

2 Methods and Results

2.1 The Cauchy Lemma

2.1.1 In Chapters 3 and 4 a unified method is used for proving all the uniqueness theorems, i.e., theorems claiming the congruence of polyhedra with coinciding data; these theorems are stated in Sections 2.3 and 2.4 below. This is the very method by which Cauchy proved his congruence theorem for closed convex polyhedra consisting of the same number of equal similarly-situated faces. Cauchy's method is one of the most impressive arguments in geometry; this is undoubtedly the prevailing opinion. The method relies upon a certain topological lemma, which will be referred to as the Cauchy Lemma. We first prove the lemma; next, we demonstrate how Cauchy applied it himself, thus clarifying the general scheme used below for proving uniqueness theorems.

The Cauchy Lemma. *Suppose that some edges of a closed convex polyhedron are labeled by plus or minus signs. Sign changes may occur in the labeled edges around a vertex. The claim is as follows: It is impossible to have at least four sign changes at every vertex. (See Fig. 60 as an illustration.)*

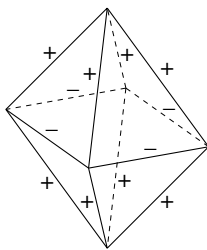


Fig. 60

The lemma can be rephrased in purely topological form:

Suppose that a “net of edges” is given on a surface homeomorphic to the sphere, i.e., suppose that finitely many “edges” (each of which is homeomorphic to a straight line segment) are given, and these edges are pairwise disjoint except possibly at their endpoints, the “vertices of the net.” (Such is

the net of labeled edges of a polyhedron.) Assume further that none of the regions separated by the net on the sphere is bounded by only two edges. (This condition reflects the fact that no two vertices of a polyhedron are joined by two edges.)¹ Then it is impossible to assign pluses and minuses to the edges of the net so that at least four sign changes occur around each vertex.

In fact, we shall prove a somewhat sharper assertion: Denote by N the total number of sign changes at all vertices and let V stand for the total number of vertices. Then

$$N \leq 4V - 8. \quad (1)$$

The claim of the lemma follows immediately from this inequality.

2.1.2 We reproduce Cauchy's proof of inequality (1). It is based on counting the sign changes that occur as one moves around each of the regions separated on the surface by the edges of the net. If two edges are adjacent at a vertex A , then they are also adjacent in moving around the region to whose boundary they belong. The converse is true as well. Therefore, if we count the sign changes occurring as one moves around the regions, then their total number will again equal N .

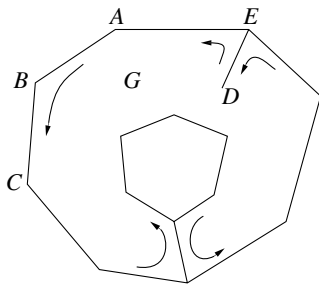


Fig. 61

This simple observation requires some justification. Orient the surface and start going around the regions of the surface in the direction prescribed by this orientation. Suppose that, while going along the contour of a region G , we passed an edge AB in the order A to B . Now, we pass by the vertex B , starting from the edge AB and going around G to the next edge BC (Fig. 61). This edge follows AB in the order around the region G as well as in the order around the vertex B . Thus, the sign change from AB to BC is counted in both cases.

¹ The lemma fails without this condition. Indeed, it suffices to take an even number $2n > 2$ of meridians on a sphere and label them cyclically by pluses and minuses. More complicated examples can also be given.

We keep going around the region G further along the edge BC from B to C , etc. We continue the procedure until we arrive at some edge that was already counted and moreover, were the motion continued, the edge would be passed in the same direction as it was at the first time.

This is because the net can contain some edges that do not separate G from other regions since the edge in question can, for instance, have a free endpoint in G or can join two closed contours that bound G (see Fig. 61). Every such edge will be passed twice: first, in one direction and, next, in the opposite direction. Thus, we can assume that each edge either belongs to the boundary between different regions or is counted twice in the boundary of a single region where it is passed in opposite directions. Therefore, if two edges, say, AB and BC separate the regions G and H , then in G they are passed from AB to BC and in H from BC to AB , which is the same as the total trip around the vertex B . If a vertex D is a free endpoint of the edge DE , then while moving around G we pass the vertex D and return to the same edge DE . This is the same as making a trip around D .

When we say that a region has n edges, we shall bear in mind that each edge not separating the region in question from another region is counted twice. The number of sign changes in moving around a region cannot be greater than the number n of its edges; furthermore, this number is always even since, when we complete this trip, we return to the initial sign. Therefore, if F_n is the number of regions with n edges, then for the total number N of sign changes we obtain the estimate

$$N \leq 2F_3 + 4F_4 + 4F_5 + \dots \quad (2)$$

Here we have used the fact that there is no region bounded by two edges, i.e., that $F_2 = 0$.

Now we transform estimate (2) by applying the generalized Euler formula:

If V , E , and F are the numbers of vertices, edges, and regions in a net, then

$$V - E + F \geq 2. \quad (3)$$

In accordance with Euler's Theorem, $V - E + F = 2$ for a connected net, while $V - E + F = 1 + H$ for a net with H connected components, as proved in Section 1.7 of Chapter 1 (Theorem 5).

Since each edge either belongs to two regions or is counted twice for a single region, we have

$$2E = \sum_n nF_n. \quad (4)$$

The total number F of regions is

$$F = \sum_n F_n. \quad (5)$$

Formula (3) implies

$$4V - 8 \geq 4E - 4F.$$

Substituting E and F from (4) and (5) in this inequality, we obtain

$$4V - 8 \geq \sum_n 2(n-2)F_n = 2F_3 + 4F_4 + 6F_6 + \dots \quad (6)$$

The right-hand side of (6) is not less than the right-hand side of (2); therefore, for $n \geq 3$ the number $2(n-2)$ is not less than the even number nearest to n from below. Consequently, (2) and (6) imply

$$N \leq 4V - 8, \quad (1)$$

as desired.

From inequality (1) we can infer a slight strengthening of the Cauchy Lemma:

Under the conditions of the Cauchy Lemma, it is impossible to have at least four sign changes for all vertices except three of them with two sign changes, or except two of them, one with no sign changes and the other with exactly two sign changes, or except one of them with the number of sign changes less than four.

Were it otherwise, we would have $N \geq 4V - 6$, contradicting inequality (1).

We shall have the opportunity to use this refined version of the Cauchy Lemma later.

2.1.3 The above proof of the Cauchy Lemma, computational by nature, looks somewhat mysterious and does not reveal the visual-geometric basis of the lemma. It therefore seems that a purely geometric proof of the lemma will not be amiss.

The proof is by contradiction. Suppose that some edges in a net on a sphere are labeled by pluses or minuses so that at each vertex there are at least four sign changes.

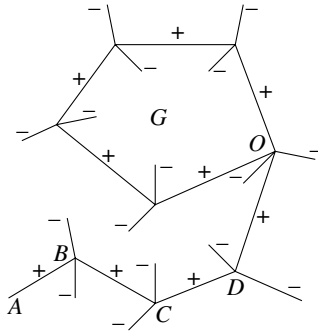


Fig. 62

Take an edge AB of the net labeled by a plus. The vertex B also has other incident edges so that there are at least four sign changes around B . Hence, there is at least one edge BC incident to B , labeled by a plus, and such that, together with AB , it separates edges incident to B and labeled by minuses. The vertex C too has an incident edge CD labeled by a plus which, together with BC , separates edges labeled by minuses. Continuing in this manner, we go from AB along the edges BC , CD , etc. Since the number of edges is finite, eventually we return to a previously visited vertex O and thus obtain a closed contour L (see Fig. 62). In particular, O may coincide with A . The contour L divides the sphere into two regions; consider one of them, G .

All edges of the contour L are labeled by pluses, and each pair of adjacent edges separates edges labeled by minuses, with the possible exception of edges incident to O . Therefore, the region G includes edges which are labeled by minuses and are incident to each of the vertices, except possibly O .

Let us prove that the region G contains at least one vertex of the net. Were it otherwise, the endpoints of any edge in G would be at the vertices of G . However, all vertices of G , except possibly O , have incident edges going inside G ; but then two neighboring vertices must be joined by such an edge. (To verify this, it is sufficient to imagine G as a polygon; the edges inside G are disjoint lines connecting its vertices. However, disjoint diagonals can be drawn from all but two of the vertices. If the edges go from all but one of the vertices, then among them there is an edge joining two neighboring vertices. A rigorous proof is carried out by induction on the number of vertices.)

If two neighboring vertices X and Y of the contour L are joined by an edge inside G , then we obtain a digon bounded by two edges XY . Furthermore, if there are no vertices inside G , then we obtain a digon in our net. By assumption, the net has no such digons. Consequently, the assumption that there are vertices of the net inside G is invalid.

Let A_1 be a vertex inside G ; there is an incident edge A_1B_1 labeled by a plus. Arguing as above, we conclude that the vertex B_1 also has an incident edge B_1C_1 labeled by a plus which, together with A_1B_1 , separates some edges incident to B_1 and labeled by minuses.

Thus, we move along the edges A_1B_1 , B_1C_1 , C_1D_1 , etc. until we arrive at a previously visited vertex within G or reach the boundary of the region G at some vertex N .

In the first case we obtain a closed contour L_1 that cuts out a new simply connected region G_1 from G . To this region G_1 we can apply the same arguments as to G .

In the second case, we move from the vertex N in the opposite direction towards the initial vertex A_1 and beyond it. If the resulting path is closed, we obtain the first case again. But if the path terminates at some vertex M of the region G , we obtain a line MN that divides G into two simply connected parts G' and G'' .

If the vertices M and N differ from O , then there are edges labeled by minuses, issuing from each of these vertices inside G , i.e., inside G' or G'' . Therefore, at least one of the regions G' and G'' is such that edges with minuses issue from all but possibly one of its vertices. (Three possible situations are shown in Figs. 63(a), 63(b), and 63(c); a region with the required properties and its exceptional vertex are indicated below each separate picture.) If, say, N coincides with O , then for G_1 we take one of the regions G' and G'' that includes an edge incident to M and labeled by a minus. Thus, in each case we can cut out a region G_1 from G which enjoys the same properties as G . But then, repeating our arguments, we can cut out a region G_2 from G_1 , etc. ad infinitum. This is, however, impossible since the number of edges in the net is finite. Consequently, there is no net whatsoever with edges labeled as assumed, which was to be proved.

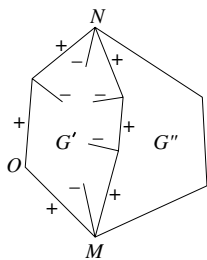


Fig. 63(a) $G = G'$,
exceptional point: O

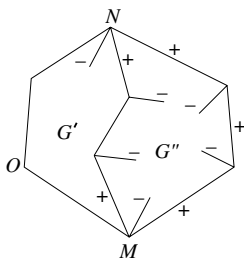


Fig. 63(b) $G = G''$,
exceptional point: N

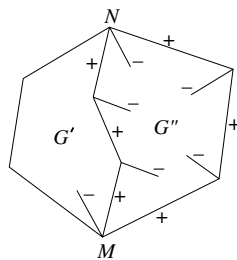


Fig. 63(c) $G = G''$,
exceptional point: M

2.1.4 Now we show how the Cauchy Lemma applies to the proof of his theorem of *the congruence of closed convex polyhedra composed of the same number of equal similarly-situated faces*.

Cauchy proves the following lemma:

If two convex polyhedral angles consist of the same number of corresponding planar angles of equal measure which follow each other in the same order, then either these polyhedral angles are congruent or the differences of the corresponding dihedral angles change sign at least four times around the vertex. (This lemma is proved in Section 3.1 of Chapter 3.)

Now, let P_1 and P_2 be two polyhedra satisfying the assumption of the Cauchy Lemma. If all their dihedral angles are equal, then the polyhedra are congruent. When not all dihedral angles are equal, label by a plus (minus) each edge of P_1 at which the dihedral angle is larger (smaller) than the corresponding dihedral angle of P_2 . In accordance with the lemma, at each vertex with a labeled edge there are at least four sign changes. This is, however, impossible by the Cauchy Lemma. Hence, it is impossible for the polyhedra P_1

and P_2 to have nonequal corresponding dihedral angles, which completes the proof of the Cauchy Theorem.²

Certainly, the application of the Cauchy Lemma is not as straightforward as this in all cases. Nevertheless, simple additional arguments, if needed, always lead to our goal, provided that lemmas ensuring the necessary properties of the labeling of the net in question have been proved.

That is why we begin Chapters 3 and 4, which are devoted to theorems concerning the congruence of polyhedra, precisely by proving such lemmas.

2.2 The Mapping Lemma³

2.2.1 All (except the two or three simplest) existence theorems for convex polyhedra to be formulated in Sections 2.3–2.5 will be proved by a unified method based on a certain topological lemma first established by Brouwer in 1912 and known as the *Domain Invariance Theorem*:

*If a domain, i.e., an open set G in n -dimensional Euclidean space is mapped homeomorphically onto some subset G' of the same space, then G' is also a domain.*⁴

From the theorem we infer a lemma that is referred to as the *Mapping Lemma*. Careful application of this lemma constitutes the method for proving our existence theorems.

By an *n -dimensional manifold* we mean a topological space in which each point has a neighborhood homeomorphic to the interior of an n -dimensional cube and in which any two points possess disjoint neighborhoods.⁵

Recall that a set M (in a topological space) is said to be *connected* if it cannot be split into two disjoint nonempty subsets closed in M (i.e., such subsets M^i that every point adhering to M^i and lying in M belongs to M^i as well).

² Historical information about the Cauchy Theorem can be found in [Fen2], [G], [Co6]. In the last article there is also a discussion of a “smooth analog” of the Cauchy Lemma. Versions of the proof of the lemma on the deformation of convex polyhedral angles can be found in [St1], [Eg], [SZ], [Mi2], [Mi7]. An assertion replacing the Cauchy Lemma on sign disposition is presented in [Tr]. On these grounds, various proofs of the Cauchy Theorem are given; see [P7], [Se1], [Tr]. In a radically different fashion, it is proved in [AS, Appendix]. – *V. Zalgaller*

³ In this section we use some results from topology. The reader unfamiliar with this field should first look through Section 2.8 of the current chapter, which contains most of the necessary information in simplest form. Incidentally, the Mapping Lemma is used only in Chapters 4, 7, and 9.

⁴ This theorem is proved in Section 2.9 of the current chapter. However, its proof, as such, is of no importance for us.

⁵ Connectedness and the possibility of partitioning into simplices is usually included in the concept of manifold. We omit these requirements.

A *connected component* of a set M is a connected subset M' of M that is not contained in any connected subset other than M' itself. Each set M is the union of its disjoint connected components.

2.2.2 The Mapping Lemma. *Let \mathbf{A} and \mathbf{B} be two manifolds of dimension n . Let φ be a mapping from \mathbf{A} into \mathbf{B} satisfying the following conditions:*

- (1) *each connected component of \mathbf{B} contains images of points in \mathbf{A} ;*
- (2) *φ is one-to-one;*
- (3) *φ is continuous;*
- (4) *if points B_m ($m = 1, 2, \dots$) of the manifold \mathbf{B} are images of points A_m of the manifold \mathbf{A} and the sequence B_m converges to a point B , then there is a point A in \mathbf{A} whose image is B and for which there exists a subsequence A_{m_i} of A_m converging to A .*

Under these conditions, $\varphi(\mathbf{A}) = \mathbf{B}$, i.e., all points of the manifold \mathbf{B} are images of some points of the manifold \mathbf{A} .

By condition (4), the mapping φ is continuous in both directions, i.e., φ and its inverse are continuous. Indeed, let $B_m = \varphi(A_m)$, $B = \varphi(A)$, and assume that the sequence B_m converges to B ($m = 1, 2, \dots$). Suppose that the sequence A_m does not converge to A . Then there is a neighborhood of A whose complement contains infinitely many points A_m , say A_{m_1}, A_{m_2}, \dots . We observe that⁶

$$B_{m_j} = \varphi(A_{m_j}); \quad B_{m_j} \rightarrow B; \quad B = \varphi(A),$$

and no subsequence consisting of the points A_{m_j} converges to A . However, this contradicts condition (4), since the injectivity of φ means that the only point mapped to B is the initial point A . Hence, the points A_m must converge to A , i.e., φ is continuous in both directions.

Since the mapping φ is one-to-one and continuous in both directions, it is a homeomorphism. The Domain Invariance Theorem now implies that the image $\varphi(\mathbf{A})$ of the manifold \mathbf{A} is an open set in \mathbf{B} . Indeed, assume $B \in \varphi(\mathbf{A})$, i.e., suppose B is the image of some point A in \mathbf{A} . Let U_B denote a neighborhood of $B \in \mathbf{B}$ that is homeomorphic to the interior of a cube, i.e., homeomorphic to n -dimensional Euclidean space. By the continuity of φ , the point A has a neighborhood V_A whose image is contained in U_B . The point A also has a neighborhood homeomorphic to an n -dimensional cube; the intersection of this neighborhood with V_A is also a neighborhood of A , being mapped into U_B . This neighborhood W_A is obviously homeomorphic to an open set in n -dimensional Euclidean space. Therefore, the Domain Invariance Theorem implies that the homeomorphic image $\varphi(W_A)$ is an open

⁶ The arrow indicates convergence: the points B_{m_j} converge to B , i.e., for every, "arbitrarily small," neighborhood of B one can find an index such that all points B_{m_j} with larger indices j belong to the neighborhood.

set in U_B , i.e., a neighborhood of B . Hence, every point $B \in \varphi(\mathbf{A})$ has a neighborhood contained in $\varphi(\mathbf{A})$, which means that $\varphi(\mathbf{A})$ is an open set.

