_k} \right| \right) .$$

4. For robots having only revolute joints there exists a number $k'_M > 0$ such that

$$\|M(\boldsymbol{x})\boldsymbol{y}\| \leq k'_M \|\boldsymbol{y}\|$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{IR}^n$.

The centrifugal and Coriolis forces matrix

Property 4.2 —Coriolis matrix $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$

The *centrifugal and Coriolis* forces matrix $C(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathrm{IR}^{n\times n}$ satisfies the following.

1. For a given manipulator, the matrix $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ may be not unique but the vector $C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ is unique.

2. $C(\boldsymbol{q}, \boldsymbol{0}) = 0$ for all vectors $\boldsymbol{q} \in \mathrm{IR}^n$.

3. For all vectors $\boldsymbol{q}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathrm{IR}^n$ and scalars α we have that

$$\begin{aligned} C(\boldsymbol{q}, \boldsymbol{x})\boldsymbol{y} &= C(\boldsymbol{q}, \boldsymbol{y})\boldsymbol{x} \\ C(\boldsymbol{q}, \boldsymbol{z} + \alpha\boldsymbol{x})\boldsymbol{y} &= C(\boldsymbol{q}, \boldsymbol{z})\boldsymbol{y} + \alpha C(\boldsymbol{q}, \boldsymbol{x})\boldsymbol{y} \,. \end{aligned}$$

4. The vector $C(\boldsymbol{q},\boldsymbol{x})\boldsymbol{y}$ may be written in the form

$$C(\boldsymbol{q},\boldsymbol{x})\boldsymbol{y} = \begin{bmatrix} \boldsymbol{x}^T C_1(\boldsymbol{q})\boldsymbol{y} \\ \boldsymbol{x}^T C_2(\boldsymbol{q})\boldsymbol{y} \\ \vdots \\ \boldsymbol{x}^T C_n(\boldsymbol{q})\boldsymbol{y} \end{bmatrix}$$

where $C_k(\boldsymbol{q})$ are symmetric matrices of dimension $n \times n$ for all $k = 1, 2, \cdots, n$. The ij-th element $C_{k_{ij}}(\boldsymbol{q})$ of the matrix $C_k(\boldsymbol{q})$ corresponds to the so-called Christoffel symbol of the first kind $c_{jik}(\boldsymbol{q})$ and which is defined as $c_{ijk}(\boldsymbol{q}) = \frac{1}{2}\left[\frac{\partial M_{kj}(\boldsymbol{q})}{\partial q_i} + \frac{\partial M_{ki}(\boldsymbol{q})}{\partial q_j} - \frac{\partial M_{ij}(\boldsymbol{q})}{\partial q_k}\right]$.

5. For robots having exclusively revolute joints, there exists a number $k_{C_1} > 0$ such that

$$\|C(\boldsymbol{q}, \boldsymbol{x})\boldsymbol{y}\| \leq k_{C_1} \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|$$

for all $\boldsymbol{q}, \boldsymbol{x}, \boldsymbol{y} \in \mathrm{IR}^n$.

6. For robots having exclusively revolute joints, there exist numbers $k_{C_1} > 0$ and $k_{C_2} > 0$ such that

$$\begin{aligned}\|C(\boldsymbol{x},\boldsymbol{z})\boldsymbol{w} - C(\boldsymbol{y},\boldsymbol{v})\boldsymbol{w}\| &\leq k_{C_1}\|\boldsymbol{z}-\boldsymbol{v}\|\ \|\boldsymbol{w}\| \\ &\quad + k_{C_2}\|\boldsymbol{x}-\boldsymbol{y}\|\,\|\boldsymbol{w}\|\,\|\boldsymbol{z}\|\end{aligned}$$

for all vector $\boldsymbol{v},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w} \in \mathrm{IR}^n$.

7. The matrix $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$, defined using the Christoffel symbol is related to the inertia matrix $M(\boldsymbol{q})$ by the expression

$$\boldsymbol{x}^T \left[\frac{1}{2}\dot{M}(\boldsymbol{q}) - C(\boldsymbol{q}, \dot{\boldsymbol{q}})\right] \boldsymbol{x} = 0 \qquad \forall\ \boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{x} \in \mathbb{R}^n$$

and as a matter of fact, $\frac{1}{2}\dot{M}(\boldsymbol{q}) - C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is skewsymmetric. Equivalently, the matrix $\dot{M}(\boldsymbol{q}) - 2C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is antisymmetric, and it is also true that

$$\dot{M}(\boldsymbol{q}) = C(\boldsymbol{q}, \dot{\boldsymbol{q}}) + C(\boldsymbol{q}, \dot{\boldsymbol{q}})^T .$$

Independently of the way in which $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is derived, it always satisfies

$$\dot{\boldsymbol{q}}^T \left[\frac{1}{2} \dot{M}(\boldsymbol{q}) - C(\boldsymbol{q}, \dot{\boldsymbol{q}}) \right] \dot{\boldsymbol{q}} = 0 \qquad \forall\ \boldsymbol{q}, \dot{\boldsymbol{q}} \in \mathrm{IR}^n .$$

The vector of gravitational torques

Property 4.3 —Gravity vector $g(q)$

The *gravitational torques vector* $\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^n$ is continuous and therefore bounded for each bounded $\boldsymbol{q}$ and also satisfies the following.

1. The vector $\boldsymbol{g}(\boldsymbol{q})$ and the velocity vector $\dot{\boldsymbol{q}}$ are correlated as

$$\int_0^T \boldsymbol{g}(\boldsymbol{q}(t))^T \dot{\boldsymbol{q}}(t)\, dt = \mathcal{U}(\boldsymbol{q}(T)) - \mathcal{U}(\boldsymbol{q}(0))$$

for all $T \in \mathbb{R}_+$.

2. For robots having only revolute joints there exists a number $k_{\mathcal{U}}$ such that

$$\int_0^T \boldsymbol{g}(\boldsymbol{q}(t))^T \dot{\boldsymbol{q}}(t)\ dt + \mathcal{U}(\boldsymbol{q}(0)) \geq k_{\mathcal{U}}$$

for all $T \in \mathbb{R}_+$ and where $k_{\mathcal{U}} = \min_{q} \{\mathcal{U}(\boldsymbol{q})\}$.

3. For robots having only revolute joints, the vector $\boldsymbol{g}(\boldsymbol{q})$ is Lipschitz, that is, there exists a constant $k_g > 0$ such that

$$\|\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y})\| \leq k_g \|\boldsymbol{x} - \boldsymbol{y}\|$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{IR}^n$. A simple way to compute k_g is by evaluating its partial derivative

$$k_g \geq n \left(\max_{i,j,q} \left| \frac{\partial g_i(\boldsymbol{q})}{\partial q_j} \right| \right) .$$

Furthermore, k_g satisfies

$$k_g \geq \left\| \frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} \right\| \geq \lambda_{\mathrm{Max}} \left\{ \frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} \right\} .$$

4. For robots having only revolute joints there exists a constant k' such that

$$\|\boldsymbol{g}(\boldsymbol{q})\| \leq k'$$

for all $\boldsymbol{q} \in \mathrm{IR}^n$.

Residual dynamics

The residual dynamics $\boldsymbol{h}(\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}})$ is defined as follows.

$$\begin{aligned}\boldsymbol{h}(\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) &= [M(\boldsymbol{q}_d) - M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}})]\,\ddot{\boldsymbol{q}}_d \\ &\quad + \left[C(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d) - C(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}_d - \dot{\tilde{\boldsymbol{q}}})\right]\dot{\boldsymbol{q}}_d \\ &\quad + \boldsymbol{g}(\boldsymbol{q}_d) - \boldsymbol{g}(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}),\end{aligned}$$

and with an abuse of notation it may be written as

$$\boldsymbol{h}(\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) = [M(\boldsymbol{q}_d) - M(\boldsymbol{q})]\,\ddot{\boldsymbol{q}}_d + [C(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d) - C(\boldsymbol{q}, \dot{\boldsymbol{q}})]\,\dot{\boldsymbol{q}}_d + \boldsymbol{g}(\boldsymbol{q}_d) - \boldsymbol{g}(\boldsymbol{q}).$$

Definition 4.1 —Vectorial tangent hyperbolic function

$$\mathbf{tanh}(\boldsymbol{x}) := \begin{bmatrix} \tanh(x_1) \\ \vdots \\ \tanh(x_n) \end{bmatrix} \tag{12}$$

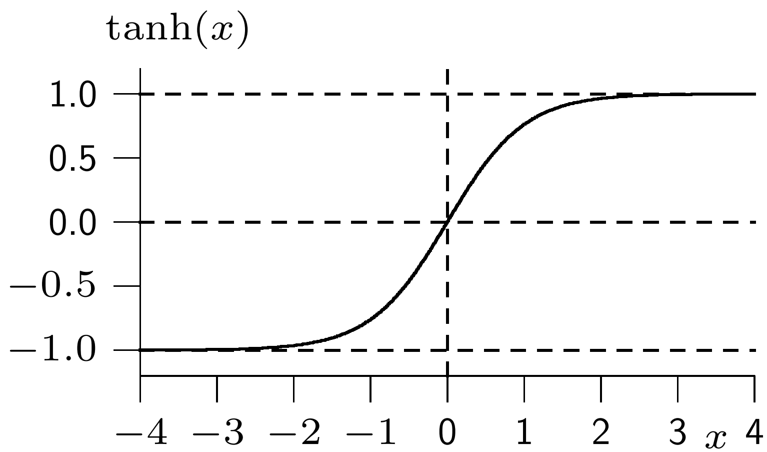

Figure 30: Tangent hyperbolic $\tanh(x)$.

The first partial derivative of $\mathbf{tanh}(\boldsymbol{x})$ is given by

$$\frac{\partial \mathbf{tanh}}{\partial \boldsymbol{x}}(\boldsymbol{x}) =: \mathsf{Sech}^2(\boldsymbol{x}) = \mathsf{diag}\{\mathsf{sech}^2(x_i)\}$$

where

$$\mathsf{sech}(x_i) := \frac{1}{e^{x_i} - e^{-x_i}}\,.$$

The vectorial tangent hyperbolic function satisfies the following properties. For any $\boldsymbol{x}, \dot{\boldsymbol{x}} \in \mathrm{IR}^n$

- $\|\mathbf{tanh}(\boldsymbol{x})\| \leq \alpha_1 \|\boldsymbol{x}\|$
- $\|\mathbf{tanh}(\boldsymbol{x})\| \leq \alpha_2$
- $\|\mathbf{tanh}(\boldsymbol{x})\|^2 \leq \alpha_3 \, \mathbf{tanh}(\boldsymbol{x})^T \boldsymbol{x}$
- $\left\|\mathsf{Sech}^2(\boldsymbol{x})\dot{\boldsymbol{x}}\right\| \leq \alpha_4 \|\dot{\boldsymbol{x}}\|$

where $\alpha_1, \cdots, \alpha_4 > 0$. With $\mathbf{tanh}(\boldsymbol{x})$ defined as in (12), the constants $\alpha_1 = 1, \alpha_2 = \sqrt{n}, \alpha_3 = 1, \alpha_4 = 1$.

Property 4.4 —Vector residual dynamics $\boldsymbol{h}(\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}})$

The *vector of residual dynamics* $\boldsymbol{h}(\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}})$ of $n \times 1$ depends on

- $\tilde{\boldsymbol{q}}$, $\dot{\tilde{\boldsymbol{q}}}$,
- $\boldsymbol{q}_d$, $\dot{\boldsymbol{q}}_d$, and $\ddot{\boldsymbol{q}}_d$ (supposed to be bounded)
  - We denote by $\|\dot{\boldsymbol{q}}_d\|_{\mathrm{M}}$ and $\|\ddot{\boldsymbol{q}}_d\|_{\mathrm{M}}$ the supreme values over the norms of the desired velocity and acceleration.

In addition, $\boldsymbol{h}(\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}})$ has the following property:

1. There exist constants $k_{h1}, k_{h2} \geq 0$ such that the norm of the residual dynamics satisfies

$$\left\| \boldsymbol{h}(\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) \right\| \leq k_{h1} \|\dot{\tilde{\boldsymbol{q}}}\| + k_{h2} \left\| \mathbf{tanh}(\tilde{\boldsymbol{q}}) \right\|$$

for all $\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}} \in \mathrm{IR}^n$, where $\mathbf{tanh}(\tilde{\boldsymbol{q}})$ is the vectorial tangent hyperbolic function introduced in the latter Definition.

where

$$k_{h2} \geq \frac{s_2}{\tanh\left(\frac{s_2}{s_1}\right)},$$

$$k_{C_1} \|\dot{\boldsymbol{q}}_d\|_{\mathrm{M}} \leq k_{h1},$$

with

$$s_1 = \left[k_g + k_M \|\ddot{\boldsymbol{q}}_d\|_{\mathrm{M}} + k_{C_2} \|\dot{\boldsymbol{q}}_d\|_{\mathrm{M}}^2\right],$$

and

$$s_2 = 2\left[k' + k'_M \|\ddot{\boldsymbol{q}}_d\|_{\mathrm{M}} + k_{C_1} \|\dot{\boldsymbol{q}}_d\|_{\mathrm{M}}^2\right].$$

Ch. 5. Case study: The Pelican prototype robot

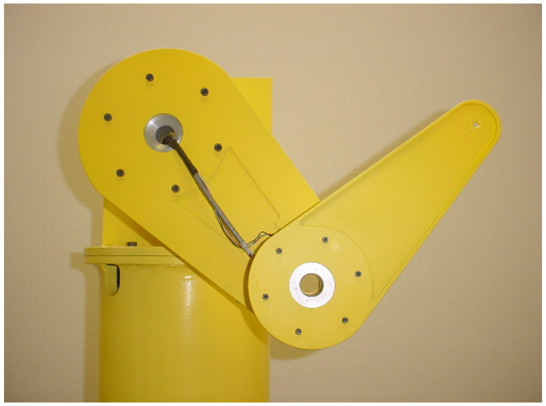

Figure 31: Pelican: experimental prototype from CICESE, Robotics lab.

The prototype is a

- vertical planar manipulator
- two links connected to revolute joints.

The links are direct driven by two brushless DC motors.

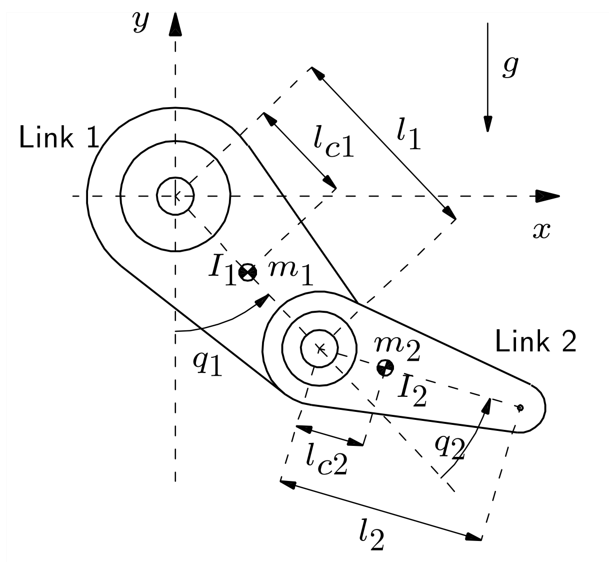

Figure 32: Diagram of the Pelican prototype robot with 2 degrees of freedom.

Table 1: Physical parameters of Pelican prototype manipulator.

Description	*Notation*	*Value*	*Units*
Length of Link 1	l_1	0.26	m
Length of Link 2	l_2	0.26	m
Distance to the center of mass (Link 1)	l_{c1}	0.0983	m
Distance to the center of mass (Link 2)	l_{c2}	0.0229	m
Mass of Link 1	m_1	6.5225	kg
Mass of Link 2	m_2	2.0458	kg
Inertia rel. to center of mass (Link 1)	I_1	0.1213	kg m^2
Inertia rel. to center of mass (Link 2)	I_2	0.0116	kg m^2
Gravity acceleration	g	9.81	m/sec^2

The chapter is organized as follows:

- Direct kinematics.
- Inverse kinematics.
- Dynamic model.
- Model properties of the dynamic model.
- Reference trajectories.

Direct kinematics

For this case, the problem consists on expressing

$$\begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{\varphi}(q_1, q_2)\,,$$

where $\boldsymbol{\varphi} : \mathrm{I\!R}^2 \rightarrow \mathrm{I\!R}^2$.

The direct kinematic model is given by

$$\begin{aligned} x &= l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) \\ y &= -l_1 \cos(q_1) - l_2 \cos(q_1 + q_2)\,. \end{aligned}$$

It may also be obtained the relation between the velocities,

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} l_1\cos(q_1) + l_2\cos(q_1+q_2) & l_2\cos(q_1+q_2) \\ l_1\sin(q_1) + l_2\sin(q_1+q_2) & l_2\sin(q_1+q_2) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \\ &= J(\boldsymbol{q}) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \end{aligned}$$

where

- $J(\boldsymbol{q}) = \dfrac{\partial \boldsymbol{\varphi}(\boldsymbol{q})}{\partial \boldsymbol{q}} \in \mathrm{IR}^{2\times 2}$ is called 'Jacobian matrix' or, 'Jacobian' of the robot, and the Relation of accelerations

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \left[\frac{d}{dt}J(\boldsymbol{q})\right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + J(\boldsymbol{q}) \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix},$$

where we have defined

- $\dfrac{d}{dt}J(\boldsymbol{q}) := \sum_{i=1}^{n} \dfrac{\partial \boldsymbol{\varphi}(\boldsymbol{q})}{\partial q_i}\dot{q}_i.$

Inverse kinematics

It has the form $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \boldsymbol{\varphi}^{-1}(x,y)$, where $\boldsymbol{\varphi}^{-1} : \Theta \to \mathrm{IR}^2$ and $\Theta \subseteq \mathrm{IR}^2$.

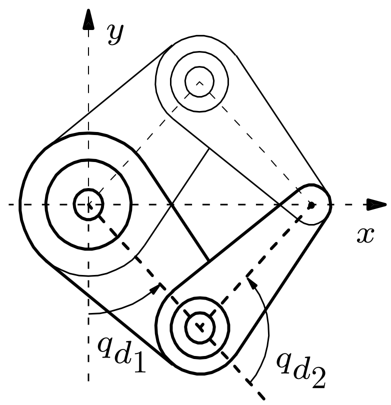

Figure 33: Two solutions to the inverse kinematics problem.

- It may have multiple solutions or no solution at all!

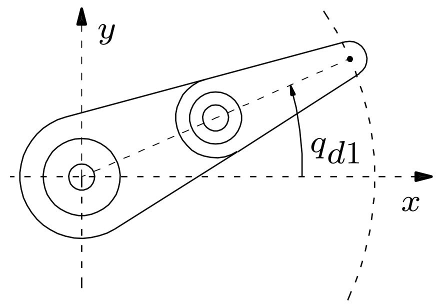

Figure 34: No solution to the inverse kinematics problem.

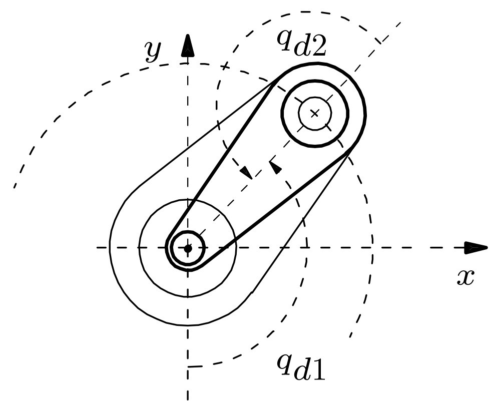

Figure 35: No solution to the inverse kinematics problem.

- A practical interest relies on its utility to
  - define $\boldsymbol{q}_d = [q_{d_1}\ q_{d_2}]^T$ from specified desired positions x_d and y_d.
  - $\begin{bmatrix} q_{d_1} \\ q_{d_2} \end{bmatrix} = \boldsymbol{\varphi}^{-1}(x, y)$.

$$
\begin{aligned}
q_{d_1} &= \tan^{-1}\left(\frac{x_d}{-y_d}\right) - \tan^{-1}\left(\frac{l_2 \sin(q_{d_2})}{l_1 + l_2 \cos(q_{d_2})}\right) \\
q_{d_2} &= \cos^{-1}\left(\frac{x_d^2 + y_d^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right) .
\end{aligned}
$$

  - The desired joint velocities may be obtained as

$$
\begin{bmatrix} \dot{q}_{d_1} \\ \dot{q}_{d_2} \end{bmatrix} = J^{-1}(\boldsymbol{q}_d) \begin{bmatrix} \dot{x}_d \\ \dot{y}_d \end{bmatrix}
$$

– and the desired joint accelerations as

$$\begin{bmatrix} \ddot{q}_{d_1} \\ \ddot{q}_{d_2} \end{bmatrix} = \underbrace{-J^{-1}(\boldsymbol{q}_d)\left[\frac{d}{dt}J(\boldsymbol{q}_d)\right]J^{-1}(\boldsymbol{q}_d)}_{\frac{d}{dt}\left[J^{-1}(\boldsymbol{q}_d)\right]}\begin{bmatrix} \dot{x}_d \\ \dot{y}_d \end{bmatrix} + J^{-1}(\boldsymbol{q}_d)\begin{bmatrix} \ddot{x}_d \\ \ddot{y}_d \end{bmatrix}$$

where

$$* \;\; J^{-1}(\boldsymbol{q}_d) = \begin{bmatrix} \dfrac{\mathrm{S}_{12}}{l_1\mathrm{S}_2} & -\dfrac{\mathrm{C}_{12}}{l_1\mathrm{S}_2} \\ \dfrac{-l_1\mathrm{S}_1 - l_2\mathrm{S}_{12}}{l_1 l_2 \mathrm{S}_2} & \dfrac{l_1\mathrm{C}_1 + l_2\mathrm{C}_{12}}{l_1 l_2 \mathrm{S}_2} \end{bmatrix},$$

and

$$* \;\; \frac{d}{dt}\left[J(\boldsymbol{q}_d)\right] = \begin{bmatrix} -l_1\mathrm{S}_1\dot{q}_{d_1} - l_2\mathrm{S}_{12}(\dot{q}_{d_1}+\dot{q}_{d_2}) & -l_2\mathrm{S}_{12}(\dot{q}_{d_1}+\dot{q}_{d_2}) \\ \\ l_1\mathrm{C}_1\dot{q}_{d_1} + l_2\mathrm{C}_{12}(\dot{q}_{d_1}+\dot{q}_{d_2}) & l_2\mathrm{C}_{12}(\dot{q}_{d_1}+\dot{q}_{d_2}) \end{bmatrix},$$

where for simplicity

- $S_1 = \sin(q_{d_1})$, $S_2 = \sin(q_{d_2})$, $C_1 = \cos(q_{d_1})$,
- $S_{12} = \sin(q_{d_1} + q_{d_2})$, $C_{12} = \cos(q_{d_1} + q_{d_2})$.

- Typically, singular configurations are those in which the end-effector of the robot is located at the physical boundary of the workspace

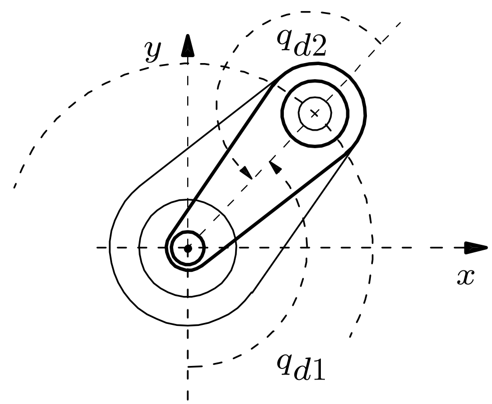

Figure 36: “Bent–over” singular configuration.

Dynamic model

Lagrangian equations

We start by writing the kinetic energy:

$$\mathcal{K}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \mathcal{K}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) + \mathcal{K}_2(\boldsymbol{q},\dot{\boldsymbol{q}})$$

where

- $\mathcal{K}_1(\boldsymbol{q},\dot{\boldsymbol{q}})$ is the kinetic energy associated to the mass m_1
- $\mathcal{K}_2(\boldsymbol{q},\dot{\boldsymbol{q}})$ is associated to the mass m_2

with

$$\begin{aligned}\mathcal{K}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \frac{1}{2}m_1\boldsymbol{v}_1^T\boldsymbol{v}_1+\frac{1}{2}I_1\dot{q}_1^2\\ &= \frac{1}{2}m_1 l_{c1}^2\dot{q}_1^2+\frac{1}{2}I_1\dot{q}_1^2\,.\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}_2(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \frac{1}{2}m_2\boldsymbol{v}_2^T\boldsymbol{v}_2+\frac{1}{2}I_2(\dot{q}_1+\dot{q}_2)^2\\ &= \frac{m_2}{2}l_1^2\dot{q}_1^2+\frac{m_2}{2}l_{c2}^2\left(\dot{q}_1^2+2\dot{q}_1\dot{q}_2+\dot{q}_2^2\right)\\ &\quad +m_2l_1l_{c2}\left(\dot{q}_1^2+\dot{q}_1\dot{q}_2\right)\ \cos(q_2)\\ &\quad +\frac{1}{2}I_2(\dot{q}_1+\dot{q}_2)^2.\end{aligned}$$

Similarly the potential energy is given by

$$\mathcal{U}(\boldsymbol{q}) = \mathcal{U}_1(\boldsymbol{q}) + \mathcal{U}_2(\boldsymbol{q})$$

where, assuming that the potential energy is zero at $y = 0$, we obtain that

- $\mathcal{U}_1(\boldsymbol{q}) = -m_1 l_{c1} g \cos(q_1)$
- $\mathcal{U}_2(\boldsymbol{q}) = -m_2 l_1 g \cos(q_1) - m_2 l_{c2} g \cos(q_1 + q_2)$.

The Lagrangian is

$$\begin{aligned}
\mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \mathcal{K}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q}) \\
&= \frac{1}{2}[m_1 l_{c1}^2 + m_2 l_1^2]\dot{q}_1^2 + \frac{1}{2} m_2 l_{c2}^2 \left[\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dot{q}_2^2\right] \\
&\quad + m_2 l_1 l_{c2} \cos(q_2) \left[\dot{q}_1^2 + \dot{q}_1\dot{q}_2\right] \\
&\quad + [m_1 l_{c1} + m_2 l_1] g \cos(q_1) \\
&\quad + m_2 g l_{c2} \cos(q_1 + q_2) \\
&\quad + \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2} I_2 [\dot{q}_1 + \dot{q}_2]^2 .
\end{aligned}$$

Applying the Lagrange's equations $\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right] - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i \qquad i = 1, 2$ we get that

$$
\begin{aligned}
\tau_1 \quad = \quad & \left[m_1 l_{c1}^2 + m_2 l_1^2 + m_2 l_{c2}^2 + 2 m_2 l_1 l_{c2} \cos(q_2) + I_1 + I_2\right] \ddot{q}_1 \\
& + \left[m_2 l_{c2}^2 + m_2 l_1 l_{c2} \cos(q_2) + I_2\right] \ddot{q}_2 \\
& - 2 m_2 l_1 l_{c2} \sin(q_2) \dot{q}_1 \dot{q}_2 - m_2 l_1 l_{c2} \sin(q_2) \dot{q}_2^2 \\
& + [m_1 l_{c1} + m_2 l_1] g \sin(q_1) \\
& + m_2 g l_{c2} \sin(q_1 + q_2)
\end{aligned}
$$

and

$$
\begin{aligned}
\tau_2 \quad = \quad & \left[m_2 l_{c2}^2 + m_2 l_1 l_{c2} \cos(q_2) + I_2\right] \ddot{q}_1 + [m_2 l_{c2}^2 + I_2] \ddot{q}_2 \\
& + m_2 l_1 l_{c2} \sin(q_2) \dot{q}_1^2 + m_2 g l_{c2} \sin(q_1 + q_2)
\end{aligned}
$$

Model in compact form

$$\underbrace{\begin{bmatrix} M_{11}(\boldsymbol{q}) & M_{12}(\boldsymbol{q}) \\ M_{21}(\boldsymbol{q}) & M_{22}(\boldsymbol{q}) \end{bmatrix}}_{M(\boldsymbol{q})} \ddot{\boldsymbol{q}} + \underbrace{\begin{bmatrix} C_{11}(\boldsymbol{q},\dot{\boldsymbol{q}}) & C_{12}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ C_{21}(\boldsymbol{q},\dot{\boldsymbol{q}}) & C_{22}(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{bmatrix}}_{C(\boldsymbol{q},\dot{\boldsymbol{q}})} \dot{\boldsymbol{q}} + \underbrace{\begin{bmatrix} g_1(\boldsymbol{q}) \\ g_2(\boldsymbol{q}) \end{bmatrix}}_{\boldsymbol{g}(\boldsymbol{q})} \quad = \quad \boldsymbol{\tau},$$

where

$$
\begin{aligned}
M_{11}(\boldsymbol{q}) &= m_1 l_{c1}^2 + m_2 \left[l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos(q_2) \right] + I_1 + I_2 \\
M_{12}(\boldsymbol{q}) &= m_2 \left[l_{c2}^2 + l_1 l_{c2} \cos(q_2) \right] + I_2 \\
M_{21}(\boldsymbol{q}) &= m_2 \left[l_{c2}^2 + l_1 l_{c2} \cos(q_2) \right] + I_2 \\
M_{22}(\boldsymbol{q}) &= m_2 l_{c2}^2 + I_2 \\
C_{11}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -m_2 l_1 l_{c2} \sin(q_2) \dot{q}_2 \\
C_{12}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -m_2 l_1 l_{c2} \sin(q_2) \left[\dot{q}_1 + \dot{q}_2 \right] \\
C_{21}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= m_2 l_1 l_{c2} \sin(q_2) \dot{q}_1 \\
C_{22}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= 0 \\
g_1(\boldsymbol{q}) &= \left[m_1 l_{c1} + m_2 l_1 \right] g \sin(q_1) + m_2 l_{c2} g \sin(q_1 + q_2) \\
g_2(\boldsymbol{q}) &= m_2 l_{c2} g \sin(q_1 + q_2) .
\end{aligned}
$$

In terms of these state variables, the dynamic model of the robot may be written as

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ M(\boldsymbol{q})^{-1}\left[\boldsymbol{\tau}(t) - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right] \end{bmatrix}.$$

The formulas and numeric values of the constants $\lambda_{\mathrm{Max}}\{M\}$, k_M, k_{C_1}, k_{C_2} and k_g are summarized in Tables below

Table 2: Parameters for the Pelican prototype.

$\lambda_{\mathrm{Max}}\{M\}$	$n \left(\max\limits_{i,j,\boldsymbol{q}} \left\| M_{ij}(\boldsymbol{q}) \right\| \right)$
k_M	$n^2 \left(\max\limits_{i,j,k,\boldsymbol{q}} \left\| \frac{\partial M_{ij}(\boldsymbol{q})}{\partial q_k} \right\| \right)$
k_{C_1}	$n^2 \left(\max\limits_{i,j,\boldsymbol{q}} \left\| C_{k_{ij}}(\boldsymbol{q}) \right\| \right)$
k_{C_2}	$n^3 \left(\max\limits_{i,j,k,l,\boldsymbol{q}} \left\| \frac{\partial C_{k_{ij}}(\boldsymbol{q})}{\partial q_l} \right\| \right)$
k_g	$n \left(\max\limits_{i,j,\boldsymbol{q}} \left\| \frac{\partial g_i(\boldsymbol{q})}{\partial q_j} \right\| \right)$

Table 3: Numeric values of the parameters for the CICESE prototype.

Parameter	*Value*	*Units*
$\lambda_{\mathrm{Max}}\{M\}$	0.7193	$\mathrm{kg\ m^2}$
k_M	0.0974	$\mathrm{kg\ m^2}$
k_{C_1}	0.0487	$\mathrm{kg\ m^2}$
k_{C_2}	0.0974	$\mathrm{kg\ m^2}$
k_g	23.94	$\mathrm{kg\ m^2/sec^2}$

Desired reference trajectories

We have selected the following reference trajectories in *joint space*:

$$\begin{bmatrix} q_{d1} \\ \\ q_{d2} \end{bmatrix} = \begin{bmatrix} b_1[1 - e^{-2.0\ t^3}] + c_1[1 - e^{-2.0\ t^3}]\sin(\omega_1 t) \\ \\ b_2[1 - e^{-2.0\ t^3}] + c_2[1 - e^{-2.0\ t^3}]\sin(\omega_2 t) \end{bmatrix} \quad [\mathrm{rad}] \qquad (13)$$

where

- $b_1 = \pi/4$ [rad], $c_1 = \pi/9$ [rad] and $\omega_1 = 4 \quad [\mathrm{rad/sec}]$, and
- $b_2 = \pi/3$ [rad], $c_2 = \pi/6$ [rad] and $\omega_2 = 3 \quad [\mathrm{rad/sec}]$.

The figure depicts the graphs of above reference trajectories against time.

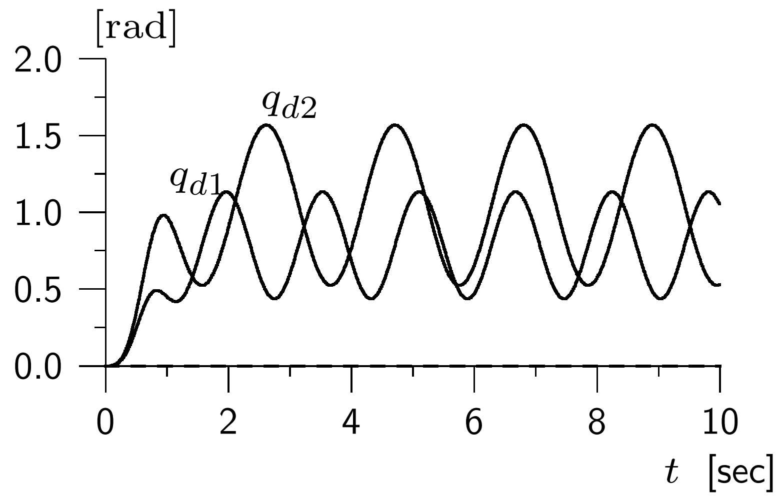

Figure 37: Desired reference trajectories.

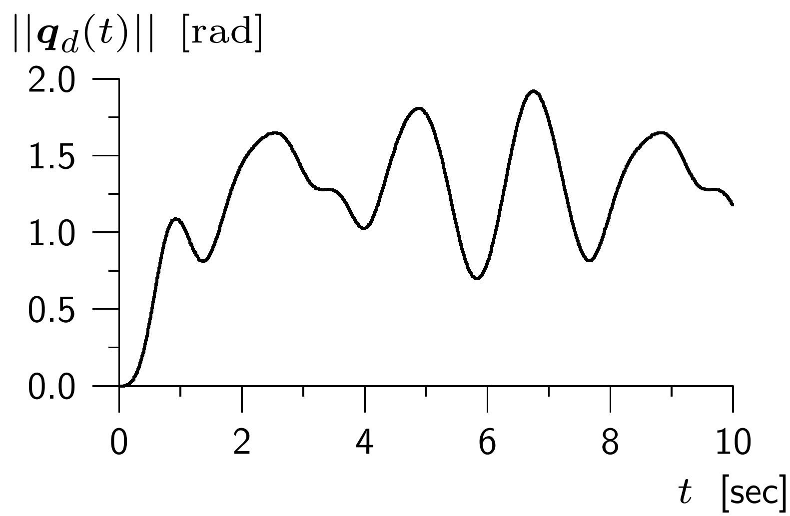

Figure 38: Norm of the desired reference positions.

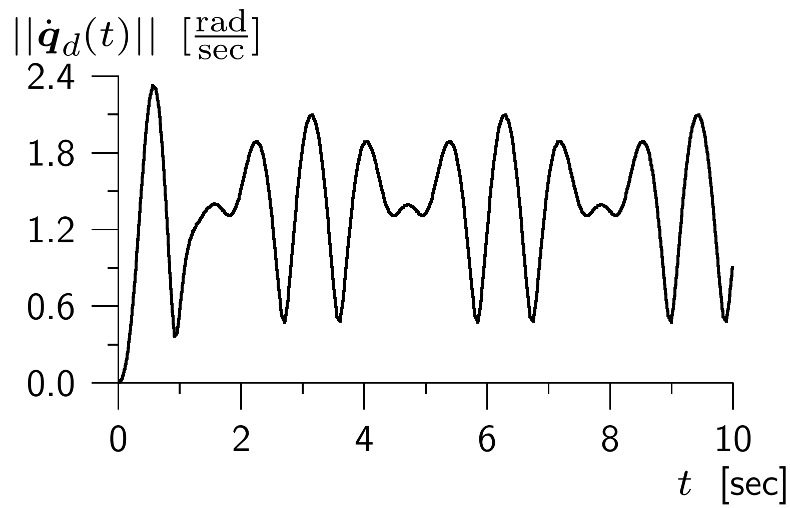

Figure 39: Norm of the desired reference velocities vector.

$$\dot{q}_{d1} = 6b_1 t^2 e^{-2.0\ t^3} + 6c_1 t^2 e^{-2.0\ t^3} \sin(\omega_1 t) + [c_1 - c_1 e^{-2.0\ t^3}] \cos(\omega_1 t)\omega_1,$$
$$\dot{q}_{d2} = 6b_2 t^2 e^{-2.0\ t^3} + 6c_2 t^2 e^{-2.0\ t^3} \sin(\omega_2 t) + [c_2 - c_2 e^{-2.0\ t^3}] \cos(\omega_2 t)\omega_2\,.$$
$$\text{in } [\,\text{rad/sec}\,]. \tag{14}$$

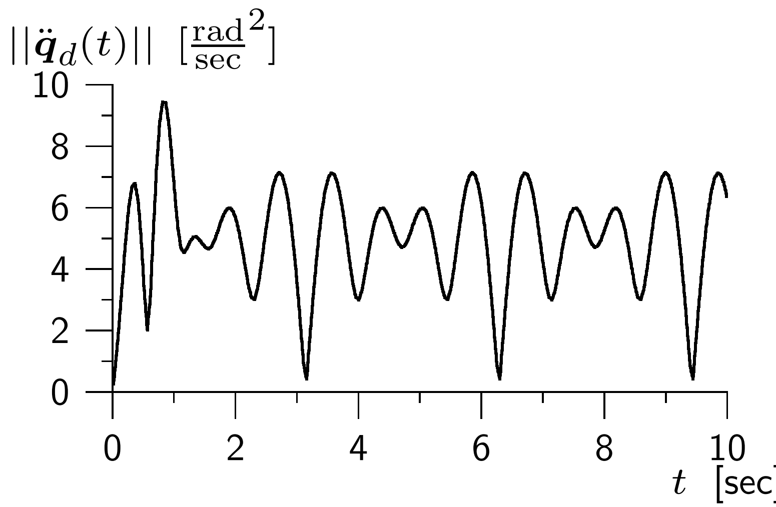

Figure 40: Norm of the desired reference accelerations vector.

Reference accelerations:

$$\begin{aligned}
\ddot{q}_{d1} \quad = \quad & 12b_1 t e^{-2.0\ t^3} - 36b_1 t^4 e^{-2.0\ t^3} + 12c_1 t e^{-2.0\ t^3} \mathrm{sen}(\omega_1 t) \\
& - 36c_1 t^4 e^{-2.0\ t^3} \mathrm{sen}(\omega_1 t) + 12c_1 t^2 e^{-2.0\ t^3} \cos(\omega_1 t)\omega_1 \\
& - [c_1 - c_1 e^{-2.0\ t^3}] \mathrm{sen}(\omega_1 t)\omega_1^2 \qquad \left[\mathrm{rad}\ /\ \mathrm{sec}^2\right],
\end{aligned}$$

$$\begin{aligned}
\ddot{q}_{d2} \quad = \quad & 12b_2 t e^{-2.0\ t^3} - 36b_2 t^4 e^{-2.0\ t^3} + 12c_2 t e^{-2.0\ t^3} \sin(\omega_2 t) \\
& - 36c_2 t^4 e^{-2.0\ t^3} \sin(\omega_2 t) + 12c_2 t^2 e^{-2.0\ t^3} \cos(\omega_2 t)\omega_2 \\
& - [c_2 - c_2 e^{-2.0\ t^3}] \sin(\omega_2 t)\omega_2^2 \qquad \left[\mathrm{rad}\ /\ \mathrm{sec}^2\right].
\end{aligned} \tag{15}$$

From above figures we deduce the following upperbounds on the norms:

$$\begin{aligned} \|\boldsymbol{q}_d\|_{\mathrm{Max}} &\leq 1.92 \text{ [rad]} \\ \|\dot{\boldsymbol{q}}_d\|_{\mathrm{Max}} &\leq 2.33 \text{ [rad/sec]} \\ \|\ddot{\boldsymbol{q}}_d\|_{\mathrm{Max}} &\leq 9.52 \text{ [rad/sec]}^2 . \end{aligned}$$

Part II

SET-POINT CONTROL

Introduction

Motion controllers are clasified

- *Set-point* controllers ($\boldsymbol{q}_d(t) = \boldsymbol{q}_d$, is a constant vector),
- and *tracking controllers* ($\boldsymbol{q}_d(t)$ is a time variant vector).

