$$P_t x_n(t,\omega) = \sum I_{C_{nk}}(t) P_t \xi_{nk} \ ,$$

and to this end, for the processes $P_t\xi_{nk}$.

The process $P_t\xi_{nk} = \mathbf{E}(\xi_{nk}|\mathcal{F}_t)$ is a martingale (see Sect. 4.4 below); and so on the basis of Sect. 4.4.3, it has a right-continuous modification. It is $\mathcal{W}$-measurable by virtue of Theorem 4.3.3.

2. Write $x^{(h)}(t,\omega) = P_{t-h\vee 0}x(t,\omega)$. This is also measurable in H. Since by hypothesis

$$\lim_{h\to 0} \mathbf{E}|x(t,\omega) - x^{(h)}(t,\omega)|^2 = 0 \ ,$$

it suffices to prove the existence of a $\mathcal{P}$-measurable modification for $x^h(t,\omega)$ and to this end for $P_{t-h}\xi_{nk}$. This is a martingale adapted to the flow $\mathcal{F}_{t-h\vee 0}$ and its right-continuous modification is predictable since every $\mathcal{F}_{t-h\vee 0}$-adapted step-process has that property. $\square$

4.4 Martingales

4.4.1 Definition and Simplest Properties

Let $T \subset R$ and to each $t \in T$, let there correspond a σ-algebra $\mathcal{F}_t$ so that $\mathcal{F}_t \subset \mathcal{F}_s$ when $t < s$. A family of numerical random variables $\{\xi_t, t \in T\}$ is a *martingale* with respect to $\{\mathcal{F}_t\}$ if: (i) ξ_t is $\mathcal{F}_t$-measurable for all $t \in T$; (ii) $\mathbf{E}\xi_t$ exists; (iii) $\mathbf{E}(\xi_t|\mathcal{F}_s) = \xi_s$ for $s < t$ (equality of random variables is understood everywhere to be with probability 1) Sometimes, one says that $\{\xi_t, \mathcal{F}_t, t \in T\}$ is a martingale. It is possible to speak about martingales without mentioning σ-algebras. It is then kept in mind that the $\mathcal{F}_t$'s are the σ-algebras generated by $\{\xi_s, s \le t, s \in T\}$. We shall be primarily interested in three cases: T is a finite set, T is the set Z_+ of nonnegative integers and $T = R_+$.

A very simple example of a martingale is a process with independent increments in which the expectation of an increment is zero. Less trivial are the following examples.

Example 4.4.1. Let $\eta(t), t \in R_+$ be a process with stationary independent increments for which $\mathbf{E}\exp\{\lambda\eta(t)\}$ exists for some λ. Then $\mathbf{E}\exp\{\lambda\eta(t)\} = \exp\{ta(\lambda)\}$, where $a(\lambda)$ is some number. The process

$$\xi(t) = \exp\{\lambda\eta(t) - ta(\lambda)\}$$

is a martingale with respect to the flow $\{\mathcal{F}_t\}$ generated by $\eta(s)$, $s \le t$.

Example 4.4.2. Let T be arbitrary and let $\{\mathcal{F}_t\}$ satisfy the monotonicity condition. Suppose that η is an arbitrary random variable in R for which $\mathbf{E}|\eta| < \infty$. If $\xi_t = \mathbf{E}(\eta|\mathcal{F}_t)$, then $\{\xi_t, \mathcal{F}_t, t \in T\}$ is a martingale.

If $\{\xi_t\}$ obeys (i), (ii) and (iii′) $\mathbf{E}(\xi_t|\mathcal{F}_s) \le \xi_s$ for $s < t$, then it is called a *supermartingale*. And if it obeys (i), (ii) and (iii″) $\mathbf{E}(\xi_t|\mathcal{F}_s) \ge \xi_s$ for $s < t$, it is called a *submartingale*. Both are also termed *semimartingales*. It is easy to verify the following properties.