Further, condition (4) implies that $\varphi(\mathbf{A})$ is also closed in \mathbf{B} . Indeed, if some points $B_n = \varphi(A_n)$ converge to B , then by condition (4) there is a point A in \mathbf{A} mapped to B , i.e., B belongs to $\varphi(\mathbf{A})$ and consequently $\varphi(\mathbf{A})$ is closed.

However, since $\varphi(\mathbf{A})$ is open and closed in \mathbf{B} and has a nonempty intersection with each connected component of \mathbf{B} , it must contain all of \mathbf{B} . Indeed, since $\varphi(\mathbf{A})$ is open, its complement $\mathbf{B} \setminus \varphi(\mathbf{A})$ is closed. Therefore, the intersections of the sets $\varphi(\mathbf{A})$ and $\mathbf{B} \setminus \varphi(\mathbf{A})$ with any set \mathbf{M} in \mathbf{B} are closed relative to \mathbf{M} .⁷ Take as \mathbf{M} a connected component \mathbf{B}' of the manifold \mathbf{B} . Then the formula

$$\mathbf{B}' = (\varphi(\mathbf{A}) \cap \mathbf{B}') \cup ((\mathbf{B} \setminus \varphi(\mathbf{A})) \cap \mathbf{B}')$$

gives a decomposition of the component \mathbf{B}' into two sets closed in \mathbf{B}' . This is impossible because \mathbf{B}' is connected; therefore, one of the sets must be empty. By condition (1), \mathbf{B}' contains points in $\varphi(\mathbf{A})$, i.e., $\mathbf{B}' \cap \varphi(\mathbf{A})$ is nonempty; thus it is $(\mathbf{B} \setminus \varphi(\mathbf{A})) \cap \mathbf{B}'$ that is empty. Hence, $\mathbf{B}' = \varphi(\mathbf{A}) \cap \mathbf{B}'$. Taking the union over all connected components, we obtain $\mathbf{B} = \varphi(\mathbf{A})$, which completes the proof.

2.2.3 Now, we demonstrate how the Mapping Lemma is used to prove existence theorems. For example, we sketch the proof of Minkowski's Theorem on the existence of a closed convex polyhedron with given directions⁸ and areas of faces. (For a detailed proof see Section 7.1 of Chapter 7.)

First we specify the necessary conditions that the unit vectors of outward face normals $\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_n$ and face areas F_1, F_2, \dots, F_n must satisfy. These conditions are as follows:

- (1) The vectors \mathbf{n}_i are not coplanar.
- (2) All F_i are positive.
- (3) $\sum_{i=1}^n \mathbf{n}_i F_i = 0$.

The necessity of the first two conditions is obvious. The third condition means that the vector area of a closed polyhedron equals zero. This is easy to prove.

Let T be any plane and let \mathbf{n} be a unit normal to it. Then the inner product $\mathbf{n}\mathbf{n}_i$ is the cosine of the angle between \mathbf{n} and the normal \mathbf{n}_i to the i th face, the i th face normal. Therefore, $(\mathbf{n}\mathbf{n}_i)F_i$ is nothing but the area of the projection of the face to the plane T taken with the appropriate sign depending on the angle between \mathbf{n} and \mathbf{n}_i .

⁷ By definition, each closed set contains all its points of adherence. Therefore, the common part of, say, the sets $\varphi(\mathbf{A})$ and \mathbf{M} contains all its adherence points that lie in \mathbf{M} ; consequently, it is closed relative to \mathbf{M} .

⁸ The direction of a face is determined by the outward unit normal to the face.

If we consider the projection of a closed convex polyhedron to the plane T , then we see that it covers some polygon twice: “positively” and “negatively.” As a result, the sum of projections of the faces counted with sign equals zero, i.e.,

$$\sum_{i=1}^n \mathbf{n}_i F_i = \mathbf{n} \sum_{i=1}^n \mathbf{n}_i F_i = 0.$$

Since the choice of the plane T and hence of the vector \mathbf{n} is arbitrary, we necessarily have

$$\sum_{i=1}^n \mathbf{n}_i F_i = 0.$$

Minkowski’s Theorem asserts that the three conditions are not only necessary but also sufficient for the existence of a convex polyhedron with prescribed face normals \mathbf{n}_i and face areas F_i , i.e., *given unit vectors \mathbf{n}_i and numbers F_i satisfying these conditions, there exists a closed convex polyhedron with outward face normals \mathbf{n}_i and face areas F_i .*

Denote by \mathbf{B} the set of all collections of numbers F_1, F_2, \dots, F_n (for some fixed vectors $\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_n$) that satisfy the stated conditions. The set \mathbf{B} can be represented as a subset of the n -dimensional space \mathbb{R}^n with coordinates F_1, F_2, \dots, F_n . The vector relation $\sum_{i=1}^n \mathbf{n}_i F_i = 0$ is equivalent to three scalar equations, thus determining an $(n-3)$ -dimensional hyperplane in \mathbb{R}^n , the vector space \mathbb{R}^{n-3} . In this vector space, the inequalities $F_i > 0$ determine some open set. As the intersection of \mathbb{R}^{n-3} and the open half-spaces $F_i > 0$, it is convex and consequently connected. It is this set that we take for the manifold \mathbf{B} . (It is assumed that \mathbf{B} is nonempty: the existence theorem asserts that if there are F_i ’s satisfying the hypotheses, i.e., if \mathbf{B} is nonempty, then a polyhedron exists as well.)

Now, consider all closed convex polyhedra P with given face normals \mathbf{n}_i . Such polyhedra exist. Indeed, by assumption the vectors \mathbf{n}_i are not coplanar and there are numbers $F_i > 0$ satisfying $\sum_{i=1}^n \mathbf{n}_i F_i = 0$. Therefore, the vectors \mathbf{n}_i do not point to a single half-space. In this case the planes that have normals \mathbf{n}_i and are tangent to some ball form a bounded convex (solid) polyhedron. Its boundary is the required closed convex polyhedron with face normals \mathbf{n}_i .

Each polyhedron P is determined by fixing its support numbers h_1, h_2, \dots, h_n (see Section 1.2 of Chapter 1). Hence we can represent the set of all polyhedra P as a subset of the n -dimensional space \mathbb{R}_1^n with coordinates h_1, h_2, \dots, h_n . It is easy to verify that this subset is open, because faces do not disappear under small shifts of the planes of the faces.

Now we combine all congruent and parallel polyhedra P into a single class. Since any parallel translation is determined by three parameters, each class is determined by $n-3$ variables. (For instance, we can take as the representative of each class a polyhedron with barycenter at the origin.) The set of all these classes \mathbf{A} constitutes an $(n-3)$ -dimensional manifold \mathbf{A} .

Thus, the manifolds \mathbf{A} and \mathbf{B} have the same dimension $n - 3$.

Since the face areas of a polyhedron always meet the conditions of the theorem, we have a natural mapping from \mathbf{A} into \mathbf{B} . If we show that it satisfies the assumptions of the Mapping Lemma, then we can infer that each point of \mathbf{B} is the image of some point of \mathbf{A} , i.e., for every collection of numbers F_1, F_2, \dots, F_n that meets the conditions of the theorem, there exists a polyhedron with these face areas.

Since \mathbf{B} is connected, condition (1) of the Mapping Lemma holds automatically. The injectivity of φ is a consequence of the following uniqueness theorem, which was also discovered by Minkowski (see a proof in Section 6.3 of Chapter 6): *a closed convex polyhedron with face normals \mathbf{n}_i and face areas F_i is unique up to translation*. In the language of the manifolds \mathbf{A} and \mathbf{B} , this means that to each given point $B \in \mathbf{B}$ there may correspond only one point $A \in \mathbf{A}$. The continuity of φ is evident from the continuous dependence of the face areas on the disposition of faces, i.e., on the support numbers. This leaves the fourth condition, which can be proved without much effort (see Section 7.1 of Chapter 7).

Thus, we have checked all the conditions of the Mapping Lemma, thus completing the proof of the theorem.

2.2.4 The example considered above reveals the main conditions for the applicability of our method. It applies to the cases in which the existence theorem has the following form: “given a , there is a b ; and, conversely, given b , there is an a .”⁹ In our example, a denotes a closed convex polyhedron and b denotes the collection of numbers F_i and the corresponding unit vectors \mathbf{n}_i ; moreover, only those F_i and \mathbf{n}_i are admissible that satisfy the conditions of the theorem. Minkowski’s Theorem asserts that for every polyhedron a there exists a collection b of face areas and face normals and simultaneously for every collection b of numbers F_i and vectors \mathbf{n}_i meeting the relevant conditions there exists a corresponding polyhedron a . Such assertions are theorems on necessary and sufficient conditions. In our example we talk about necessary and sufficient conditions for given numbers F_i and unit vectors \mathbf{n}_i to be face areas and face normals of a closed convex polyhedron. The necessity of the conditions is often obvious and the crux of the matter is in proving their sufficiency. This feature is characteristic of all existence theorems or theorems on necessary and sufficient conditions proved in this book.

In the proposed method, the proof of the uniqueness theorem always precedes the existence. The actual mapping φ between the objects under consideration is usually defined in a natural way. To ensure its injectivity, we must construct the manifold of objects in a special way, relying upon the “precision” of the corresponding uniqueness theorem. To this end, when we

⁹ This fact, of course, does not provide a one-to-one correspondence between the objects in question.

define the elements constituting the manifold, we must unite the appropriate objects into classes or endow them with additional features.

For instance, in the proof of Minkowski's Theorem, the manifold **A** consists of the classes of polyhedra that differ by translations. Or another example: in the proof of the existence of a polyhedron with a prescribed development, the correspondence between the development and the polyhedron becomes one-to-one in a natural way when we consider polyhedra endowed with the net of edges of some development. We then restrict ourselves to dealing with developments and nets of fixed structure. Finally, in order to make the correspondence one-to-one, we unite into a single class polyhedra superposable with their prescribed nets by a motion or a motion and a reflection. The manifold **A** under consideration consists of these classes. Here the manifold **A** may be empty. However, the proof of the first requirement of the Mapping Lemma obviously contains the proof of the nonemptiness of **A**. At the same time, formally speaking, the empty set is also an n -dimensional manifold, since the definition of manifold does not require nonemptiness.

2.2.5 The general method above can certainly be used not only to prove existence theorems for polyhedra. Using it, we can prove, for example, the fundamental theorem of algebra, which claims the existence of n roots for any polynomial of n th degree.¹⁰ For the manifold **B** we take the manifold of polynomials of n th degree with complex coefficients and leading coefficient equal to 1, excluding polynomials with multiple roots. For **A** we take the manifold of unordered n -tuples A of complex numbers. The two manifolds are $2n$ -dimensional. There are other examples of the applicability of this method to algebraic problems. An application to an existence theorem in the theory of functions of a complex variable was given by G. M. Goluzin [Go]. Some other, possibly earlier, applications of this method are probably known. (For example, in the trivial one-dimensional case, the method corresponds to the familiar method of continuation in a parameter already used by S. N. Bernshteĭn [Be, p. 79].) It is worth noting that so far there are no other methods, except a similar one described in Subsections 2.6.7 and 2.6.8, which makes it possible to prove all the existence theorems of the present book. Incidentally, these are all the general existence theorems of this type, known so far, for convex polyhedra (assuming that we talk about data determining a polyhedron). However, for Minkowski's theorem and similar ones (Theorems 9 and 11 of Section 2.4) other methods of proof are known.¹¹

¹⁰Such a proof is given in my article [A6].

¹¹Alternative proofs of some theorems are pointed out in further footnotes. –
V. Zalgaller

2.3 Determining a Polyhedron from a Development (Survey of Chapters 3, 4, and 5)

2.3.1 We are concerned with gluing a polyhedron from some development as described in Section 1.6 of Chapter 1.

Conditions that must be satisfied by the development of a closed convex polyhedron were found in Sections 1.6 and 1.7:

- (1) “The positive curvature condition”: for each vertex of a development, the sum of the angles glued together at this vertex must be at most 2π .

The sum of the angles contiguous to a single vertex of a convex polyhedron is always less than 2π . The possibility of the sum of angles to equal 2π means only that, after gluing, the vertex of the development may become an interior point of a face or an edge rather than a true vertex of the polyhedron.

Therefore, the condition is necessary for any development of every convex polyhedron.

- (2) “The Euler condition”: if f , e , and v denote the number of faces, edges, and vertices in a development, then the equality $f - e + v = 2$ must hold.

The necessity of this condition for each development of a closed convex polyhedron was proved in Theorem 6, Section 1.7 in Chapter 1. (We recall that glued edges or vertices are regarded respectively as the same edge or vertex of a development and that all polygons of a development are simple by assumption.)

One might think that we should add the condition that a development include no unbounded polygons and have empty boundary. However, it was established in Theorems 7 and 8 of Section 1.7 in Chapter 1 that for such developments we always have $f - e + v < 2$, and hence the Euler condition alone guarantees boundedness of the polygons and emptiness of the boundary.

2.3.2 If we regard as a closed convex polyhedron any doubly-covered convex polygon obtained by gluing together two superposed equal polygons, then we can assert the following:

Theorem 1. *The stated conditions are not only necessary but also sufficient for a given development to define a closed convex polyhedron by gluing. Moreover, there may be only two such polyhedra: one is the mirror image of the other or, which is the same, one of them is the other “inside-out.” (If a polyhedron has a plane of symmetry, then this yields the same polyhedron. Polyhedra differing only in their position in space are not regarded as distinct.)*

We have to take into consideration the case of a doubly-covered polygon, since a development may consist, for example, of two squares that must be

glued together along their sides. It is obvious that the conditions stated above are satisfied in that case. Despite the degenerate character of this case, we do not exclude it, since this would complicate the formulation of our result. Further in this section as well as in Chapters 3 and 4 devoted to the proof of this and related results, we regard any doubly-covered convex polygon as a convex polyhedron, unless the contrary is stated explicitly.

Actually, we will prove Theorem 1 in a somewhat different formulation. As was proved in Section 1.7 of Chapter 1 (Theorem 6), a development satisfying the Euler condition $f - e + v = 2$ is homeomorphic to the sphere. Therefore, Theorem 1 can be rephrased as

Theorem 1*. *Each development homeomorphic to the sphere and having the sum of angles at most 2π at each vertex defines a closed convex polyhedron by gluing. Moreover, the polyhedron is unique up to a motion or up to a motion and a reflection.*

2.3.3 The advantage of the Euler condition is that, like other conditions imposed on developments by definition, the condition can be readily verified for any given development; as a result, it is always possible to find out whether or not the given development defines a closed convex polyhedron by gluing. Moreover, it is not assumed in advance that some polyhedron is actually produced from the given development by gluing; this can be checked under the indicated conditions. Since there may be only one such polyhedron (up to a motion or a motion and a reflection “turning it inside out”), in the course of gluing together a development the polyhedron appears without extra effort.¹² However, we cannot predict how the polyhedron will look. The thing is that the polygons and edges of a development might in no way correspond to the faces and edges of the polyhedron; a priori it is only known that the vertices of the polyhedron result from vertices of the development where the sum of angles is less than 2π . Some examples of developments whose edges do not correspond to edges of the polyhedron are shown in Section 1.6 of Chapter 1 (Figs. 37 and 38).

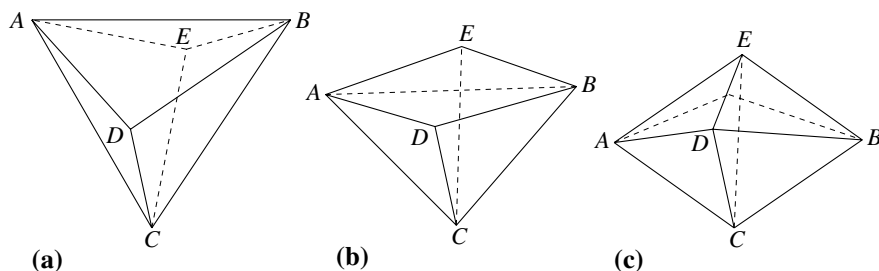
To determine the structure of a polyhedron from a development, i.e., to indicate its genuine edges in the development, is a problem whose general solution seems hopeless. It admits a simple solution only for the tetrahedron.¹³ The point is that, with the growth of the number of vertices, the number of

¹²Given a development of a convex polyhedron, it is of course possible to glue together as many nonconvex polyhedra from it as we please. It suffices, for instance, to bend the polyhedron inward in a neighborhood of one of its vertices. For that reason, the words “without extra effort” should be understood in the sense that, while gluing, one must only control convexity.

¹³The edges of a convex polyhedron must be shortest arcs joining the vertices of the development at which the sum of angles is less than 2π . Moreover, the planar angles at one vertex must satisfy the inequalities asserting that each of the angles is less than or equal to the sum of the others. If one knows how to draw shortest arcs in a development (and an explicit method for this can be indicated), then

possible different polyhedron structures increases extremely rapidly. If $n(v)$ denotes the number of structures of polyhedra with v vertices, then $n(4) = 1$, $n(5) = 2$, $n(6) = 7$, $n(7) = 34$, $n(8) = 257$.¹⁴ Moreover, when the development varies continuously, the structure of the corresponding polyhedron can change.

A simple example is given in Figs. 64(a)–(c), where the polyhedra (b) and (c) can be obtained from (a) by continuously lowering the vertices A and B . The dashed line AB in the polyhedra (b) and (c) corresponds to an edge in their developments which have the same structure as the natural development of the polyhedron (a).



Figs. 64(a)–(c)

2.3.4 Since the development of a polyhedron is not uniquely determined by the polyhedron, it is natural to ask what intrinsic characteristics of a polyhedron are encoded in its development.