Consider the dynamic model of a robot manipulator $M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$, which can be written as:

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ M(\boldsymbol{q})^{-1}\left[\boldsymbol{\tau}(t) - C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right] \end{bmatrix}$$

where

- $M(\boldsymbol{q}) \in \mathrm{IR}^{n\times n}$ is the inertia matrix,
- $C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \in \mathrm{IR}^{n}$ is the vector of centrifugal and Coriolis forces,
- $\boldsymbol{g}(\boldsymbol{q}) \in \mathrm{IR}^{n}$ is el vector of gravitational forces and torques and
- $\boldsymbol{\tau} \in \mathrm{IR}^{n}$ is a vector of external forces and torques applied at the joints.
- The vectors $\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}} \in \mathrm{IR}^{n}$ denote the position, velocity and joint acceleration respectively.

The *objective of set-point control* consists on finding $\boldsymbol{\tau}$ such that

$$\lim_{t \to \infty} \boldsymbol{q}(t) = \boldsymbol{q}_d$$

where $\boldsymbol{q}_d \in \mathrm{IR}^n$ is a *given* constant vector which represents the desired set-point.

It is convenient to rewrite the set point control objective as

$$\lim_{t \to \infty} \tilde{\boldsymbol{q}}(t) = \mathbf{0}$$

where $\tilde{\boldsymbol{q}} \in \mathrm{IR}^n$ stands for the joint position errors

$$\tilde{\boldsymbol{q}}(t) := \boldsymbol{q}_d - \boldsymbol{q}(t)\,.$$

In general, a control law may be expressed as

$$\boldsymbol{\tau} = \boldsymbol{\tau}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \boldsymbol{q}_d, M(\boldsymbol{q}), C(\boldsymbol{q}, \dot{\boldsymbol{q}}), \boldsymbol{g}(\boldsymbol{q})\right).$$

However, for practical purposes, it is desirable that the controller does not depend on the joint acceleration $\ddot{\boldsymbol{q}}$.

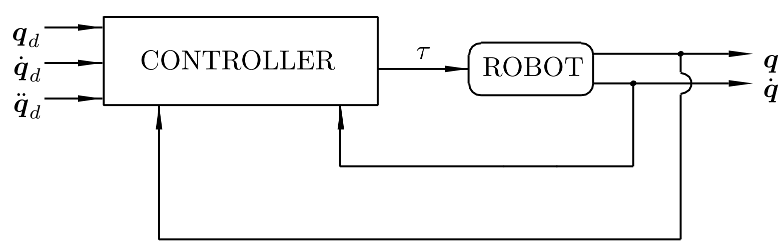

Figure 41: Set-point control: closed-loop system.

A methodology to analyze the stability may be summarized in:

1. Derivation of the closed loop dynamic equation.

2. Representation of the closed loop equation in the state-space form, i.e.,

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{q}_d - \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{q}_d, M(\boldsymbol{q}), C(\boldsymbol{q}, \dot{\boldsymbol{q}}), \boldsymbol{g}(\boldsymbol{q}))\,.$$

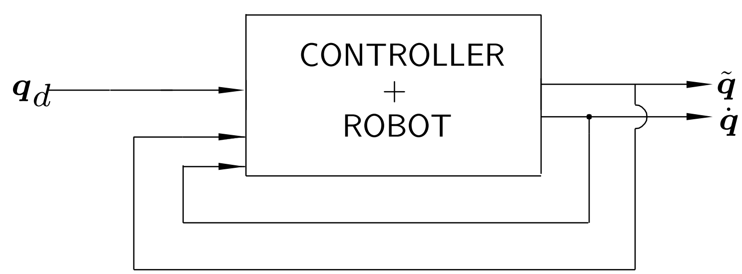

Figure 42: Set-point control closed-loop system: Input-output representation.

3. Study of the existence and possible unicity of the equilibrium for the closed loop equation.

4. Proposition of a Lyapunov function candidate to study the stability

5. Alternatively to step 4), determine the qualitative behavior of the solutions of the closed loop equation.

The controllers that we present may be called "conventional" (commonly used in industrial robots).

- Velocity feedback Proportional control and Proportional Derivative (PD) control
- PD control with gravity compensation
- PD control with precalculated gravity compensation
- Proportional Integral Derivative (PID) control

Ch. 6. Velocity feedback proportional control and PD control

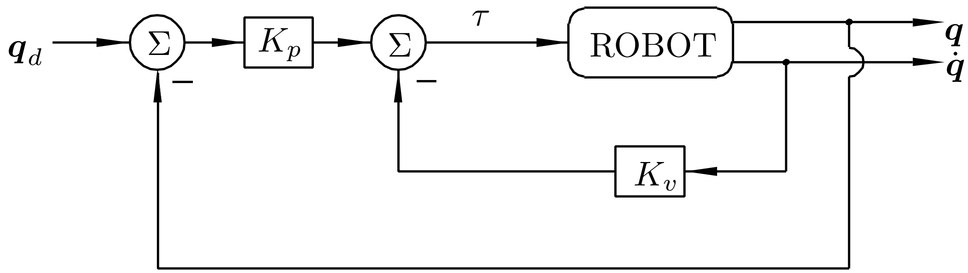

Figure 43: Velocity feedback proportional control.

Control law given by $\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} - K_v\dot{\boldsymbol{q}}$, where

- $K_p, K_v \in \mathrm{IR}^{n\times n}$ are symmetric positive definite matrices
- $\boldsymbol{q}_d \in \mathrm{IR}^n$ corresponds to the desired joint position,
- $\tilde{\boldsymbol{q}} = \boldsymbol{q}_d - \boldsymbol{q} \in \mathrm{IR}^n$ is called position error.

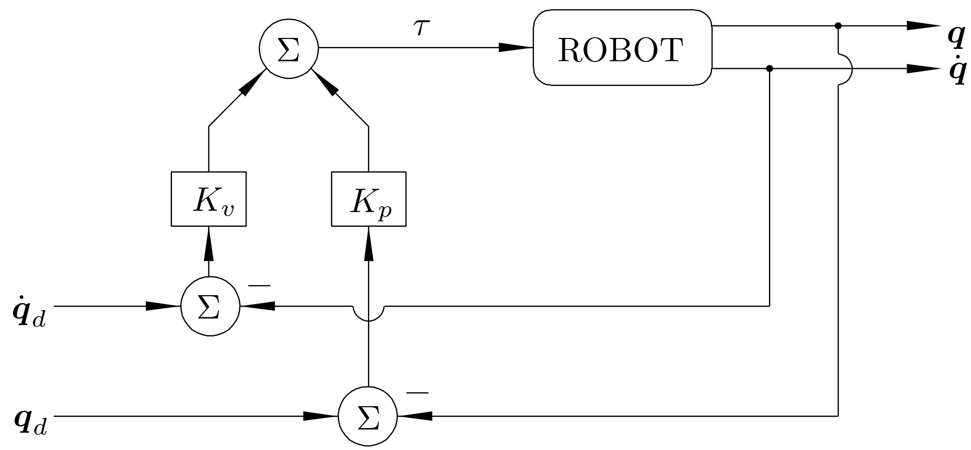

Figure 44: PD Control.

The PD control law is given by: $\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}}$, where

- $K_p, K_v \in \mathrm{IR}^{n\times n}$ are also symmetric positive definite

- When $\boldsymbol{q}_d$, is constant, then, $\dot{\tilde{\boldsymbol{q}}} = -\dot{\boldsymbol{q}}$ (both control laws are identical).

We assume a constant $\boldsymbol{q}_d$, then, the closed loop equation may be rewritten

$$\frac{d}{dt}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \\ \dot{\boldsymbol{q}}\end{bmatrix} = \begin{bmatrix}-\dot{\boldsymbol{q}}\\ \\ M(\boldsymbol{q})^{-1}\left[K_p\tilde{\boldsymbol{q}} - K_v\dot{\boldsymbol{q}} - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right]\end{bmatrix}.$$

- It may have multiple equilibria given by
  - ⋆ $\begin{bmatrix}\tilde{\boldsymbol{q}}^T & \dot{\boldsymbol{q}}^T\end{bmatrix}^T = \begin{bmatrix}\boldsymbol{s}^T & \boldsymbol{0}^T\end{bmatrix}^T$ where $\boldsymbol{s} \in \mathbb{R}^n$ is solution of

$$K_p\boldsymbol{s} - \boldsymbol{g}(\boldsymbol{q}_d - \boldsymbol{s}) = \boldsymbol{0}.$$

Robots without gravity term

Dynamic model given by

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{\tau}.$$

Assuming that $\boldsymbol{q}_d$ is constant, the closed loop equation becomes

$$\frac{d}{dt}\begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} -\dot{\boldsymbol{q}} \\ M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}})^{-1}\left[K_p\tilde{\boldsymbol{q}} - K_v\dot{\boldsymbol{q}} - C(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}\right] \end{bmatrix}$$

- It represents an autonomous differential equation (since $\boldsymbol{q}_d$ is constant).
- The origin $\begin{bmatrix}\tilde{\boldsymbol{q}}^T & \dot{\boldsymbol{q}}^T\end{bmatrix}^T = \boldsymbol{0}$ is the only equilibrium of this equation.

To study the stability of the equilibrium, consider

- Lyapunov function candidate

$$
\begin{aligned}
V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) &= \frac{1}{2}\begin{bmatrix} \tilde{\boldsymbol{q}} \\ \\ \dot{\boldsymbol{q}} \end{bmatrix}^T \begin{bmatrix} K_p & 0 \\ \\ 0 & M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}) \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \\ \dot{\boldsymbol{q}} \end{bmatrix} \\
&= \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}}\,.
\end{aligned}
$$

⋆ $V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})$ is positive definite since $M(\boldsymbol{q}) > 0$ and $K_p > 0$.

- The total derivative of $V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})$ results

$$\begin{aligned}\dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) &= -\dot{\boldsymbol{q}}^T K_v \dot{\boldsymbol{q}} \\ &= -\begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & K_v \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} \leq 0,\end{aligned}$$

  – By Theorem 2.3 we conclude that the origin is stable and,
  – the solutions $\tilde{\boldsymbol{q}}(t)$ and $\dot{\boldsymbol{q}}(t)$ are bounded.

- By applying the La Salle's theorem (Theorem 2.7) we have that

– the set Ω is given by

$$\begin{aligned}\Omega &= \left\{ \boldsymbol{x} \in \mathrm{IR}^{2n} : \dot{V}(\boldsymbol{x}) = 0 \right\} \\ &= \left\{ \boldsymbol{x} = \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} \in \mathrm{IR}^{2n} : \dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) = 0 \right\} \\ &= \{ \tilde{\boldsymbol{q}} \in \mathrm{IR}^n, \dot{\boldsymbol{q}} = \mathbf{0} \in \mathrm{IR}^n \} \ ,\end{aligned}$$

* where $\begin{bmatrix} \tilde{\boldsymbol{q}}(0)^T & \dot{\boldsymbol{q}}(0)^T \end{bmatrix}^T = \mathbf{0} \in \mathrm{IR}^{2n}$ is the only initial condition in Ω for which $\boldsymbol{x}(t) \in \Omega$ for all $t \geq 0$.

– This is enough to establish global asymptotic stability of the origin, $\begin{bmatrix} \tilde{\boldsymbol{q}}^T & \dot{\boldsymbol{q}}^T \end{bmatrix}^T = \mathbf{0} \in \mathrm{IR}^{2n}$.

Robots with gravity term

The closed loop equation is

$$\frac{d}{dt}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\boldsymbol{q}}\end{bmatrix} = \begin{bmatrix} -\dot{\boldsymbol{q}} \\ M(\boldsymbol{q})^{-1}\left[K_p\tilde{\boldsymbol{q}} - K_v\dot{\boldsymbol{q}} - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right]\end{bmatrix}. \tag{16}$$

The study of this section is limited to robots having only revolute joints.

- It may have multiple equilibria given by
  - ⋆ $\left[\tilde{\boldsymbol{q}}^T \;\; \dot{\boldsymbol{q}}^T\right]^T = \left[\boldsymbol{s}^T \;\; \boldsymbol{0}^T\right]^T$ where $\boldsymbol{s} \in \mathbb{R}^n$ is solution of

$$K_p\boldsymbol{s} - \boldsymbol{g}(\boldsymbol{q}_d - \boldsymbol{s}) = \boldsymbol{0}\,.$$

Example 6.1

Consider the model of an ideal pendulum

$$J\ddot{q} + mgl\,\sin(q) = \tau\,.$$

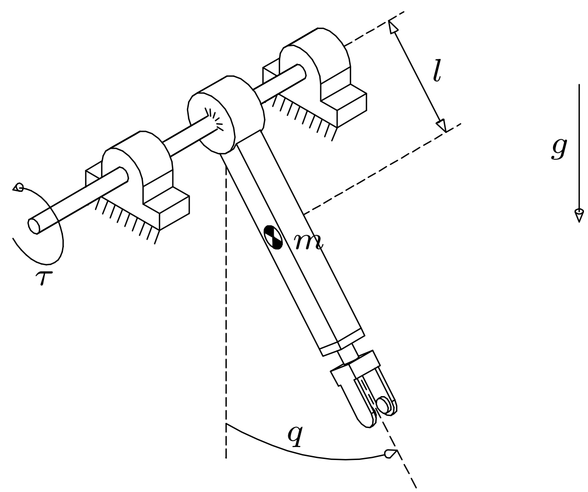

Figure 45: Pendulum.

The equilibria expression takes the form

$$k_p s - mgl\,\sin(q_d - s) = 0\,.$$

For the sake of illustration consider the following numeric values

$$\begin{array}{ll} J = 1 & mgl = 1 \\ k_p = 0.25 & q_d = \pi/2. \end{array}$$

The closed loop system under PD control has the equilibria

$$\begin{bmatrix} \tilde{q} \\ \dot{q} \end{bmatrix} \in \left\{ \begin{bmatrix} 1.25 \\ 0 \end{bmatrix}, \begin{bmatrix} -2.13 \\ 0 \end{bmatrix}, \begin{bmatrix} -3.56 \\ 0 \end{bmatrix} \right\}.$$

Unicity of the equilibrium. The equilibria of the closed loop equation (16) satisfy

- $\left[\tilde{\boldsymbol{q}}^T \ \dot{\boldsymbol{q}}^T\right]^T = \left[\boldsymbol{s}^T \ \ \boldsymbol{0}^T\right]^T$, where
  - ⋆ $\boldsymbol{s} \in \mathrm{IR}^n$ is solution of $\boldsymbol{s} = K_p^{-1}\boldsymbol{g}(\boldsymbol{q}_d - \boldsymbol{s}) = \boldsymbol{f}(\boldsymbol{s}, \boldsymbol{q}_d)$.
- For all vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{q}_d \in \mathrm{IR}^n$, we get that

$$\|\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{q}_d) - \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{q}_d)\| \leq \frac{k_g}{\lambda_{\min}\{K_p\}} \|\boldsymbol{x} - \boldsymbol{y}\|$$

- If $\lambda_{\min}\{K_p\} > k_g$ we have (by contraction mapping theorem)
  - ⋆ the only equilibrium is $\left[\tilde{\boldsymbol{q}}^T \ \ \dot{\boldsymbol{q}}^T\right] = \left[\boldsymbol{s}^{*T} \ \ \boldsymbol{0}^T\right]^T$.

Arbitrarily bounded position and velocity error. Case where

- K_p is not restricted to satisfy $\lambda_{\min}\{K_p\} > k_g$,
- but it is enough that K_p be positive definite.

Define the following non-negative function

$$V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}} + \mathcal{U}(\boldsymbol{q}) - k_U \geq 0\,.$$

where $k_U = \min_{\boldsymbol{q}} \{\mathcal{U}(\boldsymbol{q})\}$

The time derivative of $V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})$ yields

$$
\begin{aligned}
\dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) &= \dot{\boldsymbol{q}}^T M(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T \dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + \tilde{\boldsymbol{q}}^T K_p \dot{\tilde{\boldsymbol{q}}} + \dot{\boldsymbol{q}}^T \boldsymbol{g}(\boldsymbol{q}) \,, \\
&= -\dot{\boldsymbol{q}}^T \overbrace{K_v}^{Q} \dot{\boldsymbol{q}} \\
&= -\begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & K_v \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} \leq 0 \,.
\end{aligned}
$$

- Invoking Lemma 2.2, we conclude
  - ⋆ $\dot{\boldsymbol{q}}(t)$ and $\tilde{\boldsymbol{q}}(t)$ are bounded for all t, and
  - ⋆ the velocities vector is square integrable, that is, $\int_0^\infty \|\dot{\boldsymbol{q}}(t)\|^2 dt < \infty$.

⋆ Moreover, the exact explicit bounds hold, for all $t \geq 0$:

$$\begin{aligned} \|\tilde{\boldsymbol{q}}(t)\|^2 &\leq \frac{2V(\tilde{\boldsymbol{q}}(0), \dot{\boldsymbol{q}}(0))}{\lambda_{\min}\{K_p\}} \\ &= \frac{\dot{\boldsymbol{q}}(0)^T M(\boldsymbol{q}(0))\dot{\boldsymbol{q}}(0) + \tilde{\boldsymbol{q}}(0)^T K_p \tilde{\boldsymbol{q}}(0) + 2\mathcal{U}(\boldsymbol{q}(0)) - 2k_U}{\lambda_{\min}\{K_p\}} \end{aligned} \tag{17}$$

$$\begin{aligned} \|\dot{\boldsymbol{q}}(t)\|^2 &\leq \frac{2V(\tilde{\boldsymbol{q}}(0), \dot{\boldsymbol{q}}(0))}{\lambda_{\min}\{M(\boldsymbol{q})\}} \\ &= \frac{\dot{\boldsymbol{q}}(0)^T M(\boldsymbol{q}(0))\dot{\boldsymbol{q}}(0) + \tilde{\boldsymbol{q}}(0)^T K_p \tilde{\boldsymbol{q}}(0) + 2\mathcal{U}(\boldsymbol{q}(0)) - 2k_U}{\lambda_{\min}\{M(\boldsymbol{q})\}} \end{aligned} \tag{18}$$

Example 6.2

Consider again the ideal pendulum from Example 6.1

$$J\ddot{q} + mgl\,\sin(q) = \tau,$$

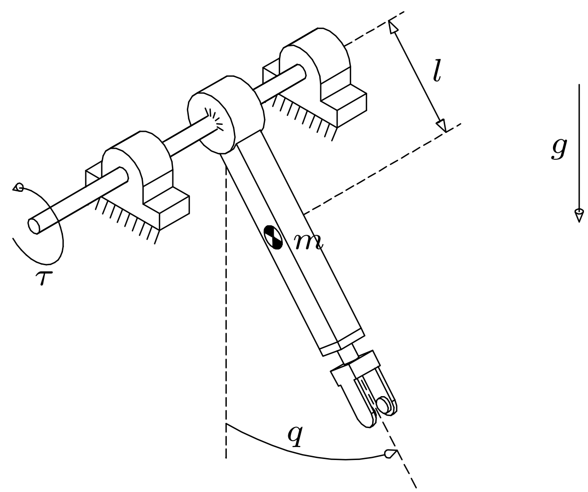

Figure 46: Pendulum.

The potential energy function is $\mathcal{U}(q) = mgl(1 - \cos(q))$ and the constant k_U is zero.

- Consider next the numeric values

$$\begin{array}{ll} J = 1 & mgl = 1 \\ k_p = 0.25 & k_v = 0.50 \\ q_d = \pi/2. & \end{array}$$

- Assume that we apply the PD controller to drive it from the initial conditions $q(0) = 0$ and $\dot{q}(0) = 0$

– According to the bounds (17) and (18), we get

$$\tilde{q}^2(t) \leq \tilde{q}^2(0) = 2.46 \ \ \mathrm{rad}^2 \quad \forall \ \ t \geq 0$$

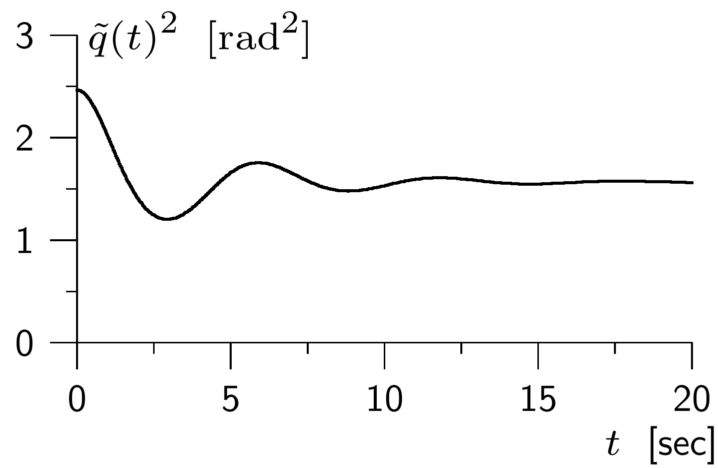

Figure 47: Graph of $\tilde{q}(t)^2$.

– and

$$\dot{q}^2(t) \leq \frac{k_p}{ml^2}\tilde{q}^2(0) = 0.61 \quad \left(\frac{\mathrm{rad}}{\mathrm{sec}}\right)^2 \qquad \forall \quad t \geq 0$$

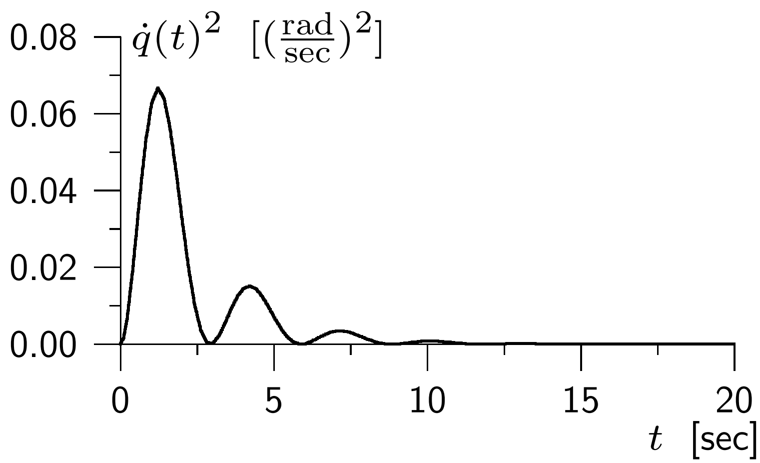

Figure 48: Graph of $\dot{q}(t)^2$

- Is interesting to observe

$$\lim_{t \to \infty} \begin{bmatrix} \tilde{q}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 1.25 \\ 0 \end{bmatrix} .$$

That is, the solutions tend to one of the three equilibria determined in Example 6.1

Example 6.3

Consider the 2-DOF *prototype robot* studied in Chapter 5,

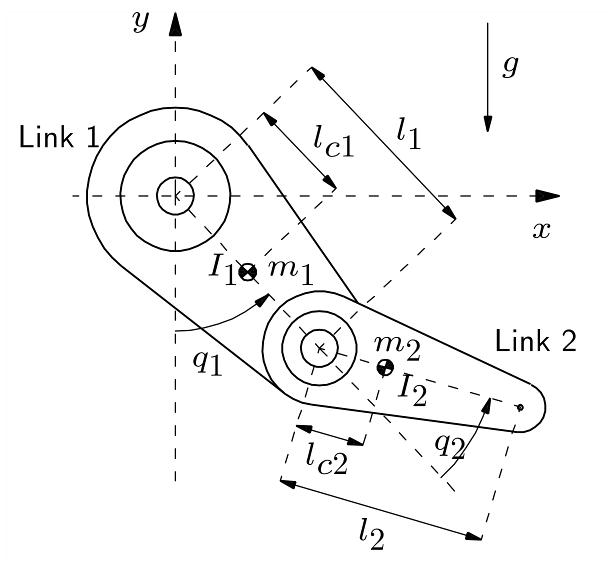

Figure 49: Diagram of the Pelican prototype robot with 2 degrees of freedom.

whose vector of gravitational torques $\boldsymbol{g}(\boldsymbol{q})$ is

$$\begin{aligned} g_1(\boldsymbol{q}) &= (m_1 l_{c1} + m_2 l_1) g \sin(q_1) + m_2 g l_{c2} \sin(q_1 + q_2) \\ g_2(\boldsymbol{q}) &= m_2 g l_{c2} \sin(q_1 + q_2). \end{aligned}$$

The control objective consists on making that

$$\lim_{t \to \infty} \boldsymbol{q}(t) = \boldsymbol{q}_d = \begin{bmatrix} \pi/10 \\ \pi/30 \end{bmatrix} \quad [\mathrm{rad}].$$

- It may be verified that $\boldsymbol{g}(\boldsymbol{q}_d) \neq \mathbf{0}$
- The origin $\begin{bmatrix} \tilde{\boldsymbol{q}}^T & \dot{\boldsymbol{q}}^T \end{bmatrix}^T = \mathbf{0} \in \mathrm{IR}^4$ of the closed loop equation with the PD controller, is not an equilibrium.

- Consider the PD controller

$$\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} - K_v\dot{\boldsymbol{q}}$$

with the following numeric values

$$K_p = \begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix}, \qquad K_v = \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}.$$

  – The initial conditions are fixed to $\boldsymbol{q}(0) = \mathbf{0}$ and $\dot{\boldsymbol{q}}(0) = \mathbf{0}$.

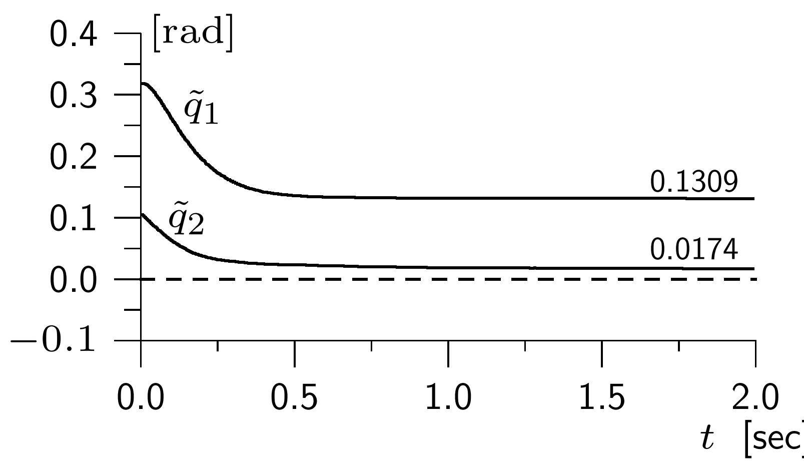

Figure 50: Position errors $\tilde{q}_1$ and $\tilde{q}_2$.

– One may appreciate that $\lim_{t\to\infty} \tilde{q}_1(t) = 0.1309$ and $\lim_{t\to\infty} \tilde{q}_2(t) = 0.0174$
– therefore, as it was expected, the control objective is not achieved

Ch. 7. PD control with gravity compensation

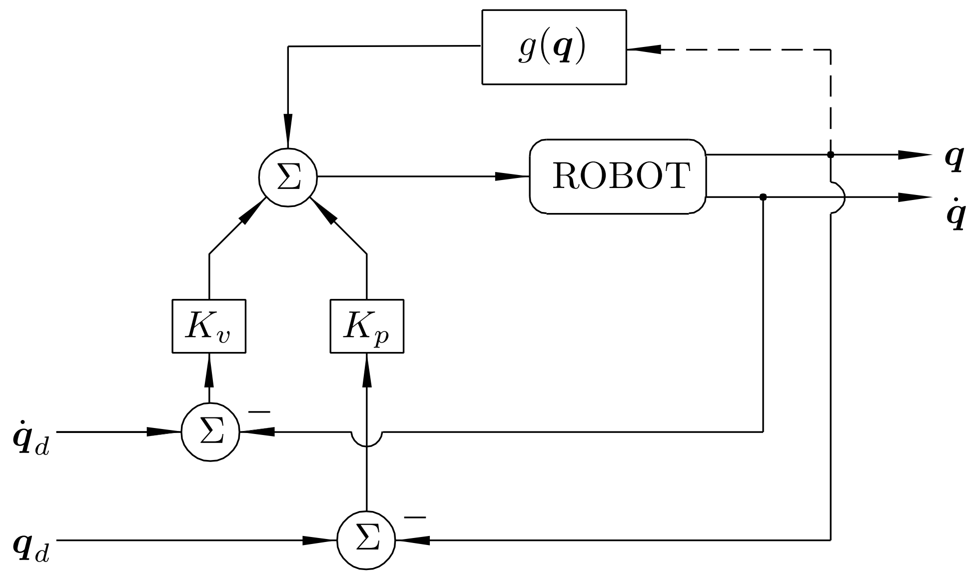

Figure 51: PD control with gravity compensation.

The control law is given by $\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + \boldsymbol{g}(\boldsymbol{q})$ where

- $K_p, K_v \in \mathrm{IR}^{n\times n}$ are positive definite symmetric matrices.

Global asymptotic stability by La Salle's theorem

Considering $\boldsymbol{q}_d$ as constant, the closed loop equation may be written as

$$\frac{d}{dt}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\boldsymbol{q}}\end{bmatrix} = \begin{bmatrix} -\dot{\boldsymbol{q}} \\ M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}})^{-1}\left[K_p\tilde{\boldsymbol{q}} - K_v\dot{\boldsymbol{q}} - C(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}\right]\end{bmatrix}$$

- The origin $\begin{bmatrix}\tilde{\boldsymbol{q}}^T & \dot{\boldsymbol{q}}^T\end{bmatrix}^T = \boldsymbol{0} \in \mathbb{R}^{2n}$ is the unique equilibrium.

To study the stability of the origin, consider

- Lyapunov function candidate

$$\begin{aligned} V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) &= \frac{1}{2} \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix}^T \begin{bmatrix} K_p & 0 \\ 0 & M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}) \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} \\ &= \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}} \end{aligned}$$

⋆ $V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})$ is positive definite since $M(\boldsymbol{q}) > 0$ and $K_p > 0$

- Its total derivative with respect to time is

$$\begin{aligned}\dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) &= -\dot{\boldsymbol{q}}^T K_v \dot{\boldsymbol{q}} \\ &= -\begin{bmatrix}\tilde{\boldsymbol{q}} \\ \\ \dot{\boldsymbol{q}}\end{bmatrix}^T \begin{bmatrix}0 & 0 \\ \\ 0 & K_v\end{bmatrix} \begin{bmatrix}\tilde{\boldsymbol{q}} \\ \\ \dot{\boldsymbol{q}}\end{bmatrix} \leq 0\end{aligned}$$

  ⋆ $\dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) \leq 0$ because $K_v > 0$

- Consequently, by Theorem 2.3 the origin is stable and all the solutions $\tilde{\boldsymbol{q}}(t)$ and $\dot{\boldsymbol{q}}(t)$ are bounded.

- By applying the La Salle's theorem (Theorem 2.7) we have that
  - the set Ω is given by

$$\begin{aligned}\Omega &= \left\{ \boldsymbol{x} \in \mathrm{IR}^{2n} : \dot{V}(\boldsymbol{x}) = 0 \right\} \\ &= \left\{ \boldsymbol{x} = \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} \in \mathrm{IR}^{2n} : \dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) = 0 \right\} \\ &= \{ \tilde{\boldsymbol{q}} \in \mathrm{IR}^n, \dot{\boldsymbol{q}} = \mathbf{0} \in \mathrm{IR}^n \} \ ,\end{aligned}$$

    * where $\begin{bmatrix} \tilde{\boldsymbol{q}}(0)^T & \dot{\boldsymbol{q}}(0)^T \end{bmatrix}^T = \mathbf{0} \in \mathrm{IR}^{2n}$ is the only initial condition in Ω for which $\boldsymbol{x}(t) \in \Omega$ for all $t \geq 0$.
  - This is enough to establish global asymptotic stability of the origin, $\begin{bmatrix} \tilde{\boldsymbol{q}}^T & \dot{\boldsymbol{q}}^T \end{bmatrix}^T = \mathbf{0} \in \mathrm{IR}^{2n}$.
  - Hence, $\lim_{t\to\infty} \tilde{\boldsymbol{q}}(t) = \mathbf{0}$ and $\lim_{t\to\infty} \dot{\boldsymbol{q}}(t) = \mathbf{0}$ (the set-point control objective is achieved).

Example 7.1

Consider the Pelican robot studied in Chapter 5

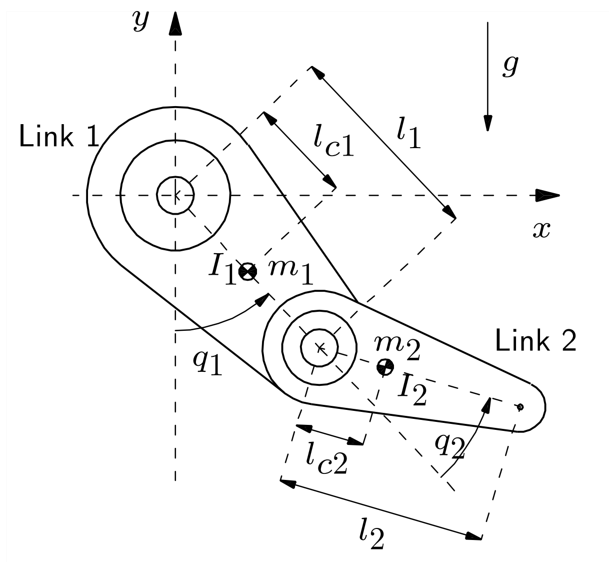

Figure 52: Diagram of the Pelican robot.

The components of the vector of gravitational torques $\boldsymbol{g}(\boldsymbol{q})$ are given by

$$\begin{aligned} g_1(\boldsymbol{q}) &= (m_1 l_{c1} + m_2 l_1) g \sin(q_1) + m_2 l_{c2} g \sin(q_1+q_2) \\ g_2(\boldsymbol{q}) &= m_2 l_{c2} g \sin(q_1+q_2) \, . \end{aligned}$$

- Consider the PD control with gravity compensation, where

$$\begin{aligned} K_p &= \mathrm{diag}\{k_p\} = \mathrm{diag}\{30\} \quad [\mathrm{Nm/rad}] \\ K_v &= \mathrm{diag}\{k_v\} = \mathrm{diag}\{7, \ 3\} \quad [\mathrm{Nm\ sec/rad}] \, . \end{aligned}$$

  – The components of $\boldsymbol{\tau}$, are given by

$$\begin{aligned} \tau_1 &= k_p \tilde{q}_1 - k_v \dot{q}_1 + g_1(\boldsymbol{q}) \\ \tau_2 &= k_p \tilde{q}_2 - k_v \dot{q}_2 + g_2(\boldsymbol{q}) \, . \end{aligned}$$

– The initial conditions are chosen as

$$\begin{aligned} q_1(0) = 0, \quad q_2(0) = 0 \\ \dot{q}_1(0) = 0, \quad \dot{q}_2(0) = 0\,, \end{aligned}$$

– and the desired joint positions as

$$q_{d1} = \pi/10, \quad q_{d2} = \pi/30 \quad [\mathrm{rad}]\,,$$

- hence, the initial state is taken to be

$$\begin{bmatrix} \tilde{\boldsymbol{q}}(0) \\ \\ \dot{\boldsymbol{q}}(0) \end{bmatrix} = \begin{bmatrix} \pi/10 \\ \pi/30 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.3141 \\ 0.1047 \\ 0 \\ 0 \end{bmatrix}.$$

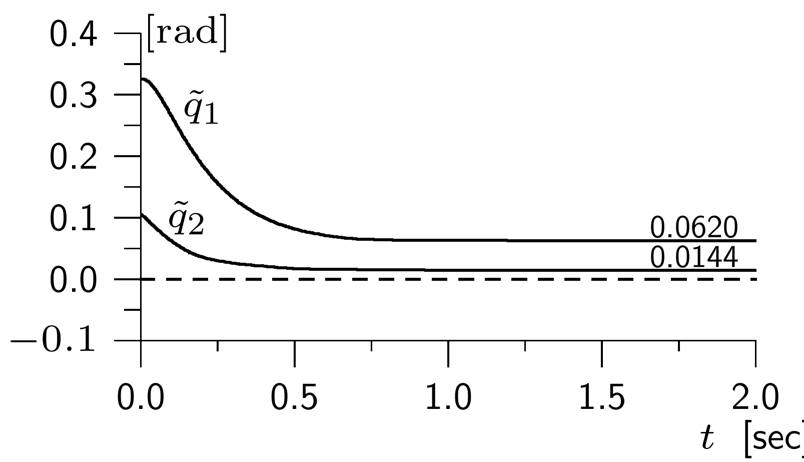

Figure 53: Position errors $\tilde{q}_1$ and $\tilde{q}_2$.

- From Figure above, steady state position errors due to to unmodelled friction phenomenon, can be seen.

Lyapunov function for global asymptotic stability

It is convenient to cite some properties of the vectorial function.

$$\mathbf{tanh}(\boldsymbol{x}) = \left[\tanh(x_1) \;\; \tanh(x_2) \;\; \cdots \;\; \tanh(x_n)\right]^T$$

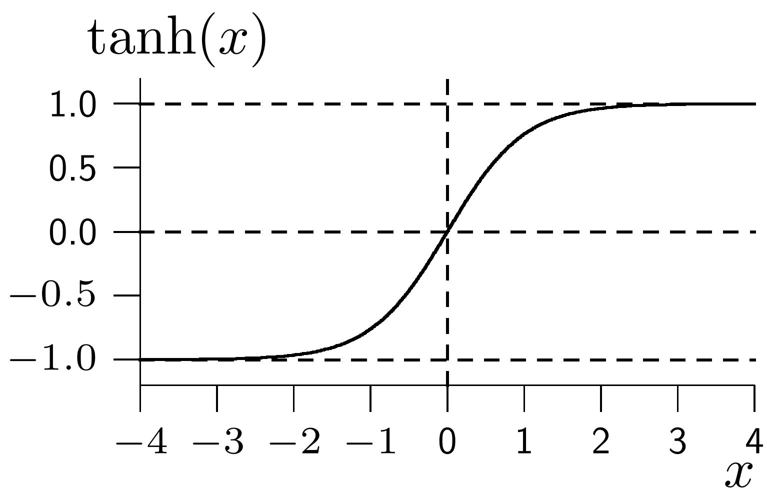

Figure 54: Tangent hyperbolic function, $\tanh(x)$

where $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

- $|x| \geq |\tanh(x)|$, and $1 \geq |\tanh(x)|$ for all $x \in \mathrm{IR}$ therefore,
  - $\star$
    $$\|\mathbf{tanh}(\boldsymbol{x})\| \leq \begin{cases} \|\boldsymbol{x}\| & \forall \;\; \boldsymbol{x} \in \mathrm{IR}^n \\ \sqrt{n} & \forall \;\; \boldsymbol{x} \in \mathrm{IR}^n \end{cases}$$
  - $\star$ $\mathbf{tanh}(\boldsymbol{x}) = \mathbf{0}$ if and only if $\boldsymbol{x} = \mathbf{0}$.
  - $\star$ For a symmetric positive definite matrix A, it has
    $$\frac{1}{2}\tilde{\boldsymbol{q}}^T A \tilde{\boldsymbol{q}} \geq \frac{1}{2}\lambda_{\min}\{A\} \|\mathbf{tanh}(\tilde{\boldsymbol{q}})\|^2 \quad \forall \; \tilde{\boldsymbol{q}} \in \mathrm{IR}^n$$
  - $\star$ If moreover, A is diagonal, then
    $$\mathbf{tanh}(\tilde{\boldsymbol{q}})^T A \tilde{\boldsymbol{q}} \geq \lambda_{\min}\{A\} \|\mathbf{tanh}(\tilde{\boldsymbol{q}})\|^2 \quad \forall \; \tilde{\boldsymbol{q}} \in \mathrm{IR}^n \, .$$

We present now an alternative stability analysis.

- Consider the Lyapunov function candidate

$$V(\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T K_p\tilde{\boldsymbol{q}} - \gamma\,\mathbf{tanh}(\tilde{\boldsymbol{q}})^T M(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad (19)$$

where

⋆ $\gamma > 0$ is a constant sufficiently small so as to satisfy simultaneously,

$$\frac{\lambda_{\min}\{K_p\}\lambda_{\min}\{M\}}{\lambda^2_{\mathrm{Max}}\{M\}} > \gamma^2 \qquad (20)$$

$$\frac{4\lambda_{\min}\{K_p\}\lambda_{\min}\{K_v\}}{\lambda^2_{\mathrm{Max}}\{K_v\} + 4\lambda_{\min}\{K_p\}[\sqrt{n}\,k_{C_1} + \lambda_{\mathrm{Max}}\{M\}]} > \gamma\,. \qquad (21)$$

⋆ There always exists $\gamma > 0$ arbitrarily small satisfying (20) and (21).

⋆ $V(\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}}) > 0$ because $K_p > 0$ and γ satisfies (20).