I. If $\{\xi_t\}$ is a martingale and $g(x)$ is convex down, then $g(\xi_t)$ is a submartingale (for instance, $|\xi_t|$ and ξ_t^2).
II. If $\{\xi_t\}$ is a supermartingale and $g(x)$ is convex up and increasing, then $g(\xi_t)$ is also a supermartingale.

More important is that (iii), (iii′) and (iii″) continue to hold for stopping times under certain additional restrictions. These restrictions disappear if T is a finite set. Stopping times are understood here to be random variables τ taking values in $T \cup \{\infty\}$ for which $\{\tau \le t\} \in \mathcal{F}_t$. When T is at most countable, it means that $\{\tau = t\} \in \mathcal{F}_t$ for all $t \in T$. We shall say that τ is a stopping time in T.

Theorem 4.4.1. *Suppose that T is a finite set and τ_1 and τ_2 are stopping times in T. Then on the set $\{\tau_1 \le \tau_2\}$, $\mathbf{E}(\xi_{\tau_2}|\mathcal{F}_{\tau_1}) \le \xi_{\tau_1}$ if $\{\xi_t\}$ is a supermartingale, $\mathbf{E}(\xi_{\tau_2}|\mathcal{F}_{\tau_1}) = \xi_{\tau_1}$ if $\{\xi_t\}$ is a martingale, and $\mathbf{E}(\xi_{\tau_2}|\mathcal{F}_{\tau_1}) \ge \xi_{\tau_1}$ if $\{\xi_t\}$ is a submartingale. $\mathcal{F}_{\tau_1}$ is the σ-algebra of sets A such that $A \cap \{\tau_1 = t\} \in \mathcal{F}_t$ for all $t \in T$.*

Proof. All three relations are proved in similar fashion. Let $\{\xi_t\}$ be a martingale. Notice that $\{\tau_1 < \tau_2\} \in \mathcal{F}_{\tau_1}$ since $\{\tau_1 < \tau_2\} \cap \{\tau_1 = t\} = \{\tau_1 = t\} \cap \{\tau_2 > t\}$. We have to show that

$$\mathbf{E} I_A \xi_{\tau_2} = \mathbf{E} I_A \xi_{\tau_1}$$

for all $A \in \mathcal{F}_{\tau_1}$, $A \subset \{\tau_1 < \tau_2\}$. There is no loss of generality in assuming that $T = \{0, 1, \ldots, n\}$ and $A \subset \{\tau_1 = k\} \cap \{\tau_2 > k\}$. Then

$$I_A \xi_{\tau_2} = I_A \sum_{m=k}^{n-1} (\xi_{m+1} - \xi_m) I_{\{\tau_2 > m\}} + I_A \xi_{\tau_1} \ .$$

Since $I_A I_{\{\tau_2 > m\}}$ is $\mathcal{F}_m$-measurable if $m \ge k$, it follows that $\mathbf{E} I_A I_{\{\tau_2>m\}} \xi_{m+1} = \mathbf{E} I_A I_{\{\tau_2>m\}} \xi_m$. Taking the expectation, we arrive at our required result. □

4.4.2 Inequalities. Existence of the Limit

We concentrate first on finite sequences $\xi_1, \ldots, \xi_n$ of random variables forming a martingale, a submartingale or a supermartingale. $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ are the corresponding σ-algebras.