As already pointed out in Subsection 1.6.2 of Chapter 1, the development determines the “intrinsic metric” of a polyhedron. Given two points X and Y in a polyhedron P , the distance between them in P is defined to be the greatest lower bound of the lengths of polygonal lines in P joining the points. The distance function $\rho_P(XY)$ is called the *intrinsic metric* of the polyhedron P .

Let R be a development of a polyhedron P . If X' and Y' are two points of R , then we can connect them in R by polygonal lines composed of line segments lying in the faces of the development and jointed at the identified points of the boundaries of the faces. It is natural to regard the greatest lower bound for the lengths of such polygonal lines as the distance $\rho_R(X'Y')$ between the points X' and Y' in the development R . If, in the course of constructing the polyhedron P from the development R by gluing, the points

fininitely many cases occur, from which we must choose the one that corresponds to the structure of the polyhedron. Consequently, it is possible in principle to effectively solve the problem of constructing a polyhedron by gluing. However, in my opinion, effectiveness is of no interest; more important is the fact that because of uniqueness, the polyhedron is obtained naturally in the required form by gluing.

¹⁴ $n(9) = 2606$; see [F] for how the subsequent numbers grow. – V. Zalgaller

X' and Y' become the points X and Y in P , then obviously $\rho_R(X', Y') = \rho_P(X, Y)$. In other words, making a polyhedron P from a development R by gluing produces an isometric mapping of R onto P , i.e., a mapping that preserves distances. All developments of the same polyhedron are obviously isometric (i.e., they can be mapped onto each other by distance-preserving maps).

A development of a polyhedron can be treated as a concrete way of specifying the intrinsic metric of the polyhedron. As was already mentioned, by virtue of the Euler condition the development of a closed convex polyhedron is homeomorphic to the sphere. By mapping a development R onto a sphere S , we carry over the metric ρ_R of the development R to S , i.e., to each pair of points X and Y of S , we assign the distance $\rho_R(XY)$ equal to the distance between the corresponding points in R . (This distance function on the sphere obviously has nothing in common with the intrinsic metric of the sphere; here the sphere is regarded as an abstract manifold.) In this metric ρ_R , every point of S possesses a neighborhood which is similar to one about a point of the development, i.e., a neighborhood isometric to a polyhedral angle, in particular to a part of the plane. The latter case holds certainly for points in the interiors of the faces of the development or of its edges: the neighborhood of a point of the second kind is the union of two half-neighborhoods that lie on faces glued together along the edge in question. If the sum of angles at a vertex A of the development differs from 2π , then the neighborhood of A is isometric to a polyhedral angle. From the standpoint of the intrinsic metric, it is fully characterized by the sum of angles at A or, in other words, by the complete angle θ at A . The difference $2\pi - \theta$ is referred to as the *curvature* at A . The condition imposed on the angles of a development of a convex polyhedron asserts that the curvature is nonnegative at all vertices.¹⁵

A metric definable by means of a development will be called *polyhedral*, and if the curvature is greater than or equal to 0 at all vertices, then we call the metric in question a *metric of positive curvature* (excluding the case in which there are no vertices at all with nonzero curvature; certainly, this case is impossible for the developments homeomorphic to the sphere.)

The existence theorem for a polyhedron with a given development stated above can now be rephrased as follows:

¹⁵Conversely, if a sphere S is endowed with a metric ρ such that every point of S has a neighborhood isometric to a polyhedral angle and the distance between two arbitrary points X and Y equals the greatest lower bound the lengths of curves joining X and Y (the length being measured in the metric ρ), then such a metric can be defined by means of a development. It suffices to surround each point O by a neighborhood composed of triangles with vertices at O and choose finitely many such neighborhoods covering the entire sphere S . The resulting triangles give a finite number of polygons constituting the faces of the required development.

Theorem 1.** *For every polyhedral metric of positive curvature given on a sphere, there exists a closed convex polyhedron realizing this metric; moreover, the polyhedron is unique up to a motion or a motion and a reflection.*

(The assertion that a polyhedron P realizes a metric ρ_R given on S means that P admits a mapping onto S which is isometric with respect to the metric ρ_R .)

There is no freedom of choice in this formulation: a polyhedron and its metric are in one-to-one correspondence provided that we do not distinguish congruent polyhedra.

Thus, Theorem 1** answers the question that was raised at the beginning of the subsection. What is encoded in each development of a polyhedron and corresponds to the polyhedron in a one-to-one fashion is the intrinsic metric of the polyhedron.

2.3.5 The uniqueness claim of Theorem 1* is proved in Section 3.3 of Chapter 3 in the following somewhat stronger form:

Theorem 2. *Each isometric mapping of one closed convex polyhedron onto another can be realized as a motion or a motion and a reflection.*

In particular, the claim applies to mappings of a polyhedron onto itself.¹⁶ This yields a strengthening of Cauchy's Theorem mentioned already in Section 2.1: two closed convex polyhedra composed of the same number of equal similarly-situated faces are congruent. The proof of our theorem is reduced to Cauchy's Theorem with the only generalization that one deals with arbitrary polygons composing the polyhedron rather than with the genuine faces, notwithstanding the fact that some of the polygons might represent parts of faces rather than entire faces. We obtain this generalization of the Cauchy Theorem as a direct corollary of the following theorem:

Theorem 3. *If two closed convex polyhedra are composed of the same number of similarly-situated planar polygons with corresponding angles of equal measure, then all corresponding dihedral angles of these polyhedra are equal.*

The reader familiar with Cauchy's proof can observe that it is the very theorem that Cauchy himself had actually proved, but not formulated, with the only difference that he only considered the faces of the polyhedron. Theorem 3 is of interest in its own right, since polyhedra with equal planar angles

¹⁶We will prove in Section 3.5 of Chapter 3 that the same is valid not only for closed but also for arbitrary convex polyhedra of total curvature 4π (the same curvature as that of a closed polyhedron). For example, an arbitrarily thin slice cut from the surface of a cube, containing all the vertices of the cube, admits no flexes preserving convexity!

can be far from isometric; a simple example is provided by all rectangular parallelepipeds.¹⁷

We shall prove an existence theorem for a closed convex polyhedron with given development in Sections 4.1–4.3 of Chapter 4, relying on Theorem 2 in particular.

2.3.6 Besides the conditions indicated in Subsection 2.3.1 and common to all developments, the development of an unbounded convex polyhedron must satisfy two additional conditions:

- (1) the development is homeomorphic to the plane;
- (2) the sum of the angles at any vertex is at most 2π .

The necessity of the first condition is obvious from the fact that every unbounded polyhedron is homeomorphic to the plane. The second condition is necessary for the same reason as it is for the development of a closed convex polyhedron.

If we adjoin doubly-covered unbounded convex polygons to unbounded convex polyhedra, then we can assert the following:

Theorem 4. *Each development homeomorphic to the plane and having the sum of angles at most 2π at each vertex defines an unbounded convex polyhedron by gluing.*

In the language of metrics, the theorem can be reformulated as follows: every complete polyhedral metric of positive curvature given on the plane is realizable as an unbounded convex polyhedron.¹⁸

This theorem will be proved in Section 4.4 of Chapter 4.

The condition of being homeomorphic to the plane is equivalent to the following two conditions, easily verified for any given development:

- (1a) the development contains at least one unbounded polygon;
- (1b) if f , e , and v denote the number of polygons, edges, and vertices in the development, then $f - e + v = 1$.

The equivalence of these two conditions to the requirement that a development be homeomorphic to the plane was established in Section 1.7 of Chapter 1 (Theorem 9).

Generally speaking, an unbounded polyhedron is not uniquely determined by each of its developments. This can be seen just from the example of a polyhedral angle: it can always be deformed without changing its faces if there are

¹⁷Stoker [Sto] conjectures the validity of the converse assertion: in the class of convex polyhedra with the same combinatorial structure, if the corresponding dihedral angles are equal, then so are the corresponding planar angles. This conjecture has been established in several particular cases [Kar], [Mi3], [And]. – V. Zalgaller

¹⁸Which means that it admits an isometric embedding into three-dimensional Euclidean space as a convex polyhedron. – V. Zalgaller

at least three of them. The only case in which uniqueness holds is described by the following theorem:

Theorem 5. *If the total curvature of a development, i.e., the sum of curvatures at all vertices, equals 2π , then exactly one (up to a motion or a motion and a reflection) unbounded convex polyhedron can be produced from it by gluing. Or: if the curvature of an unbounded convex polyhedron equals 2π , then every isometric mapping of this polyhedron onto another convex polyhedron is a motion or a motion and a reflection.*

The total curvature of an unbounded convex polyhedron never exceeds 2π . For polyhedra with total curvature less than 2π , we have the following theorem (see the proof in Section 4.5 of Chapter 4):

Theorem 6. *Let R be a development satisfying the necessary conditions indicated above and having total curvature $\omega < 2\pi$. Let R be supplied with an orientation. Let L be a ray in one of its unbounded polygons. Let V be a convex polyhedral angle of curvature ω with a given orientation, i.e., an oriented circuit around its vertex, and let L_1 be a generatrix of V . Then R defines by gluing a convex polyhedron P with limit angle V such that under the infinite similarity contraction of P to V the line on P corresponding to L transforms into L_1 and the orientation on P induced by the orientation of R transforms into the given orientation of V . Such a polyhedron P is unique up to translation if the angle V is fixed in space.*

Consequently, for the polyhedron to be unique, it is necessary to specify, together with the development R , the limit angle V , the corresponding rays L and L_1 , and the corresponding orientations on R and V . The change of orientation on V cannot be achieved simply by reflection, since under reflection of the polyhedron P its limit angle is reflected too, and consequently becomes different from the given angle V . If we fix R , V , and L and rotate L_1 around the angle V , then the polyhedron P will, in a certain sense, turn around the angle V ; moreover, its structure will in general change. In exactly the same manner, under a continuous variation of the angle V , the polyhedron P will also deform. The equality between the curvatures of the development R and the angle V is required because of Theorem 3 of Section 1.5 in Chapter 1, according to which the curvature of an unbounded convex polyhedron equals that of its limit angle.

Theorems 5 and 6 are due to S. P. Olovyanishnikov [Ol1]. We prove Theorem 5 and the uniqueness part of Theorem 6 in Chapter 3 (Sections 3.3–3.4). In the same chapter (Section 3.2) we prove the following theorem on unbounded polyhedra, which is analogous to Theorem 3:

Theorem 7. *If two unbounded convex polyhedra are composed of the same number of similarly-situated polygons with corresponding angles of equal*

measure and the limit angles of the polyhedra are equal, then the dihedral angles of the polyhedra are equal as well.

(If the limit angle of one of the polyhedra degenerates into a ray, then the second condition is redundant, for in this case the equality of the corresponding angles of the polygons implies the equality of the limit angles.)

Finally, we establish similar results for polyhedra bounded by a single polygonal line. In this case, in order to obtain the assertions of Theorems 2 and 3, it suffices to impose the following additional condition on the boundaries of the polyhedra: the corresponding angles between adjacent boundary edges of the two polyhedra must be equal (see Section 3.5 of Chapter 3).

However, no effective necessary and sufficient conditions are known under which a development defines by gluing a convex polyhedron with boundary. The trivial condition that the development possess an extension to a development of a closed (or unbounded) polyhedron is ineffective. Nevertheless, in some cases it allows us to formulate easily verifiable sufficient conditions. The corresponding results are presented in Section 5.1 of Chapter 5.

The question when is a polyhedron with boundary determined by its development (metric) and when does it admit flexes (i.e., continuous deformations preserving the metric) which preserve convexity is discussed in Section 3.5 of Chapter 3 and Section 5.2 of Chapter 5. A number of interesting results have been obtained in this direction, but an exhaustive solution to the problem is still unavailable.¹⁹

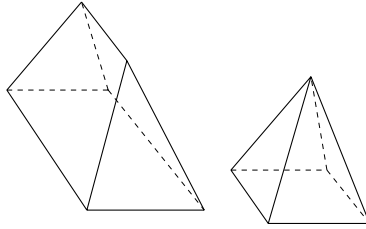
2.4 Polyhedra with Prescribed Face Directions (Survey of Chapters 6, 7, and 8)

2.4.1 The direction of a face of a convex polyhedron is determined by the outward normal to the face. We consider faces and support planes of polyhedra as parallel only if so are their outward normals.²⁰

Actually, problems concerning polyhedra with prescribed face directions are nontrivial because the face directions fail to determine the structure of a polyhedron to any extent, as can be seen from the simple example of the polyhedra in Fig. 65.

¹⁹At present, effectively verifiable necessary and sufficient conditions are known for the rigidity of a convex polyhedron with one or several components of the boundary. They were established by L. A. Shor in the article [Sho1] (see the Supplement to Chapter 5 in the present book) and the note [Sho4]. – V. Zalgaller

²⁰Here and in the sequel, the phrase “vectors are parallel to one another” means that they have the same direction. In this sense, if a polyhedron degenerates into a doubly-covered polygon, then its two faces are not considered parallel to one another, since they have opposite outward normals.

**Fig. 65**

As applied to convex polygons, all theorems stated below are either simply trivial, as our main Theorem 8, or have very simple proofs; with the exception of Theorems 10 and 11, whose proofs are not simple even in the case of polygons.

In Sections 6.1–6.3 of Chapter 6 we prove the following:

Theorem 8. *If we are given two closed convex polyhedra such that to each face of one of them there corresponds a parallel face of the other and vice versa, and moreover the corresponding faces cannot be placed inside one another by translation (i.e., they either protrude from one another or coincide), then the polyhedra are translates of one another (i.e., can be obtained from one another by translation).*

We can even assume that if one of the polyhedra lacks a face with the same outward normal as some face of the other, then we treat such a face as if it exists but degenerates into an edge lying in the corresponding support plane. In this case the edge in question cannot be placed in the corresponding face of the other polyhedron.

This theorem can also be rephrased as follows:

Two closed convex polyhedra are either translates of one another or one of them has a face Q for which the other polyhedron possesses a boundary element (face, edge, or vertex) lying in a support plane parallel to Q , and this boundary element can be placed into Q by translation, but will not coincide with Q .

The already-mentioned Minkowski Uniqueness Theorem is a partial corollary to Theorem 8:

Theorem 9. *If the faces two closed convex polyhedra are pairwise parallel and of equal area, then the polyhedra are translates of one another.*

In other words, a closed convex polyhedron is uniquely determined by its face directions and face areas. Indeed, polygons of equal area cannot be placed in one another (if they are not congruent), and this theorem is therefore contained in Theorem 8.

One can formulate many such corollaries to Theorem 8. For instance, call a function $f(Q)$ defined on the set of convex polyhedra Q *monotone* if $f(Q_1) < f(Q_2)$ whenever Q_1 is contained in but does not coincide with Q_2 . If two closed convex polyhedra have corresponding parallel faces and, for each pair of parallel faces $Q_i^{(1)}$ and $Q_i^{(2)}$, there is a monotone function f_i such that $f_i(Q_i^{(1)}) = f_i(Q_i^{(2)})$, then the polyhedra are translates of one another. Indeed, since the functions f_i are monotone, the equality $f_i(Q_i^{(1)}) = f_i(Q_i^{(2)})$ implies that the parallel faces $Q_i^{(1)}$ and $Q_i^{(2)}$ cannot be placed in one another, so that the congruence of the polyhedra follows from Theorem 8.

Taking area as the value of all the f_i 's, we obtain Theorem 9; taking the perimeter as the value of all the f_i 's, we obtain the theorem asserting that a polyhedron is uniquely determined by the directions and perimeters of its faces, etc.

Theorem 8 is applicable to some less general but important problems that will be considered in Chapter 8. For example, from Theorem 8 we will infer Lindelöf's Theorem [Lin]:²¹

Theorem 10. *Among all closed convex polyhedra with given face directions and total area, the greatest volume is bounded by a polyhedron circumscribing a ball.*

This theorem is a partial corollary to a certain theorem due to Minkowski, which is of prime interest in its own right, a theorem about the so-called mixing of convex bodies and their mixed volumes (see Section 6.2 of Chapter 6 and Sections 7.2 and 7.3 of Chapter 7). Another application of Theorem 8 consists in finding all possible parallelohedra, i.e., convex (solid) polyhedra whose translates tile the entire space if fitted together along their faces like holes in a honeycomb (see Section 8.1 of Chapter 8).

2.4.2 Minkowski proved not only the uniqueness, but also the existence of a closed convex polyhedron with prescribed face directions and face areas:

Theorem 11. *If the unit vectors $\mathbf{n}_1, \dots, \mathbf{n}_m$ and the numbers F_1, \dots, F_m satisfy the conditions:*

- (1) *the vectors $\mathbf{n}_1, \dots, \mathbf{n}_m$ are all distinct and not coplanar;*
- (2) *all the F_i are positive;*
- (3) $\sum_{i=1}^m \mathbf{n}_i F_i = 0$,

then there exists a closed convex polyhedron with outward face normals \mathbf{n}_i and face areas F_i .

²¹Another proof was given by Minkowski in 1897 in the same article [Min1] where he proved Theorems 9 and 11.

We prove this theorem in Section 7.1 of Chapter 7 by a method based on the Mapping Lemma; in Section 7.2 of Chapter 7 we also reproduce Minkowski's original proof.

Other existence theorems similar to Theorem 11 are not known. For example, we have a uniqueness theorem for polyhedra with given face perimeters; however, we have no corresponding existence theorem for the simple reason that the appropriate necessary conditions for the face perimeters of the closed convex polyhedron are not known. If one manages to find such conditions, our method will probably allow to find the corresponding existence theorem.²² This remark also relates to all other monotone functions on faces.