- The time derivative of the Lyapunov function candidate yields (19).

$$\begin{aligned}\dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) \quad = \quad & -\dot{\boldsymbol{q}}^T K_v \dot{\boldsymbol{q}} + \gamma \dot{\boldsymbol{q}}^T \mathsf{Sech}^2(\tilde{\boldsymbol{q}})^T M(\boldsymbol{q}) \dot{\boldsymbol{q}} - \gamma \mathbf{tanh}(\tilde{\boldsymbol{q}})^T K_p \tilde{\boldsymbol{q}} \\ & + \gamma \mathbf{tanh}(\tilde{\boldsymbol{q}})^T K_v \dot{\boldsymbol{q}} - \gamma \mathbf{tanh}(\tilde{\boldsymbol{q}})^T C(\boldsymbol{q}, \dot{\boldsymbol{q}})^T \dot{\boldsymbol{q}}\,,\end{aligned}$$

⋆ and is upperbounded by

$$\dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) \leq -\gamma \begin{bmatrix} \|\mathbf{tanh}(\tilde{\boldsymbol{q}})\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix}^T Q \begin{bmatrix} \|\mathbf{tanh}(\tilde{\boldsymbol{q}})\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix}$$

where

$$Q = \begin{bmatrix} \lambda_{\min}\{K_p\} & -\frac{1}{2}\lambda_{\mathrm{Max}}\{K_v\} \\ -\frac{1}{2}\lambda_{\mathrm{Max}}\{K_v\} & \frac{1}{\gamma}\lambda_{\min}\{K_v\} - \sqrt{n}\, k_{C1} - \lambda_{\mathrm{Max}}\{M\} \end{bmatrix}.$$

- $\dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})$ is a negative definite function because

$$\lambda_{\min}\{K_p\} > 0$$

and

$$\frac{4\lambda_{\min}\{K_p\}\lambda_{\min}\{K_v\}}{\lambda^2_{\mathrm{Max}}\{K_v\} + 4\lambda_{\min}\{K_p\}[\sqrt{n}k_{C1} + \lambda_{\mathrm{Max}}\{M\}]} > \gamma.$$

- Finally, Theorem 2.4 allows to establish global asymptotic stability of the origin.

Ch. 8. PD control with desired gravity compensation

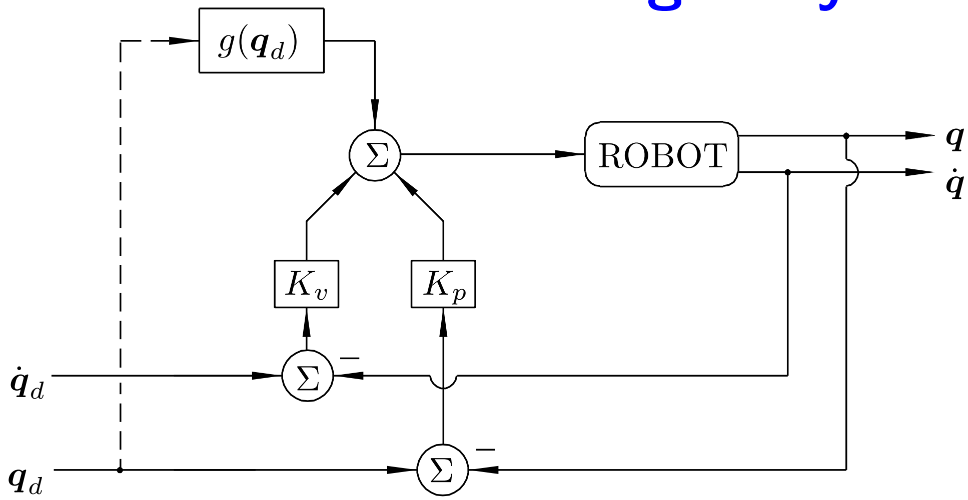

Figure 55: PD Control with desired gravity compensation.

Control law given by $\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + \boldsymbol{g}(\boldsymbol{q}_d)$, where

- $K_p, K_v \in \mathrm{IR}^{n\times n}$ are symmetric positive definite matrices
- $\boldsymbol{g}(\boldsymbol{q}_d)$ may be evaluated offline, it is not necessary anymore to evaluate $\boldsymbol{g}(\boldsymbol{q})$ in real time.

Considering the desired position $\boldsymbol{q}_d$ to be constant, the closed-loop equation may be written as

$$\frac{d}{dt}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \\ \dot{\boldsymbol{q}}\end{bmatrix} = \begin{bmatrix}-\dot{\boldsymbol{q}}\\ \\ M(\boldsymbol{q})^{-1}\left[K_p\tilde{\boldsymbol{q}} - K_v\dot{\boldsymbol{q}} - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d) - \boldsymbol{g}(\boldsymbol{q})\right]\end{bmatrix} \tag{22}$$

- $\begin{bmatrix}\tilde{\boldsymbol{q}}^T & \dot{\boldsymbol{q}}^T\end{bmatrix}^T = \boldsymbol{0} \in \mathbb{R}^{2n}$ is an equilibrium point.

- There are as many equilibria as solutions in $\tilde{\boldsymbol{q}}$, may have the equation $K_p\tilde{\boldsymbol{q}} = \boldsymbol{g}(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q}_d)$.

- If $K_p >> 0$ (sufficiently), then $\tilde{\boldsymbol{q}} = \boldsymbol{0} \in \mathbb{R}^n$ is the unique solution.

Example 8.1

Consider the model of an ideal pendulum

$$J\ddot{q} + mgl\,\sin(q) = \tau$$

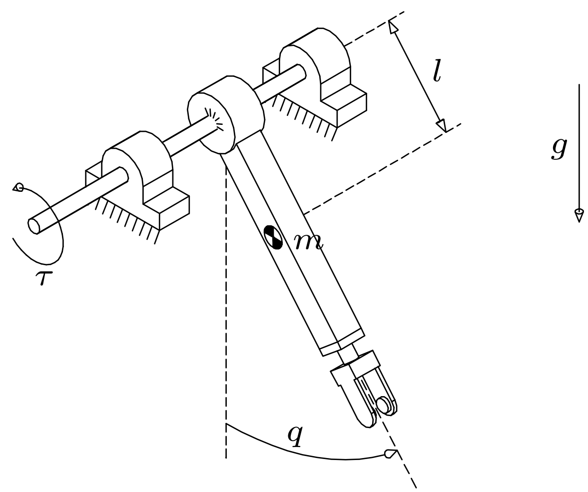

Figure 56: Pendulum.

where $g(q) = mgl\ \sin(q)$.

- The equilibria equation takes the form:

$$k_p\tilde{q} = mgl\ \left[\sin(q_d - \tilde{q}) - \sin(q_d)\right] .$$

- Consider the following numeric values,

$$\begin{array}{ll} J = 1 & mgl = 1 \\ k_p = 0.25 & q_d = \pi/2 \,. \end{array}$$

- Either via a graphical method or numeric algorithms, one may verify the equilibria,

$$\begin{bmatrix} \tilde{q} \\ \dot{q} \end{bmatrix} \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -0.51 \\ 0 \end{bmatrix}, \begin{bmatrix} -4.57 \\ 0 \end{bmatrix} \right\} .$$

- Consider now a larger value for k_p (sufficiently "large"), e.g.,

$$k_p = 1.25$$

In this scenario the origin is the unique equilibrium, i.e.,

$$\begin{bmatrix} \tilde{q} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathrm{IR}^2 .$$

Boundedness of position and velocity errors, $\tilde{q}$ and $\dot{q}$

Assume that

- K_p and K_v are positive definite (without assuming that K_p is sufficiently "large").

Then

- For a desired constant position $\boldsymbol{q}_d$,
- the closed loop equation has an equilibrium at the origin, but there might also be other equilibria.
- In spite of this, $\tilde{\boldsymbol{q}}(t)$ and $\dot{\boldsymbol{q}}(t)$ remain bounded for all initial conditions.

Define the following non-negative function:

$$
\begin{aligned}
V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) \;\; = \;\; & \mathcal{K}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathcal{U}(\boldsymbol{q}) - k_{\mathcal{U}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}} \\
& + \tilde{\boldsymbol{q}}^T \boldsymbol{g}(\boldsymbol{q}_d) + \frac{1}{2}\boldsymbol{g}(\boldsymbol{q}_d)^T K_p^{-1} \boldsymbol{g}(\boldsymbol{q}_d)
\end{aligned}
$$

where

- $\mathcal{K}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}}$ denotes the kinetic energy
- $\mathcal{U}(\boldsymbol{q})$ denotes the potential energy
- and $k_{\mathcal{U}} = \min_q\{\mathcal{U}(\boldsymbol{q})\}$.

The time derivative of $V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})$ along the trajectories of the closed loop system results

$$\dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) = -\dot{\boldsymbol{q}}^T \overbrace{K_v}^{Q} \dot{\boldsymbol{q}}\,.$$

- By invoking Lemma 2.2, we conclude that
  - ⋆ $\dot{\boldsymbol{q}}(t)$ and $\tilde{\boldsymbol{q}}(t)$ are bounded
  - ⋆ $\dot{\boldsymbol{q}}(t)$, is square integrable, that is,

$$\int_0^\infty \|\dot{\boldsymbol{q}}(t)\|^2 dt < \infty\,.$$

⋆ The explicit upper-bounds on the position and velocity errors are:

$$\|\tilde{\boldsymbol{q}}(t)\| \leq \frac{\|\boldsymbol{g}(\boldsymbol{q}_d)\| + \sqrt{\|\boldsymbol{g}(\boldsymbol{q}_d)\|^2 + 2\lambda_{\min}\{K_p\}V(\tilde{\boldsymbol{q}}(0), \dot{\boldsymbol{q}}(0))}}{\lambda_{\min}\{K_p\}} \tag{23}$$

for all $t \geq 0$, and

$$\|\dot{\boldsymbol{q}}(t)\|^2 \leq \frac{2V(\tilde{\boldsymbol{q}}(0), \dot{\boldsymbol{q}}(0))}{\lambda_{\min}\{M(\boldsymbol{q})\}} \tag{24}$$

for all $t \geq 0$.

Example 8.2

Consider again the model of the ideal pendulum

$$J\ddot{q} + mgl\,\sin(q) = \tau,$$

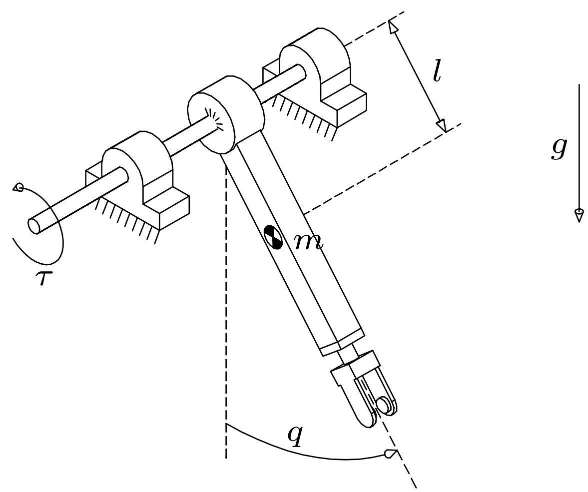

Figure 57: Pendulum.

The potential energy function is $\mathcal{U}(q) = mgl[1 - \cos(q)]$ and $k_U = 0$.

- Consider the numeric values

$$\begin{array}{ll} J = 1 & mgl = 1 \\ k_p = 0.25 & k_v = 0.50 \\ q_d = \pi/2\,. & \end{array}$$

- Assume that we use the PD control with desired gravity compensation to control it from the initial conditions $q(0) = 0$ and $\dot{q}(0) = 0$.

- It is easy to verify that

$$\begin{aligned} g(q_d) &= mgl\,\sin(\pi/2) = 1 \\ V(\tilde{q}(0), \dot{q}(0)) &= \frac{1}{2}k_p\tilde{q}^2(0) + mgl\tilde{q}(0) + \frac{1}{2k_p}(mgl)^2 = 3.87\,. \end{aligned}$$

– According to the bounds (23) and (24), we get

$$\tilde{q}^2(t) \quad \leq \quad \left[\frac{mgl + \sqrt{[mgl + k_p\tilde{q}(0)]^2 + (mgl)^2}}{k_p} \right]^2 \leq 117.79 \;\left[\,\mathrm{rad}^2\,\right]$$

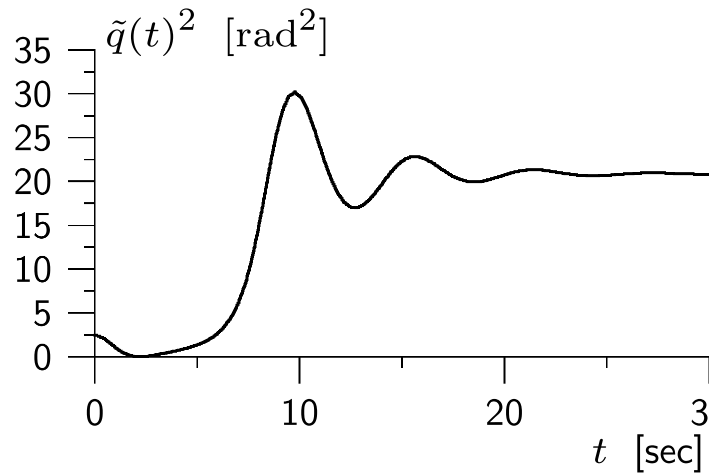

Figure 58: Graph of position errors, $\tilde{q}(t)^2$.

– and

$$\dot{q}^2(t) \quad \leq \quad \frac{2}{J}\left[\frac{k_p}{2}\tilde{q}^2(0) + mgl\tilde{q}(0) + \frac{1}{2k_p}(mgl)^2\right] \leq 7.75 \quad \left[\frac{\text{rad}}{\text{sec}}\right]^2.$$

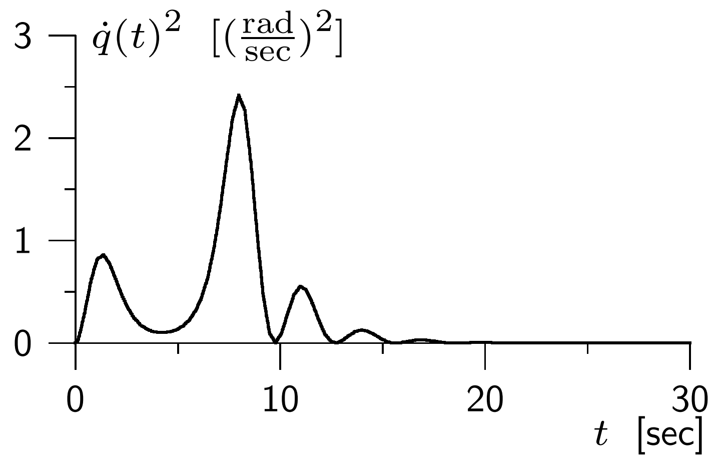

Figure 59: Graph of velocities, $\dot{q}(t)^2$.

- Evidence from simulation shows

$$\lim_{t\to\infty} \begin{bmatrix} \tilde{q}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} -4.57 \\ 0 \end{bmatrix} .$$

- The solutions tend precisely to one among the three equilibria which do not correspond to the origin.

- PD control with desired gravity compensation may fail to verify the set-point control objective.

Unicity of equilibrium

For robots having only revolute joints and considering the closed-loop equation (22), we have.

- The equilibria satisfy $\left[\tilde{\boldsymbol{q}}^T \;\; \dot{\boldsymbol{q}}^T\right]^T = \left[\tilde{\boldsymbol{q}}^T \;\; \mathbf{0}^T\right]^T \in \mathrm{I\!R}^{2n}$ where
  - ⋆ $\tilde{\boldsymbol{q}} \in \mathrm{I\!R}^n$ is solution of $\tilde{\boldsymbol{q}} = K_p^{-1}\left[\boldsymbol{g}(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q}_d)\right] = \boldsymbol{f}(\tilde{\boldsymbol{q}}, \boldsymbol{q}_d)$.

- For all vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{I\!R}^n$, we have that

$$\|\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{q}_d) - \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{q}_d)\| \leq \frac{k_g}{\lambda_{\min}\{K_p\}} \|\boldsymbol{x} - \boldsymbol{y}\|,$$

- If $\lambda_{\min}\{K_p\} > k_g$ we have (by contraction mapping theorem)
  - ⋆ the unique equilibrium is the origin, $\left[\tilde{\boldsymbol{q}}^T \;\; \dot{\boldsymbol{q}}^T\right]^T = \left[\mathbf{0}^T \;\; \mathbf{0}^T\right]^T \in \mathrm{I\!R}^{2n}$.

Global asymptotic stability

To study stability of the equilibrium of closed loop equation, consider

- $\lambda_{\min}\{K_p\} > k_g$ (the origin is the unique equilibrium)
- Lyapunov function candidate

$$\begin{aligned} V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) &= \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}})\dot{\boldsymbol{q}} + \mathcal{U}(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q}_d) \\ &\quad + \boldsymbol{g}(\boldsymbol{q}_d)^T \tilde{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}}\,. \end{aligned}$$

  ⋆ $V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})$ is radially unbounded and positive definite because $M(\boldsymbol{q}) > 0$ and $\lambda_{\min}\{K_p\} > k_g$.

- The time derivative of $V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})$ results in $\dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) = -\dot{\boldsymbol{q}}^T K_v \dot{\boldsymbol{q}} \leq 0$
- By applying the La Salle's Theorem 2.7, we have
  - The set Ω is given by

$$\begin{aligned}\Omega &= \left\{\boldsymbol{x} \in \mathrm{IR}^{2n} : \dot{V}(\boldsymbol{x}) = 0\right\} \\ &= \left\{\boldsymbol{x} = \begin{bmatrix}\tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}\end{bmatrix} \in \mathrm{IR}^{2n} : \dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) = 0\right\} \\ &= \{\tilde{\boldsymbol{q}} \in \mathrm{IR}^n, \dot{\boldsymbol{q}} = \boldsymbol{0} \in \mathrm{IR}^n\} \ .\end{aligned}$$

    * $\left[\tilde{\boldsymbol{q}}(0)^T \quad \dot{\boldsymbol{q}}(0)^T\right]^T = \boldsymbol{0} \in \mathrm{IR}^{2n}$ is the unique initial condition in Ω for which $\boldsymbol{x}(t) \in \Omega$ for all $t \geq 0$.
  - This is enough to guarantee global asymptotic stability of the origin $\left[\tilde{\boldsymbol{q}}^T \quad \dot{\boldsymbol{q}}^T\right]^T = \boldsymbol{0} \in \mathrm{IR}^{2n}$.

Example 8.3

Consider the 2-DOF *prototype robot* studied in Chapter 5,

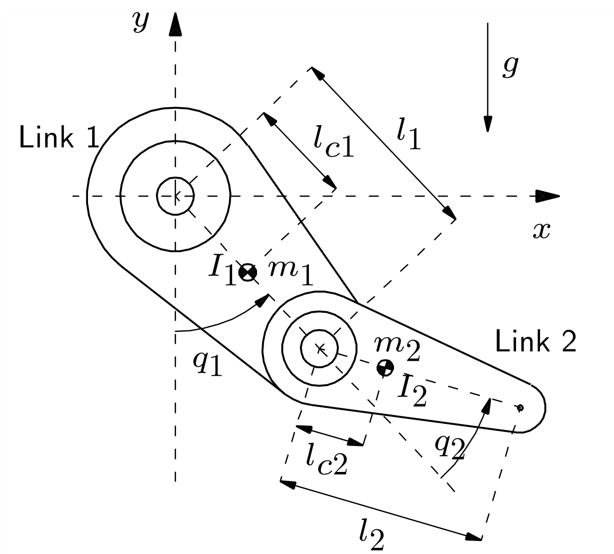

Figure 60: Diagram of the Pelican prototype robot with 2 degrees of freedom.

whose components of the gravitational torques vector $\boldsymbol{g}(\boldsymbol{q})$ are given by

$$\begin{aligned} g_1(\boldsymbol{q}) &= [m_1 l_{c1} + m_2 l_1] g \sin(q_1) + m_2 l_{c2} g \sin(q_1 + q_2) \\ g_2(\boldsymbol{q}) &= m_2 l_{c2} g \sin(q_1 + q_2) \, . \end{aligned}$$

The constant k_g may be obtained as

$$\begin{aligned} k_g &= n \left[\max{}_{i,j,q} \left| \frac{\partial g_i(\boldsymbol{q})}{\partial q_j} \right| \right] \\ &= n \left[[m_1 l_{c1} + m_2 l_1] g + m_2 l_{c2} g \right] \\ &= 23.94 \quad \left[\mathrm{kg\ m^2/sec^2} \right] . \end{aligned}$$

- Consider the PD controller with desired gravity compensation where

$$\lambda_{\min}\{K_p\} > k_g \, .$$

  - In particular, we pick

$$\begin{aligned} K_p &= \operatorname{diag}\{k_p\} = \operatorname{diag}\{30\} \quad [\mathrm{Nm/rad}] \, , \\ K_v &= \operatorname{diag}\{k_v\} = \operatorname{diag}\{7, \; 3\} \quad [\mathrm{Nm\ sec/rad}] \; . \end{aligned}$$

  - The components of the control input $\boldsymbol{\tau}$ are given by

$$\begin{aligned} \tau_1 &= k_p\tilde{q}_1 - k_v\dot{q}_1 + g_1(\boldsymbol{q}_d) \, , \\ \tau_2 &= k_p\tilde{q}_2 - k_v\dot{q}_2 + g_2(\boldsymbol{q}_d) \, . \end{aligned}$$

- The initial conditions corresponding to the positions and velocities, are set to

$$\begin{aligned} q_1(0) = 0, \quad q_2(0) = 0\,, \\ \dot{q}_1(0) = 0, \quad \dot{q}_2(0) = 0\,. \end{aligned}$$

- The desired joint positions are chosen as

$$q_{d1} = \pi/10 \;\; [\mathrm{rad}] \quad q_{d2} = \pi/30 \;\; [\mathrm{rad}]\,,$$

- hence, the initial state is set to

$$\begin{bmatrix} \tilde{\boldsymbol{q}}(0) \\ \\ \dot{\boldsymbol{q}}(0) \end{bmatrix} = \begin{bmatrix} \pi/10 \\ \pi/30 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.3141 \\ 0.1047 \\ 0 \\ 0 \end{bmatrix} \;\; [\,\mathrm{rad}\,]\,.$$

- Figure shows the experimental results

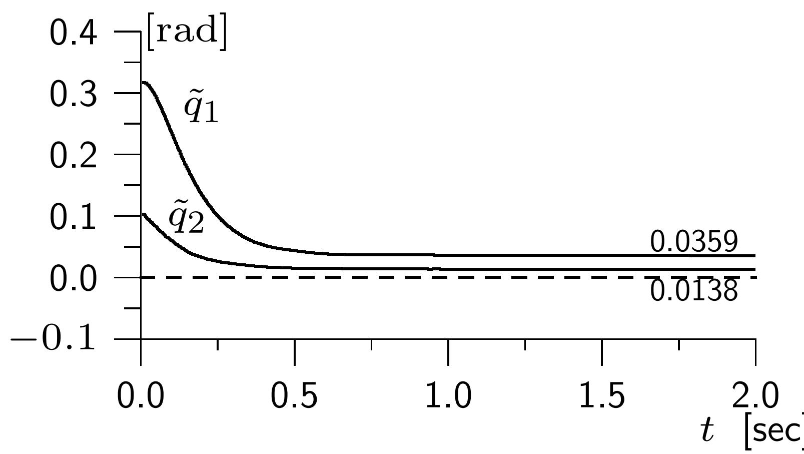

Figure 61: Position errors $\tilde{q}_1$ and $\tilde{q}_2$

  – The position errors do not vanish due to non–modeled friction effects.

Lyapunov function for global asymptotic stability

We show now a global asymptotic stability analysis without using La Salle's theorem

- Consider the following Lyapunov function candidate,

$$V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) = \frac{1}{2}\begin{bmatrix}\tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}\end{bmatrix}^T \overbrace{\begin{bmatrix}\frac{2}{\varepsilon_2}K_p & -\frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}M(\boldsymbol{q}) \\ -\frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}M(\boldsymbol{q}) & M(\boldsymbol{q})\end{bmatrix}}^{P}\begin{bmatrix}\tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}\end{bmatrix} \underbrace{+\mathcal{U}(\boldsymbol{q}) - \mathcal{U}(\boldsymbol{q}_d) + \boldsymbol{g}(\boldsymbol{q}_d)^T\tilde{\boldsymbol{q}} + \frac{1}{\varepsilon_1}\tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}}}_{f(\tilde{\boldsymbol{q}})},$$

which can be rewritten as

$$\begin{aligned} V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) &= \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}} + \mathcal{U}(\boldsymbol{q}) - \mathcal{U}(\boldsymbol{q}_d) + \boldsymbol{g}(\boldsymbol{q}_d)^T\tilde{\boldsymbol{q}} \\ & \quad + \left[\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right] \tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}} - \frac{\varepsilon_0}{1 + \|\tilde{\boldsymbol{q}}\|} \tilde{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}} \end{aligned}$$

where $\varepsilon_0 > 0$, $\varepsilon_1 > 2$ and $\varepsilon_2 > 2$ are chosen so that

$$\frac{2\lambda_{\min}\{K_p\}}{k_g} > \varepsilon_1 > 2$$

$$\varepsilon_2 = \frac{2\varepsilon_1}{\varepsilon_1 - 2} > 2$$

$$\sqrt{\frac{2\lambda_{\min}\{K_p\}}{\varepsilon_2 \beta}} > \varepsilon_0 > 0 \quad \text{with} \quad \beta \ (\geq \lambda_{\text{Max}}\{M(\boldsymbol{q})\} \ ) .$$

– To show positive definitness, we rearrange

$$
\begin{aligned}
V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) \quad = \quad & \frac{1}{2}\left[-\dot{\boldsymbol{q}}+\varepsilon \tilde{\boldsymbol{q}}\right]^T M(\boldsymbol{q})\left[-\dot{\boldsymbol{q}}+\varepsilon \tilde{\boldsymbol{q}}\right]+\frac{1}{2} \tilde{\boldsymbol{q}}^T\left[\frac{2}{\varepsilon_2} K_p-\varepsilon^2 M(\boldsymbol{q})\right] \tilde{\boldsymbol{q}} \\
& +\underbrace{\mathcal{U}(\boldsymbol{q})-\mathcal{U}(\boldsymbol{q}_d)+\boldsymbol{g}(\boldsymbol{q}_d)^T \tilde{\boldsymbol{q}}+\frac{1}{\varepsilon_1} \tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}}}_{f(\tilde{\boldsymbol{q}})},
\end{aligned}
$$

with $\varepsilon=\frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}$.

* $V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})$ is positive definite because $M(\boldsymbol{q})>0$, and
* $\left[\frac{2}{\varepsilon_2} K_p-\varepsilon^2 M(\boldsymbol{q})\right]>0$ and $\lambda_{\min}\{K_p\}>k_g$.

- The time derivative of the above Lyapunov function candidate takes the form

$$\begin{aligned}\dot V(\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}}) &= -\dot{\boldsymbol{q}}^T K_v \dot{\boldsymbol{q}} + \varepsilon \dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}} - \varepsilon \tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}} + \varepsilon \tilde{\boldsymbol{q}}^T K_v \dot{\boldsymbol{q}} \\ &\quad - \varepsilon \dot{\boldsymbol{q}}^T C(\boldsymbol{q},\dot{\boldsymbol{q}})\tilde{\boldsymbol{q}} - \varepsilon \tilde{\boldsymbol{q}}^T [\boldsymbol{g}(\boldsymbol{q}_d) - \boldsymbol{g}(\boldsymbol{q})] - \dot{\varepsilon} \tilde{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}}\,,\end{aligned}$$

  – which is upperbounded by

$$\begin{aligned}\dot V(\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}}) &\le -\,\varepsilon \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix}^T \overbrace{\begin{bmatrix} \lambda_{\min}\{K_p\} - k_g & -\frac{1}{2}\lambda_{\mathrm{Max}}\{K_v\} \\ -\frac{1}{2}\lambda_{\mathrm{Max}}\{K_v\} & \frac{1}{2\varepsilon_0}\lambda_{\min}\{K_v\} \end{bmatrix}}^{Q} \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix} \\ &\quad - \frac{1}{2}\underbrace{\left[\lambda_{\min}\{K_v\} - 2\varepsilon_0(k_{C_1} + 2\beta)\right]}_{\delta} \|\dot{\boldsymbol{q}}\|^2 \,.\end{aligned}$$

– The matrix Q is positive definite if it holds that

$$\begin{aligned} \lambda_{\min}\{K_p\} &> k_g \,, \\ \frac{2\lambda_{\min}\{K_v\}(\lambda_{\min}\{K_p\} - k_g)}{\lambda^2_{\mathrm{Max}}\{K_v\}} &> \varepsilon_0 \,, \end{aligned}$$

and we have that $\delta > 0$ if

$$\frac{\lambda_{\min}\{K_v\}}{2[k_{C_1} + 2\beta]} > \varepsilon_0 \,.$$

– Under this scenario, we get

$$\begin{aligned}\dot{V}(\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}}) &\leq -\frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}\lambda_{\min}\{Q\}\left[\|\tilde{\boldsymbol{q}}\|^2+\|\dot{\boldsymbol{q}}\|^2\right]-\frac{\delta}{2}\|\dot{\boldsymbol{q}}\|^2,\\ &\leq -\varepsilon_0\lambda_{\min}\{Q\}\frac{\|\tilde{\boldsymbol{q}}\|^2}{1+\|\tilde{\boldsymbol{q}}\|}-\frac{\delta}{2}\|\dot{\boldsymbol{q}}\|^2,\end{aligned}$$

which is a negative definite function.

- We conclude that the origin $\begin{bmatrix}\tilde{\boldsymbol{q}}^T & \dot{\boldsymbol{q}}^T\end{bmatrix}^T = \mathbf{0} \in \mathbb{R}^{2n}$ is a globally asymptotically stable (Theorem 2.4).

Ch. 9. PID control

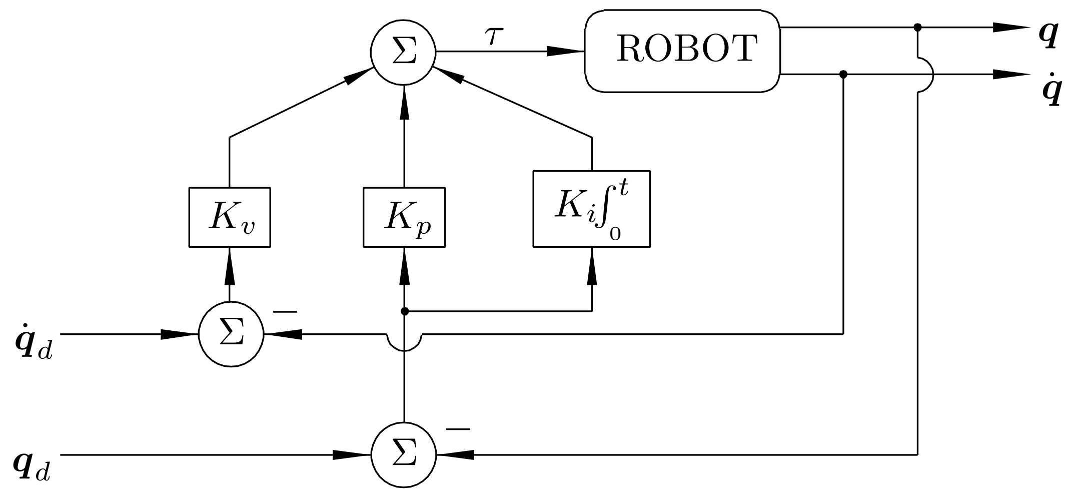

Figure 62: PID control.

The PID control law is given by $\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + K_i \int_0^t \tilde{\boldsymbol{q}}(\sigma)\, d\sigma$, where

- $K_p, K_v, K_i \in \mathrm{IR}^{n\times n}$, are symmetric positive definite matrices.

- The control law may be expressed via the two following equations,

$$\begin{aligned} \boldsymbol{\tau} &= K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + K_i\boldsymbol{\xi} \\ \dot{\boldsymbol{\xi}} &= \tilde{\boldsymbol{q}}\,. \end{aligned}$$

The closed loop equation is

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{\xi} \\ \tilde{\boldsymbol{q}} \\ \dot{\tilde{\boldsymbol{q}}} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\tilde{\boldsymbol{q}}} \\ \ddot{\boldsymbol{q}}_d - M(\boldsymbol{q})^{-1}\left[K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + K_i\boldsymbol{\xi} - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right] \end{bmatrix}.$$

- If the desired position $\boldsymbol{q}_d(t)$ is constant, the equilibrium is

$$\begin{bmatrix} \boldsymbol{\xi} \\ \\ \tilde{\boldsymbol{q}} \\ \\ \dot{\tilde{\boldsymbol{q}}} \end{bmatrix} = \begin{bmatrix} {K_i}^{-1} \, \boldsymbol{g}(\boldsymbol{q}_d) \\ \\ \mathbf{0} \\ \\ \mathbf{0} \end{bmatrix} \in \mathrm{IR}^{3n} \, .$$

This equilibrium may be translated to the origin (via change of variable)

$$\boldsymbol{z} = \boldsymbol{\xi} - {K_i}^{-1}\boldsymbol{g}(\boldsymbol{q}_d) \, .$$

The corresponding closed loop equation is

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{z} \\ \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{q}} \\ -\dot{\boldsymbol{q}} \\ M(\boldsymbol{q})^{-1}\left[K_p\tilde{\boldsymbol{q}} - K_v\dot{\boldsymbol{q}} + K_i\boldsymbol{z} + \boldsymbol{g}(\boldsymbol{q}_d) - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right] \end{bmatrix}. \tag{25}$$

- Above equation is autonomous and its unique equilibrium is the origin $\left[\boldsymbol{z}^T \;\; \tilde{\boldsymbol{q}}^T \;\; \dot{\boldsymbol{q}}^T\right]^T = \boldsymbol{0} \in \mathbb{R}^{3n}$.

For the sequel,we adopt the following global change of variables,

$$\begin{bmatrix} \boldsymbol{w} \\ \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \alpha\, I & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} \quad \text{with} \quad \alpha > 0.$$

The closed loop equation (25) may be expressed as

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{w}\\ \tilde{\boldsymbol{q}}\\ \dot{\boldsymbol{q}}\end{bmatrix} = \begin{bmatrix}\alpha\,\tilde{\boldsymbol{q}}-\dot{\boldsymbol{q}}\\ -\dot{\boldsymbol{q}}\\ M(\boldsymbol{q})^{-1}\left[\left[K_p-\frac{1}{\alpha}K_i\right]\tilde{\boldsymbol{q}}-K_v\dot{\boldsymbol{q}}+\frac{1}{\alpha}K_i\boldsymbol{w}+\boldsymbol{g}(\boldsymbol{q}_d)-C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}-\boldsymbol{g}(\boldsymbol{q})\right]\end{bmatrix}. \tag{26}$$

- Above equation is autonomous
- The origin of the state space, $\begin{bmatrix} \boldsymbol{w}^T & \tilde{\boldsymbol{q}}^T & \dot{\boldsymbol{q}}^T \end{bmatrix}^T = \mathbf{0} \in \mathbb{R}^{3n}$ is the unique equilibrium.
- If K_p and K_v are sufficiently "large" and K_i sufficiently "small" in the following sense

$$\frac{\lambda_{\min}\{M\}\lambda_{\min}\{K_v\}}{\lambda_{\mathrm{Max}}{}^2\{M\}} > \frac{\lambda_{\mathrm{Max}}\{K_i\}}{\lambda_{\min}\{K_p\} - k_g},$$

and moreover

$$\lambda_{\min}\{K_p\} > k_g\,,$$

then, the set-point control objective is achieved locally.

Lyapunov function candidate

- A positive definite Lyapunov function candidate is

$$
\begin{aligned}
V(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}, \boldsymbol{w}) \;=\;& \frac{1}{2}\begin{bmatrix} \boldsymbol{w} \\ \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix}^T \begin{bmatrix} \frac{1}{\alpha}K_i & 0 & 0 \\ 0 & \alpha K_v & -\alpha M(\boldsymbol{q}) \\ 0 & -\alpha M(\boldsymbol{q}) & M(\boldsymbol{q}) \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} \\
& + \frac{1}{2}\tilde{\boldsymbol{q}}^T \left[K_p - \frac{1}{\alpha}K_i\right]\tilde{\boldsymbol{q}} + \mathcal{U}(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q}_d) + \tilde{\boldsymbol{q}}^T \boldsymbol{g}(\boldsymbol{q}_d)
\end{aligned}
$$

  - ⋆ $\mathcal{U}(\boldsymbol{q})$ denotes as usual, the robot's potential energy, and
  - ⋆ α is a positive constant satisfying

$$
\frac{\lambda_{\min}\{M\}\lambda_{\min}\{K_v\}}{{\lambda_{\mathrm{Max}}}^2\{M\}} > \alpha > \frac{\lambda_{\mathrm{Max}}\{K_i\}}{\lambda_{\min}\{K_p\} - k_g}\,.
$$

Time derivative of the Lyapunov function candidate

- It may be written as

$$\begin{aligned}\dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}, \boldsymbol{w}) &= -\dot{\boldsymbol{q}}^T \left[K_v - \alpha M(\boldsymbol{q})\right] \dot{\boldsymbol{q}} - \tilde{\boldsymbol{q}}^T \left[\alpha K_p - K_i\right] \tilde{\boldsymbol{q}} \\ &\quad - \alpha \tilde{\boldsymbol{q}}^T C(\boldsymbol{q}, \dot{\boldsymbol{q}})^T \dot{\boldsymbol{q}} - \alpha \tilde{\boldsymbol{q}}^T \left[\boldsymbol{g}(\boldsymbol{q}_d) - \boldsymbol{g}(\boldsymbol{q})\right]\end{aligned}$$

  - It can be upperbounded by

$$\dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}, \boldsymbol{w}) \leq - \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix}^T \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22}(\dot{\boldsymbol{q}}) \end{bmatrix} \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix}$$

  where

$$\begin{aligned}Q_{11} &= \alpha \ \left[\lambda_{\min}\{K_p\} - k_g\right] - \lambda_{\mathrm{Max}}\{K_i\}, \\ Q_{22}(\dot{\boldsymbol{q}}) &= \lambda_{\min}\{K_v\} - \alpha \ \left[\lambda_{\mathrm{Max}}\{M\} + k_{C_1} \|\dot{\boldsymbol{q}}\|\right].\end{aligned}$$

– It is possible to prove that there exists a ball $\mathcal{D}$

$$\mathcal{D} := \left\{ \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}, \boldsymbol{w} \in \mathbb{R}^n : \left\| \begin{matrix} \boldsymbol{w} \\ \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{matrix} \right\| < \eta \right\}$$

on which $\dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}, \boldsymbol{w})$ is negative semidefinite.

– Therefore, according with Theorem 2.2, the origin of the closed loop equation (26), is a stable equilibrium.

Asymptotic stability

We may use La Salle's theorem (Theorem 2.7).

- The set Ω is given by

$$\begin{aligned}
\Omega &= \left\{ \boldsymbol{x} \in \mathrm{IR}^{3n} : \dot{V}(\boldsymbol{x}) = 0 \right\} \\
&= \left\{ \boldsymbol{x} = \begin{bmatrix} \boldsymbol{w} \\ \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} \in \mathrm{IR}^{3n} \ : \ \dot{V}(\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}, \boldsymbol{w}) = 0 \right\} \\
&= \left\{ \boldsymbol{w} \in \mathrm{IR}^{n}, \tilde{\boldsymbol{q}} = \mathbf{0} \in \mathrm{IR}^{n}, \dot{\boldsymbol{q}} = \mathbf{0} \in \mathrm{IR}^{n} \right\} .
\end{aligned}$$

⋆ $\left[\boldsymbol{w}(0)^T \quad \tilde{\boldsymbol{q}}(0)^T \quad \dot{\boldsymbol{q}}(0)^T \right]^T = \mathbf{0} \in \mathrm{IR}^{3n}$ is the only initial condition in Ω for which $\boldsymbol{x}(t) \in \Omega$ for all $t \geq 0$.

⋆ We conclude that the origin is locally asymptotically stable.

Tuning procedure

The preceding stability analysis allows to extract a simple tuning procedure.

- $\lambda_{\mathrm{Max}}\{K_i\} \geq \lambda_{\min}\{K_i\} > 0$
- $\lambda_{\mathrm{Max}}\{K_p\} \geq \lambda_{\min}\{K_p\} > k_g$
- $\lambda_{\mathrm{Max}}\{K_v\} \geq \lambda_{\min}\{K_v\} > \dfrac{\lambda_{\mathrm{Max}}\{K_i\}}{\lambda_{\min}\{K_p\} - k_g} \cdot \dfrac{\lambda_{\mathrm{Max}}{}^2\{M\}}{\lambda_{\min}\{M\}}.$
  - It requires the knowledge of the structure of $M(\boldsymbol{q})$ and $\boldsymbol{g}(\boldsymbol{q})$.
  - Nonetheless, it is sufficient to have upper bounds on $\lambda_{\mathrm{Max}}\{M(\boldsymbol{q})\}$ and k_g, and a lower bound for $\lambda_{\min}\{M(\boldsymbol{q})\}$.