(a) *Inequalities for the maximum.*

Theorem 4.4.2. 1. *If $\{\xi_k, \mathcal{F}_k, k = x_1, \ldots, x_n\}$ is a submartingale, then*

$$a\mathbf{P}\{\max_k \xi_k \geq a\} \leq \mathbf{E}(\xi_n \vee 0) . \tag{4.4.1}$$

2. *If $\{\xi_k, \mathcal{F}_k, k = x_1, \ldots, x_n\}$ is a supermartingale, then*

$$a\mathbf{P}\{\max_k \xi_k \geq a\} \leq \mathbf{E}\xi_0 - \mathbf{E}(\xi_n \wedge 0) . \tag{4.4.2}$$

Proof. 1. Let $\tau = k < n$ if $\xi_k \geq a, \xi_{k-1} < a, \ldots, \xi_1 < a$ and let $\tau = n$ if $\xi_{n-1} < a, \ldots, \xi_1 < a$. Then τ is a stopping time relative to $\{\mathcal{F}_k\}$ since $\{\tau = k\} \in \mathcal{F}_k$. Let A be the event $\{\max_k \xi_k \geq a\}$. $A \in \mathcal{F}_\tau$ because $A \cap \{\tau = k\} \in \mathcal{F}_k$. Hence on the basis of Theorem 4.4.1,

$$\mathbf{E}\xi_n I_A \geq \mathbf{E}\xi_\tau I_A \geq a\mathbf{P}(A) .$$

It remains to observe that $\mathbf{E}\xi_n I_A \leq \mathbf{E}(\xi_n \vee 0) I_A \leq \mathbf{E}(\xi_n \vee 0)$.

2. Retaining the preceding notation, we have $\mathbf{E}\xi_\tau \leq \mathbf{E}\xi_0$ and

$$\mathbf{E}\xi_\tau = \mathbf{E}\xi_\tau I_A + \mathbf{E}\xi_\tau(1 - I_A) \geq a\mathbf{P}(A) + \mathbf{E}\xi_n(1 - I_A) .$$

Therefore

$$a\mathbf{P}(A) \leq \mathbf{E}\xi_\tau - \mathbf{E}\xi_n(1 - I_A) \leq \mathbf{E}\xi_\tau - \mathbf{E}(\xi_n \wedge 0)(1 - I_A) \leq \mathbf{E}\xi_0 - \mathbf{E}(\xi_n \wedge 0).$$

□

Corollary. *If $\{\xi_k, \mathcal{F}_k, k = 1, \ldots, n\}$ is a supermartingale, then*

$$a\mathbf{P}\{\sup_k |\xi_k \geq a\} \leq \mathbf{E}|\xi_0| + \mathbf{E}|\xi_n|.$$

(b) *Inequalities for the number of crossings.* A numerical sequence $x_1, x_2, \ldots, x_n$ is said to cross the band $[a, b] (a < b)$ at least k times downward from above if there are numbers $i_1 < i_2 < \ldots < i_{2k-1} < i_{2k}$ such that $x_{i_1} \leq a, x_{i_3} \leq a, \ldots, x_{i_{2k-1}} \leq a, x_{i_2} \geq b, x_{i_4} \geq b, \ldots, x_{i_{2k}} \geq b$. The number of crossings upward from below is defined in similar fashion and their sum is the number of crossings of $[a, b]$ by the sequence $x_1, \ldots, x_n$.

Theorem 4.4.3. *Let $\{\xi_k, \mathcal{F}_k, k = 1, \ldots, n\}$ be a supermartingale. If $\nu_+[a, b]$ is the number of times the sequence $\xi_1, \ldots, \xi_n$ crosses $[a, b]$ upward from below, then*

$$(b - a)\mathbf{E}\nu_+[a, b] \leq \mathbf{E}(a - \xi_n) \vee 0 \tag{4.4.3}$$

Proof. Consider stopping times in $\{1, 2, \ldots, n\}$ defined by $\tau_1 = \inf(\{i : \xi_i \leq a\} \cup \{n\})$, $\tau_2 = \inf(\{i \geq \tau_1 : \xi_1 \geq b\} \cup \{n\})$ and so on. Let $A_m = \{\nu_+[a,b] \geq m\} = \{\tau_{2m-1} < n\} \cap \{\xi_{2m} \geq b\}, \{\tau_{2m-1} < n\} \in \mathcal{F}_{\tau_{2m-1}}$. Therefore