Clearly, some nontrivial conditions must always hold here. A closed convex polyhedron having m faces with given directions is determined by the m distances from the planes of its faces to some point in its interior. However, the specification of, say, m face perimeters l_1, \dots, l_m determines a polyhedron only up to translation, which reduces the degree of freedom from m to $m - 3$, since each translation is determined by the three components of the translation vector. Therefore, the perimeters must satisfy some conditions that, in the m -dimensional domain of positive values l_1, \dots, l_m , should distinguish an $(m - 3)$ -dimensional subset of actually feasible values for the perimeters of the faces. The main difficulty consists in finding conditions distinguishing this set of feasible values l_1, \dots, l_m for the face perimeters.

2.4.3 The congruence condition for closed polyhedra which was stipulated in Theorem 8 becomes meaningless for unbounded polyhedra when applied to their unbounded faces; it fails if the polyhedra are translates of one another, since one of two congruent parallel unbounded polygons can be placed inside the other, as can be seen from Fig. 66. To require the validity of this condition only for bounded faces is obviously insufficient; this is clear from the simplest example shown in Fig. 67.

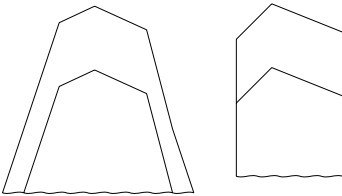


Fig. 66

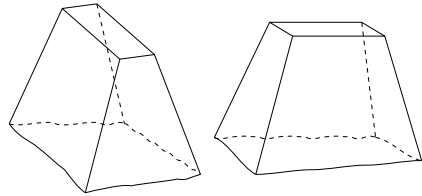


Fig. 67

²²An algorithm for finding conditions for given numbers to serve as the face perimeters of a closed convex polyhedron is proposed in [Z4] – V. Zalgaller

Therefore, an additional condition on unbounded faces is in order:

Unbounded polyhedra must have “unbounded parts as translates”, i.e., a (sufficiently large) bounded part can be cut out from each of them so that the remaining parts will coincide after translation.

This condition is obviously equivalent to the requirement that the planes of all unbounded faces of polyhedra coincide after a common translation. Introducing this condition, we arrive at a theorem which will be proved in Section 6.4 of Chapter 6:

Theorem 12. *If two unbounded convex polyhedra have unbounded parts as translates and all pairs of bounded parallel faces are such that no face of the pair can be placed inside the other by translation, then the polyhedra are translates of one another.*

In the case of bounded nonclosed polyhedra, we obtain similar theorems in Section 6.5 of Chapter 6 by introducing certain conditions on the extreme faces (i.e., those adhering to the boundary) and on the boundary edges. Theorem 12 on unbounded polyhedra can easily be restated for bounded polyhedra if we recall the remark of Subsection 1.1.6 on the replacement of unbounded polyhedra by bounded ones whose extreme faces can be extended. In this connection, it should be noted that theorems of such a type can be regarded as concerning polyhedra whose boundary is fixed and which are subject to some additional conditions. This is nothing more than a “boundary value problem” now posed in the framework of elementary geometry. In boundary value problems, one generally deals with a function (or, perhaps, a surface) which is fixed on the boundary of some domain and must satisfy some condition inside the domain, usually the requirement of being a solution of a differential equation. (The connection between our theorems on polyhedra and such boundary value problems for differential equations will be shown in the sections “Generalizations” of Chapters 6 and 7.)

2.4.4 From Theorem 12 on the congruence of unbounded polyhedra we can infer the following corollary, which is similar to Minkowski’s Theorem 9:

If two unbounded convex polyhedra have unbounded parts as translates and if their bounded faces are pairwise parallel to one another and have equal areas, then the polyhedra are translates of one another.

We obtain a similar result if we take arbitrary monotone functions on faces instead of area; the argument will be the same as in Subsection 2.4.1 above.

For unbounded polyhedra we can also formulate theorems that are analogous to Theorem 11 on the existence of a closed convex polyhedron with prescribed face directions and face areas. To this end, we first clarify the necessary conditions that must be satisfied by the face normals and face areas of an unbounded polyhedron with parallel unbounded edges.

The unbounded part of such a polyhedron is a semi-infinite convex prism Π . If \mathbf{n} is a vector along its edge, pointing to the bounded part of the polyhedron, then the outward normals to bounded faces of the polyhedron must form acute angles with \mathbf{n} (Fig. 68).

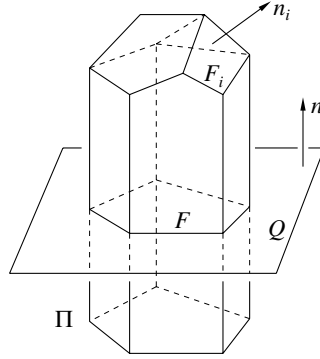


Fig. 68

The projections of the bounded faces to the plane Q perpendicular to \mathbf{n} cover the section of the prism Π by this plane. Therefore, if \mathbf{n}_i and F_i are the outward normals and the areas of the bounded faces, respectively, and F is the area of the section, then

$$F = \sum_i (\mathbf{n}\mathbf{n}_i) F_i,$$

since $(\mathbf{n}\mathbf{n}_i)F_i$ is exactly the area of the projection of the i th face to the plane Q .

It turns out that these conditions are also sufficient for the existence of a convex polyhedron with given unbounded part and prescribed normals and areas of bounded faces. We now state the corresponding theorem, which will be proved in Section 7.3 of Chapter 7:

Theorem 13. *Given an unbounded convex prism Π , let \mathbf{n} be the unit vector along an edge of Π . Let F denote the area of the section of the prism by a plane perpendicular to \mathbf{n} . Let $\mathbf{n}_1, \dots, \mathbf{n}_m$ be unit vectors forming acute angles with \mathbf{n} . Finally, let F_1, \dots, F_m be positive numbers such that*

$$\sum_{i=1}^m (\mathbf{n}\mathbf{n}_i) F_i = F.$$

Then there exists a convex polyhedron with unbounded part Π whose bounded faces have outward normals \mathbf{n}_i and areas F_i .

By virtue of the Corollary to Theorem 12 above, such a polyhedron is unique up to translation along the vector \mathbf{n} , since such a translation maps the prism Π into itself.

For unbounded polyhedra with nonparallel unbounded edges, the following much more general result is valid:

Theorem 14. *Let Q be an unbounded part of some unbounded convex polyhedron with nonparallel unbounded edges (Fig. 69). Let S denote the spherical polygon that is spanned by the spherical images of the faces of Q and let $\mathbf{n}_1, \dots, \mathbf{n}_m$ be vectors from the origin to interior points of S . Finally, let f_1, \dots, f_m be arbitrary monotone continuous functions, defined on the set of polygons, which take the values zero and infinity when the area of a polygon becomes zero or infinity. Then, for arbitrary positive numbers a_1, \dots, a_m , there exists an unbounded convex polyhedron, with unbounded part Q , whose bounded faces have outward normals $\mathbf{n}_1, \dots, \mathbf{n}_m$ and, moreover, the functions f_1, \dots, f_m take the values a_1, \dots, a_m at these faces, so that $f_i = a_i$ for the face with normal \mathbf{n}_i .*

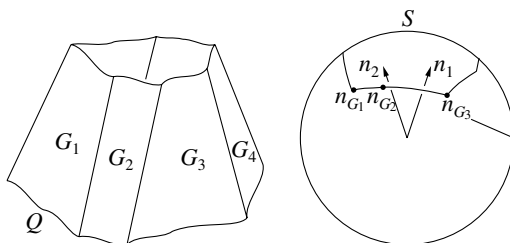


Fig. 69

From the corollary to Theorem 12 stated at the beginning of this subsection, it follows that such a polyhedron is unique, since no translation is possible when the unbounded part Q is fixed.

The necessity of the conditions of Theorem 14 imposed on the normals has been already observed in Section 2.2. The spherical image of an unbounded convex polyhedron is a convex spherical polygon spanned by the spherical images of the unbounded faces of the polyhedron in the sense that its vertices are the ends of the outward normals from the center of the sphere to the faces with nonparallel unbounded edges, whereas the ends of the normals to the faces with parallel unbounded edges lie on its sides. The ends of the normals to the bounded faces lie in the interior of this spherical polygon.

Specifying the unbounded part Q is equivalent to fixing the planes of its faces. However, not all planes can serve as face planes of some unbounded polyhedron. For them to do so, in accordance with what was said above, it is necessary that the normals to the faces drawn from the center of the sphere pass through all the vertices and, perhaps, through some points on the

sides of a certain convex spherical polygon. Moreover, even this is insufficient and additional conditions on the distances from the planes to the origin are needed. They will be specified further in Subsection 2.4.7.

At first glance, Theorem 14 is rather surprising owing to its generality. However, it is in fact a much more superficial observation than Theorems 11 and 13, which involve only area. This will become apparent from the proof of Theorem 14 proposed by A. V. Pogorelov and described in Section 7.4 of Chapter 7. Incidentally, we shall obtain a still more general result by waiving the condition that the functions f_i vanish together with the area.

As to polyhedra with boundary, no similar theorems are known, except for rather trivial consequences of Theorems 8, 13, and 14. The difficulty is in formulating conditions to be imposed both on the areas of the faces and on the boundary of the polyhedron, since there is obviously a dependence between them.²³

2.4.5 Theorems 9–14 stated above are all generalized word for word to spaces of arbitrary dimension, although we have to change the method for proving Theorems 9, 10, and 12.²⁴ Theorem 8 on the general congruence condition for closed polyhedra is an exception. The fact that it fails in four-dimensional space is apparent from the following simple example: A cube with edge 2 and a rectangular parallelepiped with edges 1, 1, 3, 3 (in four-dimensional space) have faces that do not fit into one another, but are not congruent.

2.4.6 A closed or unbounded convex polyhedron is completely determined by fixing the planes of its faces: it is the boundary of the intersection of the half-spaces bounded by these planes. Given the unit vectors \mathbf{n}_i of outward normals to the faces, the face areas will be determined if we specify the distances from the faces to the origin of coordinates; here the distance is considered positive when the direction from the origin to the plane in question coincides with the direction of the outward face normal; in the opposite case, the distance is negative. With this agreement on signs, we call these distances the *support numbers* of the polyhedron.²⁵ The support number h_i is the right-hand side of the normal equation of the face plane:

$$\mathbf{n}_i \mathbf{x} = h_i,$$

²³This problem remains unsolved. – V. Zalgaller

²⁴Such a general proof for Theorems 9 and 10 was given by Minkowski. It is presented in Section 8.3 of Chapter 8. The general proof for Theorem 12 is due to Pogorelov and will be described in Section 7.5 of Chapter 7.

²⁵The reader familiar with the theory of convex bodies can readily observe that the support numbers are nothing more than the values of the support function of the polyhedron at the unit vectors of the outward face normals. Their specification replaces that of the whole support function. This is the reason for their names. Minkowski, in his article [Min1], called them tangential parameters. The notion of support function was introduced by Minkowski.

where \mathbf{n}_i is the unit outward normal and \mathbf{x} is the vector from the origin to an arbitrary point of the plane.

The half-space containing the polyhedron is determined by the inequality

$$\mathbf{n}_i \mathbf{x} \leq h_i$$

(and not by the opposite one, since \mathbf{n}_i is the outward normal, i.e., the one pointing to the half-space not containing the polyhedron). The polyhedron itself is the boundary of the intersection of these half-spaces.

Support numbers are not arbitrary. Let the vectors of outward normals be $\mathbf{n}_1, \dots, \mathbf{n}_m$ and let h_1, \dots, h_m be the support numbers.

Suppose that the vector \mathbf{n}_k is a linear combination of the other vectors \mathbf{n}_i with nonnegative coefficients:

$$\mathbf{n}_k = \sum_{i \neq k} \nu_{ki} \mathbf{n}_i \quad (\nu_{ki} \geq 0). \quad (1)$$

If \mathbf{x} is the position vector of a point X in the interior of the k th face, then

$$\mathbf{n}_k \mathbf{x} = h_k. \quad (2)$$

At the same time, since the point X belongs to the polyhedron and lies on none of the planes of the other faces, for $i \neq k$ we have

$$\mathbf{n}_i \mathbf{x} < h_i.$$

Multiplying these inequalities by the numbers $\nu_{ki} \geq 0$, we obtain

$$\sum \nu_{ki} \mathbf{n}_i \mathbf{x} < \sum \nu_{ki} h_i,$$

or by virtue of (1) and (2)

$$h_k < \sum_{i \neq k} \nu_{ki} h_i. \quad (3)$$

If the vector \mathbf{n}_k is represented as a linear combination with nonpositive coefficients of the other vectors \mathbf{n}_i , then

$$\mathbf{n}_k = \sum_{i \neq k} \nu_{ki} \mathbf{n}_i \quad (\nu_{ki} \leq 0), \quad (4)$$

and in exactly the same way we obtain

$$h_k > \sum_{i \neq k} \nu_{ki} h_i. \quad (5)$$

Consequently: in order that the given numbers h_1, \dots, h_m be the support numbers of a convex polyhedron with outward normals $\mathbf{n}_1, \dots, \mathbf{n}_m$, it is necessary that, for every expansion (1) or (4) of each of the vectors \mathbf{n}_k whose

coefficients ν_{ki} are all nonnegative or all nonpositive, inequalities (3) or (5) hold.

In Section 7.5 and Section 7.6 of Chapter 7, we shall prove that this condition is also sufficient:

Theorem 15. *Let $\mathbf{n}_1, \dots, \mathbf{n}_m$ be m distinct noncoplanar unit vectors. Assume that m respective numbers h_1, \dots, h_m are associated with the vectors \mathbf{n}_k so that if a vector \mathbf{n}_k is represented as a linear combination of the other vectors,*

$$\mathbf{n}_k = \sum_{i \neq k} \nu_{ki} \mathbf{n}_i,$$

with all coefficients nonnegative or nonpositive, then

$$h_k < \sum_{i \neq k} \nu_{ki} h_i \quad \text{if} \quad \nu_{ki} \geq 0,$$

$$h_k > \sum_{i \neq k} \nu_{ki} h_i \quad \text{if} \quad \nu_{ki} \leq 0.$$

Then there exists a convex polyhedron with outward face normals $\mathbf{n}_1, \dots, \mathbf{n}_m$ and support numbers h_1, \dots, h_m . If all the vectors \mathbf{n}_i point to a single half-space, then the polyhedron is unbounded. In the opposite case it is bounded.

(We assume that each half-space includes its boundary plane.)²⁶

It suffices to require the validity for only those inequalities on h_k that correspond to the expansions of each vector \mathbf{n}_k in terms of three other vectors \mathbf{n}_i .

The last remark is essential for the following reason. If a vector \mathbf{n}_k admits an expansion in terms of four or more given vectors with positive (negative) coefficients, then it admits infinitely many such expansions. At the same time, there can be only one expansion in terms of three vectors. Hence, the last remark allows us to confine our consideration to finitely many inequalities on the numbers h_k , thus making the conditions of the theorem effectively verifiable.

The proof of Theorem 15 might seem rather simple at first sight: “Intersect the half-spaces $\mathbf{n}_i \mathbf{x} \leq h_i$ and the desired polyhedron results.” However, to

²⁶In my article [A6] the formulation of the theorem is erroneous: the inequalities for h_k corresponding to expansions of the vectors \mathbf{n}_k with nonpositive ν_{ki} ’s were forgotten. The inequalities with nonnegative ν_{ki} ’s alone are insufficient as can be seen from the following simple example. Take the vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$, and \mathbf{n}_4 of the outward face normals of a right tetrahedron. Each of them expands in terms of the others with only negative coefficients. Therefore, the conditions on h_k with positive coefficients ν_{ki} are left out and, dropping the conditions with negative ν_{ki} , we impose no conditions whatever. At the same time, it is easy to translate the face planes of the right tetrahedron so that they bound no polyhedron with the corresponding outward normals.

claim this would be rather hasty, since we do not know if this intersection is nonempty.²⁷ This is the first difficulty in proving Theorem 15.

2.4.7 From Theorem 15 it is easy to deduce a necessary and sufficient condition for given planes to be the planes of the unbounded faces of some convex polyhedron.

Theorem 16. *In order that the planes P_1, \dots, P_m with normals $\mathbf{n}_1, \dots, \mathbf{n}_m$ be the planes of the unbounded faces of some convex polyhedron (with outward face normals $\mathbf{n}_1, \dots, \mathbf{n}_m$), it is necessary and sufficient, first, that the normals $\mathbf{n}_1, \dots, \mathbf{n}_m$ point to vertices and, perhaps, to points on the sides of a convex spherical polygon (namely the polygon which is the spherical image of the entire polyhedron with given unbounded faces) and, second, that the support numbers h_1, \dots, h_m of the planes P_1, \dots, P_m satisfy the conditions of Theorem 15.*

We already know that these conditions are necessary. They are also sufficient. Indeed, under the assumptions of Theorem 15, there exists a polyhedron with normals \mathbf{n}_i and support numbers h_i . It will be unbounded, since the normals \mathbf{n}_i point to a single half-space; and all its faces will be unbounded, since the normal to a bounded face goes to the interior of the spherical polygon spanned by the ends of the normals to unbounded faces, whereas all the normals \mathbf{n}_i lie on the boundary of the polygon by assumption.

If the vectors \mathbf{n}_i point to the boundary of the convex polygon or, equivalently, lie on the surface of a convex polyhedral angle, then it is obvious that none of them has an expansion in terms of the others with negative coefficients. For this reason, the corresponding condition of Theorem 15 simply disappears.