Example 9.2

Consider the 2-DOF *prototype robot* showed in Figure 63.

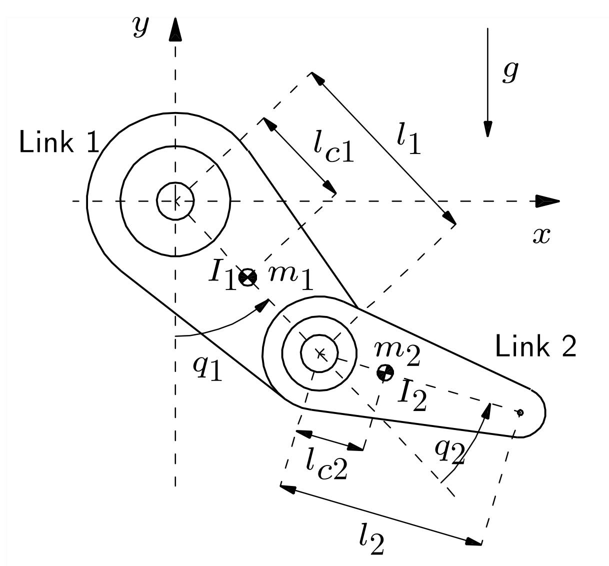

Figure 63: Diagram of the Pelican prototype robot.

The elements of the inertia matrix $M(\boldsymbol{q})$ are

$$\begin{aligned}
M_{11}(\boldsymbol{q}) &= m_1 l_{c1}^2 + m_2 \left(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos(q_2) \right) + I_1 + I_2 \\
M_{12}(\boldsymbol{q}) &= m_2 \left(l_{c2}^2 + l_1 l_{c2} \cos(q_2) \right) + I_2 \\
M_{21}(\boldsymbol{q}) &= m_2 \left(l_{c2}^2 + l_1 l_{c2} \cos(q_2) \right) + I_2 \\
M_{22}(\boldsymbol{q}) &= m_2 l_{c2}^2 + I_2 .
\end{aligned}$$

The components of the gravitational torques vector $\boldsymbol{g}(\boldsymbol{q})$, are given by

$$\begin{aligned}
g_1(\boldsymbol{q}) &= (m_1 l_{c1} + m_2 l_1) g \sin(q_1) + m_2 l_{c2} g \sin(q_1 + q_2) \\
g_2(\boldsymbol{q}) &= m_2 l_{c2} g \sin(q_1 + q_2) .
\end{aligned}$$

- Firstly, we compute the value of k_g using the numeric values listed in Table 5.1

$$\begin{aligned} k_g &= n\left(\text{Max}_{\ i,j,q}\left|\frac{\partial g_i(\boldsymbol{q})}{\partial q_j}\right|\right) \\ &= n(m_1 l_{c1} + m_2 l_1 + m_2 l_{c2})g \\ &= 23.94 \quad \left[\text{kg m}^2/\text{sec}^2\right] \end{aligned}$$

- We proceed now to compute numerically $\lambda_{\min}\{M(\boldsymbol{q})\}$ and $\lambda_{\text{Max}}\{M(\boldsymbol{q})\}$

$$\begin{aligned} \lambda_{\min}\{M(\boldsymbol{q})\} &= 0.011 \quad \left[\text{kg m}^2\right], \\ \lambda_{\text{Max}}\{M(\boldsymbol{q})\} &= 0.361, \quad \left[\text{kg m}^2\right] \end{aligned}$$

which correspond to $q_2 = 0$.

- By following the tuning procedure, we finally determine the following matrices,

$$
\begin{aligned}
K_i &= \operatorname{diag}\{1.5\} \quad [\mathrm{Nm}\,/\,(\mathrm{rad\ sec})]\,, \\
K_p &= \operatorname{diag}\{30\} \quad [\mathrm{Nm}\,/\,\mathrm{rad}]\,, \\
K_v &= \operatorname{diag}\{7, 3\} \quad [\mathrm{Nm\ sec}\,/\,\mathrm{rad}]\,.
\end{aligned}
$$

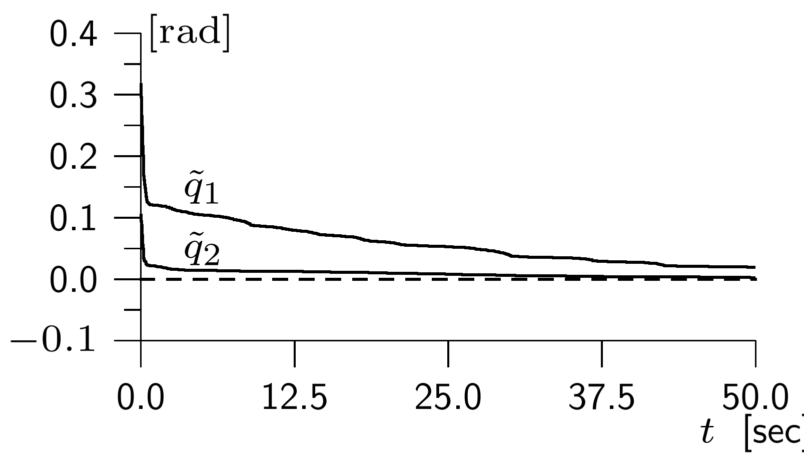

Figure 64: Position errors $\tilde{q}_1$ and $\tilde{q}_2$

From Figure we may conclude that the transient response is slower

- This is due to the fact that the tuning procedure limits $\lambda_{\min}\{K_i\}$ by a relatively small upperbound.

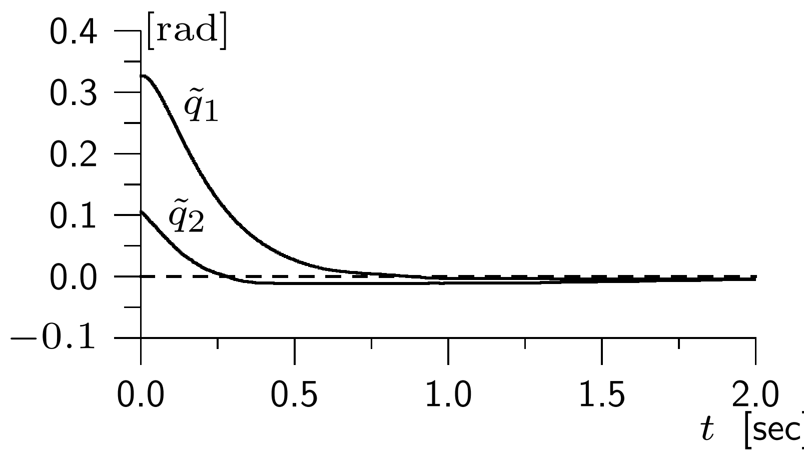

Figure 65: Position errors $\tilde{q}_1$ and $\tilde{q}_2$

If the tuning procedure is violated the performance of the PID controller improves up.

- The latter results have been obtained increasing the values of K_i to

$$K_i = \mathrm{diag}\{70, 100\} \quad [\mathrm{Nm} \,/\, (\mathrm{rad\ sec})] \,.$$

Part III

TRACKING CONTROL

Introduction

Consider the dynamic model

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}\ .$$

In terms of the state vector $\begin{bmatrix} \boldsymbol{q}^T & \dot{\boldsymbol{q}}^T \end{bmatrix}^T$ gives

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ M(\boldsymbol{q})^{-1}\left[\boldsymbol{\tau}(t) - C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right] \end{bmatrix}$$

where

- $M(\boldsymbol{q}) \in \mathrm{IR}^{n\times n}$ is the inertia matrix,
- $C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \in \mathrm{IR}^n$ is the vector of centrifugal and Coriolis forces,
- $\boldsymbol{g}(\boldsymbol{q}) \in \mathrm{IR}^n$ is el vector of gravitational forces and torques and
- $\boldsymbol{\tau} \in \mathrm{IR}^n$ is a vector of external forces and torques applied at the joints.
- The vectors $\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}} \in \mathrm{IR}^n$ denote the position, velocity and joint acceleration respectively.

Given a set of vectorial bounded functions $\boldsymbol{q}_d$, $\dot{\boldsymbol{q}}_d$ and $\ddot{\boldsymbol{q}}_d$ referred to as *desired* joint positions, velocities and accelerations.

The *objective of tracking control* consists on finding $\boldsymbol{\tau}$ such that

$$\lim_{t \to \infty} \tilde{\boldsymbol{q}}(t) = \mathbf{0}$$

where

- $\tilde{\boldsymbol{q}} \in \mathbb{R}^n$ stands for the joint position errors vector

$$\tilde{\boldsymbol{q}}(t) := \boldsymbol{q}_d(t) - \boldsymbol{q}(t)\,,$$

- $\dot{\tilde{\boldsymbol{q}}}(t) = \dot{\boldsymbol{q}}_d(t) - \dot{\boldsymbol{q}}(t)$ stands for the velocity error.

In general, a control law may be expressed as

$$\boldsymbol{\tau} = \boldsymbol{\tau}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d, M(\boldsymbol{q}), C(\boldsymbol{q}, \dot{\boldsymbol{q}}), \boldsymbol{g}(\boldsymbol{q})\right).$$

For practical purposes it is desirable that the controller does not depend on the joint acceleration $\ddot{\boldsymbol{q}}$.

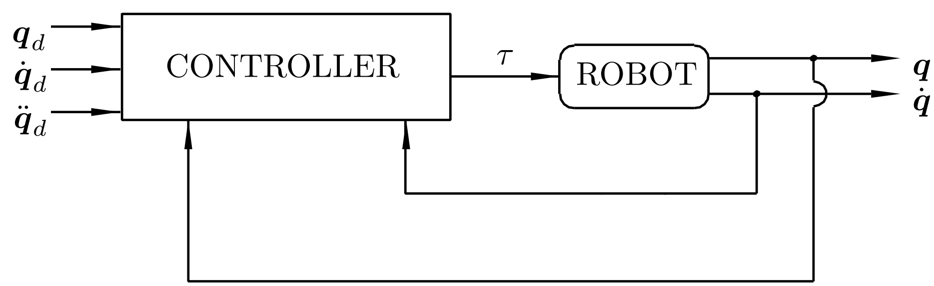

Figure 66: Tracking control: closed loop system.

A methodology to analyze the stability may be summarized in:

1. Derivation of the closed loop dynamic equation.

2. Representation of the closed loop equation in the state-space form,

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{q}_d - \boldsymbol{q} \\ \dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}} \end{bmatrix} = \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d, M(\boldsymbol{q}), C(\boldsymbol{q}, \dot{\boldsymbol{q}}), \boldsymbol{g}(\boldsymbol{q}))\,.$$

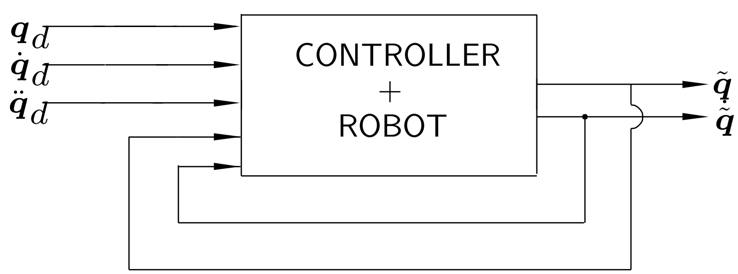

Figure 67: Tracking control closed-loop system: Input-output representation.

3. Study of the existence and possible unicity of the equilibrium

4. Proposition of a Lyapunov function candidate to study the stability of any equilibrium of interest

5. Alternatively to step 4), determine the qualitative behavior of the solutions of the closed loop equation.

The controllers that we consider are, in order,

- Computed torque control and Computed torque+ control.
- PD control with compensation and PD+ control.
- Feedforward control and PD plus feedforward control.

Ch. 10. Computed-torque control and Computed-torque+ control

In this chapter we study two controllers that do not present explicitly the linear PD term

- Computed–torque control,
- Computed–torque+ control.

Computed-torque control

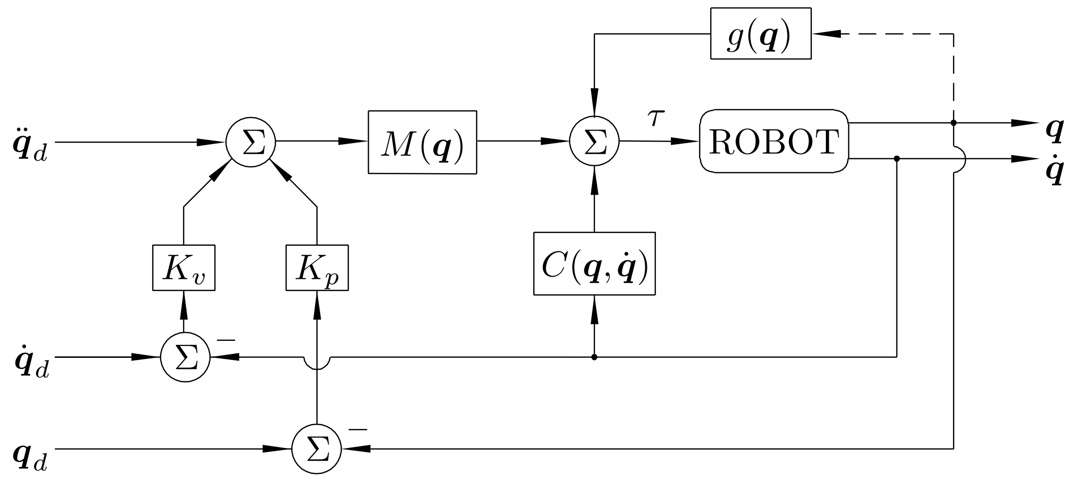

Figure 68: Computed-torque control.

The corresponding equation to computed-torque control is given by

$$\boldsymbol{\tau} = M(\boldsymbol{q}) \left[\ddot{\boldsymbol{q}}_d + K_v \dot{\tilde{\boldsymbol{q}}} + K_p \tilde{\boldsymbol{q}} \right] + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) \,, \tag{27}$$

where K_p and K_v are symmetric positive definite matrices.

The closed loop equation is

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} = M(\boldsymbol{q})\left[\ddot{\boldsymbol{q}}_d + K_v\dot{\tilde{\boldsymbol{q}}} + K_p\tilde{\boldsymbol{q}}\right] .$$

Above equation reduces to

$$\ddot{\tilde{\boldsymbol{q}}} + K_v\dot{\tilde{\boldsymbol{q}}} + K_p\tilde{\boldsymbol{q}} = \boldsymbol{0}$$

which in turn, may be expressed in terms of the state vector $\left[\tilde{\boldsymbol{q}}^T \quad \dot{\tilde{\boldsymbol{q}}}^T\right]^T$ as

$$\frac{d}{dt}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix} = \begin{bmatrix}\dot{\tilde{\boldsymbol{q}}}\\ -K_p\tilde{\boldsymbol{q}} - K_v\dot{\tilde{\boldsymbol{q}}}\end{bmatrix} = \begin{bmatrix}0 & I\\ -K_p & -K_v\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix}, \qquad (28)$$

where I is the identity matrix of dimension n.

It is important to remark that

- the closed loop equation (28) is represented by a linear autonomous differential equation,
- whose unique equilibrium point is given by $\begin{bmatrix}\tilde{\boldsymbol{q}}^T & \dot{\tilde{\boldsymbol{q}}}^T\end{bmatrix}^T = \mathbf{0} \in \mathrm{I\!R}^{2n}$.

Consider next

- the globally positive definite Lyapunov function candidate

$$\begin{aligned} V(\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) &= \frac{1}{2}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix}^T \begin{bmatrix} K_p+\varepsilon K_v & \varepsilon I\\ \varepsilon I & I\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix}\\ &= \frac{1}{2}\Big[\dot{\tilde{\boldsymbol{q}}}+\varepsilon\tilde{\boldsymbol{q}}\Big]^T\Big[\dot{\tilde{\boldsymbol{q}}}+\varepsilon\tilde{\boldsymbol{q}}\Big]+\frac{1}{2}\tilde{\boldsymbol{q}}^T\big[K_p+\varepsilon K_v-\varepsilon^2 I\big]\,\tilde{\boldsymbol{q}} \quad (29)\end{aligned}$$

where the constant ε satisfies,

$$K_v - \varepsilon I > 0\,.$$

$$K_p + \varepsilon K_v - \varepsilon^2 I > 0\,.$$

- Evaluating the total time derivative of $V(\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}})$ we get that

$$\begin{aligned}
\dot{V}(\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) &= -\dot{\tilde{\boldsymbol{q}}}^T [K_v - \varepsilon I]\, \dot{\tilde{\boldsymbol{q}}} - \varepsilon \tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}} \\
&= -\begin{bmatrix} \tilde{\boldsymbol{q}} \\ \\ \dot{\tilde{\boldsymbol{q}}} \end{bmatrix}^T \begin{bmatrix} \varepsilon K_p & 0 \\ \\ 0 & K_v - \varepsilon I \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \\ \dot{\tilde{\boldsymbol{q}}} \end{bmatrix} .
\end{aligned} \tag{30}$$

⋆ $\dot{V}(\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}})$ in (30) is globally negative definite.

- In view of Theorem 2.4, we get that the origin $\begin{bmatrix}\tilde{\boldsymbol{q}}^T & \dot{\tilde{\boldsymbol{q}}}^T\end{bmatrix}^T = \boldsymbol{0} \in \mathrm{IR}^{2n}$ is globally uniformly asymptotically stable and therefore

$$\lim_{t\to\infty} \dot{\tilde{\boldsymbol{q}}}(t) = \boldsymbol{0}$$

$$\lim_{t\to\infty} \tilde{\boldsymbol{q}}(t) = \boldsymbol{0}\,.$$

- For practical purposes, we can choose,

$$\begin{aligned} K_p &= \operatorname{diag}\left\{\omega_1^2, \cdots, \omega_n^2\right\} \\ K_v &= \operatorname{diag}\left\{2\omega_1, \cdots, 2\omega_n\right\}\,. \end{aligned}$$

Example 10.2

Consider the Pelican prototype robot studied in Chapter 5

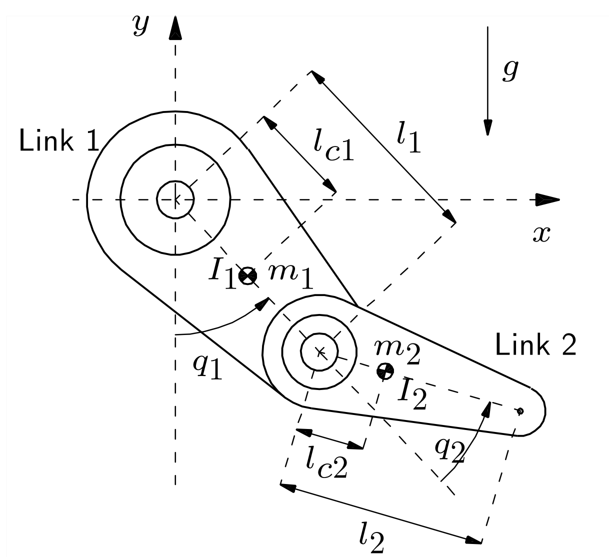

Figure 69: Diagram of the Pelican prototype.

Consider the Computed-torque control (27) on this robot for tracking control.

- The desired reference trajectory, $\boldsymbol{q}_d(t)$, is given by Equation (13).

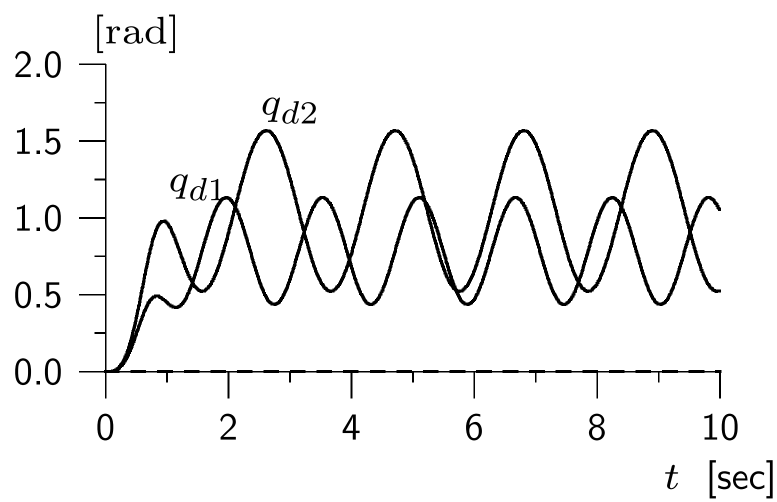

Figure 70: Desired reference trajectories.

$$\begin{bmatrix} q_{d1} \\ \\ q_{d2} \end{bmatrix} = \begin{bmatrix} b_1[1 - e^{-2.0\ t^3}] + c_1[1 - e^{-2.0\ t^3}] \sin(\omega_1 t) \\ \\ b_2[1 - e^{-2.0\ t^3}] + c_2[1 - e^{-2.0\ t^3}] \sin(\omega_2 t) \end{bmatrix} \qquad [\mathrm{rad}]$$

where

$\star$ $b_1 = \pi/4$ [rad], $c_1 = \pi/9$ [rad] and $\omega_1 = 4 \quad [\mathrm{rad/sec}]$, and

$\star$ $b_2 = \pi/3$ [rad], $c_2 = \pi/6$ [rad] and $\omega_2 = 3 \quad [\mathrm{rad/sec}]$.

- $\dot{\boldsymbol{q}}_d(t)$ and $\ddot{\boldsymbol{q}}_d(t)$, were analytically found, and they correspond to Equations (14) and (15), respectively.

- The symmetric positive definite matrices K_p and K_v are chosen as

$$
\begin{aligned}
K_p &= \operatorname{diag}\{\omega_1^2,\ \omega_2^2\} = \operatorname{diag}\{1500,\ 14000\} \quad [1\ /\ \mathrm{sec}] \\
K_v &= \operatorname{diag}\{2\omega_1,\ 2\omega_2\} = \operatorname{diag}\{77.46,\ 236.64\} \quad \left[1\ /\ \mathrm{sec}^2\right],
\end{aligned}
$$

  – where we used $\omega_1 = 38.7 \quad [\mathrm{rad}\ /\ \mathrm{sec}]$ and $\omega_2 = 118.3 \quad [\mathrm{rad}\ /\ \mathrm{sec}]$.

- The initial conditions are chosen as

$$
\begin{aligned}
q_1(0) = 0, &\quad q_2(0) = 0 \\
\dot{q}_1(0) = 0, &\quad \dot{q}_2(0) = 0\,.
\end{aligned}
$$

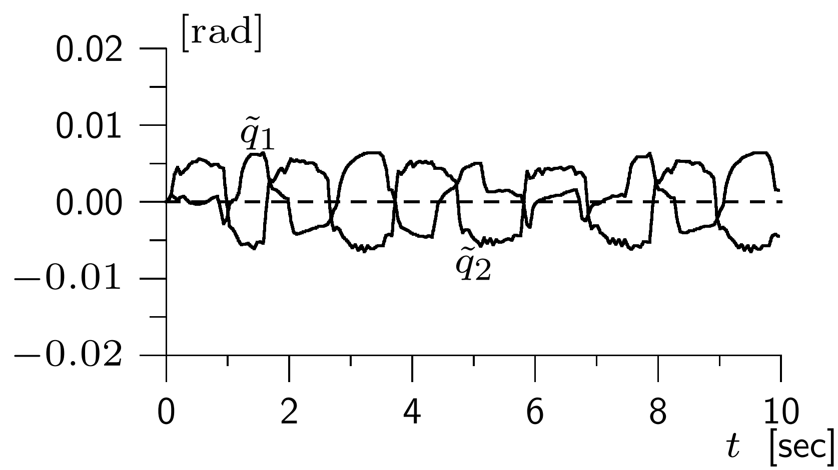

Figure 71: Position errors.

- The steady state position errors are not zero due to the friction effects of the actual robot which nevertheless, are neglected in the analysis.

Computed-torque+ control

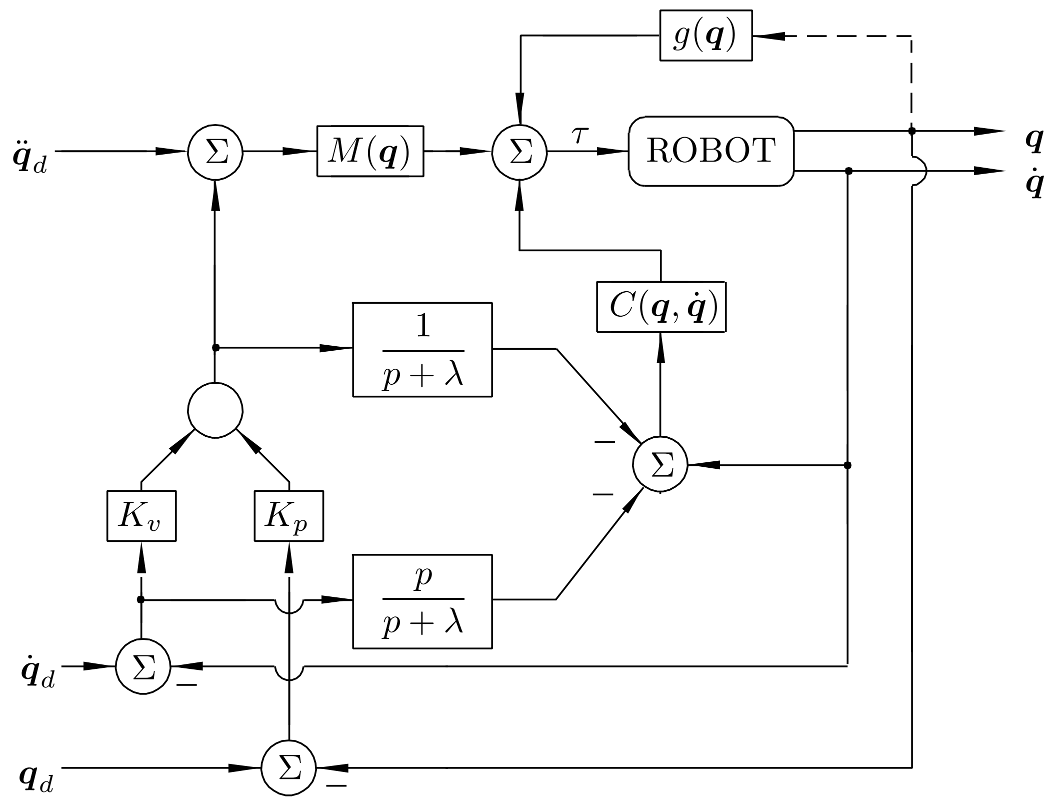

Figure 72: Computed-torque+ control.

The equation corresponding to the computed-torque+ controller is given by

$$\boldsymbol{\tau} = M(\boldsymbol{q})\left[\ddot{\boldsymbol{q}}_d + K_v\dot{\tilde{\boldsymbol{q}}} + K_p\tilde{\boldsymbol{q}}\right] + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) - C(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{\nu} \tag{31}$$

where

- K_v and K_p are symmetric positive definite design matrices,
- $\boldsymbol{\nu} \in \mathrm{IR}^n$ is obtained by filtering the errors of position $\tilde{\boldsymbol{q}}$ and velocity $\dot{\tilde{\boldsymbol{q}}}$, that is,

$$\boldsymbol{\nu} = -\frac{bp}{p+\lambda}\dot{\tilde{\boldsymbol{q}}} - \frac{b}{p+\lambda}\left[K_v\dot{\tilde{\boldsymbol{q}}} + K_p\tilde{\boldsymbol{q}}\right], \tag{32}$$

  ⋆ p is the differential operator (i.e., $p := \frac{d}{dt}$)
  ⋆ λ, b are positive design constants.

The computed-torque+ control law is *dynamic* The expression (32) in the state space form is a linear autonomous system given by

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{\xi}_1\\ \\ \boldsymbol{\xi}_2\end{bmatrix} = \begin{bmatrix}-\lambda I & 0\\ \\ 0 & -\lambda I\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}_1\\ \\ \boldsymbol{\xi}_2\end{bmatrix} + \begin{bmatrix}K_p & K_v\\ \\ 0 & -\lambda I\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix} \tag{33}$$

$$\boldsymbol{\nu} = \begin{bmatrix}-I & -I\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}_1\\ \\ \boldsymbol{\xi}_2\end{bmatrix} - \begin{bmatrix}0 & I\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix} \tag{34}$$

- where $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathrm{IR}^n$ are the new state variables.

Now, we have the following closed loop equation

$$\frac{d}{dt}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\\ \boldsymbol{\xi}_1\\ \boldsymbol{\xi}_2\end{bmatrix}=\begin{bmatrix}\dot{\tilde{\boldsymbol{q}}}\\ -M(\boldsymbol{q})^{-1}C(\boldsymbol{q},\dot{\boldsymbol{q}})\left[\boldsymbol{\xi}_1+\boldsymbol{\xi}_2+\dot{\tilde{\boldsymbol{q}}}\right]-K_v\dot{\tilde{\boldsymbol{q}}}-K_p\tilde{\boldsymbol{q}}\\ -\lambda\boldsymbol{\xi}_1+K_p\tilde{\boldsymbol{q}}+K_v\dot{\tilde{\boldsymbol{q}}}\\ -\lambda\boldsymbol{\xi}_2-\lambda\dot{\tilde{\boldsymbol{q}}}\end{bmatrix}, \qquad (35)$$

- the origin $\left[\tilde{\boldsymbol{q}}^T \;\; \dot{\tilde{\boldsymbol{q}}}^T \;\; \boldsymbol{\xi}_1^T \;\; \boldsymbol{\xi}_2^T\right]^T=\mathbf{0}\in \mathrm{IR}^{4n}$ is an equilibrium point.

The study of global asymptotic stability of the origin of the closed loop equation (35)

- is actually an open problem.
- Nevertheless, using Lemma 2.2 and Corollary A.2 we will show that
  - the functions $\tilde{\boldsymbol{q}}(t)$, $\dot{\tilde{\boldsymbol{q}}}(t)$ and $\boldsymbol{\nu}(t)$ are bounded, and
  - that the motion control objective is verified.

Toward this end, we can get an equivalent closed loop equation

$$M(\boldsymbol{q})\left[\dot{\boldsymbol{\nu}} + \lambda \boldsymbol{\nu}\right] + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{\nu} = \mathbf{0}\,.$$

- Consider now the following non-negative function

$$V(t, \boldsymbol{\nu}, \tilde{\boldsymbol{q}}) = \frac{1}{2}\boldsymbol{\nu}^T \underbrace{M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}})}_{P} \boldsymbol{\nu} \geq 0$$

- The derivative with respect to time of $V(\boldsymbol{\nu}, \tilde{\boldsymbol{q}})$ is given by

$$\dot{V}(\boldsymbol{\nu}, \tilde{\boldsymbol{q}}) = -\boldsymbol{\nu}^T \underbrace{\lambda M(\boldsymbol{q})}_{Q} \boldsymbol{\nu} \leq 0$$

We conclude

- $\star$ $\boldsymbol{\nu} \in L_\infty^n \cap L_2^n$
- $\star$ and

$$\underbrace{\boldsymbol{\nu}(t)^T \boldsymbol{\nu}(t)}_{\|\boldsymbol{\nu}(t)\|^2} \leq \frac{2V(\boldsymbol{\nu}(0), \tilde{\boldsymbol{q}}(0))}{\alpha} e^{-2\lambda t}.$$

This means that that $\boldsymbol{\nu}(t) \to \mathbf{0}$ exponentially.

- Making use of the latter and of Corollary A.2 we get that

$$\tilde{\boldsymbol{q}}, \ \dot{\tilde{\boldsymbol{q}}} \in L_2^n \cap L_\infty^n$$

$$\lim_{t\to\infty} \tilde{\boldsymbol{q}}(t) = \mathbf{0},$$

Example 10.3

Consider the 2-DOF *prototype robot* studied in Chapter 5

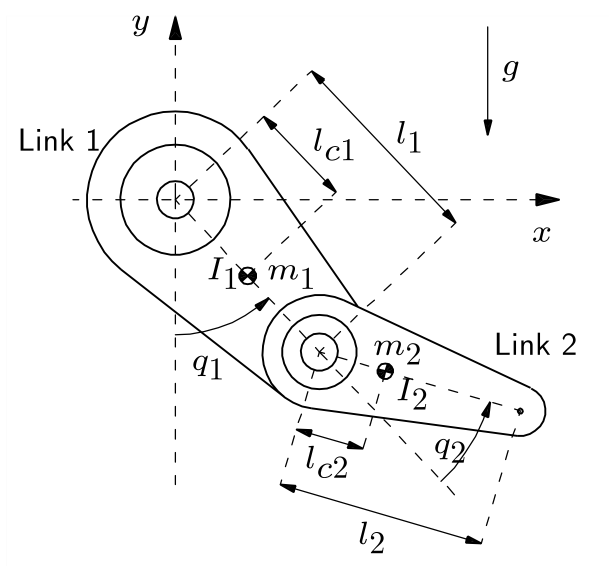

Figure 73: Diagram of the Pelican prototype.

Consider the Computed-torque+ control described by (31), (33) and (34) applied to this robot.

- $\boldsymbol{q}_d(t)$, $\dot{\boldsymbol{q}}_d(t)$ and $\ddot{\boldsymbol{q}}_d(t)$ are those used in the previous example.

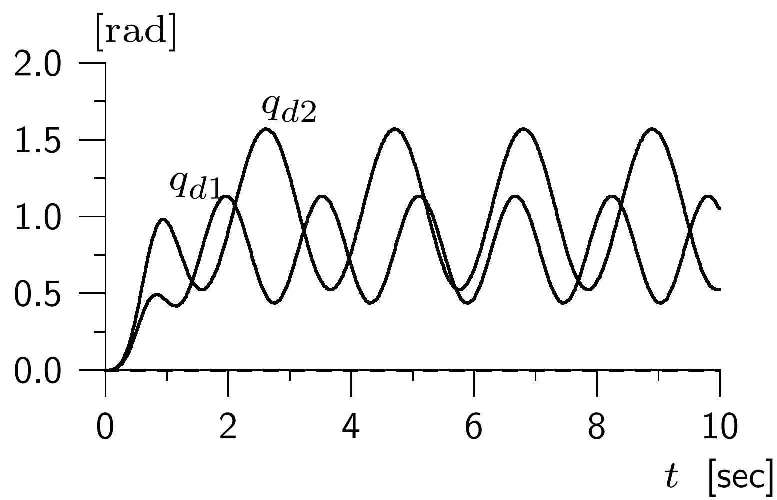

Figure 74: Desired reference trajectories.

- K_p and K_v, and the constant λ are taken as

$$\begin{aligned} K_p &= \operatorname{diag}\{\omega_1^2,\ \omega_2^2\} = \operatorname{diag}\{1500,\ 14000\} \quad [1\ /\ \mathrm{sec}] \\ K_v &= \operatorname{diag}\{2\omega_1,\ 2\omega_2\} = \operatorname{diag}\{77.46,\ 236.64\} \quad \left[1\ /\ \mathrm{sec}^2\right] \\ \lambda &= 60\,. \end{aligned}$$

- The initial conditions of the controller state variables are fixed at

$$\boldsymbol{\xi}_1(0) = 0, \quad \boldsymbol{\xi}_2(0) = 0\,.$$

- The initial conditions of the actual positions and velocities are set to

$$\begin{aligned} q_1(0) = 0, &\quad q_2(0) = 0 \\ \dot{q}_1(0) = 0, &\quad \dot{q}_2(0) = 0\,. \end{aligned}$$

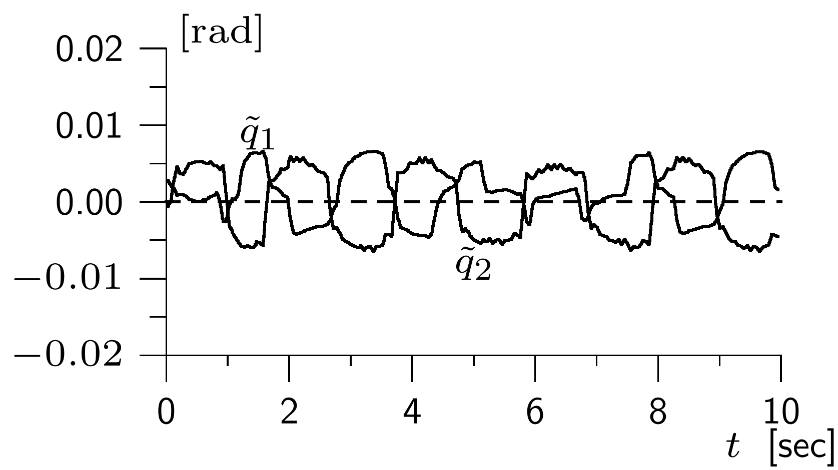

Figure 75: Position errors.

- It is interesting to remark that the plots obtained with the Computed-torque control, present a considerable similarity to those of figure 75.

Ch. 11. PD+ Control and PD Control with Compensation

We present two controllers whose control laws are based on the dynamic equations of the system but which also involve certain nonlinearities that are evaluated along the desired trajectories.

- PD control with compensation
- PD+ control.

PD Control with Compensation

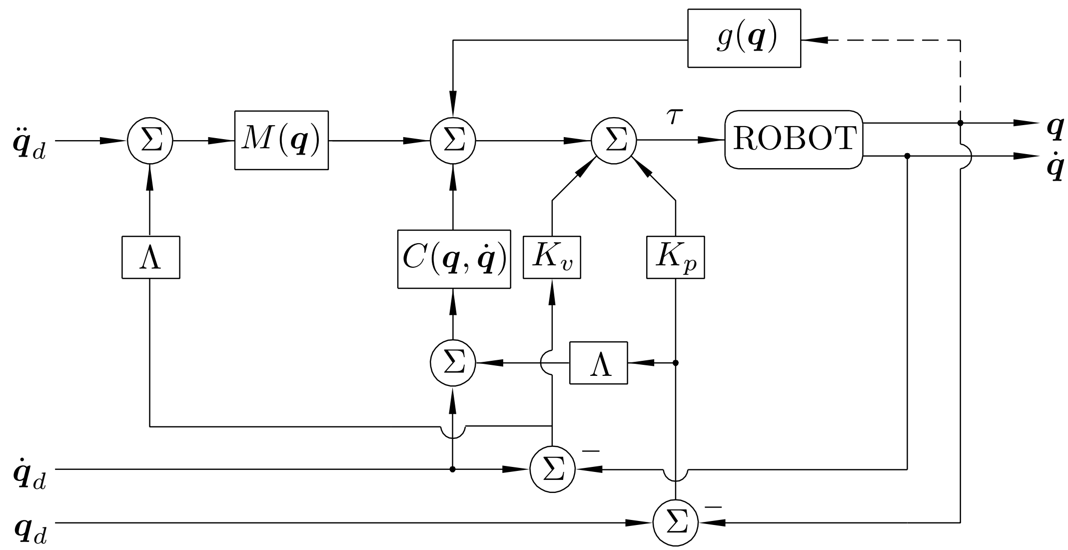

Figure 76: PD control with compensation.

The PD control law with compensation may be written as

$$\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + M(\boldsymbol{q})\left[\ddot{\boldsymbol{q}}_d + \Lambda\dot{\tilde{\boldsymbol{q}}}\right] + C(\boldsymbol{q},\dot{\boldsymbol{q}})\left[\dot{\boldsymbol{q}}_d + \Lambda\tilde{\boldsymbol{q}}\right] + \boldsymbol{g}(\boldsymbol{q}), \quad (36)$$

- where $K_p = K_p^T > 0,\ K_v = K_v^T > 0\ \in \mathbb{R}^{n\times n}$ and $\Lambda = {K_v}^{-1}K_p\,.$

The equation of closed loop is

$$M(\boldsymbol{q})\left[\ddot{\tilde{\boldsymbol{q}}}+\Lambda\dot{\tilde{\boldsymbol{q}}}\right]+C(\boldsymbol{q},\dot{\boldsymbol{q}})\left[\dot{\tilde{\boldsymbol{q}}}+\Lambda\tilde{\boldsymbol{q}}\right]=-K_p\tilde{\boldsymbol{q}}-K_v\dot{\tilde{\boldsymbol{q}}},$$

which may be expressed in terms of the state vector $\begin{bmatrix}\tilde{\boldsymbol{q}}^T & \dot{\tilde{\boldsymbol{q}}}^T\end{bmatrix}^T$ as

$$\frac{d}{dt}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix}=\begin{bmatrix}\dot{\tilde{\boldsymbol{q}}}\\ M(\boldsymbol{q})^{-1}\left[-K_p\tilde{\boldsymbol{q}}-K_v\dot{\tilde{\boldsymbol{q}}}-C(\boldsymbol{q},\dot{\boldsymbol{q}})\left[\dot{\tilde{\boldsymbol{q}}}+\Lambda\tilde{\boldsymbol{q}}\right]\right]-\Lambda\dot{\tilde{\boldsymbol{q}}}\end{bmatrix},$$

- It is non-autonomous and
- has the origin $\begin{bmatrix}\tilde{\boldsymbol{q}}^T & \dot{\tilde{\boldsymbol{q}}}^T\end{bmatrix}^T=\boldsymbol{0}\in \mathrm{IR}^{2n}$ as an equilibrium point.