$$\begin{aligned}0 &\geq \mathbf{E}(\xi_{\tau_{2m}} - \xi_{\tau_{2m-1}}) I_{\{\tau_{2m-1}<n\}} \\ &\geq (b-a)\mathbf{P}(A_m) + \mathbf{E}(\xi_{\tau_{2m}} - a) I_{\{\tau_{2m-1}<n, \xi_{\tau_{2m}}<b\}} \\ &\geq (b-a)\mathbf{P}(A_m) + \mathbf{E}(\xi_n - a) I_{\{\tau_{2m-1}<n, \xi_{\tau_{2m}}<b\}}.\end{aligned}$$

Since the events $\{\tau_{2m-1} < n, \xi_{\tau_{2m}} < b\} \subset \{\tau_{2m-1} < n, \tau_{2m} = n\}$ are mutually exclusive, it follows that

$$\begin{aligned}(b-a) \sum \mathbf{P}(A_m) &\leq \mathbf{E}[(a - \xi_n) \vee 0] \\ &\quad \times \sum I_{\{\tau_{2m-1}<n, \tau_{2m}=n\}} \leq \mathbf{E}[(a-\xi_n) \vee 0].\end{aligned}$$

But

$$\sum \mathbf{P}(A_m) = \mathbf{E}\nu_+[a,b]. \qquad \square$$

(c) *Limit theorem.* We next consider an infinite sequence $\{\xi_k, \mathcal{F}_k, k = 1, 2, \ldots\}$ which is either a super- or submartingale. We wish to find conditions under which the sequence has a limit with probability 1.

Theorem 4.4.4. *Let $\{\xi_k, \mathcal{F}_k, k = 1, 2, \ldots\}$ be a supermartingale for which $\inf_n \mathbf{E}(\xi_n \wedge 0) > -\infty$. Then $\lim\limits_{n\to\infty} \xi_n$ exists with probability* 1.

Proof. If $\{x_n\}$ is a numerical sequence, then it has a limit if (a) it is bounded and (b) it crosses any $[r_1, r_2]$, with $r_1 < r_2$ rational numbers, finitely many times. Write $\eta_n^+ = \max\{\xi_1, \ldots, \xi_n\}, \eta_n^- = \max\{-\xi_1, \ldots, -\xi_n\}, \eta^+ = \lim\limits_{n\to\infty} \eta_n^+$, and $\eta^- = \lim\limits_{n\to\infty} \eta_n^-$. Let $\nu^n[r_1, r_2]$ be the number of times the sequence $\xi_1, \ldots, \xi_n$ crosses $[r_1, r_2]$ and $\nu[r_1, r_2] = \lim\limits_{n\to\infty} \nu^n[r_1, r_2]$ the number of times that the infinite sequence crosses $[r_1, r_2]$. Since $\{-\xi_k\}$ is a submartingale, Theorem 4.4.2 implies for positive a that

$$\begin{aligned}&a\mathbf{P}\{\eta_n^+ \geq a\} \leq \mathbf{E}\xi_0 - \mathbf{E}(\xi_n \wedge 0), \quad a\mathbf{P}\{\eta^+ \geq a\} \leq \mathbf{E}\xi_0 - \inf_n \mathbf{E}(\xi_n \wedge 0), \\ &a\mathbf{P}\{\eta_n^- \geq a\} \leq \mathbf{E}(-\xi_n) \vee 0 = -\mathbf{E}(\xi_n \wedge 0), \\ &a\mathbf{P}\{\eta^- \geq a\} \leq -\inf_n \mathbf{E}(\xi_n \wedge 0).\end{aligned}$$

Hence, $\mathbf{P}\{\sup\limits_k |\xi_k| \geq a\} \leq \frac{1}{a}(\mathbf{E}\xi_0 - 2\inf\limits_n \mathbf{E}(\xi_n \wedge 0))$. The sequence $\{\xi_k\}$ is bounded almost surely. Since $\nu^n[r_1, r_2] \leq 2\nu_+[r_1, r_2] + 1$,

$$\mathbf{E}\nu[r_1, r_2] \leq \frac{1}{(b-a)} \Big(1 + 2(|a| - \inf_n \mathbf{E}(\xi_n \wedge 0))\Big) < \infty$$

on the basis of Theorem 4.4.3. $\square$

Corollary. *A nonnegative supermartingale has a limit with probability* 1.