The expansion of a vector \mathbf{n} in the three vectors \mathbf{n}_1 , \mathbf{n}_2 , and \mathbf{n}_3 with positive coefficients means that \mathbf{n} lies inside the trihedral angle with edges \mathbf{n}_1 , \mathbf{n}_2 , and \mathbf{n}_3 . Therefore, for the case in which all the vectors \mathbf{n}_i lie on the surface of a convex polyhedral angle, none of them can be expanded with positive coefficients in terms of three (or more) other vectors.

Only one possibility remains open: a vector \mathbf{n}_k lies in the same plane as two other vectors \mathbf{n}_i and \mathbf{n}_j and is located within the angle between them. Then it is a vector that expands in the others with positive coefficients. This means that the end of the vector \mathbf{n}_k lies in the interior of a side of the spherical polygon spanned by the ends of the three vectors under consideration. Such a vector, as we know, is a normal to an unbounded face with parallel unbounded edges.

In consequence, the assumptions of Theorem 15 become much simpler and in this case need to be verified only for vectors pointing to the interiors

²⁷This aspect is essential not only for geometry. It constitutes an important step in solving many problems of linear programming. We shall return to this question in a more detailed setting in Section 7.5 of Chapter 7. – V. Zalgaller

of the sides of the spherical polygon. If there are no such vectors at all, then none of the expansions mentioned in Theorem 15 for the vectors \mathbf{n}_i is possible, and therefore all the restrictions are absent, i.e., any collection of planes with these normals \mathbf{n}_i will be the planes of the unbounded faces of some polyhedron.

An exceptional case occurs when all the vectors \mathbf{n}_i are coplanar, so that the spherical polygon spanned by their ends turns out to be a hemisphere. This corresponds to the case in which the unbounded part of the polyhedron is a prism. Here all the conditions on the numbers h_i must be taken into account. However, we can regard the h_i 's as the support numbers of a polygon which is a section of the prism. Then in Theorem 15 it suffices to take expansions in two vectors.

Theorem 15 and 16 can be generalized to spaces of arbitrary dimension n with the only provision that in n -dimensional space one considers expansions in terms of n vectors.

2.5 Polyhedra with Vertices on Prescribed Rays (Survey of Chapter 9)

2.5.1 Take some point O in space and draw rays l_1, \dots, l_m from it which do not all go inside a single half-space. We will consider closed convex polyhedra with vertices on these rays. The point O must lie inside such a polyhedron by assumption. Such polyhedra exist: for instance, any polyhedron inscribed in a ball centered at O .

Let r_1, \dots, r_m stand for the distances from the vertices of such a polyhedron to O . The numbers r_1, \dots, r_m are not arbitrary, as established in the following theorem:

Theorem 17. *The following condition is necessary and sufficient for the positive numbers r_1, \dots, r_m to serve as distances from the vertices to O for a closed convex polyhedron with vertices on the rays l_1, \dots, l_m :*

If \mathbf{e}_i are the unit vectors along the rays l_i and $\mathbf{e}_k = \sum \nu_{ki} \mathbf{e}_i$ is an expansion of some \mathbf{e}_k in terms of the other vectors with nonnegative coefficients, then

$$\frac{1}{r_k} < \sum \frac{\nu_{ki}}{r_i}.$$

It suffices that these inequalities be satisfied for similar expansions of each of the vectors \mathbf{e}_k in terms of three other vectors.

The proof of the theorem is elementary. Taking points A_i on the rays, consider their convex hull. This is the required polyhedron P . A point A_i is a vertex of P if and only if it lies beyond the convex hull of the other points.

As will be shown in Section 9.1, this yields the necessity and sufficiency of the conditions of the theorem.

There is a formal analogy between Theorem 17 and Theorem 15 of the previous section. This analogy has a simple geometric foundation: a polar transformation takes the polyhedron with vertices on the rays l_i to a polyhedron with faces having outward normals \mathbf{e}_i . Here, if r_i are the distances from the vertices of the given polyhedron to the point O , then the support numbers of the latter polyhedron are $h_i = 1/r_i$. (Polar transformations were considered in Subsection 1.5.4.) However, a complete analogy between Theorem 17 to Theorem 15 is impossible, since in Theorem 17 all the r_i are possible, whereas in Theorem 15 negative values of h_i are admissible.

2.5.2 We raise the following questions: If l_1, \dots, l_m are rays issuing from O and not contained inside a single half-space, then what conditions are necessary and sufficient for given numbers $\omega_1, \dots, \omega_m$ to serve as the curvatures (the areas of spherical images) at the vertices of a closed convex polyhedron with vertices on the rays l_1, \dots, l_m ? (The polyhedron is assumed to have no other vertices.) To what extent is a polyhedron determined by the conditions that all its vertices lie on given rays and have given curvatures? The answer to the first question is provided by the following theorem.

Theorem 18. *Suppose the rays l_1, \dots, l_m issue from a point O and do not lie inside a single half-space. Let Ω_{i_1, \dots, i_k} be the area of the spherical image of the solid angle presenting the convex hull of the rays l_{i_1}, \dots, l_{i_k} . Then the following conditions are necessary and sufficient for a set of numbers $\omega_1, \dots, \omega_m$ to serve as the areas of the spherical images of the vertices of a convex polyhedron with vertices on the rays l_1, \dots, l_m :*

- (1) *all the ω_i are positive;*
- (2) $\sum_{i=1}^m \omega_i = 4\pi$;
- (3) *for every collection of rays l_{i_1}, \dots, l_{i_k} we have*

$$\sum \omega_{j_p} > \Omega_{i_1, \dots, i_k},$$

where the sum is taken over all rays l_{j_p} lying beyond the convex hull of the rays l_{i_1}, \dots, l_{i_k} .

Of course, we only consider those collections of rays l_{i_1}, \dots, l_{i_k} whose convex hull does not coincide with the whole space, since the last condition no longer makes sense otherwise.

The necessity of conditions (1) and (2) is obvious: condition (2) means that the spherical image of the closed convex polyhedron covers all of the sphere.

Let us prove the necessity of the third condition. Let P be a closed convex polyhedron with vertices on the rays l_1, \dots, l_m . Take rays l_{i_1}, \dots, l_{i_k} and consider their convex hull V , assuming that V does not cover the whole

space. The convex hull of every finite collection of rays issuing from a single point O is a convex solid polyhedral angle with vertex O .

Suppose that the rays l_{j_1}, \dots, l_{j_l} and accordingly the vertices A_{j_1}, \dots, A_{j_l} lie outside V . Each support plane of V intersects the polyhedron P , and an appropriate parallel translation of the plane from the point O takes it to a position where it is the support plane at some vertex of P that lies outside V . (Fig. 70 shows a similar situation in the case of a polygon.) Consequently, each support plane of V has a translate which is a support plane of P at one of the vertices A_{j_1}, \dots, A_{j_l} . Actually, there are other support planes at these vertices, for instance the planes of faces disjoint from V . Therefore, the spherical image of V is contained in the spherical image of the vertices A_{j_1}, \dots, A_{j_l} , while differs from the latter. This yields the inequality $\sum_{p=1}^l \omega_{j_p} > \Omega_{i_1, \dots, i_k}$.

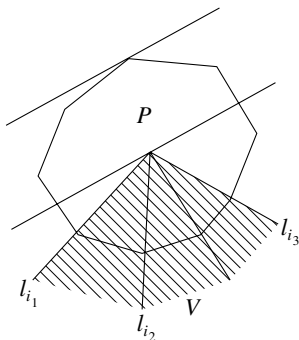


Fig. 70

The sufficiency of the conditions of Theorem 18 will be proved in Section 9.1 of Chapter 9; there we prove the existence of a convex polyhedron whose vertices lie on given rays and have given areas of spherical images.

Under a positive dilation (a homothety transformation) from O , the vertices remain on the same rays and their spherical images obviously remain the same. Other transformations are impossible, i.e., a polyhedron is determined from the rays l_i through its vertices and the curvatures ω_i at the vertices to within a similarity transformation with center the common origin of the rays.

This is a consequence of the following general theorem:

Theorem 19. *If the vertices of two bounded convex solid polyhedra lie on the same rays issuing from their common interior point O , then either the polyhedra are homothetic to each other with positive coefficient and center O , or they have a pair of vertices lying on the same ray and the spherical image of one of them is a proper subset of the spherical image of the other.*

If the areas of the spherical images are equal, then the second possibility is excluded and the polyhedra must be similar. In place of area, we can take

an arbitrary monotone function of spherical polygons (or, equivalently, of polyhedral angles).

The proof of Theorem 19 is very simple. Suppose that the polyhedra P_1 and P_2 have vertices on common rays issuing from a common interior point O . Contract P_2 homothetically to O so that P_2 is taken inside P_1 . The spherical images do not change.

Now, begin increasing the polyhedron P_2 homothetically from the point O until P_2 touches P_1 at some point. If this point is not a vertex, then P_1 and P_2 touch at a face or an edge (the polyhedra do not cross, since P_2 is still inside P_1). However, in that case, recalling that the vertices of P_1 and P_2 lie on the same rays, we see that P_1 and P_2 have at least one common vertex belonging to the face or edge at which the polyhedra touch. Thus, at the moment when the polyhedra P_1 and P_2 touch, some their vertices A_1 and A_2 coincide. Since P_2 is inside P_1 , the polyhedral angle V_2 at A_2 is contained in the polyhedral angle V_1 at A_1 . Therefore, every support plane of V_1 is a support plane of V_2 , i.e., the spherical image of the vertex A_2 includes the spherical image of the vertex A_1 .

If the areas of all the spherical images are assumed equal, then the spherical images of the vertices A_2 and A_1 coincide. Therefore the polyhedral angles V_1 and V_2 also coincide, i.e., the edges of P_1 and P_2 touching at the common vertex $A_1 = A_2$ overlap. Since the endpoints of the edges lie on common rays, the endpoints of these edges coincide as well. Applying the same arguments to these endpoints and continuing in a similar way, we see that the polyhedra P_1 and P_2 coincide. Hence, before the transformation the polyhedra were already similar, with O the center of similarity. This concludes the proof.

2.5.3 We now turn to unbounded polyhedra, assuming them to be solid. We could consider unbounded convex polyhedra with vertices on given rays issuing from some point O ; however, we restrict ourselves to polyhedra with vertices on given parallel rays; this case corresponds to an infinitely distant point O . All results that will be obtained can easily be converted to the case of a finite point O .

For an infinitely distant point O , the requirement that O lie inside the polyhedron amounts to the condition that every straight line parallel to the given rays is either disjoint from the polyhedron or has a common half-line with it.

Specifying parallel rays through the vertices of a polyhedron is equivalent to specifying a plane T perpendicular to the rays together with the projections A_1, \dots, A_m of the vertices to this plane. The polyhedron P in question must be situated so that every straight line perpendicular to T is either disjoint from P or has a common half-line with P . Then the projection of P to T either covers all of T once or covers some (bounded or unbounded) convex polygon in T whose interior is covered once, while its sides are the projections of the faces of P perpendicular to T .

Since to determine an unbounded convex polyhedron we need to know not only its vertices, but also its limit angle, the limit angle of the polyhedron must be involved in analysis as well. Here we also treat the limit angle of a polyhedron as a solid polyhedral angle. (As mentioned above, this angle may degenerate into a half-line or a flat angle.)

The limit angle of a polyhedron is defined up to a parallel translation. To be definite, we assume that the vertex of the angle coincides with the vertex O_1 of the polyhedron that projects to the point A_1 .

The limit angle V of the polyhedron P is situated relative to the plane T just like P itself: every straight line perpendicular to T either is disjoint from V or has a common half-line with V . Indeed, suppose that a straight line L is perpendicular to T and has a nonempty intersection with V . Since the vertex O_1 of V belongs to the polyhedron, we can homothetically contract P to O_1 , obtaining polyhedra P' contained in P and containing V . In the limit, the polyhedra P' converge to V . Since the straight line L meets V , it meets all the polyhedra P' and hence intersects each of them in a half-line. (Otherwise, increasing P' to P and accordingly translating the straight line L , we would obtain a straight line L_1 whose intersection with P is not a half-line, in contradiction to the condition imposed on the disposition of the polyhedron P .) The limit of these half-lines is a half-line common to the angle V and the straight line L .

From now on the conditions imposed on the disposition of the polyhedron and its limit angle relative to the plane T are assumed satisfied.

We now state a theorem similar to Theorem 18:

Theorem 20. *Let points A_1, \dots, A_m with associated numbers $\omega_1, \dots, \omega_m$ be given on a plane T . The following conditions are necessary and sufficient for $\omega_1, \dots, \omega_m$ to serve as the areas of the spherical images of the vertices of an unbounded polyhedron whose vertices project to the points A_1, \dots, A_m :*

- (1) *all the ω_i are positive and less than π ;*
- (2) *$\sum_{i=1}^m \omega_i \leq 2\pi$.*

The necessity of the first condition is evident. That of the second was proved in Subsection 1.5.3: the spherical image of an unbounded polyhedron is included in a hemisphere and hence its area is at most 2π . Theorem 20 is interesting because the curvatures at the vertices need only obey the above trivial conditions.

Under a translation of a polyhedron in a direction perpendicular to the plane T , the projections of the vertices of the polyhedron and their spherical images remain the same. However, in general, these data do not determine the polyhedron uniquely up to translation, as is seen from the following theorem.

Theorem 21. *An unbounded convex polyhedron for given A_1, \dots, A_m and $\omega_1, \dots, \omega_m$ is unique up to translation if and only if $\sum_{i=1}^m \omega_i = 2\pi$, i.e., if the area of the spherical image of the polyhedron equals 2π .*

It was proved in Subsection 1.5.3 that the spherical image of the limit angle of a polyhedron coincides with the spherical image of the polyhedron. Therefore, if $\sum_{i=1}^m \omega_i = 2\pi$, then the limit angle degenerates into a half-line and, consequently, is given in advance. If the area of the spherical image of a polyhedron, and hence of its limit angle, is less than 2π , then the limit angle may still vary. This leads to the following general theorem:

Theorem 22. *Let points A_1, \dots, A_m with associated numbers $\omega_1, \dots, \omega_m$ be given on a plane T so that (1) $0 < \omega_i < 2\pi$ for all i and (2) $\sum_{i=1}^m \omega_i \leq 2\pi$. Further, assume that V is a convex polyhedral angle for which the area of the spherical image is equal to $\sum_{i=1}^m \omega_i$ and which is positioned so that every straight line perpendicular to T is either disjoint from V or has a common half-line with V .²⁸*

Then there is an unbounded convex polyhedron P for which:

- (1) *the points A_1 are the projections of its vertices;*
- (2) *the numbers ω_i are the areas of the spherical images of the corresponding vertices of P ;*
- (3) *the angle V is the limit angle of P .*

Such a polyhedron is unique up to translation in a direction perpendicular to the plane T .

Theorem 22 is proved in Section 9.2 of Chapter 9. The uniqueness part of Theorem 22 is a consequence of a certain general theorem similar to Theorem 19, and has an equally simple proof (see Section 9.2).

Analogous existence and uniqueness theorems also hold for polyhedra with boundary.

2.5.4 If the limit angle V and the projections A_1, \dots, A_m of the vertices of a polyhedron to a plane T are given, then, to determine the polyhedron, it only remains to specify the heights p_1, \dots, p_m of the vertices above the plane T , the heights being positive from one side of T and negative from the other. Conditions that are necessary and sufficient for numbers p_1, \dots, p_m to serve as the heights of the vertices of a polyhedron given the limit angle V and the projections A_1, \dots, A_m of the vertices are rather cumbersome and will not be given here. They can easily be derived from the fact that a point B_k of height p_k above A_k is a vertex of a polyhedron if and only if B_k lies beyond the convex hull of the limit angle and the other vertices B_i .

2.5.5 All the theorems formulated in this section can be generalized to convex polyhedra in a space of arbitrary dimension together with their proofs: it suffices to replace 4π and 2π by the area of a sphere and the area of a hemisphere of radius one in that space. In particular, these theorems are

²⁸ V may be a half-line or a flat angle.

valid for convex polygons upon replacement of 4π and 2π by 2π and π respectively. The proofs of Theorems 20, 21, and 22 for unbounded polygons are rather simple. The limit angle of a polygon determines the directions of the unbounded edges; hence, the polygon can be constructed immediately from the angles α_i at its vertices and the projections of vertices to a straight line T . (Of course, here the role of curvatures ω_i is played by the quantities $\pi - \alpha_i$.) Theorem 18 for bounded polygons is far from being so simple, and we know no other proof but the one duplicating what was done in the case of polyhedra.

2.5.6 Finally, we return to the remark concerning polar transformations of polyhedra that was made in the beginning of the section. The polar transformation with respect to the unit sphere centered at an interior point O of a convex polyhedron P associates some convex polyhedron P' with P . The faces of P' are perpendicular to the rays issuing from O through the vertices of P and conversely: the vertices of P' lie on the rays issuing from O and perpendicular to the faces of P (see Subsection 1.5.4). The relationship between the faces and vertices of the two polyhedra is reciprocal: to each face of one of the polyhedra corresponds a vertex of the other and vice versa; moreover, to vertices belonging to a given face correspond faces touching at the corresponding vertex. Take the projection of the polyhedron P' from the point O to the sphere S . The sphere splits into convex spherical polygons S'_i , which are the projections of the faces Q' of P' . The vertices of these polygons are the projections of the vertices of P' . (See Fig. 32, in which polar polygons are shown for simplicity; for them the unit circle centered at O is taken instead of the sphere.)

The rays issuing from O and passing through the vertices of some face Q' of P' are the normals to the faces of P touching at the corresponding vertex A_i . Therefore, S'_i is the convex polygon spanned by the endpoints of the normals to the faces touching at A_i . In other words, S'_i is exactly the spherical image of the vertex A_i .