The stability analysis may be carried out by considering

- Lyapunov function candidate

$$V(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) = \frac{1}{2}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix}^T \begin{bmatrix} 2K_p + \Lambda^T M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}})\Lambda & \Lambda^T M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}) \\ M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}})\Lambda & M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}) \end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix}.$$

  – We rewrite it in the following form

$$V(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) = \frac{1}{2}\left[\dot{\tilde{\boldsymbol{q}}} + \Lambda\tilde{\boldsymbol{q}}\right]^T M(\boldsymbol{q})\left[\dot{\tilde{\boldsymbol{q}}} + \Lambda\tilde{\boldsymbol{q}}\right] + \tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}} \qquad (37)$$

– which is equivalent to

$$V(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) = \frac{1}{2}\begin{bmatrix} \dot{\tilde{\boldsymbol{q}}} \\ \tilde{\boldsymbol{q}} \end{bmatrix}^T \underbrace{\begin{bmatrix} I & \Lambda^T \\ 0 & I \end{bmatrix}\begin{bmatrix} K_p & 0 \\ 0 & M(\boldsymbol{q}) \end{bmatrix}\begin{bmatrix} I & 0 \\ \Lambda & I \end{bmatrix}}_{B^TAB}\begin{bmatrix} \dot{\tilde{\boldsymbol{q}}} \\ \tilde{\boldsymbol{q}} \end{bmatrix}.$$

∗ It is easy to see

$$V(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) \geq \frac{1}{2}\lambda_{\mathrm{Min}}\{B^TAB\}\left[\|\dot{\tilde{\boldsymbol{q}}}\| + \|\tilde{\boldsymbol{q}}\|^2\right]$$

∗ It is positive definite (because $K_p > 0$, $K_v > 0$ and hence $B^TAB > 0$) and radially unbounded.

– Correspondingly, since the inertia matrix is bounded uniformly in $\boldsymbol{q}$, we have that

$$V(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) \leq \frac{1}{2}\lambda_{\mathrm{Max}}\{M\} \left\| \dot{\tilde{\boldsymbol{q}}} + \Lambda\tilde{\boldsymbol{q}} \right\|^2 + \lambda_{\mathrm{Max}}\{K_p\} \left\| \tilde{\boldsymbol{q}} \right\|^2$$

hence, $V(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}})$ is also decrescent.

- The time derivative of the Lyapunov function candidate (37) is

$$\begin{aligned}\dot{V}(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) &= -\dot{\tilde{\boldsymbol{q}}}^T K_v \dot{\tilde{\boldsymbol{q}}} - \tilde{\boldsymbol{q}}^T \Lambda^T K_v \Lambda \tilde{\boldsymbol{q}} \\ &= -\begin{bmatrix}\tilde{\boldsymbol{q}} \\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix}^T \begin{bmatrix}\Lambda^T K_v \Lambda & 0 \\ 0 & K_v\end{bmatrix} \begin{bmatrix}\tilde{\boldsymbol{q}} \\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix}.\end{aligned}$$

  – Globally negative definite function.

- From Theorem 2.4 we conclude immediately global uniform asymptotic stability of the equilibrium $\begin{bmatrix}\tilde{\boldsymbol{q}}^T & \dot{\tilde{\boldsymbol{q}}}^T\end{bmatrix}^T = \boldsymbol{0} \in \mathrm{I\!R}^{2n}$.

Example 11.1

Consider the Pelican robot presented in Chapter 5

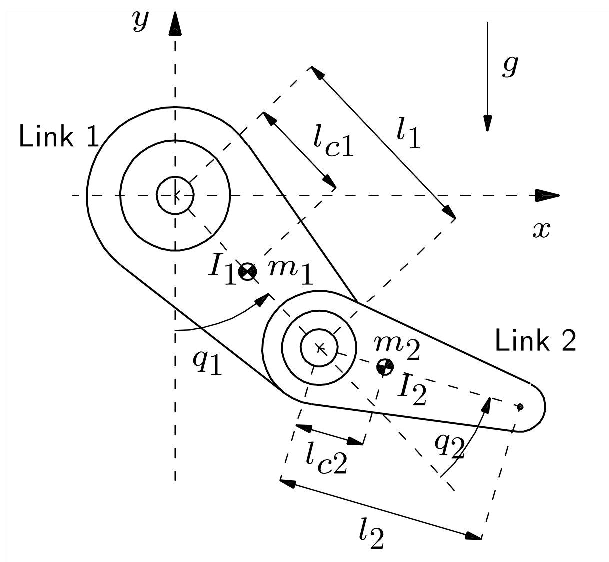

Figure 77: Diagram of the Pelican robot.

Consider this robot under PD control with compensation (36).

- It is desired that the robot tracks the trajectories $\boldsymbol{q}_d(t)$, $\dot{\boldsymbol{q}}_d(t)$ and $\ddot{\boldsymbol{q}}_d(t)$ represented by Equations (13)–(15).

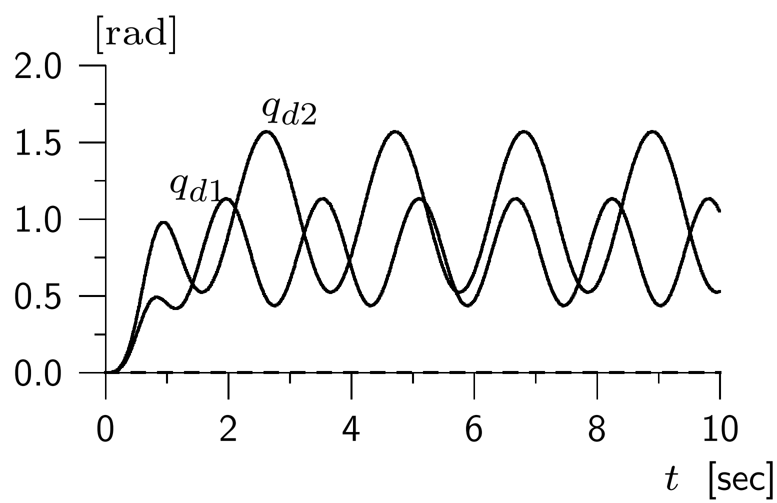

Figure 78: Desired reference trajectories.

$$\begin{bmatrix} q_{d1} \\ \\ q_{d2} \end{bmatrix} = \begin{bmatrix} b_1[1 - e^{-2.0\ t^3}] + c_1[1 - e^{-2.0\ t^3}]\ \sin(\omega_1 t) \\ \\ b_2[1 - e^{-2.0\ t^3}] + c_2[1 - e^{-2.0\ t^3}]\ \sin(\omega_2 t) \end{bmatrix} \qquad [\mathrm{rad}]$$

where

$\star$ $b_1 = \pi/4$ [rad], $c_1 = \pi/9$ [rad] and $\omega_1 = 4 \quad [\mathrm{rad/sec}]$, and

$\star$ $b_2 = \pi/3$ [rad], $c_2 = \pi/6$ [rad] and $\omega_2 = 3 \quad [\mathrm{rad/sec}]$.

- $\dot{\boldsymbol{q}}_d(t)$ and $\ddot{\boldsymbol{q}}_d(t)$, were analytically found, and they correspond to Equations (14) and (15), respectively.

- The symmetric positive definite matrices K_p and K_v are chosen so that

$$
\begin{aligned}
K_p &= \mathrm{diag}\{200,\ 150\} \quad [\mathrm{N\ m\ /\ rad}]\,, \\
K_v &= \mathrm{diag}\{3\} \quad [\mathrm{N\ m\ sec\ /\ rad}]\,,
\end{aligned}
$$

and therefore $\Lambda = K_v^{-1} K_p = \mathrm{diag}\{66.6,\ 50\}\ [1/\mathrm{sec}]$.

- The initial conditions corresponding a the positions and velocities are

$$
\begin{array}{ll}
q_1(0) = 0, & q_2(0) = 0 \\
\dot{q}_1(0) = 0, & \dot{q}_2(0) = 0\,.
\end{array}
$$

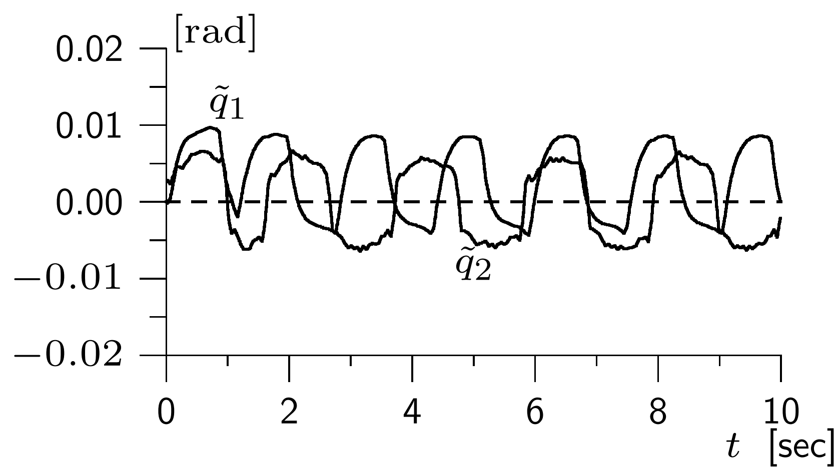

Figure 79: Position errors.

- The experimental steady state tracking position errors $\tilde{\boldsymbol{q}}(t)$, by virtue of friction phenomena in the actual robot, are not zero.

PD+ control

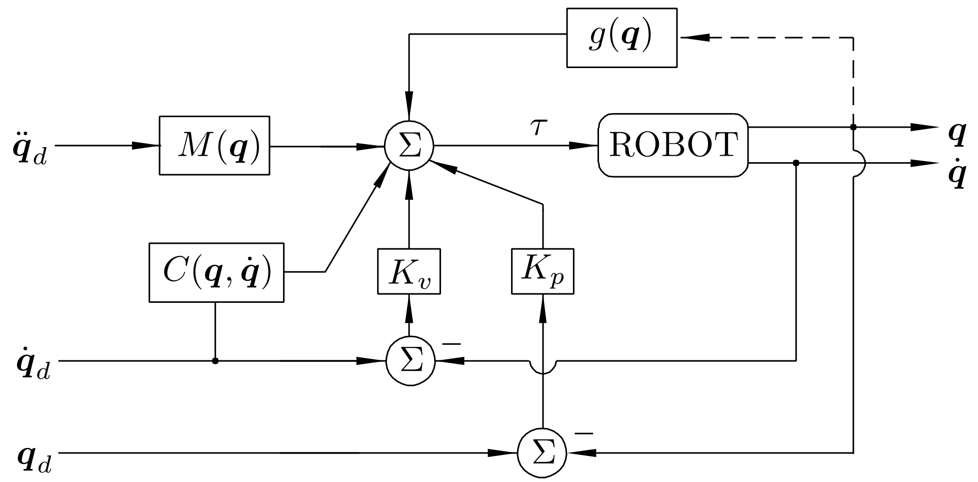

Figure 80: PD+ control.

The control law of PD+ control is given by

$$\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + M(\boldsymbol{q})\ddot{\boldsymbol{q}}_d + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_d + \boldsymbol{g}(\boldsymbol{q}) \tag{38}$$

- where $K_p, K_v \in \mathrm{IR}^{n\times n}$ are symmetric positive definite matrices

The closed loop equation may be written as

$$\frac{d}{dt}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix} = \begin{bmatrix}\dot{\tilde{\boldsymbol{q}}}\\ M(\boldsymbol{q}_d-\tilde{\boldsymbol{q}})^{-1}\left[-K_p\tilde{\boldsymbol{q}}-K_v\dot{\tilde{\boldsymbol{q}}}-C(\boldsymbol{q}_d-\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}}_d-\dot{\tilde{\boldsymbol{q}}})\dot{\tilde{\boldsymbol{q}}}\right]\end{bmatrix}.$$

- Nonlinear nonautonomous differential equation
- The only equilibrium point is the origin $\left[\tilde{\boldsymbol{q}}^T\ \ \dot{\tilde{\boldsymbol{q}}}^T\right]^T=\boldsymbol{0}\in \mathrm{IR}^{2n}$.

To analyze the stability of the origin consider now

- Lyapunov function candidate

$$
\begin{aligned}
V(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) &= \frac{1}{2}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix}^T \overbrace{\begin{bmatrix}K_p & 0\\ 0 & M(\boldsymbol{q}_d-\tilde{\boldsymbol{q}})\end{bmatrix}}^{P} \begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix}\\
&= \frac{1}{2}\dot{\tilde{\boldsymbol{q}}}^T M(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}}, \qquad (39)
\end{aligned}
$$

  – which is positive definite since $M(\boldsymbol{q}) > 0$ and $K_p > 0$.

- Taking the time derivative of (39) we obtain

$$
\begin{aligned}
\dot{V}(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) &= -\dot{\tilde{\boldsymbol{q}}}^T \overbrace{K_v}^{Q} \dot{\tilde{\boldsymbol{q}}} \\
&= -\begin{bmatrix} \tilde{\boldsymbol{q}} \\ \\ \dot{\tilde{\boldsymbol{q}}} \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ \\ 0 & K_v \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \\ \dot{\tilde{\boldsymbol{q}}} \end{bmatrix} \leq 0
\end{aligned}
$$

- From Theorem 2.3 we conclude stability of the origin.
- La Salle's theorem cannot be used to conclude global asymptotic stability.
- Alternatively, we may use Lemma 2.2
  - position and velocity errors are bounded and
  - the velocity error is square-integrable ($\int_0^\infty \left\| \dot{\tilde{\boldsymbol{q}}}(t) \right\|^2 dt < \infty$.)

Example 11.3

Consider the 2-DOF *prototype robot* studied in Chapter 5

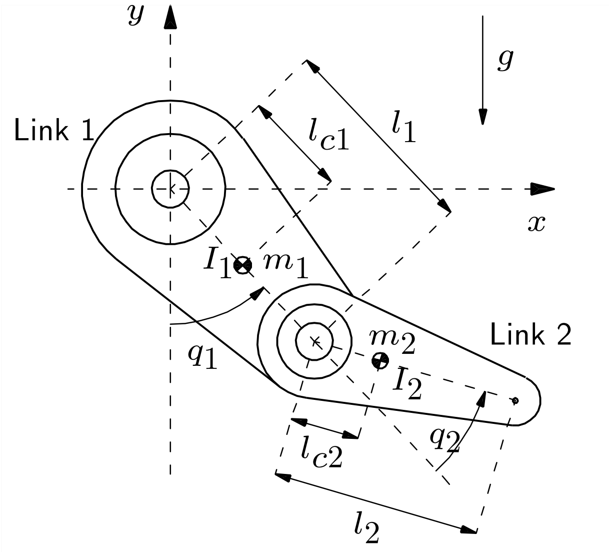

Figure 81: Diagram of the Pelican robot.

Consider the application of the PD control+ (38) on this robot.

- The joint desired trajectories of position, velocity and acceleration: $\boldsymbol{q}_d(t)$, $\dot{\boldsymbol{q}}_d(t)$ and $\ddot{\boldsymbol{q}}_d(t)$, are given by Equations (13)–(15).

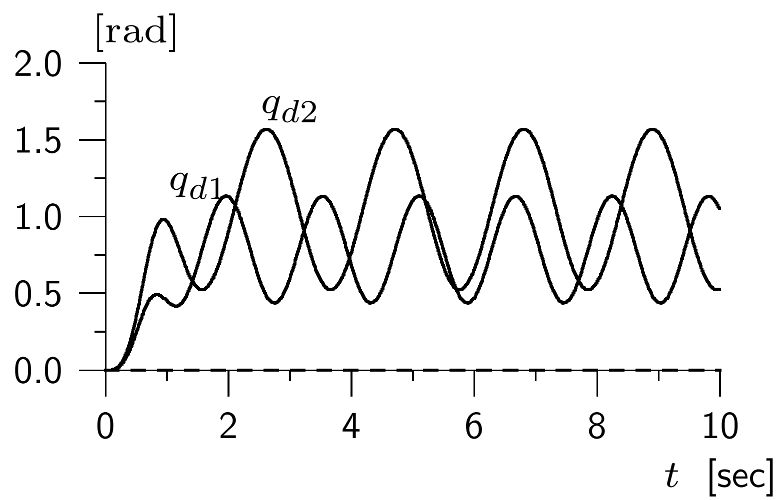

Figure 82: Desired reference trajectories.

- The symmetric positive definite matrices K_p and K_v are chosen as

$$
\begin{aligned}
K_p &= \mathrm{diag}\{200,\ 150\} \quad [\mathrm{N\ m\ /\ rad}] \\
K_v &= \mathrm{diag}\{3\} \quad [\mathrm{N\ m\ sec\ /\ rad}]\ .
\end{aligned}
$$

- The initial conditions corresponding to the positions and velocities, are fixed as

$$
\begin{array}{ll}
q_1(0) = 0, & q_2(0) = 0 \\
\dot{q}_1(0) = 0, & \dot{q}_2(0) = 0\,.
\end{array}
$$

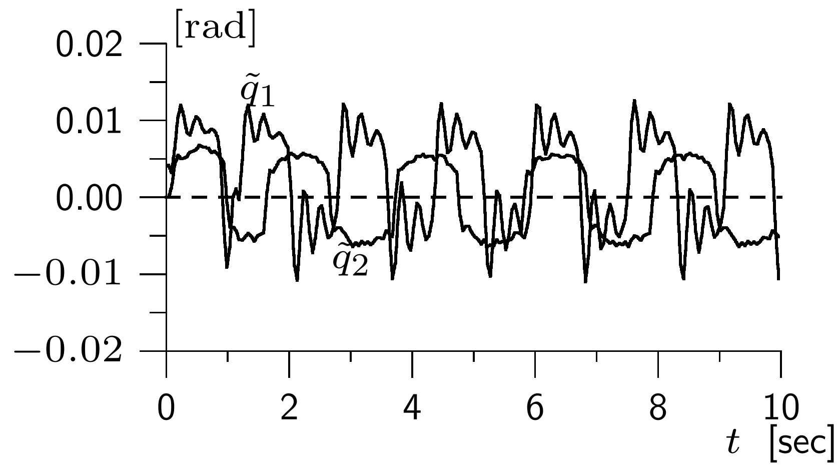

Figure 83: Position errors.

- The experimental steady state tracking position errors $\tilde{\boldsymbol{q}}(t)$, by virtue of friction phenomena (neglected in the analysis) in the actual robot, are not zero.

Lyapunov function for asymptotic stability

We present an alternative stability analysis. Consider now,

- Lyapunov function candidate,

$$
\begin{aligned}
V(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) &= \frac{1}{2}\begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\tilde{\boldsymbol{q}}} \end{bmatrix}^T \begin{bmatrix} K_p & \frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|} M(\boldsymbol{q}) \\ \frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|} M(\boldsymbol{q}) & M(\boldsymbol{q}) \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\tilde{\boldsymbol{q}}} \end{bmatrix} \qquad (40) \\
&= W(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) + \underbrace{\frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}}_{\varepsilon(\tilde{\boldsymbol{q}})} \tilde{\boldsymbol{q}}^T M(\boldsymbol{q}) \dot{\tilde{\boldsymbol{q}}}
\end{aligned}
$$

  - where $W(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) = \frac{1}{2}\dot{\tilde{\boldsymbol{q}}}^T M(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}}$ is the Lyapunov function (39).

– The positive constant ε_0 is chosen so as to satisfy simultaneously

* $\sqrt{\dfrac{\lambda_{\min}\{K_p\}}{\lambda_{\mathrm{Max}}\{M(\boldsymbol{q})\}}} > \varepsilon_0 > 0$
* $\dfrac{\lambda_{\min}\{K_v\}}{2\left(k_{C_1} + 2\lambda_{\mathrm{Max}}\{M(\boldsymbol{q})\}\right)} > \varepsilon_0 > 0$
* $\dfrac{2\lambda_{\min}\{K_p\}\lambda_{\min}\{K_v\}}{\left(\lambda_{\mathrm{Max}}\{K_v\} + k_{C_1}\,\|\dot{\boldsymbol{q}}_d\|_{\mathrm{Max}}\right)^2} > \varepsilon_0 > 0$

– This, implies that the matrix $K_p - \varepsilon^2 M(\boldsymbol{q}) > 0$

- The function (40) may be rewritten as

$$V(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) = \frac{1}{2}\Big[\dot{\tilde{\boldsymbol{q}}} + \varepsilon\tilde{\boldsymbol{q}}\Big]^T M(\boldsymbol{q})\Big[\dot{\tilde{\boldsymbol{q}}} + \varepsilon\tilde{\boldsymbol{q}}\Big] + \frac{1}{2}\tilde{\boldsymbol{q}}^T\left[K_p - \varepsilon^2 M(\boldsymbol{q})\right]\tilde{\boldsymbol{q}}$$

– which is positive definite since so is $M(\boldsymbol{q})$ as well as $K_p - \varepsilon^2 M(\boldsymbol{q})$.

- Notice that $V(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}})$ satisfies

$$V(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) \leq \frac{1}{2}\begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}^T \begin{bmatrix} \lambda_{\text{Max}}\{K_p\} & \varepsilon_0\ \lambda_{\text{Max}}\{M\} \\ \varepsilon_0\ \lambda_{\text{Max}}\{M\} & \lambda_{\text{Max}}\{M\} \end{bmatrix} \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}, \quad (41)$$

  – The matrix on the right hand side of inequality (41) is positive definite in view of the condition on ε_0,

$$\sqrt{\frac{\lambda_{\text{Max}}\{K_p\}}{\lambda_{\text{Max}}\{M(q)\}}} > \varepsilon_0 > 0 \qquad \forall\ \ \boldsymbol{q} \in \mathrm{IR}^n$$

  – Thus, the function (40) is positive definite, radially unbounded and decrescent.

- The time derivative of the Lyapunov function candidate (40) is given by

$$
\begin{aligned}
\dot{V}(\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) \;=\; & -\dot{\tilde{\boldsymbol{q}}}^T K_v \dot{\tilde{\boldsymbol{q}}} + \varepsilon(\tilde{\boldsymbol{q}})\dot{\tilde{\boldsymbol{q}}}^T M(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} + \underbrace{\varepsilon(\tilde{\boldsymbol{q}})\dot{\tilde{\boldsymbol{q}}}^T C(\boldsymbol{q},\dot{\boldsymbol{q}})\tilde{\boldsymbol{q}}}_{a(\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}})} \\
& - \varepsilon(\tilde{\boldsymbol{q}})\tilde{\boldsymbol{q}}^T \Big[K_p \tilde{\boldsymbol{q}} + K_v \dot{\tilde{\boldsymbol{q}}} \Big] + \underbrace{\dot{\varepsilon}(\tilde{\boldsymbol{q}})\tilde{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}}}_{b(\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}})} \;.
\end{aligned}
$$

– It may be upperbounded in the following manner

$$
\begin{aligned}
\dot{V}(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) \quad \leq \quad & -\varepsilon \underbrace{\begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}^T Q \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}}_{h(\|\tilde{\boldsymbol{q}}\|, \|\dot{\tilde{\boldsymbol{q}}}\|)} \\
& - \frac{1}{2} \underbrace{\left[\lambda_{\min}\{K_v\} - 2\varepsilon_0 \left(k_{C_1} + 2\lambda_{\mathrm{Max}}\{M\}\right)\right]}_{\delta} \left\|\dot{\tilde{\boldsymbol{q}}}\right\|^2,
\end{aligned}
$$

∗ where the symmetric matrix Q is given by

$$Q = \begin{bmatrix} \lambda_{\min}\{K_p\} & -\frac{1}{2}\left(\lambda_{\mathrm{Max}}\{K_v\} + k_{C_1}\left\|\dot{\boldsymbol{q}}_d\right\|\right) \\ -\frac{1}{2}\left(\lambda_{\mathrm{Max}}\{K_v\} + k_{C_1}\left\|\dot{\boldsymbol{q}}_d\right\|\right) & \frac{1}{2\varepsilon_0}\lambda_{\min}\{K_v\} \end{bmatrix}$$

∗ The function $\dot{V}(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}})$ is negative definite

- Thus, using Theorem 2.4 we conclude that the origin $\left[\tilde{\boldsymbol{q}}^T \ \dot{\tilde{\boldsymbol{q}}}^T\right]^T = \mathbf{0} \in \mathrm{IR}^{2n}$ is globally uniformly asymptotically stable.

Ch. 12. Feedforward control and PD plus feedforward control

Practical implementation of controllers is via digital technology.

- Sampling of the joint position $\boldsymbol{q}$ and of the velocity $\dot{\boldsymbol{q}}$,
- computation of the control action $\boldsymbol{\tau}$ from the control law,
- the 'order' to apply this control action is sent to the actuators.

Control strategies using precomputed terms based on $\boldsymbol{q}_d(t)$, $\dot{\boldsymbol{q}}_d(t)$ and $\ddot{\boldsymbol{q}}_d(t)$, has advantages:

- Reduction in the time of computation of $\boldsymbol{\tau}$
- Higher processing frequency (larger potential for 'fast' tasks)

Feedforward control

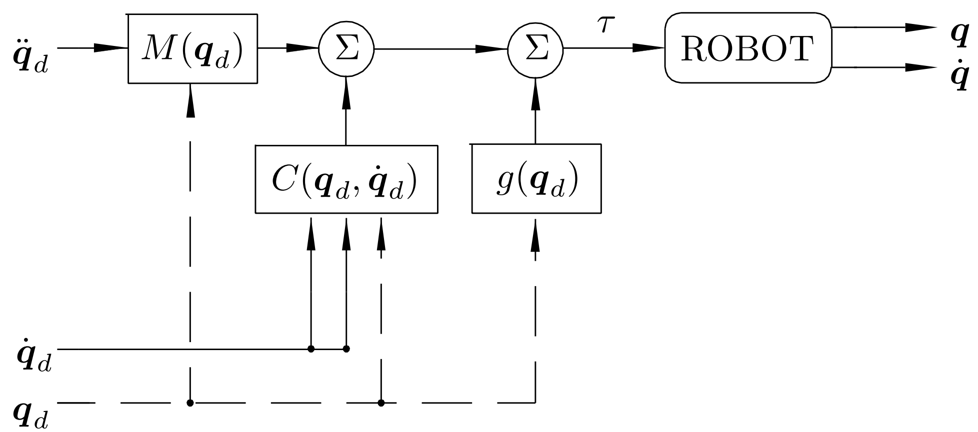

Figure 84: Feedforward control.

The feedforward controller is given by

$$\boldsymbol{\tau} = M(\boldsymbol{q}_d)\ddot{\boldsymbol{q}}_d + C(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d)\dot{\boldsymbol{q}}_d + \boldsymbol{g}(\boldsymbol{q}_d)\,. \tag{42}$$

The behavior of the control system is described by

$$\frac{d}{dt}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix} = \begin{bmatrix}\dot{\tilde{\boldsymbol{q}}}\\ -M^{-1}\left[(M_d - M)\ddot{\boldsymbol{q}}_d + C_d\dot{\boldsymbol{q}}_d - C\dot{\boldsymbol{q}} + \boldsymbol{g}_d - \boldsymbol{g}\right]\end{bmatrix},$$

where

- $M = M(\boldsymbol{q})$, $M_d = M(\boldsymbol{q}_d)$, $C = C(\boldsymbol{q}, \dot{\boldsymbol{q}})$, $C_d = C(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d)$
- $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{q})$ and $\boldsymbol{g}_d = \boldsymbol{g}(\boldsymbol{q}_d)$.
- The origin $\begin{bmatrix}\tilde{\boldsymbol{q}}^T & \dot{\tilde{\boldsymbol{q}}}^T\end{bmatrix}^T = \boldsymbol{0} \in \mathrm{IR}^{2n}$ is an equilibrium point of the previous equation but in general, it is not the only one.

Example 12.3

Consider the 2-DOF *prototype robot* studied in Chapter 5

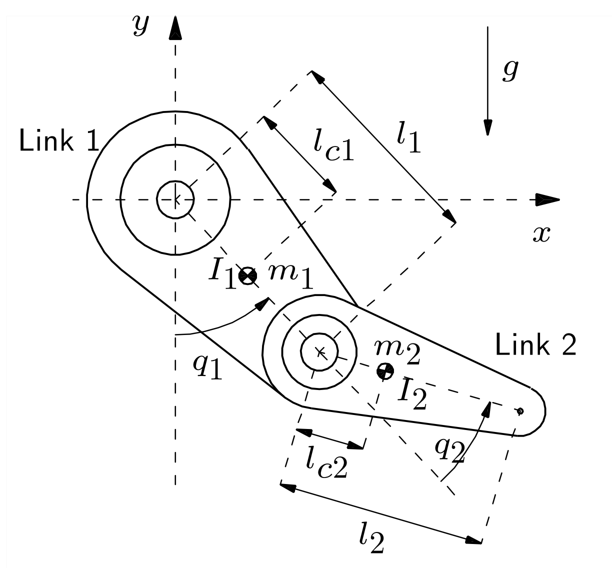

Figure 85: 2-DOF robot.

Consider the application of feedforward control (42) on this robot.

- The desired trajectory is given by $\boldsymbol{q}_d(t)$ which is defined in (13).

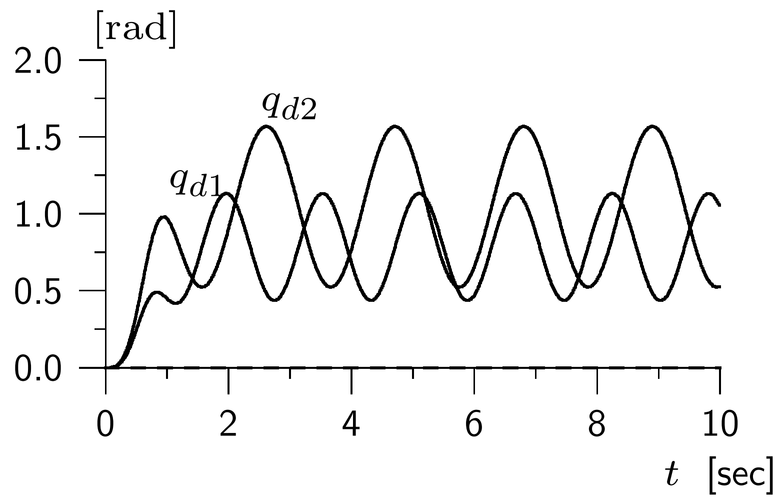

Figure 86: Desired reference trajectories.

- The initial conditions are chosen as

$$q_1(0) = 0, \quad q_2(0) = 0$$
$$\dot{q}_1(0) = 0, \quad \dot{q}_2(0) = 0\,.$$

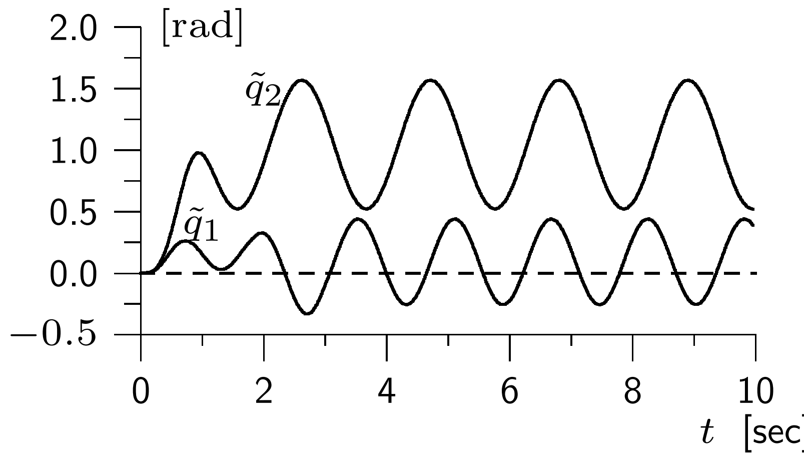

Figure 87: Position errors

- Figure shows the position errors $\tilde{\boldsymbol{q}}(t)$ tend to an oscillatory behavior. Naturally, this behavior is far from satisfactory.

- A rigorous generic analysis of stability or instability seems to be an impossible task.

PD plus feedforward control

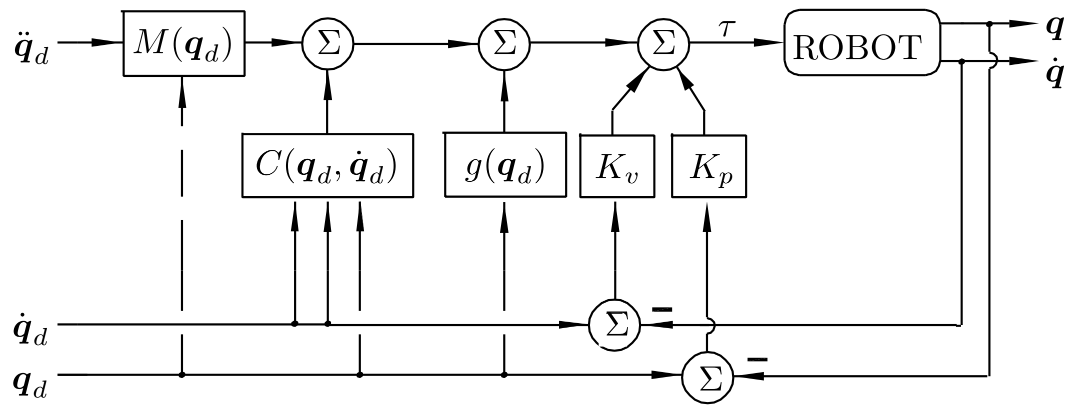

Figure 88: PD plus feedforward control.

Control law is given by

$$\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + M(\boldsymbol{q}_d)\ddot{\boldsymbol{q}}_d + C(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d)\dot{\boldsymbol{q}}_d + \boldsymbol{g}(\boldsymbol{q}_d),$$

- where $K_p = K_p^T > 0,\ K_v = K_v^T > 0 \ \in \mathbb{R}^{n\times n}$.

The closed loop equation may be written as:

$$\frac{d}{dt}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\end{bmatrix} = \begin{bmatrix}\dot{\tilde{\boldsymbol{q}}}\\ M(\boldsymbol{q})^{-1}\left[-K_p\tilde{\boldsymbol{q}} - K_v\dot{\tilde{\boldsymbol{q}}} - C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\tilde{\boldsymbol{q}}} - \boldsymbol{h}(\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}})\right]\end{bmatrix}, \qquad (43)$$

- where the so-called residual dynamics, is given by

$$\boldsymbol{h}(\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) = [M(\boldsymbol{q}_d) - M(\boldsymbol{q})]\ddot{\boldsymbol{q}}_d + [C(\boldsymbol{q}_d,\dot{\boldsymbol{q}}_d) - C(\boldsymbol{q},\dot{\boldsymbol{q}})]\dot{\boldsymbol{q}}_d + \boldsymbol{g}(\boldsymbol{q}_d) - \boldsymbol{g}(\boldsymbol{q})\,.$$

- The origin $[\tilde{\boldsymbol{q}}^T \quad \dot{\tilde{\boldsymbol{q}}}^T]^T = \boldsymbol{0} \in \mathbb{R}^{2n}$ of the state space is an equilibrium.

- However, the number of equilibria depends on the proportional gain K_p.

Unicity of the equilibrium

The equilibria are the constant vectors

- $[\tilde{\boldsymbol{q}}^T \;\; \dot{\tilde{\boldsymbol{q}}}^T]^T = [\tilde{\boldsymbol{q}}^{*T} \;\; \boldsymbol{0}^T]^T \in \mathbb{R}^{2n}$,
  - where $\tilde{\boldsymbol{q}}^* \in \mathbb{R}^n$ is a solution of $K_p\tilde{\boldsymbol{q}}^* + \boldsymbol{h}(\tilde{\boldsymbol{q}}^*, \boldsymbol{0}) = \boldsymbol{0}$.
  - It always is satisfied by the trivial solution $\tilde{\boldsymbol{q}}^* = \boldsymbol{0} \in \mathbb{R}^n$
- Explicit conditions to ensure unicity of the equilibrium are presented next.
  - Define $\boldsymbol{k}(\tilde{\boldsymbol{q}}^*) = K_p^{-1}\boldsymbol{h}(\tilde{\boldsymbol{q}}^*, \boldsymbol{0})$.
  - For all vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we have

$$\begin{aligned} \|\boldsymbol{k}(\boldsymbol{x}) - \boldsymbol{k}(\boldsymbol{y})\| &\leq \left\|K_p^{-1}\left[\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{0}) - \boldsymbol{h}(\boldsymbol{y}, \boldsymbol{0})\right]\right\| \\ &\leq \lambda_{\mathrm{Max}}\left\{K_p^{-1}\right\}\|\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{0}) - \boldsymbol{h}(\boldsymbol{y}, \boldsymbol{0})\| \,. \end{aligned}$$

– On the other hand, we have that

$$
\begin{aligned}
\|\boldsymbol{h}(\boldsymbol{x},\boldsymbol{0})-\boldsymbol{h}(\boldsymbol{y},\boldsymbol{0})\| \quad \leq \quad & \|[M(\boldsymbol{q}_d-\boldsymbol{y})-M(\boldsymbol{q}_d-\boldsymbol{x})]\,\ddot{\boldsymbol{q}}_d\| \\
& +\|[C(\boldsymbol{q}_d-\boldsymbol{y},\dot{\boldsymbol{q}}_d)-C(\boldsymbol{q}_d-\boldsymbol{x},\dot{\boldsymbol{q}}_d)]\,\dot{\boldsymbol{q}}_d\| \\
& +\|\boldsymbol{g}(\boldsymbol{q}_d-\boldsymbol{y})-\boldsymbol{g}(\boldsymbol{q}_d-\boldsymbol{x})\| \ .
\end{aligned}
$$

$$
\|\boldsymbol{h}(\boldsymbol{x},\boldsymbol{0})-\boldsymbol{h}(\boldsymbol{y},\boldsymbol{0})\| \leq \left[k_g+k_M\,\|\ddot{\boldsymbol{q}}_d\|_{\mathrm{M}}+k_{C_2}\,\|\dot{\boldsymbol{q}}_d\|_{\mathrm{M}}^2\right]\|\boldsymbol{x}-\boldsymbol{y}\| \ .
$$

– We get that

$$\|\boldsymbol{k}(\boldsymbol{x}) - \boldsymbol{k}(\boldsymbol{y})\| \leq \frac{1}{\lambda_{\min}\{K_p\}} \left[k_g + k_M \|\ddot{\boldsymbol{q}}_d\|_{\mathrm{M}} + k_{C_2} \|\dot{\boldsymbol{q}}_d\|_{\mathrm{M}}^2 \right] \|\boldsymbol{x} - \boldsymbol{y}\| \ .$$

- Invoking the contraction mapping theorem, we conclude that

$$\lambda_{\min}\{K_p\} > k_g + k_M \|\ddot{\boldsymbol{q}}_d\|_{\mathrm{M}} + k_{C_2} \|\dot{\boldsymbol{q}}_d\|_{\mathrm{M}}^2$$

– It is a sufficient condition for $\boldsymbol{k}(\tilde{\boldsymbol{q}}^*)$ to have a unique fixed point, and therefore,
– for the origin of the state space to be the unique equilibrium of the system in closed loop (43).

Global uniform asymptotic stability

We assume that given a constant $\gamma > 0$,

- K_v is chosen sufficiently "large" in the sense that

$$\lambda_{\mathrm{Max}}\{K_v\} \geq \lambda_{\min}\{K_v\} > k_{h1} + \gamma\ b\,,$$

- and so is K_p but in the sense that

$$\lambda_{\mathrm{Max}}\{K_p\} \geq \lambda_{\min}\{K_p\} > \alpha_3\left[\frac{\left[2\ \gamma\ a + k_{h2}\right]^2}{4\ \gamma\left[\lambda_{\min}\{K_v\} - k_{h1} - \gamma b\right]} + k_{h2}\right]$$

- so that

$$\lambda_{\mathrm{Max}}\{K_p\} \geq \lambda_{\min}\{K_p\} > \gamma^2 \frac{\alpha_1^2\ \lambda_{\mathrm{Max}}^2\{M\}}{\lambda_{\min}\{M\}} \tag{44}$$

  - where k_{h1} and k_{h2} are defined in Chapter 4
  - while the constants a and b are given by

$$\begin{aligned} a &= \frac{1}{2}\left[\lambda_{\mathrm{Max}}\{K_v\} + k_{C_1}\left\|\dot{\boldsymbol{q}}_d\right\|_{\mathrm{M}} + k_{h1}\right], \\ b &= \alpha_4\ \lambda_{\mathrm{Max}}\{M\} + \alpha_2\ k_{C_1}\,. \end{aligned}$$

Lyapunov function candidate. To carry out the stability analysis, consider

- Lyapunov function candidate

$$V(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) = \frac{1}{2}\dot{\tilde{\boldsymbol{q}}}^T M(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}} + \gamma \mathbf{tanh}(\tilde{\boldsymbol{q}})^T M(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} \qquad (45)$$

where $\gamma > 0$ is a given constant and

$$\mathbf{tanh}(\boldsymbol{x}) = \begin{bmatrix} \tanh(x_1) \\ \vdots \\ \tanh(x_n) \end{bmatrix}$$

with $\boldsymbol{x} \in \mathrm{IR}^n$.