Remark. Let $\{\xi_n, \mathcal{F}_n, n = 1, 2, \ldots\}$ be a uniformly integrable martingale. Then $\sup_n \mathbf{E}|\xi_n| < \infty$ and by Theorem 4.4.4, $\lim_{n\to\infty} \xi_n = \xi_\infty$ exists. If $A_m \in \mathcal{F}_m$,

$$\mathbf{E}\xi_\infty I_{A_m} = \lim_{n\to\infty} \mathbf{E}\xi_n I_{A_m} = \mathbf{E}\xi_m I_{A_m}$$

(taking the limit under the integral sign is permissible in view of the uniform integrability). Thus

$$\xi_m = \mathbf{E}(\xi_\infty|\mathcal{F}_m). \tag{4.4.4}$$

Conversely, if $\{\xi_n\}$ is representable as (4.4.4), then it is a uniformly integrable martingale. That it is a martingale is obvious. The uniform integrability is a consequence of the next assertion.

Lemma. *Let $\mathbf{E}|\xi| < \infty$ and let $\{\mathcal{F}_\theta, \theta \in \Theta\}$ be a collection of σ-algebras with $\mathcal{F}_\theta \subset \mathcal{F}$. Then the family of random variables $\{\eta_\theta = \mathbf{E}(\xi|\mathcal{F}_\theta), \theta \in \Theta\}$ is uniformly integrable.*

Proof. Clearly $\mathbf{E}|\eta_\theta| \le \mathbf{E}|\xi|$. $\mathbf{P}\{|\eta_\theta| > c^2\} \le c^{-2}\mathbf{E}|\xi|$. Therefore

$$\begin{aligned}
\mathbf{E}|\eta_\theta| I_{\{|\eta_\theta|>c^2\}} &= \mathbf{E}|\mathbf{E}(\xi|\mathcal{F}_\theta)| I_{\{|\eta_\theta|>c^2\}} \le \mathbf{E}\mathbf{E}(|\xi|\big|\mathcal{F}_\theta) I_{\{|\eta_\theta|>c^2\}} \\
&= \mathbf{E}|\xi| I_{\{|\eta_\theta|>c^2\}} = \mathbf{E}|\xi| I_{\{|\xi|\le c\}} I_{\{|\eta_\theta|>c^2\}} \\
&+ \mathbf{E}|\xi| I_{\{|\xi|>c\}} I_{\{|\eta_\theta|>c^2\}} \le c\frac{\mathbf{E}|\xi|}{c^2} + \mathbf{E}|\xi| I_{\{|\xi|>c\}}
\end{aligned}$$

The right-hand side does not depend on θ and approaches zero as $c \to \infty$. □

Corollary. *Let $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, $\mathcal{F}_\infty = \bigvee_n \mathcal{F}_n$ and $\mathbf{E}|\xi| < \infty$. Then*

$$\mathbf{E}(\xi|\mathcal{F}_\infty) = \lim_{n\to\infty} \mathbf{E}(\xi|\mathcal{F}_n)$$

with probability 1.

4.4.3 Continuous Parameter

Let $T \subset R_+$. Assume that $\{F_t, t \in R_+\}$ is a flow of σ-algebras.