Thus, *the projection of a face of P' to the sphere S is the spherical image of the corresponding vertex of P* . The roles of P and P' can be interchanged because of their reciprocity.

After these remarks, each theorem about polyhedra with vertices on given rays can be related to some theorem about polyhedra with faces perpendicular to the rays. For example, Theorem 18 becomes the following theorem.

Theorem 23. *Suppose that rays l_1, \dots, l_m issuing from a point O are not contained in a single half-space. Let Ω_{i_1, \dots, i_k} denote the area of the solid angle which is the convex hull of the rays l_{i_1}, \dots, l_{i_k} . Then the following conditions are necessary and sufficient for the existence of a closed convex polyhedron with given areas $\omega_1, \dots, \omega_m$ of the central projections of its faces to the unit sphere centered at its interior point O and with faces perpendicular to the rays l_i :*

- (1) all the ω_i are positive;
- (2) $\sum_{i=1}^m \omega_i = 4\pi$;
- (3) $\sum_j \omega_j > \Omega_{i_1, \dots, i_k}$ for every collection of rays l_{i_1}, \dots, l_{i_k} going inside one half-space, where the sum is taken over all j corresponding to rays l_j not contained in the convex hull of the rays l_{i_1}, \dots, l_{i_k} .

Theorem 19 becomes the following assertion:

Theorem 24. *If two closed convex polyhedra with a common interior point have pairwise parallel faces, then either the polyhedral angles projecting the parallel faces of the polyhedra from a common interior point O coincide and the polyhedra are homothetic with center of homothety at the point O , or among these angles there is one which projects a face of one of the polyhedra and contains as a proper subset the angle projecting the parallel face of the other polyhedron.*

In particular, the data of Theorem 23 determine a polyhedron uniquely up to similarity.

Certainly, Theorems 23 and 24 may be proved directly, in perfect analogy with Theorems 18 and 19.

The theorems on unbounded polyhedra with vertices on given parallel rays do not admit such polar transformation, since the center O in that case is at infinity. If we take a finite point O rather than one at infinity, we obtain theorems for unbounded polyhedra that admit a polar transformation.²⁹

2.6 Infinitesimal Rigidity Theorems (Survey of Chapters 10 and 11)

2.6.1 Suppose that a polyhedron P_0 is subject to some deformation, i.e., assume given a continuous family of polyhedra P_t depending on a parameter t ; moreover, let the value $t = 0$ correspond to the initial polyhedron P_0 . It is convenient to think of the parameter t as time, and so we shall speak of the

²⁹ An unbounded polyhedron with vertices on given parallel rays can be defined by an equation $z = f(x, y)$ over a plane (x, y) perpendicular to these rays. Instead of a polar transformation, we can apply the Legendre transformation (i.e., polarity with respect to an elliptic paraboloid) to such a polyhedron. To a plane $z - z_0 = p(x - x_0) + q(y - y_0)$ and a point $u_0 = p, v_0 = q, w_0 = px_0 + qy_0 - z_0$ in it, the Legendre transformation assigns the point (x_0, y_0, z_0) and the plane $w - w_0 = x_0(u - u_0) + y_0(v - v_0)$. As all polarities, the Legendre transformation maps convex polyhedra to convex polyhedra, so that to each theorem on convex polyhedra given by the equation $z = f(x, y)$ there corresponds some “dual” theorem on polar polyhedra. – V. Zalgaller

motion of elements of P_0 leading to its deformation, e.g. the motion of its vertices, the rotation of its faces, etc.³⁰

If x is some quantity connected with a polyhedron, for example, the area of faces, the length of edges, etc., then under deformation this x becomes a function of t . The derivative $\frac{dx}{dt}$ is called the *rate of variation* of x . The quantity x is *stationary* if the initial rate of its variation is zero: $(\frac{dx}{dt})_{t=0} = 0$.

We will study “initial deformations” of polyhedra, i.e., first order infinitesimal deformations at the initial moment, rather than finite deformations. In other words, we are interested in the principal terms of the first order in t of the deformations rather than the entire deformations of some elements of a polyhedron or related quantities: in the differentials dx , or equivalently in the initial rates $(\frac{dx}{dt})_{t=0}$, rather than in the increments Δx .

A polyhedron P_0 is *infinitesimally rigid* under some conditions if any initial deformation of P_0 subject to these conditions is reduced, as regards velocity, to motion of the polyhedron as a rigid body (or, in a more general case, to some other trivial transformation). This means that under the given conditions all elements of the polyhedron are stationary, provided that we exclude any motion, for example, by fixing one of the vertices of the polyhedron together with the direction of an edge issuing from it and fixing the direction of a face containing the edge.

Rigidity theorems are assertions of the following kind: a polyhedron is infinitesimally rigid if, under its deformations of some type, certain data pertinent to the polyhedron are stationary. For example, in essence Cauchy proved but did not explicitly state the following theorem³¹: *if all faces of a closed convex polyhedron are infinitesimally rigid, then the polyhedron itself is infinitesimally rigid*. That is, assume a closed polyhedron is deformed so that its structure remains the same and the planes of its faces move (undergo translations and rotations) with some velocity vectors. If all the faces are infinitesimally rigid, i.e., the lengths of edges and the angles on faces are stationary, then the obtained deformation of the polyhedron is given by a rigid motion with the same velocity vectors.

This theorem is analogous to Cauchy’s Theorem on the congruence of polyhedra with equal similarly-situated faces. The analogy becomes particularly clear if we observe that the definition of rigidity can be rephrased as follows: a polyhedron P_0 is infinitesimally rigid if, under the imposed conditions, every deformed polyhedron P_t coincides with P_0 to within first order

³⁰The main notions of the general theory of flexes and infinitesimal flexes are described, for instance, in [E2], [E3], [IS]. The last two surveys treat infinitesimal flexes of various orders in more detail. – V. Zalgaller

³¹This theorem was first explicitly stated and proved by M. Dehn [De]. Dehn overlooked the fact that the rigidity theorem follows by a word for word repetition of Cauchy’s arguments yielding the congruence theorem for polyhedra with equal faces. Dehn’s method differs drastically from that of Cauchy. Another proof was given by H. Weyl [We].

infinitesimals in t .³² The latter definition is obviously equivalent to the former. At the same time, it shows that rigidity theorems are theorems on congruence to within first order infinitesimals. This observation impels us to seek a rigidity theorem for each congruence theorem. Certainly, such passage from some theorems to others is in general far from trivial. First, rigidity theorems are assertions on congruence to within first order infinitesimals rather than claims on exact congruence. Second, the assumptions of rigidity theorems include the stationarity of elements of polyhedra, i.e., the congruence of the elements of P_0 and P_t to within first order infinitesimals in t , rather than their exact congruence. Nevertheless, to each congruence theorem (or, which is the same, to each uniqueness theorem) formulated in Sections 2.3–2.5, corresponds an analogous rigidity theorem with the assumptions and claim appropriately modified.

Below we formulate as examples some rigidity theorems which will be proved in Chapters 10 and 11.

2.6.2 The following statement corresponds to Theorem 2 of Section 2.4:

Theorem 25. *If a closed convex polyhedron is deformed so that the intrinsic distances between its vertices are stationary, then the initial deformation is given by a motion. In other words, a closed convex polyhedron with stationary intrinsic metric is infinitesimally rigid.*

This result may be rephrased as follows:

Theorem 25*. *Assume that a development R_0 of a closed convex polyhedron P_0 be given on P_0 so that all vertices of R_0 are among the vertices of P_0 or lie on edges of P_0 . If the polyhedron P_0 is deformed in such a way that no new vertices appear on it, except possibly vertices of R_0 lying on “old edges,” then the development R_0 also undergoes a deformation³³ and if the lengths of the edges and the angles of the polygons of R_0 are stationary, then the polyhedron P_0 is infinitesimally rigid, i.e., the initial deformation of P_0 is given by a rigid motion. In other words, a closed convex polyhedron with stationary development is infinitesimally rigid.*

In Theorems 25* and 25, we a priori exclude neither foldings of faces or edges nor even the violation of convexity of the given polyhedron. However,

³²That is, there are polyhedra P_0^τ congruent to P_0 and such that the maximum of the distances of their vertices from the corresponding vertices of P_t is an infinitesimal of higher order than t at small t .

³³We shall prove that on a polyhedron P_t close to P_0 there always exists a development close to R_0 . It is for this reason that we may speak of the deformation of a development. The deformation of a polyhedron P_0 obeys certain general constraints that are not formulated here. In short, they are reduced to the requirement that there appear no new vertices other than the vertices of the development.

new vertices may appear only on “old edges” of the polyhedron. If such new vertices are present, then in Theorem 25 they are taken into account among the vertices. The violation of convexity of the polyhedron may be caused by “folding the polyhedron inward” at a new vertex.

The above-mentioned theorem on the infinitesimal rigidity of a polyhedron with infinitesimally rigid faces is a corollary to Theorem 25* (or 25). Theorem 25* is surely more general, since it allows foldings of faces. For example, a cube with one face deleted is infinitesimally rigid provided that all faces are infinitesimally rigid, and it is nonrigid in case foldings of faces along diagonals are permitted. (The first assertion is straightforward from the obvious rigidity of trihedral angles with stationary planar angles. The second is verified directly.)

Besides the general differences of all rigidity theorems from congruence theorems, Theorem 25* differs from the theorem on congruence of polyhedra with equal developments in the following aspects:

(1) In the congruence theorem, the condition that the vertices of the development are among the vertices of the polyhedron or lie on the edges is absent: under gluing, these vertices may appear inside faces. In Theorem 25*, however, this requirement is necessary: the theorem fails without it, as will be shown in Section 10.1 of Chapter 10.

(2) The congruence theorem is valid for polyhedra degenerating into polygons, while Theorem 25* fails for them. (This is trivial when we allow the violation of convexity: it suffices to fold a polygon along a diagonal. However, all doubly-covered polygons (except triangles) are also nonrigid without violating convexity. For instance, take a doubly-covered square $ABCD$ with side a and lift the vertex D to height $h = vt$ from the plane of the square, where v is the velocity of motion of D . We obtain a tetrahedron with edge

$$CD = \sqrt{a^2 + h^2} \approx a + \frac{1}{2} \frac{v^2}{a} t^2.$$

We see that the variation of the distances between vertices is a second order infinitesimal in t , i.e., these distances are stationary, although the vertex D moves with a positive speed. A similar argument is obviously applicable to an arbitrary polygon with more than three sides.)

(3) In the congruence theorem, the polyhedra in question are assumed convex, whereas in Theorem 25 or 25* violation of the convexity under deformation is not excluded: it may be caused by foldings of faces when new vertices appear on edges. (However, the theorem asserts that if a development is stationary, then the movement of a point inside the polyhedron has zero initial velocity.) In this aspect, Theorem 25 is more general than Theorem 2.

By analogy to Theorem 3 of Section 2.3 on the equality of dihedral angles in polyhedra with equal planar angles, we have the following

Theorem 26. *If a closed convex polyhedron is deformed so that the angles on its faces are stationary, then the dihedral angles of the polyhedron are also stationary.*

Here by faces we may mean not only genuine faces but the parts into which they may be split by line segments not interesting in the interiors of genuine faces. The deformation of a polyhedron consists in moving the planes of these parts of faces.

For unbounded polyhedra, there also exist rigidity theorems analogous to congruence theorems stated in Section 2.3. Roughly speaking, their statements are obtained by simply substituting stationarity for congruence; the theorems will be formulated precisely in Chapter 10.

2.6.3 Theorem 26 admits of the following mechanical interpretation:

Theorem 27. *Assume given a system of rectilinear rods joined by hinges at their endpoints and constituting the edges of a closed convex polyhedron. Then the rods of the system are not strained provided that there are no external forces, the rods are not bent, and the system is in equilibrium.*

An example of a strained system without external forces can be obtained from three matches and three rubber threads (Fig. 71). The match-sticks are put on the sides of an equilateral triangle and the rubber threads connect the vertices to the center of the triangle. The rubber threads are fastened at the center and are strained. Therefore, in the system there are strains, although it is in equilibrium without the action of external forces. Theorem 27 asserts that such a strained state of equilibrium is impossible for the system of edges of a closed convex polyhedron.

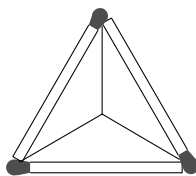


Fig. 71

The relationship between Theorems 26 and 27 will be completely revealed in Section 10.4 of Chapter 10, in which we also establish similar results for some other systems of hinged rods.

2.6.4 In Section 2.4 we presented some general theorems on the congruence of convex polyhedra with parallel faces. To these theorems certain rigidity theorems will correspond. To state the latter, we introduce the notion of

essentially monotone function of polygons. We say that a function $f(Q)$ of a convex polygon Q is *essentially monotone* if, under the translation of any side of Q outside the polygon, the function f varies at a rate greater than zero, i.e., whenever h_i is the distance from a given point inside Q to the straight line through the i th side, we have $\frac{\partial f}{\partial h_i} > 0$.

The area and perimeter of a polygon serve as examples.

We now formulate a rigidity theorem similar to Theorem 8 of Section 2.4 on the congruence of closed convex polyhedra:

Theorem 28. *If a closed convex polyhedron P is deformed by translations of the planes of the faces of P such that some essentially monotone function (in general, its own for each face) is stationary at every face of P , then the initial deformation of P is given by a translation.*

As opposed to the corollary to Theorem 8 indicated in Section 2.4, here the function in faces is assumed essentially monotone rather than simply monotone. Without this condition Theorem 28 fails. For example, put

$$f(Q) = [F(Q) - 1]^3,$$

where $F(Q)$ is the area of the polygon Q . It is easy to see that f is monotone: if Q_1 contains Q_2 , then $F(Q_1) > F(Q_2)$ and consequently $f(Q_1) > f(Q_2)$. However, since $F(Q) = 1$, the function f is not essentially monotone, because

$$\frac{\partial f}{\partial f_i} = 3[F(Q) - 1]^2 \frac{\partial F}{\partial h_i}$$

and consequently

$$\frac{\partial f}{\partial h_i} = 0 \quad \text{at} \quad F = 1.$$

Hence, if we take a cube with unit edge and push some of its faces outwards with arbitrary speed, then the function f is stationary at all faces, although the deformation is not reduced to a translation even to within first order infinitesimals.

Rigidity theorems similar to Theorem 28 are also valid for unbounded polyhedra. There the additional condition requires that the planes of the unbounded faces be invariant (or, at least, stationary), while the stationarity condition for the appropriate essentially monotone functions need be imposed only on bounded faces.

These theorems will be precisely stated and proved in Chapter 11.

2.6.5 The congruence theorems formulated in Sections 2.3 and 2.4 are proved, as we pointed out, by using the Cauchy Lemma. It turns out that this lemma is inapplicable to the rigidity theorems in the general setting in which we prove them, because its main condition (forbidding only two sign changes around a vertex) cannot be guaranteed in advance. For example, when the

planar angles of a convex polyhedral angle are stationary, the number of sign changes of the rate of variation of its dihedral angles may equal two, provided that there are dihedral angles of measure π . This corresponds exactly to the fact that we allow a genuine face of the polyhedron under study to be partitioned into separate pieces. The possibility of exactly two sign changes in this situation can be easily seen from examples (see Section 10.1). However, we will prove that in this case the sign distribution still obeys an additional condition which suffices to ensure the result of the Cauchy Lemma: in passing labeled edges around a vertex, it is impossible to have at least four sign changes or two sign changes under the above-mentioned additional condition. This “Strong Cauchy Lemma” is proved in Section 10.2 of Chapter 10 and on its basis we prove all the rigidity theorems of Chapters 10 and 11.

2.6.6 A rigidity theorem also corresponds to the general Theorem 19 on the similarity of polyhedra with vertices on given rays and spherical images that cannot be placed in one another. To formulate it, we introduce the notion of essentially monotone function of spherical polygons by analogy to the corresponding notion for plane polygons.

Namely, a convex spherical polygon is bounded by great circles through the sides of the polygon and a rotation of such a great circle about two antipodal points lying on it but not belonging to a side brings about a deformation of the polygon. A function f of spherical polygons is said to be *essentially monotone* if, under such a rotation of an arbitrary side of the polygon with positive velocity directed out of the polygon, the function f increases at a positive rate. In particular, area is easily seen to possess this property.

The rigidity theorem corresponding to Theorem 19 is:

Theorem 29. *If the deformation of a closed convex polyhedron is due to the motion of its vertices along given rays issuing from an interior point of the polyhedron and such that some essentially monotone function is stationary at the spherical image of each vertex, then the initial deformation of the polyhedron is given by a similarity transformation with center O .*

The proof of this theorem is almost as simple as that of Theorem 19 (see Section 9.1). Similar theorems are valid for unbounded polyhedra as well, but under the additional assumption of stationarity of the limit angle of the polyhedron (see Section 9.2).

2.6.7 We now establish a deep connection between the infinitesimal rigidity theorems on the one hand and the existence and uniqueness theorems on the other hand. It turns out that we can deduce, under rather general conditions, existence and uniqueness theorems from infinitesimal rigidity theorems.³⁴

³⁴Observe one more connection between rigidity and infinitesimal rigidity. As was shown by N. V. Efimov [E4], first or second order rigidity implies that a sur-

Let a_1, \dots, a_n be some parameters determining a polyhedron up to some trivial transformation and let b_1, \dots, b_n be other parameters related to the polyhedron, for which we wish to prove that the polyhedron is determined up to a transformation of the same kind (i.e., an existence and uniqueness theorem holds for polyhedra with data b_1, \dots, b_n). For example, if we consider closed polyhedra with $n+1$ vertices lying on given rays issuing from the same point O , then we can take the parameters a_1, \dots, a_n to be the ratios of the distance of the n vertices from the point O to the distance of the $(n+1)$ th vertex from O . These ratios determine the polygon up to homothety of center O . For the parameters b_1, \dots, b_n , we can take the curvatures at the n vertices, i.e., the areas of the spherical images of the vertices; the curvature at the $(n+1)$ th vertex is determined from the condition that the total curvature of the closed polyhedron equals 4π .