- The Lyapunov function candidate (45) satisfies the following inequality

$$V(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) \geq \frac{1}{2} \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}^T \begin{bmatrix} \lambda_{\min}\{K_p\} & -\gamma\ \alpha_1\ \lambda_{\mathrm{Max}}\{M\} \\ -\gamma\ \alpha_1\ \lambda_{\mathrm{Max}}\{M\} & \lambda_{\min}\{M\} \end{bmatrix} \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}$$

  - It is positive definite and radially unbounded since
    * K_p is positive definite —$\lambda_{\min}\{K_p\} > 0$—,
    * and it is chosen so as to satisfy (44).

$$\lambda_{\mathrm{Max}}\{K_p\} \geq \lambda_{\min}\{K_p\} > \gamma^2 \frac{\alpha_1^2\ \lambda_{\mathrm{Max}}^2\{M\}}{\lambda_{\min}\{M\}}.$$

- One may also show

$$V(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}) \leq \frac{1}{2} \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}^T \begin{bmatrix} \lambda_{\text{Max}}\{K_p\} & \gamma\ \alpha_1\ \lambda_{\text{Max}}\{M\} \\ \gamma\ \alpha_1\ \lambda_{\text{Max}}\{M\} & \lambda_{\text{Max}}\{M\} \end{bmatrix} \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\tilde{\boldsymbol{q}}}\| \end{bmatrix}$$

  - whose right hand side is positive definite and radially unbounded since
    - the condition

$$\lambda_{\text{Max}}\{K_p\} > \gamma^2 \alpha_1^2\ \lambda_{\text{Max}}\{M\},$$

      is satisfied under hypothesis (44) on K_p.
  - This means that $V(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}})$ is decrescent.

Time derivative.

- The time derivative of the Lyapunov function candidate yields

$$\begin{aligned}\dot{V}(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) &= -\dot{\tilde{\boldsymbol{q}}}^T K_v \dot{\tilde{\boldsymbol{q}}} - \gamma \dot{\tilde{\boldsymbol{q}}}^T \mathsf{Sech}^2(\tilde{\boldsymbol{q}})^T M(\boldsymbol{q})\dot{\tilde{\boldsymbol{q}}} - \gamma \mathbf{tanh}(\tilde{\boldsymbol{q}})^T K_p \tilde{\boldsymbol{q}} \\ &\quad - \gamma \mathbf{tanh}(\tilde{\boldsymbol{q}})^T K_v \dot{\tilde{\boldsymbol{q}}} + \gamma \mathbf{tanh}(\tilde{\boldsymbol{q}})^T C(\boldsymbol{q},\dot{\boldsymbol{q}})^T \dot{\tilde{\boldsymbol{q}}} \\ &\quad - \dot{\tilde{\boldsymbol{q}}}^T \boldsymbol{h}(\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}}) - \gamma\ \mathbf{tanh}(\tilde{\boldsymbol{q}})^T \boldsymbol{h}(\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}})\,.\end{aligned}$$

- which can be upperbounded by

$$\dot{V}(t, \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) \leq$$
$$-\gamma \begin{bmatrix} \|\mathbf{tanh}(\tilde{\boldsymbol{q}})\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix}^T \underbrace{\begin{bmatrix} \frac{\lambda_{\min}\{K_p\}}{\alpha_3} - k_{h2} & -a - \frac{1}{\gamma}\frac{k_{h2}}{2} \\ -a - \frac{1}{\gamma}\frac{k_{h2}}{2} & \frac{1}{\gamma}\left[\lambda_{\min}\{K_v\} - k_{h1}\right] - b \end{bmatrix}}_{R(\gamma)} \begin{bmatrix} \|\mathbf{tanh}(\tilde{\boldsymbol{q}})\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix}$$

  - ⋆ where

$$\begin{aligned} a &= \frac{1}{2}\left[\lambda_{\mathrm{Max}}\{K_v\} + k_{C_1}\|\dot{\boldsymbol{q}}_d\|_{\mathrm{M}} + k_{h1}\right], \\ b &= \alpha_4\, \lambda_{\mathrm{Max}}\{M\} + \alpha_2\, k_{C_1}. \end{aligned}$$

  - ⋆ $R(\gamma)$ is positive definite
  - ⋆ therefore, $\dot{V}(t, \tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}})$ is globally negative definite.

- Theorem 2.4 concludes global uniform asymptotic stability of the origin.

Tuning procedure. It can be summarized as

- Derivation of the dynamic robot model to be controlled. Particularly, computation of $M(\boldsymbol{q})$, $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\boldsymbol{g}(\boldsymbol{q})$ in closed form.

- Computation of the constants $\lambda_{\mathrm{Max}}\{M(\boldsymbol{q})\}$, $\lambda_{\mathrm{min}}\{M(\boldsymbol{q})\}$, k_M, k'_M,k_{C_1}, k_{C_2}, k' and k_g. For this, it is suggested to use the information given in Table 4.1.

- Computation of $\|\ddot{\boldsymbol{q}}_d\|_{\mathrm{Max}}$, $\|\dot{\boldsymbol{q}}_d\|_{\mathrm{Max}}$ from the specification of a given task to the robot.

- Computation of the constants s_1 and s_2 given by

$$s_1 = \left[k_g + k_M \left\|\ddot{\boldsymbol{q}}_d\right\|_{\mathrm{M}} + k_{C_2} \left\|\dot{\boldsymbol{q}}_d\right\|_{\mathrm{M}}^2\right] ,$$

and

$$s_2 = 2\left[k' + k'_M \left\|\ddot{\boldsymbol{q}}_d\right\|_{\mathrm{M}} + k_{C_1} \left\|\dot{\boldsymbol{q}}_d\right\|_{\mathrm{M}}^2\right] .$$

Computation of k_{h1} and k_{h2} given by

$\star$ $k_{h1} \geq k_{C_1} \left\|\dot{\boldsymbol{q}}_d\right\|_{\mathrm{M}}$,

$\star$ $k_{h2} \geq \dfrac{s_2}{\tanh\left(\frac{s_2}{s_1}\right)}$.

- Computation of the constants a and b given by

$$
\begin{aligned}
a &= \frac{1}{2}\left[\lambda_{\mathrm{Max}}\{K_v\} + k_{C_1}\,\|\dot{\boldsymbol{q}}_d\|_{\mathrm{M}} + k_{h1}\right], \\
b &= \alpha_4\;\lambda_{\mathrm{Max}}\{M\} + \alpha_2\;k_{C_1},
\end{aligned}
$$

where $\alpha_2 = \sqrt{n}, \alpha_4 = 1$.

- Select $\gamma > 0$ and determine the design matrices K_p and K_v so that their smallest eigenvalues satisfy

  ⋆ $\lambda_{\min}\{K_v\} > k_{h1} + \gamma\ b,$

  ⋆ $\lambda_{\min}\{K_p\} > \alpha_3 \left[\frac{\left[2\ \gamma\ a + k_{h2}\right]^2}{4\ \gamma \left[\lambda_{\min}\{K_v\} - k_{h1} - \gamma b\right]} + k_{h2} \right],$

  ⋆ $\lambda_{\min}\{K_p\} > \gamma^2 \frac{\alpha_1^2\ \lambda_{\mathrm{Max}}^2\{M\}}{\lambda_{\min}\{M\}},$

  with $\alpha_1 = 1, \alpha_3 = 1.$

Example 12.5

Consider the 2-DOF *prototype robot* showed in the Figure

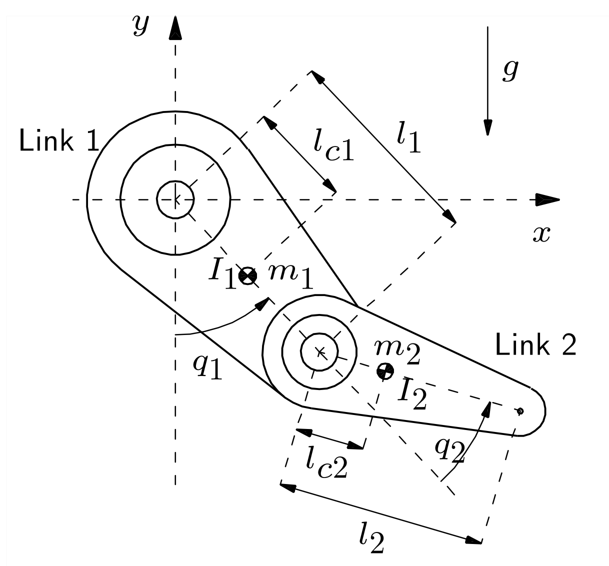

Figure 89: 2-DOF robot.

The elements of the inertia matrix $M(\boldsymbol{q})$ are

$$
\begin{aligned}
M_{11}(\boldsymbol{q}) &= m_1 l_{c1}^2 + m_2 \left(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos(q_2) \right) + I_1 + I_2 \\
M_{12}(\boldsymbol{q}) &= m_2 \left(l_{c2}^2 + l_1 l_{c2} \cos(q_2) \right) + I_2 \\
M_{21}(\boldsymbol{q}) &= m_2 \left(l_{c2}^2 + l_1 l_{c2} \cos(q_2) \right) + I_2 \\
M_{22}(\boldsymbol{q}) &= m_2 l_{c2}^2 + I_2 \,.
\end{aligned}
$$

The elements of the centrifugal and Coriolis forces matrix $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are given by

$$\begin{aligned} C_{11}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -m_2 l_1 l_{c2} \sin(q_2)\dot{q}_2 \\ C_{12}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -m_2 l_1 l_{c2} \sin(q_2)\left(\dot{q}_1 + \dot{q}_2\right) \\ C_{21}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= m_2 l_1 l_{c2} \sin(q_2)\dot{q}_1 \\ C_{22}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= 0\,. \end{aligned}$$

The elements of the vector of gravitational torques $\boldsymbol{g}(\boldsymbol{q})$ are

$$\begin{aligned} g_1(\boldsymbol{q}) &= (m_1 l_{c1} + m_2 l_1) g \sin(q_1) + m_2 l_{c2} g \sin(q_1 + q_2) \\ g_2(\boldsymbol{q}) &= m_2 l_{c2} g \sin(q_1 + q_2)\,. \end{aligned}$$

- Using the numeric values of the constants given in Table 5.1 as well as the formulas on Table 4.1, we get

$$
\begin{aligned}
k_M &= 0.0974 \;\; \left[\text{kg m}^2\right], \\
k_{C_1} &= 0.0487 \;\; \left[\text{kg m}^2\right], \\
k_{C_2} &= 0.0974 \;\; \left[\text{kg m}^2\right], \\
k_g &= 23.94 \;\; \left[\text{kg m}^2/\text{sec}^2\right], \\
k'_M &= \lambda_{\text{Max}}\{M(\boldsymbol{q})\} = 0.3614 \;\; \left[\text{kg m}^2\right], \\
\lambda_{\min}\{M(\boldsymbol{q})\} &= 0.011 \;\; \left[\text{kg m}^2\right].
\end{aligned}
$$

- Numerically, we get: $k' = 7.664 \quad [\text{N m}]$, and

$$\|\dot{\boldsymbol{q}}_d\|_{\text{Max}} = 2.33 \;\; [\text{ rad/sec}] \quad \text{and,} \quad \|\ddot{\boldsymbol{q}}_d\|_{\text{Max}} = 9.52 \;\; [\text{ rad/sec}^2].$$

- Using this information and the definitions of the constants from the tuning procedure, we get that

$$
\begin{aligned}
s_1 &= 25.385 \ \ [\mathrm{N\ m}]\,, \\
s_2 &= 22.733 \ \ [\mathrm{N\ m}]\,, \\
k_{h1} &= 0.114 \ \ \left[\mathrm{kg\ m^2/sec}\right]\,, \\
k_{h2} &= 31.834 \ \ [\mathrm{N\ m}]\,, \\
a &= 1.614 \ \ \left[\mathrm{kg\ m^2/sec}\right]\,, \\
b &= 0.43 \ \ \left[\mathrm{kg\ m^2}\right]\,.
\end{aligned}
$$

- Finally, we set $\gamma = 2$ $[\mathrm{sec}^{-1}]$,

- An appropriate choice of the gains is

$$\begin{aligned} K_p &= \operatorname{diag}\{200,\ 150\}\ [\mathrm{N\ m}], \\ K_v &= \operatorname{diag}\{3\}\ [\mathrm{N\ m\ sec/rad}]. \end{aligned}$$

- The initial conditions corresponding to the positions and velocities, are chosen as

$$\begin{aligned} q_1(0) = 0, \quad q_2(0) = 0 \\ \dot{q}_1(0) = 0, \quad \dot{q}_2(0) = 0\,. \end{aligned}$$

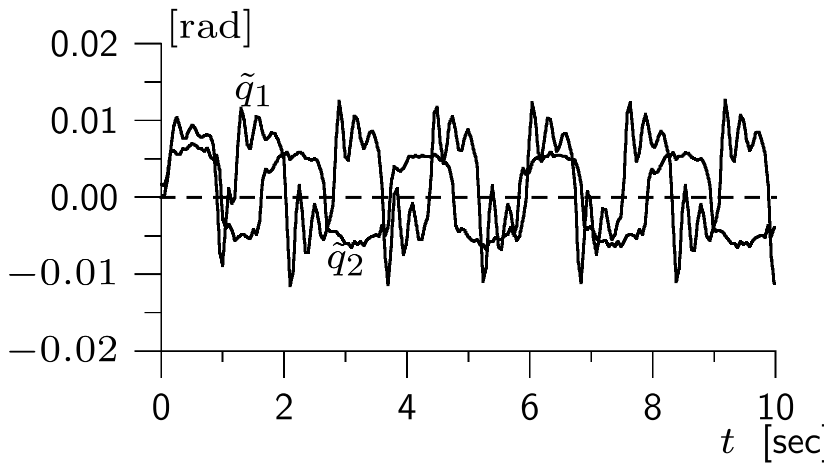

Figure 90: Position errors

- In contrast to example 12.3 where the controller did not carry the PD term, the behavior abtained here is satisfactory.

Part IV

ADVANCED TOPICS

Introduction to Part IV

We deal with a variety of topics:

- Control without velocity measurements

- Control under model uncertainty.

  Specifically:

  – P "D" control with gravity compensation and P "D" control with desired gravity compensation
  – Introduction to adaptive robot control
  – PD control with adaptive gravity compensation
  – PD control with adaptive compensation.

Ch. 13. P"D" Control with gravity compensation and P"D" Control with precalculated gravity compensation

The interest to count on controllers without measurement of velocity, is twofold:

- no poor quality for certain bands of operation
- suppression of velocity sensors (no tachometers and "resolvers")
  - reduction in the production cost
  - robot lighter.

The design of controllers that do not require velocity measurements to control robot manipulators

- is a topic of investigation broached in the decade of the 1990's
- to date, many questions remain open.
- the common idea: to propose state observers to estimate the velocity.

  In this chapter we present an alternative:

  - substituting $\dot{\boldsymbol{q}}$, by the *filtering* of $\boldsymbol{q}$
    - through a first order system of zero relative degree,
    - whose output is denoted, by $\boldsymbol{\vartheta}$.

- Specifically, $\boldsymbol{\vartheta} \in \mathrm{IR}^n$ is given by:

$$\boldsymbol{\vartheta} = \mathrm{diag}\left\{\frac{b_i p}{p + a_i}\right\}\boldsymbol{q}$$

  ⋆ where, $p = \frac{d}{dt}$, a_i and b_i are arbitrary strictly positive real constants, for $i = 1, 2, \cdots, n$.

- A state-space representation of above Equation is:

$$\begin{aligned} \dot{\boldsymbol{x}} &= -A\boldsymbol{x} - AB\boldsymbol{q} \\ \boldsymbol{\vartheta} &= \boldsymbol{x} + B\boldsymbol{q} \end{aligned}$$

  ⋆ where, $\boldsymbol{x} \in \mathrm{IR}^n$ represents the state vector of the filters,
  ⋆ $A = \mathrm{diag}\{a_i\}$ and $B = \mathrm{diag}\{b_i\}$.

In this chapter we present the study of the proposed modification for the following controllers:

- PD control with gravity compensation

- PD control with desired gravity compensation.
  - the *derivative* part of both controllers is no longer proportional to the *derivative* of the position error $\tilde{\boldsymbol{q}}$
  - this motivates the quotes in "D" in the name of the controller.

P"D" Control with gravity compensation

Control law:

$$\begin{aligned}
\boldsymbol{\tau} &= K_p\tilde{\boldsymbol{q}} + K_v\left[\dot{\boldsymbol{q}}_d - \boldsymbol{\vartheta}\right] + \boldsymbol{g}(\boldsymbol{q}) \\
\dot{\boldsymbol{x}} &= -A\boldsymbol{x} - AB\boldsymbol{q} \\
\boldsymbol{\vartheta} &= \boldsymbol{x} + B\boldsymbol{q}
\end{aligned}$$

where

- $K_p, K_v \in \mathrm{IR}^{n\times n}$ are diagonal positive definite matrices,
- $A = \mathrm{diag}\{a_i\}$ and $B = \mathrm{diag}\{b_i\}$ and a_i and b_i are arbitrary real strictly positive constants for $i = 1, 2, \cdots, n$.

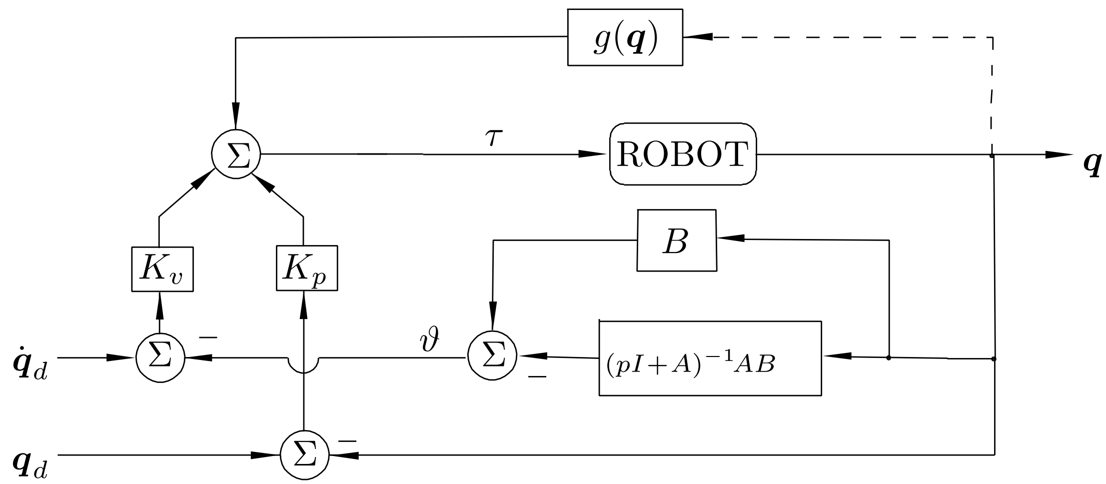

Figure 91: P"D" Control with gravity compensation.

When $\boldsymbol{q}_d$ is a constant vector

- the control law becomes

$$\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} - K_v \mathrm{diag}\left\{\frac{b_i p}{p + a_i}\right\}\boldsymbol{q} + \boldsymbol{g}(\boldsymbol{q}).$$

- it verifies the set-point control objective: $\lim_{t\to\infty} \boldsymbol{q}(t) = \boldsymbol{q}_d$ (with $\boldsymbol{q}_d \in \mathbb{R}^n$ constant)

- The closed loop equation may be rewritten as

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{\xi} \\ \\ \tilde{\boldsymbol{q}} \\ \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} -A\boldsymbol{\xi} + AB\tilde{\boldsymbol{q}} \\ \\ -\dot{\boldsymbol{q}} \\ \\ M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}})^{-1}\left[K_p\tilde{\boldsymbol{q}} - K_v[\boldsymbol{\xi} - B\tilde{\boldsymbol{q}}] - C(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}\right] \end{bmatrix}$$

  ⋆ which is an autonomous differential equation
  ⋆ $\boldsymbol{\xi} = \boldsymbol{x} + B\boldsymbol{q}_d$.
  ⋆ the origin $\begin{bmatrix} \boldsymbol{\xi}^T & \tilde{\boldsymbol{q}}^T & \dot{\boldsymbol{q}}^T \end{bmatrix}^T = \boldsymbol{0} \in \mathbb{R}^{3n}$ is the unique equilibrium.

Stability of the origin,

- Lyapunov function candidate

$$V(\boldsymbol{\xi},\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}}) = \mathcal{K}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \frac{1}{2}\tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}} + \frac{1}{2}(\boldsymbol{\xi} - B\tilde{\boldsymbol{q}})^T K_v B^{-1} (\boldsymbol{\xi} - B\tilde{\boldsymbol{q}})$$

where

⋆ $\mathcal{K}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}}$ is the kinetic energy function
⋆ $K_v B^{-1}$ is positive definite.
⋆ Hence, $V(\boldsymbol{\xi},\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}})$ is globally positive definite.

- Time derivative of the Lyapunov function candidate

$$\begin{aligned}\dot V(\boldsymbol{\xi},\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}}) &= \dot{\boldsymbol{q}}^T M(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T \dot M(\boldsymbol{q})\dot{\boldsymbol{q}} + \tilde{\boldsymbol{q}}^T K_p \dot{\tilde{\boldsymbol{q}}} \\ &\quad + \left[\boldsymbol{\xi} - B\tilde{\boldsymbol{q}}\right]^T K_v B^{-1}\left[\dot{\boldsymbol{\xi}} - B\dot{\tilde{\boldsymbol{q}}}\right].\end{aligned}$$

⋆ Using the closed loop equation we obtain

$$\begin{aligned}\dot V(\boldsymbol{\xi},\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}}) &= -\left[\boldsymbol{\xi} - B\tilde{\boldsymbol{q}}\right]^T K_v B^{-1} A\left[\boldsymbol{\xi} - B\tilde{\boldsymbol{q}}\right] \\ &= -\begin{bmatrix}\boldsymbol{\xi}\\ \tilde{\boldsymbol{q}}\\ \dot{\boldsymbol{q}}\end{bmatrix}^T \begin{bmatrix} K_v B^{-1}A & -K_v A & 0\\ -K_v A & BK_v A & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}\\ \tilde{\boldsymbol{q}}\\ \dot{\boldsymbol{q}}\end{bmatrix}^T\end{aligned}$$

⋆ $\dot V(\boldsymbol{\xi},\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}})$ is globally negative semidefinite.

- The origin is asymptotically stable (use the La Salle's Theorem).

Example 13.1

Consider the Pelican robot

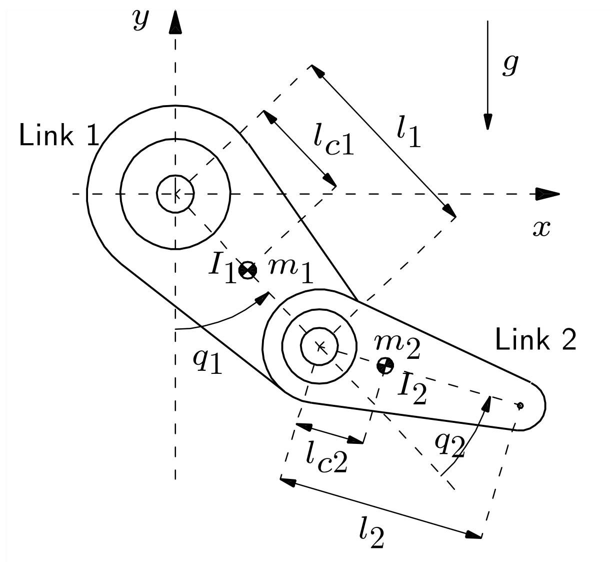

Figure 92: Diagram of the Pelican robot.

- The components of the vector of gravitational torques $\boldsymbol{g}(\boldsymbol{q})$ are given by

$$\begin{aligned} g_1(\boldsymbol{q}) &= (m_1 l_{c1} + m_2 l_1) g \sin(q_1) + m_2 l_{c2} g \sin(q_1 + q_2) \\ g_2(\boldsymbol{q}) &= m_2 l_{c2} g \sin(q_1 + q_2) \,. \end{aligned}$$

- Consider the P "D" controller with gravity compensation with

$$\begin{aligned} K_p &= \mathrm{diag}\{k_p\} = \mathrm{diag}\{30\} \quad [\mathrm{Nm/rad}] \,, \\ K_v &= \mathrm{diag}\{k_v\} = \mathrm{diag}\{7,\ 3\} \quad [\mathrm{Nm\ sec/rad}] \,, \\ A &= \mathrm{diag}\{a_i\} = \mathrm{diag}\{30,\ 70\} \quad [1/\mathrm{sec}] \,, \\ B &= \mathrm{diag}\{b_i\} = \mathrm{diag}\{30,\ 70\} \quad [1/\mathrm{sec}] \,. \end{aligned}$$

- The components of the control input $\boldsymbol{\tau}$ are given by

$$
\begin{aligned}
\tau_1 &= k_p\tilde{q}_1 - k_v\vartheta_1 + g_1(\boldsymbol{q}) \\
\tau_2 &= k_p\tilde{q}_2 - k_v\vartheta_2 + g_2(\boldsymbol{q}) \\
\dot{x}_1 &= -a_1x_1 - a_1b_1q_1 \\
\dot{x}_2 &= -a_2x_2 - a_2b_2q_2 \\
\vartheta_1 &= x_1 + b_1q_1 \\
\vartheta_2 &= x_2 + b_2q_2\,.
\end{aligned}
$$

- Initial conditions

$$
\begin{aligned}
q_1(0) = 0, \quad & q_2(0) = 0 \\
\dot{q}_1(0) = 0, \quad & \dot{q}_2(0) = 0 \\
x_1(0) = 0, \quad & x_2(0) = 0\,.
\end{aligned}
$$

- The desired joint positions are chosen as

$$q_{d1} = \pi/10, \quad q_{d2} = \pi/30 \quad [\mathrm{rad}]\,.$$

- In terms of the state vector of the closed loop equation, the initial state is

$$\begin{bmatrix} \boldsymbol{\xi}(0) \\ \tilde{\boldsymbol{q}}(0) \\ \dot{\boldsymbol{q}}(0) \end{bmatrix} = \begin{bmatrix} b_1\pi/10 \\ b_2\pi/30 \\ \pi/10 \\ \pi/30 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9.423 \\ 7.329 \\ 0.3141 \\ 0.1047 \\ 0 \\ 0 \end{bmatrix}.$$

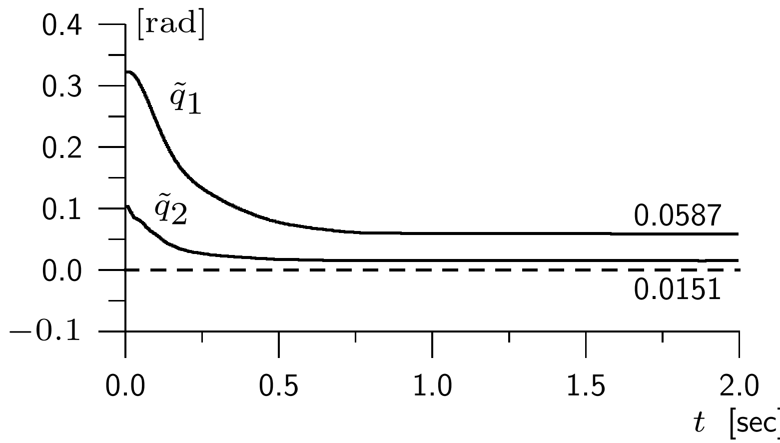

Figure 93: Position errors $\tilde{q}_1(t)$ and $\tilde{q}_2(t)$.

- Figure presents the experimental results,
  - $\star$ $\tilde{\boldsymbol{q}}(t)$ tends asymptotically to a constant nonzero value (due to the non-modeled friction effects).

P"D" Control with desired gravity compensation

Control law

$$\begin{aligned} \boldsymbol{\tau} &= K_p\tilde{\boldsymbol{q}} + K_v\left[\dot{\boldsymbol{q}}_d - \boldsymbol{\vartheta}\right] + \boldsymbol{g}(\boldsymbol{q}_d) \\ \dot{\boldsymbol{x}} &= -A\boldsymbol{x} - AB\boldsymbol{q} \\ \boldsymbol{\vartheta} &= \boldsymbol{x} + B\boldsymbol{q} \end{aligned}$$

where

- $K_p, K_v \in \mathbb{R}^{n\times n}$ are diagonal positive definite matrices,
- $A = \mathrm{diag}\{a_i\}$ and $B = \mathrm{diag}\{b_i\}$ with a_i and b_i arbitrary real strictly positive constants for all $i = 1, 2, \cdots, n$.

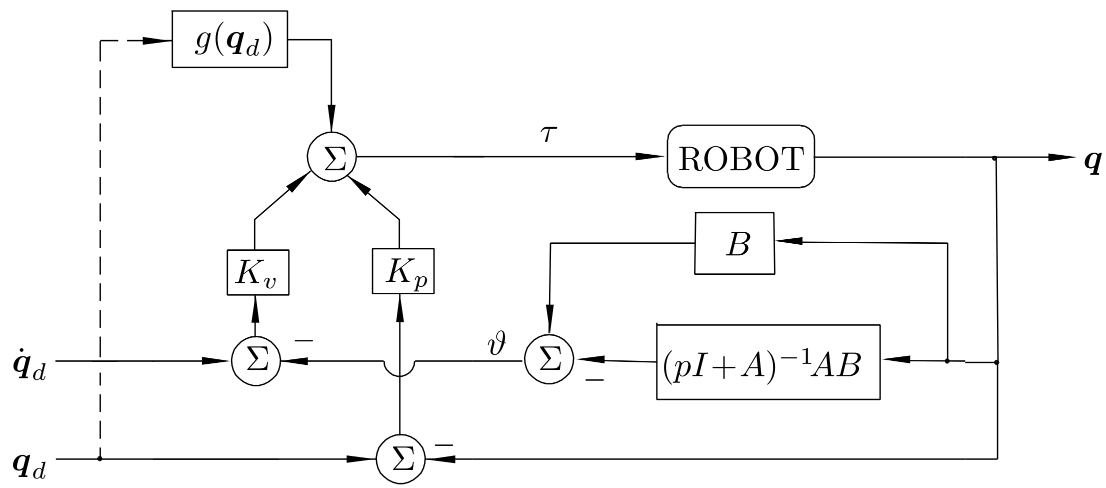

Figure 94: P"D" Control with desired gravity compensation.

When $\boldsymbol{q}_d$ is a constant vector

- the control law may be expressed by

$$\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} - K_v \text{diag}\left\{\frac{b_i p}{p + a_i}\right\}\boldsymbol{q} + \boldsymbol{g}(\boldsymbol{q}_d),$$

If $\lambda_{\min}\{K_p\} > k_g$, then

- it verifies the set-point control objective that is, $\lim_{t\to\infty} \boldsymbol{q}(t) = \boldsymbol{q}_d$
- The closed loop equation may rewritten as:

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{\xi} \\ \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} -A\boldsymbol{\xi} + AB\tilde{\boldsymbol{q}} \\ -\dot{\boldsymbol{q}} \\ M(\boldsymbol{q})^{-1}\left[K_p\tilde{\boldsymbol{q}} - K_v[\boldsymbol{\xi} - B\tilde{\boldsymbol{q}}] + \boldsymbol{g}(\boldsymbol{q}_d) - C(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}_d - \tilde{\boldsymbol{q}})\right] \end{bmatrix}$$

  ⋆ which is an autonomous differential equation.
  ⋆ $\boldsymbol{\xi} = \boldsymbol{x} + B\boldsymbol{q}_d$.
  ⋆ the origin $\begin{bmatrix} \boldsymbol{\xi}^T & \tilde{\boldsymbol{q}}^T & \dot{\boldsymbol{q}}^T \end{bmatrix}^T = \boldsymbol{0} \in \mathrm{IR}^{3n}$ is the unique equilibrium

Stability of the origin

- Consider the Lyapunov function candidate

$$V(\boldsymbol{\xi}, \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) = \mathcal{K}(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) + f(\tilde{\boldsymbol{q}}) + \frac{1}{2}(\boldsymbol{\xi} - B\tilde{\boldsymbol{q}})^T K_v B^{-1} (\boldsymbol{\xi} - B\tilde{\boldsymbol{q}})$$

where

⋆ $\mathcal{K}(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q}_d - \tilde{\boldsymbol{q}})\dot{\boldsymbol{q}}$

⋆ $f(\tilde{\boldsymbol{q}}) = \mathcal{U}(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q}_d) + \boldsymbol{g}(\boldsymbol{q}_d)^T \tilde{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}}$.

∗ Since $K_v B^{-1}$ is positive definite and

∗ $\lambda_{\min}\{K_p\} > k_g$

∗ Consequently, $V(\boldsymbol{\xi}, \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})$ is also globally positive definite.

- The time derivative of the Lyapunov function candidate yields

$$\begin{aligned}\dot{V}(\boldsymbol{\xi}, \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) &= \dot{\boldsymbol{q}}^T M(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T \dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\tilde{\boldsymbol{q}}}^T \boldsymbol{g}(\boldsymbol{q}_d - \tilde{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}_d)^T \dot{\tilde{\boldsymbol{q}}} \\ &\quad + \tilde{\boldsymbol{q}}^T K_p \dot{\tilde{\boldsymbol{q}}} + [\boldsymbol{\xi} - B\tilde{\boldsymbol{q}}]^T K_v B^{-1} \left[\dot{\boldsymbol{\xi}} - B\dot{\tilde{\boldsymbol{q}}}\right] .\end{aligned}$$

  ⋆ Using the closed loop equation we obtain

$$\dot{V}(\boldsymbol{\xi}, \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}) = -(\boldsymbol{\xi} - B\tilde{\boldsymbol{q}})^T K_v B^{-1} A (\boldsymbol{\xi} - B\tilde{\boldsymbol{q}})$$

  ⋆ $\dot{V}(\boldsymbol{\xi}, \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}})$ is a globally semidefinite negative function.

- The origin is global asymptotically stable (use La Salle's Theorem).

Example 13.2

Consider the Pelican robot

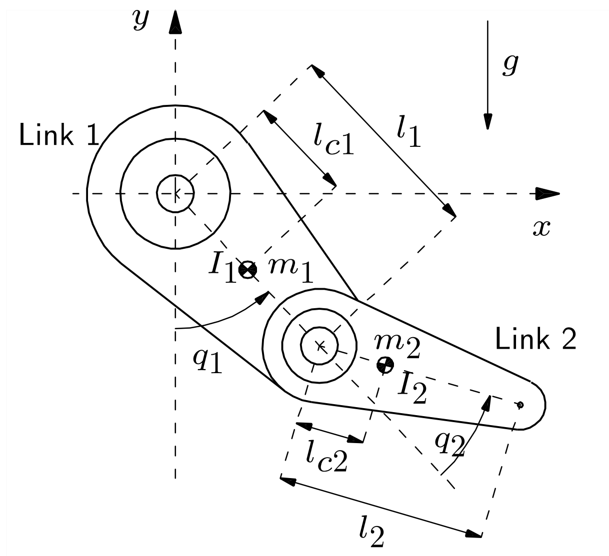

Figure 95: Diagram of the Pelican robot.

- The components of the vector of gravitational torques $\boldsymbol{g}(\boldsymbol{q})$ are given by

$$\begin{aligned} g_1(\boldsymbol{q}) &= (m_1 l_{c1} + m_2 l_1) g \sin(q_1) + m_2 l_{c2} g \sin(q_1 + q_2) \\ g_2(\boldsymbol{q}) &= m_2 l_{c2} g \sin(q_1 + q_2)\,. \end{aligned}$$

- The constant k_g may be obtained as (see Property , 4.3):

$$\begin{aligned} k_g &= n \left(\max{}_{i,j,q} \left| \frac{\partial g_i(\boldsymbol{q})}{\partial q_j} \right| \right) \\ &= n((m_1 l_{c1} + m_2 l_1) g + m_2 l_{c2} g) \\ &= 23.94 \quad \left[\mathrm{kg\ m^2/sec^2} \right]. \end{aligned}$$

- Consider the P "D" control with desired gravity compensation satisfying:

$$\lambda_{\min}\{K_p\} > k_g \,.$$

In particular, these matrices are taken to be

$$\begin{aligned} K_p &= \mathrm{diag}\{k_p\} = \mathrm{diag}\{30\} \quad [\mathrm{Nm/rad}]\,, \\ K_v &= \mathrm{diag}\{k_v\} = \mathrm{diag}\{7,\ 3\} \quad [\mathrm{Nm\ sec/rad}]\,, \\ A &= \mathrm{diag}\{a_i\} = \mathrm{diag}\{30,\ 70\} \quad [1/\mathrm{sec}]\,, \\ B &= \mathrm{diag}\{b_i\} = \mathrm{diag}\{30,\ 70\} \quad [1/\mathrm{sec}]\,. \end{aligned}$$

- The components of the control input $\boldsymbol{\tau}$ are given by

$$\begin{aligned}
\tau_1 &= k_p\tilde{q}_1 - k_v\vartheta_1 + g_1(\boldsymbol{q}) \\
\tau_2 &= k_p\tilde{q}_2 - k_v\vartheta_2 + g_2(\boldsymbol{q}) \\
\dot{x}_1 &= -a_1x_1 - a_1b_1q_1 \\
\dot{x}_2 &= -a_2x_2 - a_2b_2q_2 \\
\vartheta_1 &= x_1 + b_1q_1 \\
\vartheta_2 &= x_2 + b_2q_2 \,.
\end{aligned}$$

- Initial conditions

$$\begin{aligned}
q_1(0) = 0, &\quad q_2(0) = 0 \\
\dot{q}_1(0) = 0, &\quad \dot{q}_2(0) = 0 \\
x_1(0) = 0, &\quad x_2(0) = 0 \,.
\end{aligned}$$

- The desired joint positions are

$$q_{d1} = \pi/10, \quad q_{d2} = \pi/30 \quad [\mathrm{rad}]\,.$$

- In terms of the state vector of the closed loop equation, the initial state is

$$\begin{bmatrix} \boldsymbol{\xi}(0) \\ \tilde{\boldsymbol{q}}(0) \\ \dot{\boldsymbol{q}}(0) \end{bmatrix} = \begin{bmatrix} b_1\pi/10 \\ b_2\pi/30 \\ \pi/10 \\ \pi/30 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9.423 \\ 7.329 \\ 0.3141 \\ 0.1047 \\ 0 \\ 0 \end{bmatrix}.$$

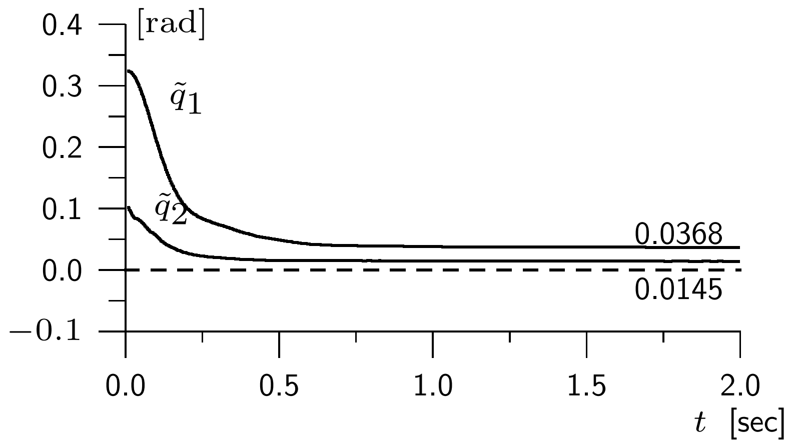

Figure 96: Position errors $\tilde{q}_1(t)$ and $\tilde{q}_2(t)$

- Figure shows the experimental results,
  - $\tilde{\boldsymbol{q}}(t)$ tends asymptotically to a constant nonzero value (due to the friction effects in the actual robot).

Ch.14. Introduction to adaptive robot control

The implicit assumptions in the preceding chapters are:

- The model is accurately known
- The constant physical parameters are accurately known.

Two general techniques deal with the absence of above considerations:

- *Robust* control aims at controlling, with a small error, a *class* of robot manipulators (model is not accurately known) with the same controller.
- *Adaptive* control deals with
  - uncertainty in the systems parameters (unknown *constant* parameters)
  - it requires the precise knowledge of the structure of the system.

In the subsequent chapters we describe and analyze two adaptive controllers for robots.

- PD control with feedforward gravity compensation.
- PD control with adaptive compensation.