Theorem 4.4.5. *Suppose that $\{\xi_t, \mathcal{F}_t, t \in R_+\}$ is a supermartingale and that the family $\{\xi_t, t \le s\}$ is uniformly integrable for any s. Then ξ_t has a right-continuous modification if $\mathbf{E}\xi_t$ is right-continuous.*

Proof. Let D_+ be the set of nonnegative rationals. Using Theorem 4.4.2 and 4.4.3, one can show that $\sup_{t\in D_+\cap[0,s]} |\xi_t|$ is finite with probability 1 for every s. The same is true for $\nu(D_+ \cap [0, s], r_1, r_2)$, the number of crossings of the band $[r_1, r_2]$ by the collection $\{\xi_t, t \in D_+ \cap [0, s]\}$; it is defined as the supremum of the number of crossings of the band $[r_1, r_2]$ by $\{\xi_1, \ldots, \xi_n\}$ over all n and all $t_1 < t_2 < \ldots t_n$ in $D_+ \cap [0, s]$. Therefore

$$\lim_{u \in D^+, u \downarrow t} \xi_u = \xi_t^*$$

exists for all t. Let us show that $\xi_t^* = \xi_t$ with probability 1. Let $u_n \downarrow t$ with $u_n \in D_+$. Then $\mathbf{E} I_A \xi_{u_n} \le \mathbf{E} I_A \xi_t$ for all $A \in \mathcal{F}_t$. Utilizing the uniform integrability, we see that $\mathbf{E} I_A \xi_t^* \le \mathbf{E} I_A \xi_t$ so that $\xi_t^* \le \xi_t$ with probability 1. But $\mathbf{E}\xi_t^* = \lim \mathbf{E}\xi_{u_n} = \mathbf{E}\xi_t$ and hence $\mathbf{P}\{\xi_t = \xi_t^*\} = 1$. □

Corollary. *If $\{\xi_t, \mathcal{F}_t, t \in R_+\}$ is a martingale, then ξ_t has a right-continuous modification.*

4.5 Stochastic Integrals and Integral Representations of Random Functions

We shall consider complex-valued random variables on a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$ that belong to $L_2(\Omega, \mathbf{P})$, as well as random functions with such values. $L_2(\Omega, \mathbf{P})$ is now a complex Hilbert space with the inner product $\langle \xi, \eta \rangle = \mathbf{E}\xi\bar{\eta}$.

4.5.1 Random Measures

Let $(X, \mathcal{B})$ be a measurable space. Consider a complex-valued function $\mu(B)$ defined on $\mathcal{B}$ that satisfies the following:

A. There exists a finite measure m on $\mathcal{B}$ such that

$$\mathbf{E}\mu(B_1)\overline{\mu(B_2)} = m(B_1 \cap B_2), \quad B_1, B_2 \in \mathcal{B} . \tag{4.5.1}$$

Then $\mu(B)$ is called a *random measure*. This term is warranted for the following reasons.

B. If B_1 and $B_2 \in \mathcal{B}$, $B_1 \cap B_2 = \emptyset$, then $\mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2)$. To see this, consider

$$\begin{aligned} &\mathbf{E}|\mu(B_1 \cup B_2) - \mu(B_1) - \mu(B_2)|^2 \\ &\quad = m(B_1 \cup B_2) - 2m(B_1) - 2m(B_2) + m(B_1) + m(B_2) = 0 . \end{aligned}$$

C. If $\{B_n, n \ge 1\} \subset \mathcal{B}$, $B_i \cap B_j = \emptyset$ for $i \ne j$, then

$$\mu\left(\bigcup_n B_n\right) = \sum_n \mu(B_n) . \tag{4.5.2}$$

This is a consequence of the relation

$$\mathbf{E}\left|\mu\left(\bigcup_n B_n\right) - \sum_{n=1}^{l} \mu(B_n)\right|^2 = m\left(\bigcup_n B_n\right) - \sum_{n=1}^{l} m(B_n) .$$

Basic Principles and Applications of Probability Theory
Skorokhod, V. - Prokhorov, Y.V. (Ed.)
2005, V, 282 p., Hardcover
ISBN: 978-3-540-54686-3