Since the parameters a_i determine a polyhedron, the quantities b_i are functions of the a_i :

$$b_i = f_i(a_1, \dots, a_n) \quad (i = 1, \dots, n). \quad (1)$$

We assume that the functions f_i are differentiable; hence,

$$db_i = \frac{\partial b_i}{\partial a_1} da_1 + \dots + \frac{\partial b_i}{\partial a_n} da_n \quad (i = 1, \dots, n). \quad (2)$$

(The differentiability condition is always satisfied for parameters $b_i = f_i(a_1, \dots, a_n)$ of common interest. In particular, in the above example the curvatures at vertices are easily seen to be differentiable functions of the distances of vertices from O .)

The theorem on infinitesimal rigidity of a polyhedron P_0 at stationary b_i obviously consists in the assertion that *if all the db_i vanish, then $da_i = 0$ at the values $a_i = a_i^0$ corresponding to P_0* . In other words, the homogeneous system (2) has the trivial solution only. This is equivalent to the fact that the determinant of system (2), i.e., the Jacobian of the system of functions (1), differs from zero. In that case, the functions (1) are well known to be invertible in a neighborhood of the values $a_i = a_i^0$ and $b_i = b_i^0$ ($i = 1, \dots, n$)

face does not admit a continuous flex analytic in a parameter even in the class of nonconvex surfaces. The following generalizations of this fact are known: If a surface admits only one linearly independent field of first order infinitesimal flex, then rigidity of order 3 implies the above-mentioned analytic rigidity [E4]; see [S4] for $n \geq 3$ and C^1 -smoothness. In the case of a polyhedron, existence of a continuous flex implies the existence of an analytic continuous flex. This algebraically natural result seems to have been first formulated by Gluck [G11]. It implies in particular that a first or second order infinitesimally rigid polyhedron does not admit a continuous flex. Using this result, Connelly [Co4] proved that an arbitrarily triangulated convex polyhedron with the new faces rigid does not admit a continuous flex in the class of all (not necessarily convex) surfaces. As was noted by I. Kh. Sabitov [IS, pp. 153–154], this result can be deduced from A. V. Pogorelov's theorem on the rigidity of general convex surfaces in the exterior of an open planar domain. – V. Zalgaller

corresponding to P_0 . Consequently, given b_i close to b_i^0 , we can uniquely find a_i close to a_i^0 ; therefore, there is a polyhedron with data b_i which is close to P_0 (up to some trivial transformation); moreover, such a polyhedron is unique up to a trivial transformation.

Thus, any rigidity theorem implies existence and uniqueness theorems in a small neighborhood of the polyhedron P_0 under study.

2.6.8 The next two simple lemmas provide conditions under which the above local result can be extended to global existence and uniqueness theorems.

Lemma A. *Let \mathbf{A} and \mathbf{B} be two manifolds and let φ be a mapping from \mathbf{A} to \mathbf{B} satisfying the following conditions:*

- (1) *if the point B in \mathbf{B} is the image of a point A in \mathbf{A} , $B = \varphi(A)$, then there is a neighborhood of B admitting the inverse mapping φ^{-1} into \mathbf{A} ;*
- (2) *every connected component of \mathbf{B} contains images of some points in \mathbf{A} ;*
- (3) *$\varphi(\mathbf{A})$ is closed, i.e., if the points B_n converge to B , then B is the image of some point in \mathbf{A} .*

Then φ is a mapping from \mathbf{A} onto \mathbf{B} .

By condition (1), if $B \in \varphi(\mathbf{A})$, then there is a neighborhood of B contained in $\varphi(\mathbf{A})$. Hence, $\varphi(\mathbf{A})$ is open.

Combining this result with conditions (2) and (3), we see that $\varphi(\mathbf{A}) = \mathbf{B}$ in exactly the same way as in the Mapping Lemma.³⁵ If \mathbf{A} is the manifold of all polyhedra considered to within a trivial transformation and \mathbf{B} is the manifold of data $B(b_1, \dots, b_n)$ to which the rigidity theorem in question applies, then this rigidity theorem yields condition (1) of Lemma A. Therefore, whenever the other two conditions are satisfied, a global existence theorem holds. Such a proof of Theorem 1 on the existence of a closed convex polyhedron from a development appears in Chapter 6 of my book [A15].

Lemma B. *Let \mathbf{A} , \mathbf{B} , and φ have the same meaning as in Lemma A, and let φ be a mapping from \mathbf{A} onto \mathbf{B} (which holds under the assumptions of Lemma A, for instance). Suppose that the following conditions are satisfied:*

- (1) *if a point B in \mathbf{B} is the image of some point A in \mathbf{A} , then there is a neighborhood U of A and a neighborhood V of B such that φ is a homeomorphism from U onto V ;*

³⁵This method is often described as the method of continuous extension. Suppose that $B_0 \in \varphi(\mathbf{A})$ and let B_1 be a point such that there is a continuous curve B_t ($0 \leq t \leq 1$) joining B_0 to B_1 . Then condition (1) of Lemma A implies that $B_t \in \varphi(\mathbf{A})$ for small t . Let T be the least upper bound of those t at which $B_t \in \varphi(\mathbf{A})$. Then $B_T \in \varphi(\mathbf{A})$ by condition (3). If we had $T < 1$, the condition (1) would give us a value $t > T$ for which $B_t \in \varphi(\mathbf{A})$. Consequently, $T = 1$ and $B_1 \in \varphi(\mathbf{A})$. Condition (2) guarantees that every point B in \mathbf{B} is joined to some point in $\varphi(\mathbf{A})$.

- (2) the manifold \mathbf{A} is connected and the manifold \mathbf{B} is simply connected³⁶, i.e., every closed curve in \mathbf{B} is contractible to a single point.

Then the mapping φ is injective.

This lemma is well known in topology. A mapping φ satisfying the conditions of Lemma B is the covering mapping of \mathbf{A} onto \mathbf{B} . When \mathbf{B} is simply connected, such a mapping is well known to be a homeomorphism; in particular, it is injective. Also, Lemma B can easily be proved without appealing to the theory of covering manifolds as follows:

Suppose that φ is not injective. Then there is a point B in \mathbf{B} which is the image of two different points A_0 and A_1 in \mathbf{A} . Since the manifold \mathbf{A} is connected, there exists a continuous curve L joining A_0 to A_1 , i.e., a point $A(t)$ can be assigned to each t in the closed interval $[0, 1]$ in such a way that $A(t)$ depends continuously on t and $A(0) = A_0$ and $A(1) = A_1$.

The image of L is some curve $M = \varphi(L)$ with points $B(t) = \varphi(A(t))$; it is closed since $\varphi(A_0) = \varphi(A_1) = B$. By condition (2), the curve M may be contracted to some point B_0 , i.e., there exists a continuous family of curves $M(s)$ ($0 \leq s \leq 1$) such that $M(0) = M$ and $M(1) = B_0$. Here the points $B(t, s)$ of the curves $M(s)$ satisfy the two following conditions:

- (1) $B(t, s)$ depends continuously on the two arguments $0 \leq t \leq 1, 0 \leq s \leq 1$;
- (2) at fixed s $B(t, s)$ is a parametric representation of the curve $M(s)$.

By virtue of the first condition of the lemma, the mapping φ is a homeomorphism in some neighborhood of each point $A \in \mathbf{A}$.

By Borel's Lemma, the curve L can be covered by finitely many such neighborhoods U_1, \dots, U_m and split into arcs L_1, \dots, L_m each lying in U_1, \dots, U_m respectively: to each arc L_i there corresponds a subinterval $\tau_i = [t_{i-1}, t_i]$ of the interval $[0, 1]$ such that if $t \in \tau_i$ then $A(t) \in U_i$.

Accordingly, the curve M will be covered by the neighborhoods $\nu_i = \varphi(U_i)$ and split into arcs $M_i = \varphi(L_i)$. Assume s_0 so small that the curve $M(s_0 t_0)$ lies in the neighborhood $\nu = \cup_i \nu_i$ of the curve $M = M(0)$. Take an arbitrary value of t ; it belongs to some interval τ_i . Then $B(t, s_0) \in \nu_i = \varphi(U_i)$, and to $B(t, s_0)$ we can associate a well-defined inverse image under φ ; denote it by $A(t, s_0)$. As the result, to the arcs of $M(s_0)$ corresponding to the intervals τ_i , we associate some curves $L_i(s_0)$ in \mathbf{A} . These curves have consecutive common endpoints belonging to the same neighborhoods over which φ is injective. Therefore, the curves $L_i(s_0)$ all together constitute a single curve $L(s_0)$ whose endpoints are obviously the points A_0 and A_1 . The image of $L(s_0)$ is the curve $B(t, s_0)$.

Proceeding similarly with the curve $L(s_0)$ replacing A_t and so forth, we eventually come to a situation in which to each curve $M(s)$ we assign some

³⁶It suffices for instance to "wind" an open rectangular strip \mathbf{A} around the lateral surface of a cylinder \mathbf{B} in such a way that the edges of the strip overlap, and as the result we obtain a noninjective mapping that satisfies all the assumptions of the lemma except the simple connectedness of \mathbf{B} .

curve $L(s)$ that joins A_0 to A_1 and is mapped onto $M(s)$.³⁷ However $M(1)$ is the point B_0 and $L(1)$ is a curve joining the two points A_0 and A_1 . Therefore, the mapping φ from $L(1)$ onto $M(1) = B_0$ is not injective in any neighborhood of each point, for example the point A_0 . This contradicts the assumptions of the lemma. Hence, the supposition that φ takes two points A_0 and A_1 into one point B_0 is invalid. Thus, the mapping φ is one-to-one, as claimed.

2.6.9 Let \mathbf{A} be the manifold of polyhedra considered up to trivial transformations and let \mathbf{B} be the manifold of data $B(b_1, \dots, b_n)$ to which the rigidity theorem applies. We have already observed that this theorem implies the first condition of Lemma B; therefore, if \mathbf{A} is connected and \mathbf{B} is simply connected, then the mapping φ from \mathbf{A} onto \mathbf{B} is one-to-one, which implies that the uniqueness theorem up to an appropriate trivial transformation holds. For example, if \mathbf{A} is the manifold of polyhedra with vertices on given rays considered up to a similarity and \mathbf{B} is the manifold of feasible values for the areas ω_i of spherical images, then \mathbf{A} is obviously connected while \mathbf{B} is simply connected, being convex (which follows from the defining conditions indicated in Theorem 18). A similar situation occurs also in other cases, for example in Minkowski's Theorem 11, where \mathbf{A} is the manifold of polyhedra with given face directions and \mathbf{B} is the manifold of feasible values of areas of faces; here \mathbf{B} is also convex and therefore simply connected. (However, I was unable to establish the simple connectedness of the manifold of feasible developments for existence theorems of polyhedra determined by a development.)

We thus have the following general method for proving existence and uniqueness theorems:

1. Prove the corresponding rigidity theorem and from it deduce existence and uniqueness "in small neighborhoods."
2. Establish the assumptions of Lemma A, thus obtaining the global existence theorem.
3. Also establish the assumptions of Lemma B, thus obtaining the global uniqueness theorem.

This method has the following advantages:

First, as shown above, the rigidity theorem is reduced to a linear problem (namely to the study of equations (2)) and therefore may be easier, especially if the question is treated analytically. This circumstance also facilitates generalizing the method to problems concerning curved surfaces, where the manifolds \mathbf{A} and \mathbf{B} are infinite-dimensional spaces.³⁸

Second, in proving existence, the global uniqueness theorem becomes redundant, as opposed to what is required by the method based on the Mapping Lemma.

³⁷A rigorous justification of the assertion is as follows: for small s , the assertion is true. Let S be the least upper bound for such values of s . Then, as $s \rightarrow S - 0$, we obtain a curve $L(s)$ with the same arguments applicable to it. Hence, $S = 1$.

³⁸This is the method that H. Weyl [W1] used to solve the existence problem of a closed convex surface with given metric.

However, if the relevant existence theorem admits a direct proof, then the Mapping Lemma provides a shorter way to proving existence. Moreover, from a purely geometric standpoint, rigidity theorems look somewhat artificial since they involve notions and theorems of the differential calculus.

2.6.10 We return to the fact that the rigidity theorem fails for polyhedra with a stationary development. We have shown that this theorem fails for polyhedra degenerating into polygons even if we require that the convexity of the polyhedron be preserved. Now, we verify this by another argument.

Let \mathbf{B} be the manifold of developments with a given number of vertices and let \mathbf{A} be the manifold of closed convex polyhedra considered up to a motion but not up to a reflection. Since the same development may be obtained by gluing two polyhedra symmetric to each other, the mapping from \mathbf{A} onto \mathbf{B} is two-to-one rather than one-to-one. The violation of injectivity due to the presence of symmetry elements in a nondegenerate polyhedron can be eliminated by labeling the vertices of this polyhedron according to the vertices of the development.³⁹

However, this is impossible for degenerate polyhedra: the reflection in the plane of a degenerate polyhedron P_0 takes all the vertices of P_0 into themselves. Arbitrarily small shifts of vertices of P_0 from different sides of the plane of P_0 lead to symmetric polyhedra, i.e., they induce the passage from one sheet of the manifold \mathbf{A} to the other. The set of degenerate polyhedra is mapped into \mathbf{B} bijectively and is the branching set of the mapping from \mathbf{A} onto \mathbf{B} just like the branching point of the function $\sqrt{x+iy}$. Therefore, the mapping from \mathbf{A} onto \mathbf{B} cannot be injective in neighborhoods of degenerate polyhedra; hence, the rigidity theorem cannot hold for them.

Proving the existence of polyhedra with a given development on the base of the rigidity theorem, we must thus exclude degenerate polyhedra and their developments from our considerations.

2.6.11 Closing the section, let us make one more remark. Suppose that, as above, \mathbf{A} and \mathbf{B} are the manifold of polyhedra A determined by the parameters a_1, \dots, a_n and the manifold of data $B(b_1, \dots, b_n)$. Let a mapping φ from \mathbf{A} onto \mathbf{B} be represented by continuously differentiable functions (1). The set of those A at which the rigidity theorem fails is the set of zeros of the Jacobian of this function system. Hence, this set is closed. Whenever it has an interior point A , the functions (1) are dependent in some neighborhood

³⁹If a polyhedron P_0 has a symmetry plane and if X and Y are symmetric vertices of P_0 , then symmetric shifts of X and Y produce a pair of mutually symmetric polyhedra close to P_0 . However, the reflection in the plane of symmetry of P_0 takes the vertex X into Y and vice versa; therefore, if we only allow vertices with the same label to be superposed, then such reflections are excluded. An analogous remark relates to the case of other types of symmetry. We can make the mapping from \mathbf{A} onto \mathbf{B} one-to-one by endowing developments with an orientation and thereby doubling \mathbf{B} .

of A and consequently φ is not one-to-one. Therefore, if φ is one-to-one, i.e., the uniqueness theorem holds, then *the set on which the rigidity theorem fails is a closed nowhere dense set*.⁴⁰

2.7 Passage from Polyhedra to Curved Surfaces

2.7.1 The general theorems on convex polyhedra discussed in Sections 2.3–2.6 are of value not only in their own right but also because they admit generalizations to more or less arbitrary convex surfaces. In studying problems of the theory of convex surfaces, it is fruitful to pose and solve them first for polyhedra. In some cases, this enables us to solve the original problem by passing to the limit from polyhedra to general convex surfaces. In other cases, this approach fails, but at least we are left with some understanding of the question.

Due to lucidity and clarity of the properties of polyhedra, this approach turns out particularly useful in questions of geometry “in the large” or in applications to irregular surfaces, when the ordinary analytical methods of the theory of surfaces are no longer automatic.⁴¹

Since our main subject is the theory of convex polyhedra, we confine ourselves to formulating the corresponding theorems for other convex surfaces in the section *Generalizations* concluding almost every chapter. We leave the theorems without proofs, but give references where the proofs can be found. Here we begin with some general remarks, first of all, about what data of a general convex surface correspond to the appropriate data of polyhedra considered above.

2.7.2 In Section 2.3 we have already established that specifying a development of a polyhedron is equivalent to specifying the intrinsic metric of the polyhedron. For a general convex surface, we should speak of the intrinsic metric of the surface, i.e., of the function of pairs of points on the surface, given by the distance between the points along the surface. By definition, this distance equals the greatest lower bound for the lengths of the curves lying on the surface and joining the points.

Given a continuous function $\rho(X, Y)$ of pairs of points on the sphere S (or another manifold), we say that $\rho(X, Y)$ is the *metric* of the surface F ,

⁴⁰Implicitly, this assertion already contains the well-known theorem by Gluck [G11] claiming that almost all closed polyhedra in \mathbb{R}^3 homeomorphic to the sphere do not admit continuous flexes. A generalization of this result to closed polyhedra of arbitrary topological type is given in [Fog]. Earlier È. G. Poznyak [Poz] proved that almost all polyhedra homeomorphic to the sphere are infinitesimally rigid; it was not known then that this fact implies the absence of continuous flexes. – V. Zalgaller

⁴¹The successive approximation by polyhedra in the applications to the intrinsic geometry of arbitrary convex surfaces is described in my book [A15].

or that F realizes the metric $\rho(X, Y)$, if there is a homeomorphism h from S onto F such that $\rho(X, Y) = \rho_F(h(X)h(Y))$ for every pair of points X and Y , where ρ_F is the intrinsic metric of the surface F .