Parameterization of the dynamic model

The dynamic model of robot manipulators as we know, is given by

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

- To emphasize the dependence of the dynamic model on the dynamic parameters, we write

$$M(\boldsymbol{q}, \boldsymbol{\theta})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\theta})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}, \boldsymbol{\theta}) = \boldsymbol{\tau}\,. \tag{46}$$

  where

  ⋆ $\boldsymbol{\theta} \in \mathrm{IR}^m$ is the m vector of parameters
  ⋆ m is some known constant
  ⋆ $\boldsymbol{\theta}$, *do not* necessarily correspond to the physical parameters of the robot

Example 14.1

Consider the example of an ideal pendulum

- Dynamic model
$$ml^2\ddot{q} + mgl\sin(q) = \tau$$
  - where its mass m is concentrated at the tip,
  - at a distance l from its axis of rotation.

- We identify:
  - ⋆ $M(q,\boldsymbol{\theta}) = ml^2\ddot{q}$, and $g(q,\boldsymbol{\theta}) = mgl\sin(q)$.
  - ⋆ $\boldsymbol{\theta} = \begin{bmatrix} ml^2 \\ mgl \end{bmatrix}$ (assuming that both parameters are unknown)
  - ⋆ it is a nonlinear vectorial function of the physical parameters.

Linearity in the Dynamic Parameters

Above example also shows that

- the dynamic model is linear in the parameters $\boldsymbol{\theta}$.

$$
\begin{aligned}
ml^2\ddot{q} + mgl\sin(q) &= [\ddot{q} \quad \sin(q)] \begin{bmatrix} ml^2 \\ mgl \end{bmatrix} \\
&=: \Phi(q, \ddot{q})\boldsymbol{\theta} \,.
\end{aligned}
$$

  ⋆ Φ contains nonlinear terms of the state, and
  ⋆ $\boldsymbol{\theta}$ is the vector of dynamic parameters.

- Property known as "linearity in the parameters" or "linear parameterization".

Property 14.1. Linearity in the dynamic parameters.

1. For all $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathrm{IR}^n$ it holds that

$$M(\boldsymbol{q},\boldsymbol{\theta})\boldsymbol{u}+C(\boldsymbol{q},\boldsymbol{w},\boldsymbol{\theta})\boldsymbol{v}+\boldsymbol{g}(\boldsymbol{q},\boldsymbol{\theta}) = \Phi(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\boldsymbol{\theta}+\boldsymbol{\kappa}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$$

where $\boldsymbol{\kappa}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ is a vector of $n \times 1$, $\Phi(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ is a matrix of $n \times m$ and the vector $\boldsymbol{\theta} \in \mathrm{IR}^m$ depends only on the dynamic parameters of the manipulator and its load.

2. Moreover, if $\boldsymbol{q}$, $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in L_\infty^n$ then $\Phi(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in L_\infty^{n\times m}$.

The Nominal Model

The dynamics (46), may be expressed as

$$M(\boldsymbol{q},\boldsymbol{\theta})\boldsymbol{u} + C(\boldsymbol{q},\boldsymbol{w},\boldsymbol{\theta})\boldsymbol{v} + \boldsymbol{g}(\boldsymbol{q},\boldsymbol{\theta}) = \\ \Phi(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\boldsymbol{\theta} + M_0(\boldsymbol{q})\boldsymbol{u} + C_0(\boldsymbol{q},\boldsymbol{w})\boldsymbol{v} + \boldsymbol{g}_0(\boldsymbol{q}),$$

where

- the *nominal model* or nominal part of the model is,

$$\boldsymbol{\kappa}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = M_0(\boldsymbol{q})\boldsymbol{u} + C_0(\boldsymbol{q},\boldsymbol{w})\boldsymbol{v} + \boldsymbol{g}_0(\boldsymbol{q})\,.$$

- $M_0(\boldsymbol{q})$, $C_0(\boldsymbol{q}, \boldsymbol{w})$ and the vector $\boldsymbol{g}_0(\boldsymbol{q})$ are the parts of
  - ⋆ $M(\boldsymbol{q})$, $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\boldsymbol{g}(\boldsymbol{q})$
  - ⋆ do not depend on $\boldsymbol{\theta}$ (*unknown* dynamic parameters).

- Given a vector $\hat{\boldsymbol{\theta}} \in \mathbb{R}^m$, we have

$$\Phi(\boldsymbol{q}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})\hat{\boldsymbol{\theta}} =$$
$$M(\boldsymbol{q}, \hat{\boldsymbol{\theta}})\boldsymbol{u} + C(\boldsymbol{q}, \boldsymbol{w}, \hat{\boldsymbol{\theta}})\boldsymbol{v} + \boldsymbol{g}(\boldsymbol{q}, \hat{\boldsymbol{\theta}}) - M_0(\boldsymbol{q})\boldsymbol{u} - C_0(\boldsymbol{q}, \boldsymbol{w})\boldsymbol{v} - \boldsymbol{g}_0(\boldsymbol{q}).$$

- A particular case is when $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w} = \boldsymbol{0} \in \mathbb{R}^n$. In this scenario

$$\boldsymbol{g}(\boldsymbol{q}, \boldsymbol{\theta}) = \Phi(\boldsymbol{q}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})\boldsymbol{\theta} + \boldsymbol{g}_0(\boldsymbol{q})\,.$$

Example 14.7

Consider the Pelican robot

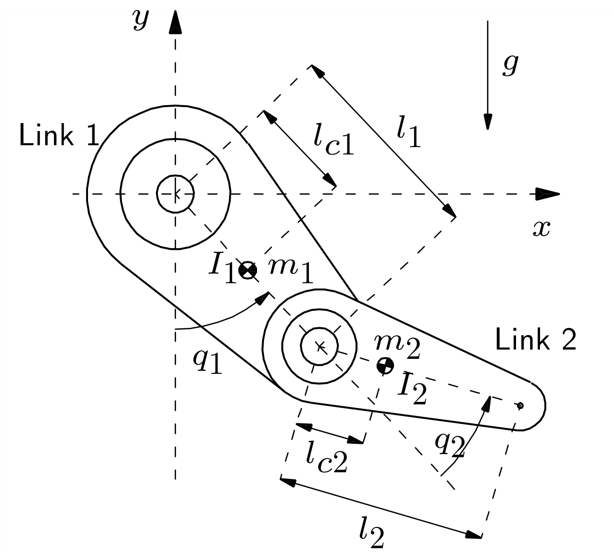

Figure 97: Diagram of the Pelican robot.

- Dynamic model

$$\underbrace{\begin{bmatrix} M_{11}(\boldsymbol{q}) & M_{12}(\boldsymbol{q}) \\ M_{21}(\boldsymbol{q}) & M_{22}(\boldsymbol{q}) \end{bmatrix}}_{M(\boldsymbol{q})} \ddot{\boldsymbol{q}} + \underbrace{\begin{bmatrix} C_{11}(\boldsymbol{q},\dot{\boldsymbol{q}}) & C_{12}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ C_{21}(\boldsymbol{q},\dot{\boldsymbol{q}}) & C_{22}(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{bmatrix}}_{C(\boldsymbol{q},\dot{\boldsymbol{q}})} \dot{\boldsymbol{q}} + \underbrace{\begin{bmatrix} g_1(\boldsymbol{q}) \\ g_2(\boldsymbol{q}) \end{bmatrix}}_{\boldsymbol{g}(\boldsymbol{q})} = \boldsymbol{\tau}$$

where

$$\begin{aligned}
M_{11}(\boldsymbol{q}) &= m_1 l_{c1}^2 + m_2 \left[l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos(q_2) \right] + I_1 + I_2 \\
M_{12}(\boldsymbol{q}) &= m_2 \left[l_{c2}^2 + l_1 l_{c2} \cos(q_2) \right] + I_2 \\
M_{21}(\boldsymbol{q}) &= m_2 \left[l_{c2}^2 + l_1 l_{c2} \cos(q_2) \right] + I_2 \\
M_{22}(\boldsymbol{q}) &= m_2 l_{c2}^2 + I_2 \\
C_{11}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= -m_2 l_1 l_{c2} \sin(q_2) \dot{q}_2
\end{aligned}$$

$$
\begin{aligned}
C_{12}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= -m_2 l_1 l_{c2}\,\sin(q_2)\,[\dot{q}_1+\dot{q}_2] \\
C_{21}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= m_2 l_1 l_{c2}\,\sin(q_2)\dot{q}_1 \\
C_{22}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= 0
\end{aligned}
$$

$$
\begin{aligned}
g_1(\boldsymbol{q}) &= [m_1 l_{c1} + m_2 l_1]\,g\,\sin(q_1) + m_2 l_{c2} g\,\sin(q_1+q_2) \\
g_2(\boldsymbol{q}) &= m_2 l_{c2} g\,\sin(q_1+q_2)\,.
\end{aligned}
$$

- We have selected as parameters of interest,
  - ⋆ m_2, I_2 and l_{c2}

- We define the vectors

$$
\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.
$$

- The parameterization leads to

$$M(\boldsymbol{q},\boldsymbol{\theta})\boldsymbol{u}+C(\boldsymbol{q},\boldsymbol{w},\boldsymbol{\theta})\boldsymbol{v}+\boldsymbol{g}(\boldsymbol{q},\boldsymbol{\theta}) =$$
$$\begin{bmatrix}\Phi_{11} & \Phi_{12} & \Phi_{13}\\ \Phi_{21} & \Phi_{22} & \Phi_{23}\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix}+M_0(\boldsymbol{q})\boldsymbol{u}+C_0(\boldsymbol{q},\boldsymbol{w})\boldsymbol{v}+\boldsymbol{g}_0(\boldsymbol{q}),$$

where

$$\begin{aligned}
\Phi_{11} &= l_1^2 u_1 + l_1 g\,\sin(q_1)\\
\Phi_{12} &= 2l_1\,\cos(q_2)u_1 + l_1\,\cos(q_2)u_2 - l_1\,\sin(q_2)w_2v_1\\
&\quad -l_1\,\sin(q_2)[w_1+w_2]v_2 + g\,\sin(q_1+q_2)\\
\Phi_{13} &= u_1+u_2\\
\Phi_{21} &= 0
\end{aligned}$$

$$
\begin{aligned}
\Phi_{22} &= l_1 \cos(q_2)u_1 + l_1 \sin(q_2)w_1 v_1 + g \sin(q_1+q_2) \\
\Phi_{23} &= u_1 + u_2 \\
\boldsymbol{\theta} &= \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} m_2 \\ m_2 l_{c2} \\ m_2 l_{c2}^2 + I_2 \end{bmatrix} \\
M_0(\boldsymbol{q}) &= \begin{bmatrix} m_1 l_{c1}^2 + I_1 & 0 \\ 0 & 0 \end{bmatrix} \\
C_0(\boldsymbol{q},\boldsymbol{w}) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\boldsymbol{g}_0(\boldsymbol{q}) &= \begin{bmatrix} m_1 l_{c1} g \sin(q_1) \\ 0 \end{bmatrix}.
\end{aligned}
$$

- $\boldsymbol{\theta}$ depends exclusively on the parameters of interest m_2, I_2 and l_{c2}.

The Adaptive Robot Control Problem

- Consider the dynamic equation

$$M(\boldsymbol{q},\boldsymbol{\theta})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{\theta})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q},\boldsymbol{\theta}) = \tau$$

or equivalently,

$$\Phi(\boldsymbol{q},\ddot{\boldsymbol{q}},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}})\boldsymbol{\theta} + M_0(\boldsymbol{q})\ddot{\boldsymbol{q}} + C_0(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}_0(\boldsymbol{q}) = \boldsymbol{\tau}\,.$$

  - Assume that $\Phi(\boldsymbol{q},\ddot{\boldsymbol{q}},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}) \in$ IR$^{n\times m}$, $M_0(\boldsymbol{q}), C_0(\boldsymbol{q},\dot{\boldsymbol{q}}) \in$ IR$^{n\times n}$ and $\boldsymbol{g}_0(\boldsymbol{q}) \in$ IR^n are **known**
  - but $\boldsymbol{\theta} \in$ IR^m is **unknown**
  - Given a set of vectorial bounded functions $\boldsymbol{q}_d$, $\dot{\boldsymbol{q}}_d$ and $\ddot{\boldsymbol{q}}_d$,
  - we seek to design controllers that achieve the position or the motion control objectives.

Parameterization of the controller

The control laws may be written in the functional form

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d, M(\boldsymbol{q}), C(\boldsymbol{q}, \dot{\boldsymbol{q}}), \boldsymbol{g}(\boldsymbol{q}))\,.$$

- Giving a little 'more' structure we have

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d) + M(\boldsymbol{q})\boldsymbol{u} + C(\boldsymbol{q}, \boldsymbol{w})\boldsymbol{v} + \boldsymbol{g}(\boldsymbol{q}),$$

  where

⋆ The first term $\boldsymbol{\tau}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d)$, usually corresponds to linear control terms of PD type, i.e.,

$$\boldsymbol{\tau}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d) = K_p[\boldsymbol{q}_d - \boldsymbol{q}] + K_v[\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}]$$

where

- K_p is the gain matrix of position
- K_v is the velocity (or derivative gain).
- It does not depend explicitly on the dynamic model

⋆ In the second term $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathrm{IR}^n$ depend on $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$ and on $\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d$ and $\ddot{\boldsymbol{q}}_d$.

- It depends explicitly on the dynamic model

- In general an adaptive controller is formed by two main parts:
  - Control law or controller.
  - Adaptation law.

- Control law in the generic form

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d) + M(\boldsymbol{q}, \hat{\boldsymbol{\theta}})\boldsymbol{u} + C(\boldsymbol{q}, \boldsymbol{w}, \hat{\boldsymbol{\theta}})\boldsymbol{v} + \boldsymbol{g}(\boldsymbol{q}, \hat{\boldsymbol{\theta}})$$

where

$\star$ $\hat{\boldsymbol{\theta}}$ is the vector of adaptive parameters

- An adaptation law (integral law or gradient type)

$$\hat{\boldsymbol{\theta}}(t) = \Gamma \int_0^t \boldsymbol{\psi}\left(s, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d\right)\ ds + \hat{\boldsymbol{\theta}}(0)$$

where

$\star$ $\Gamma = \Gamma^T \in \mathrm{IR}^{m\times m}$ and $\hat{\boldsymbol{\theta}}(0) \in \mathrm{IR}^m$ are design parameters
$\star$ Γ is usually diagonal and positive definite ('adaptation gain')
$\star$ $\boldsymbol{\psi}$ is a vectorial function to determine, of dimension m.
$\star$ $\hat{\boldsymbol{\theta}}(0)$ is an arbitrary vector
  $*$ in practice, we choose it as the best approximation available on $\boldsymbol{\theta}$.

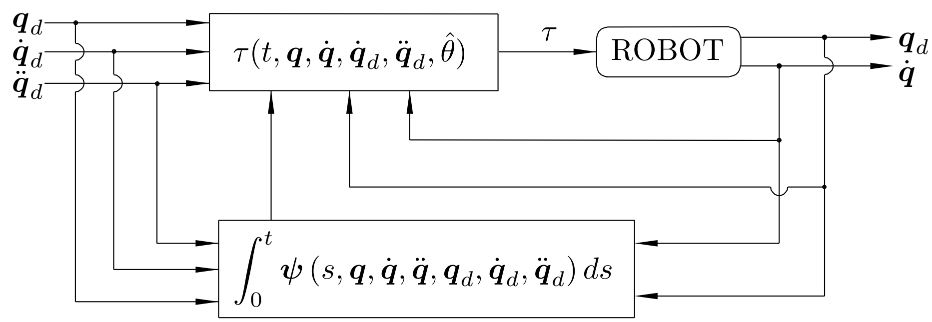

Figure 98: Adaptive control of robots: block-diagram.

- An equivalent representation of the adaptation law

$$\dot{\hat{\boldsymbol{\theta}}}(t) = \Gamma \boldsymbol{\psi}\left(s, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, \boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d\right) .$$

- It is desirable from a practical viewpoint,
  - that the control law as well as the adaptation law, do not depend explicitly on $\ddot{\boldsymbol{q}}$.

Stability and Convergence of Adaptive Control Systems

- An adaptive system guarantees the parametric convergence if
  - ⋆ the limit of $\hat{\boldsymbol{\theta}}(t)$ when $t \to \infty$ exists and is such that

$$\lim_{t\to\infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$$

  - ⋆ Parametric convergence is not an intrinsic characteristic of an adaptive controller.

- Stability analysis is based on Lyapunov theory,
  - ⋆ with the inclusion of

$$\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \qquad \text{(parametric errors vector)}$$

- General form of the closed loop equation

$$\frac{d}{dt}\begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\tilde{\boldsymbol{q}}} \\ \tilde{\boldsymbol{\theta}} \end{bmatrix} = \boldsymbol{f}\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d, \tilde{\boldsymbol{\theta}}\right)$$

  - the origin is an equilibrium point.
  - In general, is *not* the only equilibrium point

- We study only stability and convergence of the position errors: $\lim_{t\to\infty} \tilde{\boldsymbol{q}}(t) = \mathbf{0}$.

- Certainty equivalence (the achievement of the control objective under parameter uncertainty)

Ch.15. PD Control with Adaptive Desired Gravity Compensation

We consider the scenario where all the joints of the robot are revolute.

The Control and Adaptive Laws

- Making use of (47), we have that for any vector $\hat{\boldsymbol{\theta}} \in \mathrm{IR}^m$ and $x \in \mathrm{IR}^n$

$$\boldsymbol{g}(\boldsymbol{x}, \hat{\boldsymbol{\theta}}) = \Phi(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, \mathbf{0})\hat{\boldsymbol{\theta}} + \boldsymbol{g}_0(\boldsymbol{x}) \,. \tag{47}$$

  - For simplicity, in the sequel we use

$$\Phi_g(\boldsymbol{x}) = \Phi(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \,.$$

- The PD control with adaptive desired gravity compensation is described

$$
\begin{aligned}
\boldsymbol{\tau} &= K_p\tilde{\boldsymbol{q}} - K_v\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d, \hat{\boldsymbol{\theta}}) \\
&= K_p\tilde{\boldsymbol{q}} - K_v\dot{\boldsymbol{q}} + \Phi_g(\boldsymbol{q}_d)\hat{\boldsymbol{\theta}} + \boldsymbol{g}_0(\boldsymbol{q}_d),
\end{aligned}
$$

and

$$
\hat{\boldsymbol{\theta}}(t) = \Gamma\Phi_g(\boldsymbol{q}_d)^T \int_0^t \left[\frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}\tilde{\boldsymbol{q}} - \dot{\boldsymbol{q}}\right] \, ds + \hat{\boldsymbol{\theta}}(0),
$$

⋆ where $K_p, K_v \in \mathrm{IR}^{n\times n}$ and $\Gamma \in \mathrm{IR}^{m\times m}$ are
  ∗ symmetric positive definite design matrices
⋆ ε_0 is a suitable positive constant.
⋆ It was used (47) with $\boldsymbol{x} = \boldsymbol{q}_d$.

- Design parameters
  - ⋆ Only K_p and ε_0 must be chosen carefully.

- To that end, we start by defining $\lambda_{\mathrm{Max}}\{M\}$, k_{C1} and k_g as
  - ⋆ $\lambda_{\mathrm{Max}}\{M(\boldsymbol{q},\boldsymbol{\theta})\} \leq \lambda_{\mathrm{Max}}\{M\} \qquad \forall\ \boldsymbol{q} \in \mathbb{R}^n, \quad \boldsymbol{\theta} \in \Omega \subset \mathbb{R}^m$
  - ⋆ $\|C(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{\theta})\| \leq k_{C1}\|\dot{\boldsymbol{q}}\| \qquad \forall\ \boldsymbol{q},\dot{\boldsymbol{q}} \in \mathbb{R}^n, \quad \boldsymbol{\theta} \in \Omega \subset \mathbb{R}^m$
  - ⋆ $\|\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}) - \boldsymbol{g}(\boldsymbol{y},\boldsymbol{\theta})\| \leq k_g\|\boldsymbol{x}-\boldsymbol{y}\| \quad \forall\ \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n, \quad \boldsymbol{\theta} \in \Omega \subset \mathbb{R}^m.$

- The constants $\lambda_{\mathrm{Max}}\{M\}$, k_{C1} and k_g are considered known.
  - It is necessary to dispose of $M(\boldsymbol{q},\boldsymbol{\theta})$, $C(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{\theta})$, $\boldsymbol{g}(\boldsymbol{q},\boldsymbol{\theta})$ and Ω,
  - but one does not require to know $\boldsymbol{\theta}$.
  - In practice the set Ω is determined from upper and lower–bounds on the dynamic parameters

- K_p and ε_0 are chosen so that

C.1) $\lambda_{\min}\{K_p\} > k_g$,

C.2) $\sqrt{\dfrac{2\lambda_{\min}\{K_p\}}{\varepsilon_2 \lambda_{\text{Max}}\{M\}}} > \varepsilon_0$,

C.3) $\dfrac{2\lambda_{\min}\{K_v\}[\lambda_{\min}\{K_p\} - k_g]}{\lambda^2_{\text{Max}}\{K_v\}} > \varepsilon_0$,

C.4) $\dfrac{\lambda_{\min}\{K_v\}}{2\left[k_{C1} + 2\lambda_{\text{Max}}\{M\}\right]} > \varepsilon_0$

where ε_2 and ε_1 are defined so that

$$\varepsilon_2 = \frac{2\varepsilon_1}{\varepsilon_1 - 2}$$

$$\frac{2\lambda_{\min}\{K_p\}}{k_g} > \varepsilon_1 > 2\,.$$

- We define the parametric errors vector as: $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$.
  - It is introduced only for analytical purposes (not used by the controller)
- It may be verified that

$$\begin{aligned} \Phi_g(\boldsymbol{q}_d)\hat{\boldsymbol{\theta}} &= \Phi_g(\boldsymbol{q}_d)\tilde{\boldsymbol{\theta}} + \Phi_g(\boldsymbol{q}_d)\boldsymbol{\theta} \\ &= \Phi_g(\boldsymbol{q}_d)\tilde{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{q}_d, \boldsymbol{\theta}) - \boldsymbol{g}_0(\boldsymbol{q}_d), \end{aligned}$$

- The control law may be written as

$$\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} - K_v\dot{\boldsymbol{q}} + \Phi_g(\boldsymbol{q}_d)\tilde{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{q}_d, \boldsymbol{\theta})\,.$$

- Using above control law in the robot model, we obtain

$$M(\boldsymbol{q}, \boldsymbol{\theta})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\theta})\dot{\boldsymbol{q}} = K_p\tilde{\boldsymbol{q}} - K_v\dot{\boldsymbol{q}} + \Phi_g(\boldsymbol{q}_d)\tilde{\boldsymbol{\theta}} + \boldsymbol{g}(\boldsymbol{q}_d, \boldsymbol{\theta}) - \boldsymbol{g}(\boldsymbol{q}, \boldsymbol{\theta})\,.$$

- As $\dot{\tilde{\boldsymbol{\theta}}} = \dot{\hat{\boldsymbol{\theta}}}$, we get that

$$\dot{\tilde{\boldsymbol{\theta}}} = \Gamma \Phi_g(\boldsymbol{q}_d)^T \left[\frac{\varepsilon_0}{1 + \|\tilde{\boldsymbol{q}}\|} \tilde{\boldsymbol{q}} - \dot{\boldsymbol{q}} \right] .$$

- From all the above we have that the closed loop equation is formed by

$$\frac{d}{dt} \begin{bmatrix} \tilde{\boldsymbol{q}} \\ \dot{\boldsymbol{q}} \\ \tilde{\boldsymbol{\theta}} \end{bmatrix} = \begin{bmatrix} -\dot{\boldsymbol{q}} \\ M(\boldsymbol{q})^{-1} \left\{ K_p \tilde{\boldsymbol{q}} - K_v \dot{\boldsymbol{q}} + \Phi_g(\boldsymbol{q}_d) \tilde{\boldsymbol{\theta}} - C(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d) - \boldsymbol{g}(\boldsymbol{q}) \right\} \\ \Gamma \Phi_g(\boldsymbol{q}_d)^T \left[\frac{\varepsilon_0}{1 + \|\tilde{\boldsymbol{q}}\|} \tilde{\boldsymbol{q}} - \dot{\boldsymbol{q}} \right] \end{bmatrix}$$

  – Autonomous differential equation (the origin is an equilibrium point).

Stability analysis

- Lyapunov function candidate

$$V(t,\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}},\tilde{\boldsymbol{\theta}}) = \frac{1}{2}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\boldsymbol{q}}\\ \tilde{\boldsymbol{\theta}}\end{bmatrix}^T \overbrace{\begin{bmatrix}\frac{2}{\varepsilon_2}K_p & -\frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}M(\boldsymbol{q},\boldsymbol{\theta}) & 0\\ -\frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}M(\boldsymbol{q},\boldsymbol{\theta}) & M(\boldsymbol{q},\boldsymbol{\theta}) & 0\\ 0 & 0 & \Gamma^{-1}\end{bmatrix}}^{P}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\boldsymbol{q}}\\ \tilde{\boldsymbol{\theta}}\end{bmatrix}$$

$$\underbrace{+\,\mathcal{U}(\boldsymbol{q},\boldsymbol{\theta}) - \mathcal{U}(\boldsymbol{q}_d,\boldsymbol{\theta}) + \boldsymbol{g}(\boldsymbol{q}_d,\boldsymbol{\theta})^T\tilde{\boldsymbol{q}} + \frac{1}{\varepsilon_1}\tilde{\boldsymbol{q}}^T K_p\tilde{\boldsymbol{q}}}_{f(\tilde{\boldsymbol{q}})}$$

$\star$ Equivalent form

$$
\begin{aligned}
V(t,\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}},\tilde{\boldsymbol{\theta}}) \quad = \quad & \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q},\boldsymbol{\theta})\dot{\boldsymbol{q}} + \mathcal{U}(\boldsymbol{q},\boldsymbol{\theta}) - \mathcal{U}(\boldsymbol{q}_d,\boldsymbol{\theta}) + \boldsymbol{g}(\boldsymbol{q}_d,\boldsymbol{\theta})^T\tilde{\boldsymbol{q}} \\
& + \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}} - \frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}\,\tilde{\boldsymbol{q}}^T M(\boldsymbol{q},\boldsymbol{\theta})\dot{\boldsymbol{q}} \\
& + \frac{1}{2}\tilde{\boldsymbol{\theta}}^T \Gamma^{-1}\tilde{\boldsymbol{\theta}}\,,
\end{aligned}
$$

- The constants $\varepsilon_0 > 0$, $\varepsilon_1 > 2$ and $\varepsilon_2 > 2$ are chosen so that

$$\frac{2\lambda_{\min}\{K_p\}}{k_g} > \varepsilon_1 > 2 \tag{48}$$

$$\varepsilon_2 = \frac{2\varepsilon_1}{\varepsilon_1 - 2} \tag{49}$$

$$\sqrt{\frac{2\lambda_{\min}\{K_p\}}{\varepsilon_2 \lambda_{\mathrm{Max}}\{M\}}} > \varepsilon_0 > 0\,. \tag{50}$$

  - ⋆ Condition (48) guarantees that $f(\tilde{\boldsymbol{q}})$ is a positive definite function
  - ⋆ (50) ensures that P is a positive definite matrix
  - ⋆ Finally (49) implies that $\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} = \frac{1}{2}$.

- Define ε as

$$\varepsilon = \varepsilon(\|\tilde{\boldsymbol{q}}\|) := \frac{\varepsilon_0}{1 + \|\tilde{\boldsymbol{q}}\|}\,.$$

  – Inequality (50) implies that the matrix

$$\frac{2}{\varepsilon_2}K_p - \left(\frac{\varepsilon_0}{1 + \|\tilde{\boldsymbol{q}}\|}\right)^2 M(\boldsymbol{q}, \boldsymbol{\theta}) = \frac{2}{\varepsilon_2}K_p - \varepsilon^2 M(\boldsymbol{q}, \boldsymbol{\theta}) \;>\; 0$$

- The Lyapunov function candidate may be rewritten as,

$$
\begin{aligned}
V(t,\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}},\tilde{\boldsymbol{\theta}}) \;=\; & \frac{1}{2}\left[-\dot{\boldsymbol{q}}+\varepsilon\tilde{\boldsymbol{q}}\right]^T M(\boldsymbol{q},\boldsymbol{\theta})\left[-\dot{\boldsymbol{q}}+\varepsilon\tilde{\boldsymbol{q}}\right] \\
& +\frac{1}{2}\tilde{\boldsymbol{q}}^T\left[\frac{2}{\varepsilon_2}K_p-\varepsilon^2 M(\boldsymbol{q},\boldsymbol{\theta})\right]\tilde{\boldsymbol{q}} \\
& +\frac{1}{2}\tilde{\boldsymbol{\theta}}^T\Gamma^{-1}\tilde{\boldsymbol{\theta}} \\
& +\underbrace{\mathcal{U}(\boldsymbol{q},\boldsymbol{\theta})-\mathcal{U}(\boldsymbol{q}_d,\boldsymbol{\theta})+\boldsymbol{g}(\boldsymbol{q}_d,\boldsymbol{\theta})^T\tilde{\boldsymbol{q}}+\frac{1}{\varepsilon_1}\tilde{\boldsymbol{q}}^T K_p\tilde{\boldsymbol{q}}}_{f(\tilde{\boldsymbol{q}})},
\end{aligned}
$$

  - which is obviously a positive definite function since
  - $M(\boldsymbol{q},\boldsymbol{\theta})>0$, $\frac{2}{\varepsilon_2}K_p-\varepsilon^2 M(\boldsymbol{q},\boldsymbol{\theta})>0$, $\Gamma>0$, and
  - $f(\tilde{\boldsymbol{q}})$ is also a positive definite function (since $\lambda_{\min}\{K_p\}>k_g$)

- Time derivative of the Lyapunov function candidate

$$\begin{aligned}\dot{V}(t,\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}},\tilde{\boldsymbol{\theta}}) \quad = \quad & -\dot{\boldsymbol{q}}^T K_v \dot{\boldsymbol{q}} + \varepsilon \dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}} - \varepsilon \tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}} + \varepsilon \tilde{\boldsymbol{q}}^T K_v \dot{\boldsymbol{q}} \\ & - \varepsilon \dot{\boldsymbol{q}}^T C(\boldsymbol{q},\dot{\boldsymbol{q}})\tilde{\boldsymbol{q}} - \varepsilon \tilde{\boldsymbol{q}}^T [\boldsymbol{g}(\boldsymbol{q}_d) - \boldsymbol{g}(\boldsymbol{q})] \\ & - \dot{\varepsilon} \tilde{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}}\,.\end{aligned}$$

  – It can be upper bounded, getting:

$$\begin{aligned}\dot{V}(t,\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}},\tilde{\boldsymbol{\theta}}) \quad \leq \quad & -\varepsilon \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix}^T \overbrace{\begin{bmatrix} \lambda_{\min}\{K_p\} - k_g & -\frac{1}{2}\lambda_{\mathrm{Max}}\{K_v\} \\ -\frac{1}{2}\lambda_{\mathrm{Max}}\{K_v\} & \frac{1}{2\varepsilon_0}\lambda_{\min}\{K_v\} \end{bmatrix}}^{Q} \begin{bmatrix} \|\tilde{\boldsymbol{q}}\| \\ \|\dot{\boldsymbol{q}}\| \end{bmatrix} \\ & -\frac{1}{2}\underbrace{\left[\lambda_{\min}\{K_v\} - 2\varepsilon_0(k_{C_1} + 2\lambda_{\mathrm{Max}}\{M\})\right]}_{\delta}\|\dot{\boldsymbol{q}}\|^2\,.\end{aligned}$$

- Q is positive definite if

$$\begin{aligned} \lambda_{\min}\{K_p\} &> k_g \\ \frac{2\lambda_{\min}\{K_v\}(\lambda_{\min}\{K_p\} - k_g)}{\lambda^2_{\mathrm{Max}}\{K_v\}} &> \varepsilon_0 \end{aligned}$$

while we have that $\delta > 0$ if

$$\frac{\lambda_{\min}\{K_v\}}{2(k_{C_1} + 2\lambda_{\mathrm{Max}}\{M\})} > \varepsilon_0 \,.$$

- Finally we obtain that

$$
\begin{aligned}
\dot{V}(t, \tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}}, \tilde{\boldsymbol{\theta}}) &\leq -\frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}\lambda_{\min}\{Q\}\left[\|\tilde{\boldsymbol{q}}\|^2 + \|\dot{\boldsymbol{q}}\|^2\right] - \frac{\delta}{2}\|\dot{\boldsymbol{q}}\|^2 \\
&\leq -\varepsilon_0\lambda_{\min}\{Q\}\frac{\|\tilde{\boldsymbol{q}}\|^2}{1+\|\tilde{\boldsymbol{q}}\|} - \frac{\delta}{2}\|\dot{\boldsymbol{q}}\|^2 .
\end{aligned}
$$

  – It is a globally negative *semidefinite* function.

- Since moreover $V(t, \tilde{\mathrm{q}}, \dot{\boldsymbol{q}}, \tilde{\boldsymbol{\theta}}) > 0$ (globally)

  – the origin of the closed loop equation is stable, and
  – its solutions are bounded, that is: $\tilde{\boldsymbol{q}}, \dot{\boldsymbol{q}} \in L_\infty^n$ and $\tilde{\boldsymbol{\theta}} \in L_\infty^m$.

- Because:

$$\frac{d}{dt}V(t,\tilde{\boldsymbol{q}}(t),\dot{\boldsymbol{q}}(t),\tilde{\boldsymbol{\theta}}(t)) \leq -\varepsilon_0\lambda_{\min}\{Q\}\frac{\|\tilde{\boldsymbol{q}}(t)\|^2}{1+\|\tilde{\boldsymbol{q}}(t)\|},$$

  – we have

$$\frac{V_0}{\varepsilon_0\lambda_{\min}\{Q\}} \geq \int_0^\infty \frac{\|\tilde{\boldsymbol{q}}(t)\|^2}{1+\|\tilde{\boldsymbol{q}}(t)\|}dt,$$

    * where $V_0 := V(0,\tilde{\boldsymbol{q}}(0),\dot{\boldsymbol{q}}(0),\tilde{\boldsymbol{\theta}}(0))$.

  – That is: $\frac{\tilde{\boldsymbol{q}}}{\sqrt{1+\|\tilde{\boldsymbol{q}}\|}} \in L_2^n$.
  – Using Lemma A.7 we obtain that: $\tilde{\boldsymbol{q}} \in L_2^n$

- Thus, from $\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}} \in L_\infty^n$ and $\tilde{\boldsymbol{q}} \in L_2^n$, and Lemma A.5 we obtain

$$\lim_{t\to\infty}\tilde{\boldsymbol{q}}(t) = \mathbf{0} \in \mathrm{IR}^n \ \ \text{(control objective achieved)}.$$

Example 15.2

Consider the Pelican robot

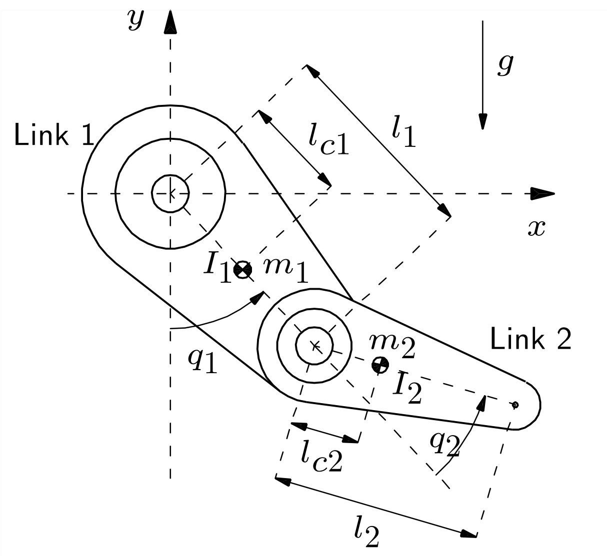

Figure 99: Diagram of the Pelican robot.

- Dynamic model:

$$\underbrace{\begin{bmatrix} M_{11}(\boldsymbol{q}) & M_{12}(\boldsymbol{q}) \\ M_{21}(\boldsymbol{q}) & M_{22}(\boldsymbol{q}) \end{bmatrix}}_{M(\boldsymbol{q})} \ddot{\boldsymbol{q}} + \underbrace{\begin{bmatrix} C_{11}(\boldsymbol{q},\dot{\boldsymbol{q}}) & C_{12}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ C_{21}(\boldsymbol{q},\dot{\boldsymbol{q}}) & C_{22}(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{bmatrix}}_{C(\boldsymbol{q},\dot{\boldsymbol{q}})} \dot{\boldsymbol{q}} + \underbrace{\begin{bmatrix} g_1(\boldsymbol{q}) \\ g_2(\boldsymbol{q}) \end{bmatrix}}_{\boldsymbol{g}(\boldsymbol{q})} = \boldsymbol{\tau}$$

where

$$\begin{aligned}
M_{11}(\boldsymbol{q}) &= m_1 l_{c1}^2 + m_2 \left[l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos(q_2) \right] + I_1 + I_2 \\
M_{12}(\boldsymbol{q}) &= m_2 \left[l_{c2}^2 + l_1 l_{c2} \cos(q_2) \right] + I_2 \\
M_{21}(\boldsymbol{q}) &= m_2 \left[l_{c2}^2 + l_1 l_{c2} \cos(q_2) \right] + I_2 \\
M_{22}(\boldsymbol{q}) &= m_2 l_{c2}^2 + I_2 \\
C_{11}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= -m_2 l_1 l_{c2} \sin(q_2) \dot{q}_2
\end{aligned}$$

$$
\begin{aligned}
C_{12}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -m_2 l_1 l_{c2} \sin(q_2)\left[\dot{q}_1 + \dot{q}_2\right] \\
C_{21}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= m_2 l_1 l_{c2} \sin(q_2)\dot{q}_1 \\
C_{22}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= 0 \\
g_1(\boldsymbol{q}) &= \left[m_1 l_{c1} + m_2 l_1\right] g \sin(q_1) + m_2 l_{c2} g \sin(q_1 + q_2) \\
g_2(\boldsymbol{q}) &= m_2 l_{c2} g \sin(q_1 + q_2)\,.
\end{aligned}
$$

- We consider parametric uncertainty in m_2, I_2 and l_{c2};
  - the numeric values of these constants are not known exactly.
  - Nevertheless, we assume to know their upper-bounds:
    - $\overline{m_2}$, $\overline{I_2}$ and $\overline{l_{c2}}$, that is,

$$
m_2 \le \overline{m_2}; \qquad I_2 \le \overline{I_2}; \qquad l_{c2} \le \overline{l_{c2}}\,.
$$

- The control problem: $\lim_{t\to\infty} \tilde{\boldsymbol{q}}(t) = 0$ (with a constant $\boldsymbol{q}_d$).

- Dynamic parameters vector $\boldsymbol{\theta} \in \mathbb{R}^3$,

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} m_2 \\ m_2 l_{c2} \\ m_2 l_{c2}^2 + I_2 \end{bmatrix} .$$

- PD control with adaptive desired gravity compensation.

$$\begin{aligned} \boldsymbol{\tau} &= K_p \tilde{\boldsymbol{q}} - K_v \dot{\boldsymbol{q}} + \Phi_g(\boldsymbol{q}_d)\hat{\boldsymbol{\theta}} + \boldsymbol{g}_0(\boldsymbol{q}_d) \\ \hat{\boldsymbol{\theta}}(t) &= \Gamma \Phi_g(\boldsymbol{q}_d)^T \int_0^t \left[\frac{\varepsilon_0}{1 + \|\tilde{\boldsymbol{q}}\|} \tilde{\boldsymbol{q}} - \dot{\boldsymbol{q}} \right] ds + \hat{\boldsymbol{\theta}}(0) \end{aligned}$$

⋆ $K_p, K_v \in \mathbb{R}^{n \times n}$ and $\Gamma \in \mathbb{R}^{m \times m}$ are symmetric positive definite design matrices

⋆ ε_0 is a suitable positive constant.

- Vector $\boldsymbol{g}_0(\boldsymbol{q}_d)$

$$\boldsymbol{g}_0(\boldsymbol{q}_d) = \begin{bmatrix} m_1 l_{c1} g \sin(q_{d1}) \\ 0 \end{bmatrix}.$$

- Matrix $\Phi_g(\boldsymbol{q}_d)$

$$\begin{aligned} \Phi_g(\boldsymbol{q}_d) &= \Phi(\boldsymbol{q}_d, \mathbf{0}, \mathbf{0}, \mathbf{0}) \\ &= \begin{bmatrix} l_1 g \sin(q_{d1}) & g \sin(q_{d1}+q_{d2}) & 0 \\ 0 & g \sin(q_{d1}+q_{d2}) & 0 \end{bmatrix}. \end{aligned}$$

- We first need to determine the numeric values of

$$\begin{array}{ll} \bullet \lambda_{\mathrm{Max}}\{M(\boldsymbol{q},\boldsymbol{\theta})\} \le \lambda_{\mathrm{Max}}\{M\} & \forall\ \boldsymbol{q} \in \mathrm{IR}^n, \quad \boldsymbol{\theta} \in \Omega \\ \bullet \|C(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{\theta})\| \le k_{C1}\,\|\dot{\boldsymbol{q}}\| & \forall\ \boldsymbol{q},\dot{\boldsymbol{q}} \in \mathrm{IR}^n, \quad \boldsymbol{\theta} \in \Omega \\ \bullet \|\boldsymbol{g}(\boldsymbol{x},\boldsymbol{\theta}) - \boldsymbol{g}(\boldsymbol{y},\boldsymbol{\theta})\| \le k_g\,\|\boldsymbol{x}-\boldsymbol{y}\| & \forall\ \boldsymbol{x},\boldsymbol{y} \in \mathrm{IR}^n, \ \ \boldsymbol{\theta} \in \Omega. \end{array}$$

- It is necessary to characterize the set $\Omega \subset \mathrm{IR}^3$, to which $\boldsymbol{\theta} \in \Omega$, as:

$$\Omega = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathrm{IR}^3 : |x_1| \le \overline{m_2}; |x_2| \le \overline{m_2}\overline{l_{c2}}; |x_3| \le \overline{m_2}\overline{l_{c2}}^2 + \overline{I_2} \right\} .$$

- Expressions for $\lambda_{\mathrm{Max}}\{M\}$, k_{C1} and k_g for the case of parametric uncertainty are:

$$\begin{aligned} \lambda_{\mathrm{Max}}\{M\} &\ge m_1 l_{c1}^2 + \overline{m_2}\left[l_1^2 + 2\overline{l_{c2}}^2 + 3 l_{c1}\overline{l_{c2}}\right] + I_1 + \overline{I_2} \\ k_{C1} &\ge n^2 \overline{m_2} l_1 \overline{l_{c2}} \\ k_g &\ge n\left[m_1 l_{c1} + \overline{m_2} l_1 + \overline{m_2}\overline{l_{c2}}\right] g \,. \end{aligned}$$

- Fixing the following values for the bounds

$$
\begin{aligned}
\overline{m_2} &= 2.898 \quad [\mathrm{kg}] \\
\overline{I_2} &= 0.0125 \quad \left[\mathrm{kg\ m}^2\right] \\
\overline{l_{c2}} &= 0.02862 \quad [\mathrm{m}],
\end{aligned}
$$

  – and considering the numeric values showed in table 5.1 we finally obtain:

$$
\begin{aligned}
\lambda_{\mathrm{Max}}\{M\} &= 0.475 \quad \left[\mathrm{kg\ m}^2\right] \\
k_{C1} &= 0.086 \quad \left[\mathrm{kg\ m}^2\right] \\
k_g &= 28.99 \quad \left[\mathrm{kg\ m}^2/\mathrm{sec}^2\right] .
\end{aligned}
$$

- The next step consists on calculating K_p, K_v and ε_0 and ε_2.
  - ⋆ Condition C.1: $\lambda_{\min}\{K_p\} > k_g$
    - ∗ As, $k_g = 28.99$, hence $K_p = \mathrm{diag}\{k_p\} = \mathrm{diag}\{30\}$.
  - ⋆ K_v is chosen arbitrarily but symmetric positive definite.
    - ∗ We may fix it to $K_v = \mathrm{diag}\{k_v\} = \mathrm{diag}\{7,\, 3\}$.
  - ⋆ We chose ε_1 according with

$$\frac{2\lambda_{\min}\{K_p\}}{k_g} > \varepsilon_1 > 2,$$

    - ∗ so an appropriate value is $\varepsilon_1 = 2.01$.
  - ⋆ ε_2 is determined from

$$\varepsilon_2 = \frac{2\varepsilon_1}{\varepsilon_1 - 2},$$

    - ∗ so we get $\varepsilon_2 = 402$.

- Using above information, it is immediate to verify that

$$\begin{aligned}
\sqrt{\frac{2\lambda_{\min}\{K_p\}}{\varepsilon_2\lambda_{\mathrm{Max}}\{M\}}} &= 0.561 \\
\frac{2\lambda_{\min}\{K_v\}[\lambda_{\min}\{K_p\}-k_g]}{\lambda^2_{\mathrm{Max}}\{K_v\}} &= 0.124 \\
\frac{\lambda_{\min}\{K_v\}}{2\left[k_{C1}+2\lambda_{\mathrm{Max}}\{M\}\right]} &= 1.448
\end{aligned}$$

  – According to conditions C.2 through C.4, ε_0 must be strictly smaller than the previous quantities. Therefore, we choose $\varepsilon_0 = 0.12$.

- Γ must be symmetric positive definite.

  – A choice is e.g., $\Gamma = \mathrm{diag}\{\gamma_1,\ \gamma_2\} = \mathrm{diag}\{500,\ 10\}$.

- The vector of initial adaptive parameters is arbitrary, and here it is taken to be: $\hat{\boldsymbol{\theta}}(0) = \mathbf{0}$.

- In summary, the control law may be written as

$$
\begin{aligned}
\tau_1 &= k_p\tilde{q}_1 - k_v\dot{q}_1 + l_1 g\,\sin(q_{d1})\hat{\theta}_1 + g\,\sin(q_{d1}+q_{d2})\hat{\theta}_2 \\
&\quad + m_1 l_{c1} g\,\sin(q_{d1}) \\
\tau_2 &= k_p\tilde{q}_2 - k_v\dot{q}_2 + g\,\sin(q_{d1}+q_{d2})\hat{\theta}_2\,.
\end{aligned}
$$

  - Notice that the control law does not depend on $\hat{\theta}_3$.

- Consequently, the adaptation law only has the following two components:

$$
\begin{aligned}
\hat{\theta}_1(t) &= \gamma l_1 g \sin(q_{d1}) \int_0^t \left[\frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}\tilde{q}_1 - \dot{q}_1\right] ds + \hat{\theta}_1(0) \\
\hat{\theta}_2(t) &= \gamma g \sin(q_{d1}+q_{d2}) \int_0^t \left[\frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}\tilde{q}_1 - \dot{q}_1\right] ds \\
&\quad + \gamma g \sin(q_{d1}+q_{d2}) \int_0^t \left[\frac{\varepsilon_0}{1+\|\tilde{\boldsymbol{q}}\|}\tilde{q}_2 - \dot{q}_2\right] ds + \hat{\theta}_2(0) \,.
\end{aligned}
$$

- Laboratory experimental results.

  – Initial conditions:

$$
\begin{aligned}
q_1(0) = 0, &\quad q_2(0) = 0 \\
\dot{q}_1(0) = 0, &\quad \dot{q}_2(0) = 0 \,.
\end{aligned}
$$

– Desired joint positions:

$$q_{d1} = \pi/10, \quad q_{d2} = \pi/30 \quad [\text{rad}]\,.$$

– In terms of the state vector of the closed loop equation, the initial state is

$$\begin{bmatrix} \tilde{\boldsymbol{q}}(0) \\ \\ \dot{\boldsymbol{q}}(0) \end{bmatrix} = \begin{bmatrix} \pi/10 \\ \pi/30 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.3141 \\ 0.1047 \\ 0 \\ 0 \end{bmatrix} \quad [\text{rad}]\,.$$

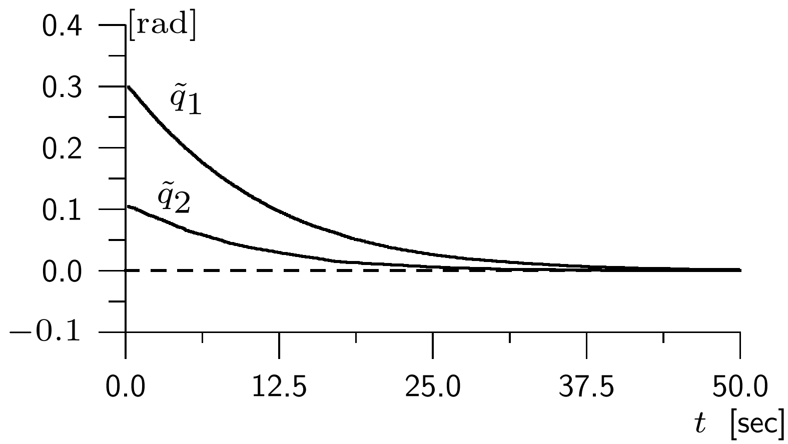

Figure 100: Position errors $\tilde{q}_1$ and $\tilde{q}_2$.

– Above Figure shows that the components of the position error $\tilde{\boldsymbol{q}}(t)$ tend asymptotically to zero in spite of the non-modeled friction phenomenon.

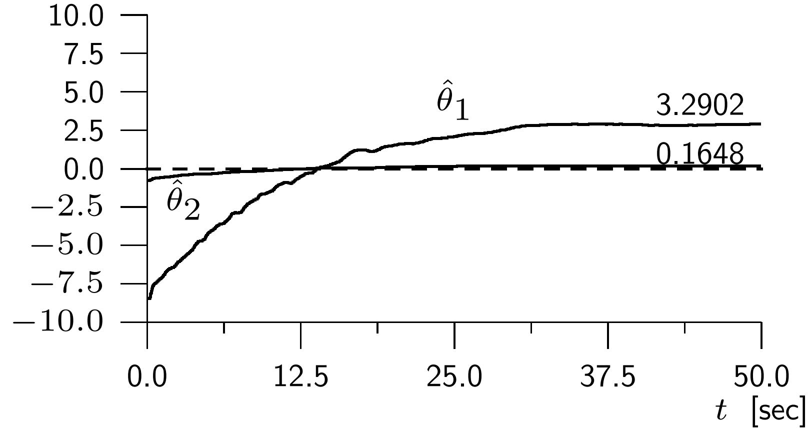

Figure 101: Estimated parameters $\hat{\theta}_1$ and $\hat{\theta}_2$.

– From Figure we appreciate that both parameters tend to values which are relatively near of the unknown values of θ_1 and θ_2, i.e.,

$$\lim_{t\to\infty}\begin{bmatrix}\hat{\theta}_1(t)\\ \hat{\theta}_2(t)\end{bmatrix}=\begin{bmatrix}3.2902\\ 0.1648\end{bmatrix}\approx\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}=\begin{bmatrix}m_2\\ m_2 l_{c2}\end{bmatrix}=\begin{bmatrix}2.0458\\ 0.047\end{bmatrix}.$$

∗ In general, the parameters do not converge to their true values (*persistency of excitation* is not verified)

– If instead of limiting the value of ε_0 we use the same gains as for the latter controllers, the performance is improved.

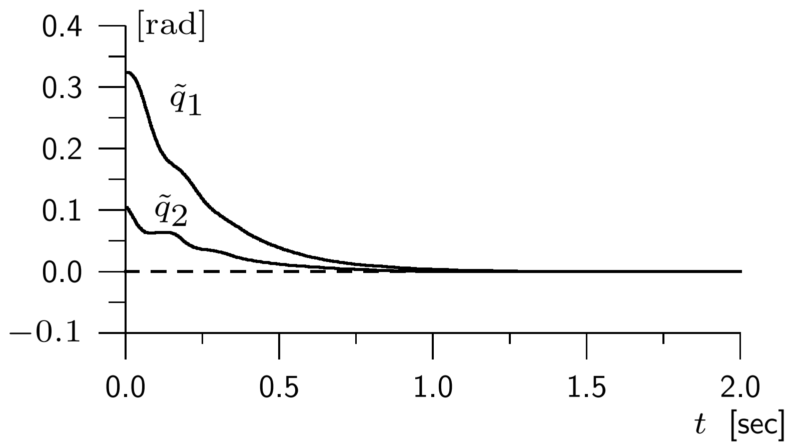

Figure 102: Position errors $\tilde{q}_1$ and $\tilde{q}_2$.

∗ For this, we set the gains to

$$K_p = \begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix} \ [\mathrm{Nm} \,/\, \mathrm{rad}]\,,$$

$$K_v = \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} \ [\mathrm{Nm\ sec} \,/\, \mathrm{rad}]\,,$$

$$\Gamma = \begin{bmatrix} 500 & 10 \\ 0 & 10 \end{bmatrix} \ [\mathrm{Nm} \,/\, (\mathrm{rad\ sec})]\,,$$

and $\varepsilon_0 = 5$, i.e., K_p and K_v have the same values as for the PD controllers.

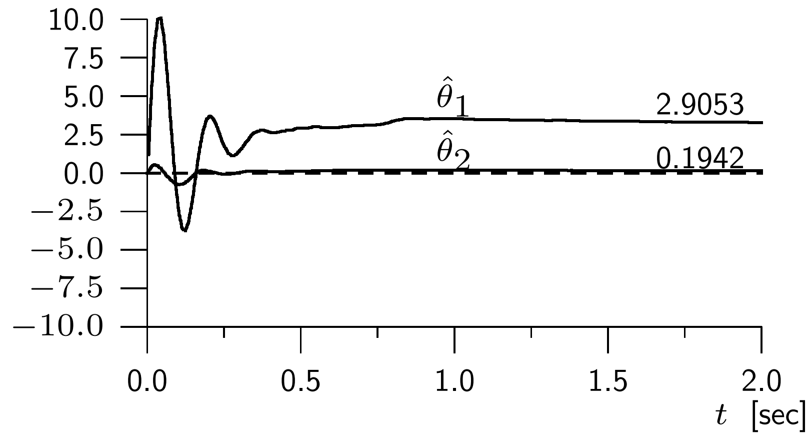

Figure 103: Estimated parameters $\hat{\theta}_1$ and $\hat{\theta}_2$.

Ch.16. PD Control with Adaptive Compensation

The control and adaptation laws

Firstly we recall that the PD controller with compensation is given by

$$\boldsymbol{\tau} = K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + M(\boldsymbol{q})\left[\ddot{\boldsymbol{q}}_d + \Lambda\dot{\tilde{\boldsymbol{q}}}\right] + C(\boldsymbol{q},\dot{\boldsymbol{q}})\left[\dot{\boldsymbol{q}}_d + \Lambda\tilde{\boldsymbol{q}}\right] + \boldsymbol{g}(\boldsymbol{q}),$$

- where $K_p, K_v \in \mathbb{R}^{n\times n}$ are symmetric positive definite design matrices,
- $\tilde{\boldsymbol{q}} = \boldsymbol{q}_d - \boldsymbol{q}$ denotes the position error,
- and Λ is defined as

$$\Lambda = {K_v}^{-1}K_p\,.$$

Now we recall the following property

- Parameterization of the dynamic model

$$M(\boldsymbol{q},\boldsymbol{\theta})\boldsymbol{u} + C(\boldsymbol{q},\boldsymbol{w},\boldsymbol{\theta})\boldsymbol{v} + \boldsymbol{g}(\boldsymbol{q},\boldsymbol{\theta}) = \\ \Phi(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\boldsymbol{\theta} + M_0(\boldsymbol{q})\boldsymbol{u} + C_0(\boldsymbol{q},\boldsymbol{w})\boldsymbol{v} + \boldsymbol{g}_0(\boldsymbol{q})$$

where

⋆ $\Phi(\boldsymbol{q},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in \mathbb{R}^{n\times m}$, $M_0(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$, $C_0(\boldsymbol{q},\boldsymbol{w}) \in \mathbb{R}^{n\times n}$, $\boldsymbol{g}_0(\boldsymbol{q}) \in \mathbb{R}^n$ and $\boldsymbol{\theta} \in \mathbb{R}^m$.

⋆ $\boldsymbol{\theta}$ contains elements that depend on the dynamic parameters.

⋆ $M_0(\boldsymbol{q})$, $C_0(\boldsymbol{q},\boldsymbol{w})$ and $\boldsymbol{g}_0(\boldsymbol{q})$ represent parts of

 ∗ $M(\boldsymbol{q})$, $C(\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\boldsymbol{g}(\boldsymbol{q})$ that do not depend on $\boldsymbol{\theta}$.

- For any vector $\hat{\boldsymbol{\theta}} \in \mathrm{IR}^m$, we obtain

$$M(\boldsymbol{q},\hat{\boldsymbol{\theta}})\left[\ddot{\boldsymbol{q}}_d+\Lambda\dot{\tilde{\boldsymbol{q}}}\right]+C(\boldsymbol{q},\dot{\boldsymbol{q}},\hat{\boldsymbol{\theta}})\left[\dot{\boldsymbol{q}}_d+\Lambda\tilde{\boldsymbol{q}}\right]+\boldsymbol{g}(\boldsymbol{q},\hat{\boldsymbol{\theta}})=$$
$$\Phi(\boldsymbol{q},\ddot{\boldsymbol{q}}_d+\Lambda\dot{\tilde{\boldsymbol{q}}},\dot{\boldsymbol{q}}_d+\Lambda\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}})\hat{\boldsymbol{\theta}}+M_0(\boldsymbol{q})\left[\ddot{\boldsymbol{q}}_d+\Lambda\dot{\tilde{\boldsymbol{q}}}\right]$$
$$+C_0(\boldsymbol{q},\dot{\boldsymbol{q}})\left[\dot{\boldsymbol{q}}_d+\Lambda\tilde{\boldsymbol{q}}\right]+\boldsymbol{g}_0(\boldsymbol{q})\,. \tag{51}$$

  – where we defined

$$\begin{aligned}\boldsymbol{u} &= \ddot{\boldsymbol{q}}_d+\Lambda\dot{\tilde{\boldsymbol{q}}}\\ \boldsymbol{v} &= \dot{\boldsymbol{q}}_d+\Lambda\tilde{\boldsymbol{q}}\\ \boldsymbol{w} &= \dot{\boldsymbol{q}}\,.\end{aligned}$$

  – In the sequel we use the abbreviation: $\Phi=\Phi(\boldsymbol{q},\ddot{\boldsymbol{q}}_d+\Lambda\dot{\tilde{\boldsymbol{q}}},\dot{\boldsymbol{q}}_d+\Lambda\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}})\,.$

We are now ready to study the PD control with adaptive compensation

- Control law

$$\begin{aligned}\boldsymbol{\tau} &= K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + M(\boldsymbol{q},\hat{\boldsymbol{\theta}})\left[\ddot{\boldsymbol{q}}_d + \Lambda\dot{\tilde{\boldsymbol{q}}}\right] + C(\boldsymbol{q},\dot{\boldsymbol{q}},\hat{\boldsymbol{\theta}})\left[\dot{\boldsymbol{q}}_d + \Lambda\tilde{\boldsymbol{q}}\right] \\ &\quad +\boldsymbol{g}(\boldsymbol{q},\hat{\boldsymbol{\theta}}) \qquad (52) \\ &= K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + \Phi\hat{\boldsymbol{\theta}} + M_0(\boldsymbol{q})\left[\ddot{\boldsymbol{q}}_d + \Lambda\dot{\tilde{\boldsymbol{q}}}\right] + C_0(\boldsymbol{q},\dot{\boldsymbol{q}})\left[\dot{\boldsymbol{q}}_d + \Lambda\tilde{\boldsymbol{q}}\right] \\ &\quad +\boldsymbol{g}_0(\boldsymbol{q}), \qquad (53)\end{aligned}$$

and

$$\hat{\boldsymbol{\theta}}(t) = \Gamma\int_0^t \Phi^T\left[\dot{\tilde{\boldsymbol{q}}} + \Lambda\tilde{\boldsymbol{q}}\right]\, ds + \hat{\boldsymbol{\theta}}(0),$$

– $K_p, K_v \in \mathrm{IR}^{n\times n}$ and $\Gamma \in \mathrm{IR}^{m\times m}$ are symmetric positive definite.
– The pass from (52) to (53) follows by using (51).

- We define the parametric errors vector as: $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$.
  - Introduced only with analytic purposes (it is not used by the controller).
  - $\tilde{\boldsymbol{\theta}}$ is unknown since it is a function of $\boldsymbol{\theta}$.

- It may be verified that

$$
\begin{aligned}
\Phi\hat{\boldsymbol{\theta}} &= \Phi\tilde{\boldsymbol{\theta}} + \Phi\boldsymbol{\theta} \\
&= \Phi\tilde{\boldsymbol{\theta}} + M(\boldsymbol{q},\boldsymbol{\theta})\left[\ddot{\boldsymbol{q}}_d + \Lambda\dot{\tilde{\boldsymbol{q}}}\right] + C(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{\theta})\left[\dot{\boldsymbol{q}}_d + \Lambda\tilde{\boldsymbol{q}}\right] + \boldsymbol{g}(\boldsymbol{q},\boldsymbol{\theta}) \\
&\quad -M_0(\boldsymbol{q})\left[\ddot{\boldsymbol{q}}_d + \Lambda\dot{\tilde{\boldsymbol{q}}}\right] - C_0(\boldsymbol{q},\dot{\boldsymbol{q}})\left[\dot{\boldsymbol{q}}_d + \Lambda\tilde{\boldsymbol{q}}\right] - \boldsymbol{g}_0(\boldsymbol{q})
\end{aligned}
$$

- The control law takes the form

$$\begin{aligned}\boldsymbol{\tau} = {} & K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + \Phi\tilde{\boldsymbol{\theta}} \\ & + M(\boldsymbol{q},\boldsymbol{\theta})\left[\ddot{\boldsymbol{q}}_d + \Lambda\dot{\tilde{\boldsymbol{q}}}\right] + C(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{\theta})\left[\dot{\boldsymbol{q}}_d + \Lambda\tilde{\boldsymbol{q}}\right] + \boldsymbol{g}(\boldsymbol{q},\boldsymbol{\theta}) \,.\end{aligned}$$

- Now the robot model can be expressed as:

$$M(\boldsymbol{q},\boldsymbol{\theta})\left[\ddot{\tilde{\boldsymbol{q}}} + \Lambda\dot{\tilde{\boldsymbol{q}}}\right] + C(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{\theta})\left[\dot{\tilde{\boldsymbol{q}}} + \Lambda\tilde{\boldsymbol{q}}\right] = -K_p\tilde{\boldsymbol{q}} - K_v\dot{\tilde{\boldsymbol{q}}} - \Phi\tilde{\boldsymbol{\theta}} \,.$$

- Considering these facts we have that

$$\dot{\tilde{\boldsymbol{\theta}}} = \Gamma\Phi^T\left[\dot{\tilde{\boldsymbol{q}}} + \Lambda\tilde{\boldsymbol{q}}\right] \,.$$

- Finally, the closed-loop equation, may be written as:

$$\frac{d}{dt}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\\ \tilde{\boldsymbol{\theta}}\end{bmatrix} = \begin{bmatrix}\dot{\tilde{\boldsymbol{q}}}\\ M(\boldsymbol{q},\boldsymbol{\theta})^{-1}\left[-K_p\tilde{\boldsymbol{q}} - K_v\dot{\tilde{\boldsymbol{q}}} - \Phi\tilde{\boldsymbol{\theta}} - C(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{\theta})\left[\dot{\tilde{\boldsymbol{q}}}+\Lambda\tilde{\boldsymbol{q}}\right]\right] - \Lambda\dot{\tilde{\boldsymbol{q}}}\\ \Gamma\Phi^T\left[\dot{\tilde{\boldsymbol{q}}}+\Lambda\tilde{\boldsymbol{q}}\right]\end{bmatrix}$$

  – which is a nonautonomous differential equation and of which the origin,

$$\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\\ \tilde{\boldsymbol{\theta}}\end{bmatrix} = \boldsymbol{0} \in \mathrm{I\!R}^{2n+m}, \quad \text{is an equilibrium point.}$$

Stability analysis

- Lyapunov function candidate

$$V(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}},\tilde{\boldsymbol{\theta}}) = \frac{1}{2}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\\ \tilde{\boldsymbol{\theta}}\end{bmatrix}^T \begin{bmatrix} 2K_p + \Lambda^T M(\boldsymbol{q},\boldsymbol{\theta})\Lambda & \Lambda^T M(\boldsymbol{q},\boldsymbol{\theta}) & 0 \\ M(\boldsymbol{q},\boldsymbol{\theta})\Lambda & M(\boldsymbol{q},\boldsymbol{\theta}) & 0 \\ 0 & 0 & \Gamma^{-1}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \dot{\tilde{\boldsymbol{q}}}\\ \tilde{\boldsymbol{\theta}}\end{bmatrix}.$$

  - which is positive definite. This, may be more clear when rewriting it as

$$V(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}},\tilde{\boldsymbol{\theta}}) = \frac{1}{2}\left[\dot{\tilde{\boldsymbol{q}}} + \Lambda\tilde{\boldsymbol{q}}\right]^T M(\boldsymbol{q},\boldsymbol{\theta})\left[\dot{\tilde{\boldsymbol{q}}} + \Lambda\tilde{\boldsymbol{q}}\right] + \tilde{\boldsymbol{q}}^T K_p \tilde{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{\theta}}^T \Gamma^{-1}\tilde{\boldsymbol{\theta}}\,.$$

- The time derivative of the Lyapunov function candidate becomes

$$\begin{aligned}\dot{V}(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}},\tilde{\boldsymbol{\theta}}) &= \left[\dot{\tilde{\boldsymbol{q}}}+\Lambda\tilde{\boldsymbol{q}}\right]^T M(\boldsymbol{q},\boldsymbol{\theta})\left[\ddot{\tilde{\boldsymbol{q}}}+\Lambda\dot{\tilde{\boldsymbol{q}}}\right]+\frac{1}{2}\left[\dot{\tilde{\boldsymbol{q}}}+\Lambda\tilde{\boldsymbol{q}}\right]^T \dot{M}(\boldsymbol{q},\boldsymbol{\theta})\left[\dot{\tilde{\boldsymbol{q}}}+\Lambda\tilde{\boldsymbol{q}}\right]\\ &\quad + 2\tilde{\boldsymbol{q}}^T K_p\dot{\tilde{\boldsymbol{q}}}+\tilde{\boldsymbol{\theta}}^T\Gamma^{-1}\dot{\tilde{\boldsymbol{\theta}}}\,.\end{aligned}$$

  – It may be reduced to

$$\dot{V}(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}},\tilde{\boldsymbol{\theta}}) = -\dot{\tilde{\boldsymbol{q}}}^T K_v\dot{\tilde{\boldsymbol{q}}}-\tilde{\boldsymbol{q}}^T\Lambda^T K_v\Lambda\tilde{\boldsymbol{q}}$$

    * $\Lambda^T K_v \Lambda$ is symmetric positive definite (because, Λ is a non-singular matrix while K_v is a symmetric positive definite matrix, cf. Lemma 2.1)
    * Therefore, $\dot{V}(t,\tilde{\boldsymbol{q}},\dot{\tilde{\boldsymbol{q}}},\tilde{\boldsymbol{\theta}})$ is a globally negative semidefinite function.

- Above result, and since, the Lyapunov function candidate is globally positive definite, radially unbounded and decrescent,
  - Theorem 2.3 guarantees that the origin of the closed-loop equation is
    - uniformly stable, and
    - all the solutions are bounded, that is,

$$\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}} \in L_\infty^n,$$

$$\tilde{\boldsymbol{\theta}} \in L_\infty^m .$$

- Because

$$\begin{aligned}\frac{d}{dt}V(t,\tilde{\boldsymbol{q}}(t),\dot{\tilde{\boldsymbol{q}}}(t),\tilde{\boldsymbol{\theta}}(t)) &\leq -\tilde{\boldsymbol{q}}(t)^T\Lambda^T K_v\Lambda\tilde{\boldsymbol{q}}(t)\\ &\leq -\lambda_{\min}\{\Lambda^T K_v\Lambda\}\left\|\tilde{\boldsymbol{q}}(t)\right\|^2\end{aligned}$$

  – we have

$$\frac{V_0}{\lambda_{\min}\{\Lambda^T K_v\Lambda\}} \geq \int_0^\infty \left\|\tilde{\boldsymbol{q}}(t)\right\|^2 \; dt$$

    * where $V_0 := (0,\tilde{\boldsymbol{q}}(0),\dot{\tilde{\boldsymbol{q}}}(0),\tilde{\boldsymbol{\theta}}(0))$

  – That is: $\tilde{\boldsymbol{q}} \in L_2^n$.

- Thus, from $\tilde{\boldsymbol{q}},\dot{\boldsymbol{q}} \in L_\infty^n$ and $\tilde{\boldsymbol{q}} \in L_2^n$, and Lemma A.5 we obtain

$$\lim_{t\to\infty}\tilde{\boldsymbol{q}}(t) = \mathbf{0} \in \mathrm{IR}^n \ \text{(control objective achieved).}$$

Example 16.3

Consider the Pelican robot

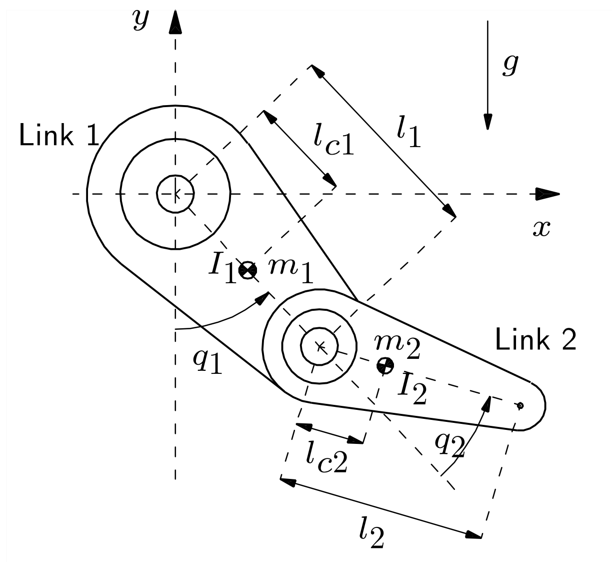

Figure 104: Diagram of the Pelican robot.

- Dynamic model:

$$\underbrace{\begin{bmatrix} M_{11}(\boldsymbol{q}) & M_{12}(\boldsymbol{q}) \\ M_{21}(\boldsymbol{q}) & M_{22}(\boldsymbol{q}) \end{bmatrix}}_{M(\boldsymbol{q})} \ddot{\boldsymbol{q}} + \underbrace{\begin{bmatrix} C_{11}(\boldsymbol{q}, \dot{\boldsymbol{q}}) & C_{12}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ C_{21}(\boldsymbol{q}, \dot{\boldsymbol{q}}) & C_{22}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{bmatrix}}_{C(\boldsymbol{q},\dot{\boldsymbol{q}})} \dot{\boldsymbol{q}} + \underbrace{\begin{bmatrix} g_1(\boldsymbol{q}) \\ g_2(\boldsymbol{q}) \end{bmatrix}}_{\boldsymbol{g}(\boldsymbol{q})} = \boldsymbol{\tau}$$

where

$$\begin{aligned} M_{11}(\boldsymbol{q}) &= m_1 l_{c1}^2 + m_2 \left[l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos(q_2) \right] + I_1 + I_2 \\ M_{12}(\boldsymbol{q}) &= m_2 \left[l_{c2}^2 + l_1 l_{c2} \cos(q_2) \right] + I_2 \\ M_{21}(\boldsymbol{q}) &= m_2 \left[l_{c2}^2 + l_1 l_{c2} \cos(q_2) \right] + I_2 \\ M_{22}(\boldsymbol{q}) &= m_2 l_{c2}^2 + I_2 \end{aligned}$$

$$
\begin{aligned}
C_{11}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -m_2 l_1 l_{c2} \sin(q_2)\dot{q}_2 \\
C_{12}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -m_2 l_1 l_{c2} \sin(q_2) \left[\dot{q}_1 + \dot{q}_2\right] \\
C_{21}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= m_2 l_1 l_{c2} \sin(q_2)\dot{q}_1 \\
C_{22}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= 0 \\
g_1(\boldsymbol{q}) &= \left[m_1 l_{c1} + m_2 l_1\right] g \sin(q_1) + m_2 l_{c2} g \sin(q_1 + q_2) \\
g_2(\boldsymbol{q}) &= m_2 l_{c2} g \sin(q_1 + q_2)\,.
\end{aligned}
$$

- Unknown parameters: m_2, I_2 and l_{c2}.

- We wish to design a controller, such that: $\lim_{t\to\infty} \tilde{\boldsymbol{q}} = 0$.
  - We use the PD Control with Adaptive Compensation.

- Parameterization of the dynamic model:

$$
\begin{aligned}
M(\boldsymbol{q},\boldsymbol{\theta})\boldsymbol{u} + C(\boldsymbol{q},\boldsymbol{w},\boldsymbol{\theta})\boldsymbol{v} + \boldsymbol{g}(\boldsymbol{q},\boldsymbol{\theta}) = \\
= \quad \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \\
+ M_0(\boldsymbol{q})\boldsymbol{u} + C_0(\boldsymbol{q},\boldsymbol{w})\boldsymbol{v} + \boldsymbol{g}_0(\boldsymbol{q})
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_{11} &= l_1^2 u_1 + l_1 g \sin(q_1) \\
\Phi_{12} &= 2l_1 \cos(q_2)u_1 + l_1 \cos(q_2)u_2 - l_1 \sin(q_2)w_2 v_1 \\
&\quad -l_1 \sin(q_2)[w_1 + w_2]v_2 + g \sin(q_1 + q_2) \\
\Phi_{13} &= u_1 + u_2
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{21} &= 0 \\
\Phi_{22} &= l_1 \cos(q_2)u_1 + l_1 \sin(q_2)w_1 v_1 + g \sin(q_1 + q_2) \\
\Phi_{23} &= u_1 + u_2 \\
\boldsymbol{\theta} &= \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} m_2 \\ m_2 l_{c2} \\ m_2 l_{c2}^2 + I_2 \end{bmatrix} = \begin{bmatrix} 2.0458 \\ 0.047 \\ 0.0126 \end{bmatrix} \\
M_0(\boldsymbol{q}) &= \begin{bmatrix} m_1 l_{c1}^2 + I_1 & 0 \\ 0 & 0 \end{bmatrix} \\
C_0(\boldsymbol{q}, \boldsymbol{w}) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\boldsymbol{g}_0(\boldsymbol{q}) &= \begin{bmatrix} m_1 l_{c1} g \sin(q_1) \\ 0 \end{bmatrix} .
\end{aligned}
$$

– Particularly:

$$M(\boldsymbol{q},\boldsymbol{\theta})\left[\ddot{\boldsymbol{q}}_d+\Lambda\dot{\tilde{\boldsymbol{q}}}\right]+C(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{\theta})\left[\dot{\boldsymbol{q}}_d+\Lambda\tilde{\boldsymbol{q}}\right]+\boldsymbol{g}(\boldsymbol{q},\boldsymbol{\theta})=$$

$$\Phi\boldsymbol{\theta}+M_0(\boldsymbol{q})\left[\ddot{\boldsymbol{q}}_d+\Lambda\dot{\tilde{\boldsymbol{q}}}\right]+\boldsymbol{g}_0(\boldsymbol{q}),$$

where

$$\begin{aligned}
\boldsymbol{u} &= \ddot{\boldsymbol{q}}_d+\Lambda\dot{\tilde{\boldsymbol{q}}}=\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}\ddot{q}_{d1}+\lambda_{11}\dot{\tilde{q}}_1+\lambda_{12}\dot{\tilde{q}}_2\\ \ddot{q}_{d2}+\lambda_{21}\dot{\tilde{q}}_1+\lambda_{22}\dot{\tilde{q}}_2\end{bmatrix}\\
\boldsymbol{v} &= \dot{\boldsymbol{q}}_d+\Lambda\tilde{\boldsymbol{q}}=\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}\dot{q}_{d1}+\lambda_{11}\tilde{q}_1+\lambda_{12}\tilde{q}_2\\ \dot{q}_{d2}+\lambda_{21}\tilde{q}_1+\lambda_{22}\tilde{q}_2\end{bmatrix}\\
\boldsymbol{w} &= \dot{\boldsymbol{q}}=\begin{bmatrix}w_1\\w_2\end{bmatrix}=\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}.
\end{aligned}$$

- The control law becomes

$$\begin{aligned}\boldsymbol{\tau} &= K_p\tilde{\boldsymbol{q}} + K_v\dot{\tilde{\boldsymbol{q}}} + \begin{bmatrix}\Phi_{11} & \Phi_{12} & \Phi_{13}\\ \Phi_{21} & \Phi_{22} & \Phi_{23}\end{bmatrix}\begin{bmatrix}\hat{\theta}_1\\ \hat{\theta}_2\\ \hat{\theta}_3\end{bmatrix}\\ &\quad + \begin{bmatrix}(m_1 l_{c1}^2 + I_1)u_1\\ 0\end{bmatrix} + \begin{bmatrix}m_1 l_{c1} g\,\sin(q_1)\\ 0\end{bmatrix},\end{aligned}$$

while the adaptation law is

$$\begin{bmatrix}\hat{\theta}_1(t)\\ \hat{\theta}_2(t)\\ \hat{\theta}_3(t)\end{bmatrix} = \Gamma\int_0^t \begin{bmatrix}\Phi_{11}[v_1 - \dot{q}_1]\\ \Phi_{12}[v_1 - \dot{q}_1] + \Phi_{22}[v_2 - \dot{q}_2]\\ \Phi_{13}[v_1 - \dot{q}_1] + \Phi_{23}[v_2 - \dot{q}_2]\end{bmatrix} ds + \begin{bmatrix}\hat{\theta}_1(0)\\ \hat{\theta}_2(0)\\ \hat{\theta}_3(0)\end{bmatrix}.$$

– Where,

$$
\begin{aligned}
K_p &= \mathrm{diag}\{200,\ 150\} \quad [\mathrm{N\ m\ /\ rad}]\,, \\
K_v &= \mathrm{diag}\{3\} \quad [\mathrm{N\ m\ sec\ /\ rad}]\,, \\
\Gamma &= \mathrm{diag}\{1.6\ \left[\mathrm{kg\ sec}^2/\mathrm{m}^2\right], 0.004\ \left[\mathrm{kg\ sec}^2\right], 0.004\left[\mathrm{kg\ m}^2\ \mathrm{sec}^2\right]\},
\end{aligned}
$$

and therefore, $\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} := K_v^{-1} K_p = \mathrm{diag}\{66.6,\ 50\}\ [1/[\mathrm{sec}]]$.

- Initial conditions, are chosen as:

$$
\begin{aligned}
&q_1(0) = 0, \quad q_2(0) = 0 \\
&\dot{q}_1(0) = 0, \quad \dot{q}_2(0) = 0 \\
&\hat{\theta}_1(0) = 0, \quad \hat{\theta}_2(0) = 0 \\
&\hat{\theta}_3(0) = 0\,.
\end{aligned}
$$

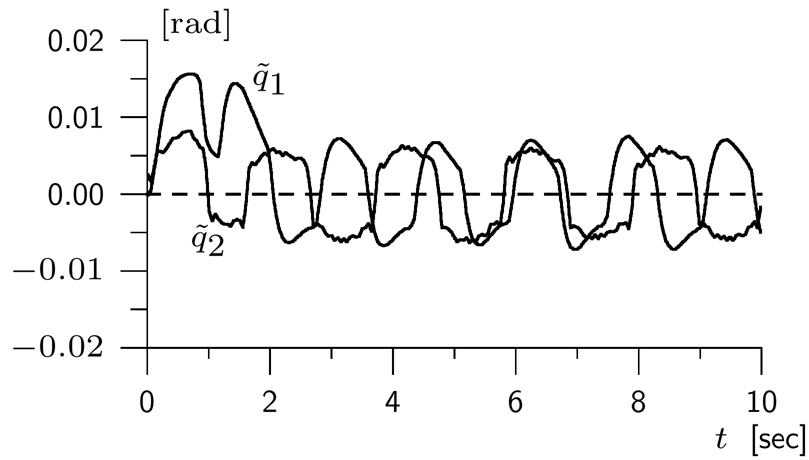

Figure 105: Position errors.

- Figure 105 shows the steady state tracking position errors $\tilde{\boldsymbol{q}}(t)$,
  - by virtue of friction phenomena in the actual robot, are not zero.

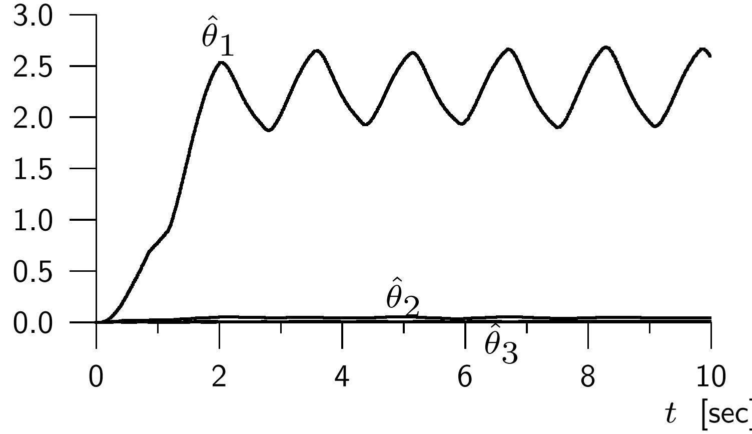

Figure 106: Estimated parameters $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$

- Figure 106 shows the evolution in time of the adaptive parameters.
  - These parameters were arbitrarily assumed to be zero at the initial instant.
  - We did not suppose having any knowledge a priori, about $\boldsymbol{\theta}$.

Thanks a lot for your attention!

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