Let F_i and \mathbf{n}_i be the areas and outward normals of the faces of a polyhedron P . Consider a unit sphere S on which we construct the spherical image of P . If M is some subset of E , then

$$F_P(M) = \sum_{\mathbf{n}_i \in M} F_i$$

is the area of the subset of P which is the inverse image of M under the spherical mapping. (The sum is taken over all faces whose normals point to M .) $F_P(M)$ is a set function on E , called the *area function* of P . Specifying the face areas and the face normals of P is obviously equivalent to specifying the area function of P . The area function of an arbitrary convex surface Φ is defined in exactly the same manner: $F_\Phi(M)$ is the area of the set N on Φ which is the inverse image of M under the spherical mapping, i.e., at least one support plane whose outward normal points to M passes through each point of N . (It can be proved that $F_\Phi(M)$ is a countably additive function defined on at least all Borel sets M .)⁴²

Let \bar{e}_i be rays issuing from a point O or, equivalently, points on the unit sphere E about O . Let ω_i be the areas of the spherical images of the vertices of a polyhedron P whose vertices lie on the rays \bar{e}_i . If M is a subset of E then

$$\omega_{P,O}(M) = \sum_{\bar{e}_i \in M} \omega_i$$

is the area of the spherical image of the subset of P whose central projection is the set M . The set function $\omega_{P,O}(M)$ depends not only on the polyhedron P but also on the choice of the point O (or of the sphere E). Therefore, we call $\omega_{P,O}(M)$ the *spherical area function relative to E* . Specifying the rays \bar{e}_i and the areas ω_i of the spherical images of vertices is obviously equivalent to specifying the spherical area function.

The spherical area function of an arbitrary convex surface Φ is defined similarly: $\omega_{P,O}(M)$ is the area of the spherical image of the subset of Φ whose central projection from O to the sphere E is the set M . (It can be proved that $\omega_{P,O}(M)$ is a countably additive set function defined for all Borel sets at least.)⁴³

For an unbounded convex surface Φ , we similarly define the function $\omega_{\Phi,T}(M)$, the area of the spherical image relative to a plane T : given a set M on T , $\omega_{\Phi,T}(M)$ is the area of the spherical image of that subset of Φ whose projection is M .

Finally, for arbitrary convex surfaces, we must consider their support functions $H(\mathbf{u})$ instead of the support numbers considered for polyhedra.

⁴²See [A3: I]. It is in this article that the notion of area function was first introduced.

⁴³See [A15, § 2 and § 4 of Chapter 5].

For a unit vector \mathbf{u} , the value of $H(\mathbf{u})$ is the distance from the origin O to the support plane Q with outward normal \mathbf{u} . Moreover, the distance is assumed positive if the direction from O to Q coincides with the direction of the vector \mathbf{u} and is negative otherwise. If \mathbf{u} is not a unit vector, then

$$H(\mathbf{u}) = |\mathbf{u}| \cdot H\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right).$$

We also note that, for a general convex surface, instead of the limit polyhedral angle we must consider the “limit cone” defined in exactly the same way.

Using the above general notions, it is easy to formulate uniqueness and rigidity theorems for general convex surfaces; they are analogous to the corresponding theorems for polyhedra. Stating existence theorems requires imposing appropriate conditions on data. For the functions $H(\mathbf{u})$, $F_\Phi(M)$, $\omega_{\Phi,E}(M)$, and $\omega_{\Phi,T}(M)$, the conditions can readily be derived from the conditions of the theorems on polyhedra. They are presented in the last sections of Chapters 7 and 9. The metric $\rho(X, Y)$ is an exception: the necessary conditions to be imposed on it cannot be obtained by simply reformulating the conditions of theorems on polyhedra.⁴⁴

2.7.3 The uniqueness and rigidity theorems cannot be carried over from polyhedra to general convex surfaces just by passing to the limit, the theorems must be supplemented by certain estimates for the deformation of a polyhedron depending on the variation of the data involved in the theorems, and these estimates must be independent of the number of vertices, or at least must not degenerate as the number of vertices tends to infinity.

For example, consider Theorem 3 on the congruence of isometric closed convex polyhedra. To this theorem must correspond the general theorem, not proved as yet, on the congruence of isometric closed convex surfaces.⁴⁵ An approach to the proof of the latter theorem via estimates for the deformation of a polyhedron depending on the variation of the intrinsic metric of the polyhedron was indicated by S. Cohn-Vossen [C-V] and consists in the following: Let r_{ij} and ρ_{ij} be the distances between the i th and j th vertices of a polyhedron, measured in the space and on the polyhedron respectively. Given two closed convex polyhedra P_1 and P_2 whose vertices are in a one-to-one correspondence, denote the above-mentioned distances between their vertices by $r_{ij}^{(1)}$, $r_{ij}^{(2)}$, $\rho_{ij}^{(1)}$, and $\rho_{ij}^{(2)}$. Then

$$\Delta r = \max |r_{ij}^{(1)} - r_{ij}^{(2)}|$$

⁴⁴These conditions can be found in the same book [A15] or, in another form, in my article [A12].

⁴⁵This general theorem is now proved by A. V. Pogorelov see [P5]. Another proof was given by Yu. A. Volkov in [Vo6]. – V. Zalgaller

obviously characterizes the magnitude of the spatial deformation upon passage from P_1 to P_2 , while

$$\Delta\rho = \max |\rho_{ij}^{(1)} - \rho_{ij}^{(2)}|$$

characterizes the magnitude of the intrinsic deformation. We must establish the estimate

$$\Delta r \leq f(\Delta\rho),$$

with a continuous function f , vanishing at $\Delta\rho = 0$, which is independent of the polyhedra P_1 and P_2 or depends on rather general characteristics of their shapes, for instance on the radii of balls containing P_1 and P_2 and of balls contained in them. Imagine this estimate established. Let F_1 and F_2 be two isometric closed convex surfaces. Take sufficiently dense nets of points A_i^1 and A_i^2 that correspond to each other under the isometric mapping of F_1 onto F_2 . Inscribe closed convex polyhedra P_1 and P_2 in F_1 and F_2 with vertices at these points. It turns out that, the denser the net of points A_i is, the closer the distance between the points measured on the polyhedron approaches the distance measured on the surface. Since the surfaces are isometric, the quantity $\Delta\rho$ for the polyhedra P_1 and P_2 tends to zero as the net of points A_i becomes infinitely dense.

Using estimate (1) with the required properties, we see that $\Delta r \rightarrow 0$ as $\Delta\rho \rightarrow 0$. However, $\Delta r = \max |r_{ij}^{(1)} - r_{ij}^{(2)}|$ is nothing but the maximum of the differences of the distances in space between the corresponding points A_i and A_j of the surfaces F_1 and F_2 . Therefore, we must have $\Delta r = 0$, i.e., the surfaces F_1 and F_2 are congruent.⁴⁶

A similar approach is a priori possible for other uniqueness theorem. However, to my knowledge, no estimate of the desired type has been proved for any of them. For this reason, such an approach to carrying uniqueness theorems from polyhedra to general convex surfaces still remains purely hypothetical.⁴⁷

The situation is the same with rigidity theorems. For this reason, proofs of these theorems for convex surfaces are carried out by other methods. Moreover, the theorems on uniqueness and rigidity of a closed convex surface given a (stationary) intrinsic metric are not proved as yet without extra assumptions on the regularity of the surface under study.⁴⁸ Thus, in the questions of uniqueness and rigidity, theorems on polyhedra may only be regarded as guidelines for the results that we could try to attain in passing to arbitrary convex surfaces.

2.7.4 The matter is simpler by far for existence theorems: all the above-stated existence theorems for convex polyhedra can be carried over to general convex

⁴⁶It is the method that was implemented by Yu. A. Volkov in [Vo6]. – V. Zalgaller

⁴⁷At present, the articles [Vo5], [K], [P10, Chap. 7], [Di] should be indicated in addition to [Vo6]. – V. Zalgaller

⁴⁸Now these theorems have been proved by A. V. Pogorelov [P10, Chap. 4].

surfaces by passing to the limit. Without going into details, this is done as follows:

Suppose that we wish to prove the existence of a convex surface with given data B_0 . Assume known the corresponding existence theorem for convex polyhedra. Then we prove the following convergence theorem: if convex polyhedra P_n converge to a surface F , then their data B_{P_n} converge, in an appropriate sense, to the data B_F of F . Here the convergence of data is only required to be single-valued, i.e., the sequence B_n cannot converge to two different limits. Next, we construct a sequence of data B_n pertinent to polyhedra which converges to B_0 . By the existence theorem for polyhedra, there are convex polyhedra P_n with data B_n . We extract a converging subsequence P_{n_i} from the sequence P_n ; fortunately this is always possible. The limit of the subsequence is some convex surface. By the convergence theorem, B_{n_i} converge to B_F . At the same time, B_{n_i} converge to B_0 by construction. Since the data B_{n_i} may have only one limit, it follows that $B_F = B_0$, i.e., the surface F has the data B .

If the data B_0 represent the support function $H(\mathbf{n})$, then the convergence of data B_m to B_0 is simply the convergence of support functions or the convergence of support numbers h_i to the values of the support function $H(\mathbf{n})$ at the vectors \mathbf{n}_i since the set of points \mathbf{n}_i on the sphere becomes infinitely dense.

If B_0 is a metric $\rho(X, Y)$ on the sphere S , then we can approximate B_0 by polyhedral metrics of positive curvature. The theorem on convergence of metrics is as follows: if closed or unbounded complete convex surfaces Φ_n converge to Φ , then $\rho_{\Phi_n}(X_n, Y_n)$ converges to $\rho_{\Phi}(X, Y)$.

If B_0 is one of the set functions $F(M)$ and $\omega(M)$ on the unit sphere E , then the convergence should be treated as weak. If $\varphi_n(M)$ and $\varphi(M)$ are countably additive set functions on E , then the functions φ_n converge weakly to φ provided

$$\lim_{n \rightarrow \infty} \int_E f(X) \varphi_n(dM) = \int_E f(X) \varphi(dM)$$

for every continuous function $f(X)$ on E , with the integral understood in the Radon sense.⁴⁹ The given functions $F(M)$ and $\omega(M)$ should be approximated, in the sense of weak convergence, by “discrete” functions $F_n(M)$ and

⁴⁹This notion of convergence, corresponding to the notion of weak convergence of functionals, can be characterized by simple geometric properties. For example, if $\varphi_n(M) \geq 0$ for all M (which is the case for our functions F and ω), then the functions φ_n converge weakly to φ if and only if

$$\lim_{n \rightarrow \infty} \varphi_n(E) = \varphi(E) \quad \text{and} \quad \lim_{n \rightarrow \infty} \varphi_n(M) \leq \varphi(M)$$

for every closed M . Proofs of this and other characteristics of weak convergence for set functions appear in my article [A10], see the résumé and Sections 15–17. Proof of the theorem on weak convergence of the functions F and ω for convergent convex surfaces is given in the articles [A5] and [A6], also see the book [A15].

$\omega_n(M)$ with finite weights at specified points and vanishing for all M not containing any of these points.

The method of passing to the limit from polyhedra to other surfaces was first used by Minkowski in his proof of the existence closed convex surfaces Φ of given Gaussian curvature $K(\mathbf{n})$ as a function of the outward normal. If M is a subset of the sphere E , then

$$F(M) = \int_M \frac{d\omega}{K(\mathbf{n})},$$

where $d\omega$ is the area element on E , is exactly the area of that subset of Φ whose spherical image is M . Consequently, the theorem of Minkowski mentioned above is a particular instance of the general existence theorem for a closed convex surface given the area function $F(M)$.

2.8 Basic Topological Notions

2.8.1 Intuitive background. Topology can roughly be defined as the part of general geometry where, when studying a figure, we only take account of the mutual adherence of parts of the figure. The general concept of “adherence” is based on the intuitive idea which lies behind the following propositions: the circle adheres to the interior of the disk; one hemisphere adheres to the other, so that the sphere may be regarded as composed of two adhering hemispheres; however, if we delete the equator, then the hemispheres no longer adhere to one another: they appear to be cut off from one another by a slit along the equator.

The author has preferred to introduce the topological structure via adherence points, guided by the perfect clarity of the main definitions – of a continuous mapping, boundary, connectedness, etc. – in this approach.

Let us try to analyze our intuitive idea of adherence. Every geometric figure is a set or, as one says in elementary geometry, a “set of points”; and in this set there is a visually clear notion of adherence of parts to each other. The principal concept is that of adherence of a point to a set.

In our intuitive understanding, a point adheres to a set if it is infinitely close to the set, as for instance a vertex of a square to the interior of the square. If a point belongs to a set, then we also consider it as adherent.

Thinking further about the notion of adherence, we see that two sets, two figures or two parts of one figure, adhere to one another if and only if at least one of them contains a point adhering to the other.

Looking at the above examples, we readily convince ourselves that it is in this sense that the circle adheres to the interior of a disk and that two hemispheres with deleted equator do not adhere to one another (since each point of one hemisphere is at a positive distance from the points of the other hemisphere not less than the distance from the point to the equator.)

2.8.2 General topological spaces. The abstract generalization of the intuitive ideas described above leads to the notion of general topological space. Namely, assume given some set R ; we call the elements of R *points*. The set R is a *general topological space* if to each part of R , i.e., to each subset M of R , we assign, according to some rule, points *adherent to* M ; these points are also referred to as *adherent points of* M , the points of M being also considered as adherent points of M . Also, we add the following natural condition: when a set increases, the number of adherent points does not decrease, i.e., speaking rigorously, if M_1 is contained in M_2 , then each adherent point of M_1 is an adherent point of M_2 .

We say that two sets in a general topological space *adhere* to one another if at least one of them contains adherent points of the other.

Clearly, these definitions simply repeat, in slightly different words, what was explained above about the adherence of figures and points.

The basic example of a general topological space is the n -dimensional “arithmetical” space or Euclidean space.⁵⁰ The points of this space are all the n -tuples (x_1, x_2, \dots, x_n) of real numbers. A point $a = (a_1, a_2, \dots, a_n)$ is an adherent point of a set A if there are points in A arbitrarily close to a , i.e., for every $\varepsilon > 0$, the set A contains a point $x = (x_1, x_2, \dots, x_n)$ such that $|x_1 - a_1| < \varepsilon, \dots, |x_n - a_n| < \varepsilon$. In that case, either a belongs to the set A or A contains a sequence of points x^i that converge to a , i.e., the differences between the coordinates of the points x^i and the point a tend to zero.

The real axis is nothing but the one-dimensional arithmetical space. Similarly, the plane is the two-dimensional arithmetical space.

If we impose the condition that all the numbers x_i lie within given limits, for example $0 < x_i < 1$ ($i = 1, \dots, n$), then the resulting space is called an n -dimensional (arithmetical) cube; more precisely, the open cube, as opposed to the closed cube which contains all its faces and is determined by the inequalities $0 \leq x_i \leq 1$.

Three-dimensional Euclidean space is also a topological space, with the following definition of adherent points: a point a is an adherent point of a set A if A contains points that are arbitrarily close to a .

An arbitrary development also provides an example of a general topological space, with the agreement that two points that will be glued are treated as the same point. If M is a subset of a development, then a point a of the development is adherent to M if there are points of M arbitrarily close to a or to some point that will be glued to a .

Each subset A of a general topological space R may itself be treated as a general topological space, by using the following definition: if M is a subset of A , then a point a of A is an adherent point of M relative to A if a is adherent to M in R . We say that A is a *subspace* of R .

⁵⁰We call the space Euclidean, abstracting however from all of its properties except the topological ones.

In this sense, every figure A in Euclidean space is a general topological space. A point a of A is adherent to a set M contained in A if there are points of M at an arbitrarily small distance from a .

In the same way, every part of n -dimensional arithmetical space is a general topological space.

2.8.3 Closed sets and open sets. Boundary. Here we discuss sets in an arbitrary topological space.

A set is *closed* if it contains all points adherent to it.

A set is *open* if its complement (i.e., the set of all points not belonging to it) is closed. This definition is equivalent to the following: a set M is open if no point of M is an adherent point of the complement of M (since in that case the complement of M itself contains all its adherent points, i.e., the complement is closed).

A point is a *boundary point* of a set M if it is adherent to M as well as to the complement of M . The *boundary* of a set M is the set of all boundary points of M .

A point of a set is called *interior* if it does not lie on the boundary of the set, i.e., it is not adherent to the complement of the set. Clearly, the definition implies that an open set is characterized by the condition that it consists only of interior points.

Examples. A half-space with its bounding plane is a closed set; without the plane, it is open; the plane is the boundary of the half-space.

The cube in n -dimensional space is defined by the inequalities $0 \leq x_i \leq 1$ ($i = 1, \dots, n$). The interior of the cube consists of the points for which $0 < x_i < 1$. The boundary of the cube consists of all the points for which $0 \leq x_i \leq 1$ ($i = 1, \dots, n$) and at least once we have the equal sign.

A *neighborhood* of a point means, as a rule, an arbitrary open set containing the point. Thus a neighborhood surrounds a point in such a way that the point is not adherent to the complementary set. This corresponds to the intuitive meaning of the term “neighborhood.”

2.8.4 Connectedness. A set is *connected* if it never splits into two parts that do not adhere to one another. By the definition of adherence, this means that the set never splits into two parts neither of which contains adherence points of the other.

If the set M itself is considered as a space (a subspace of the space under study), then *relatively closed sets* and *relatively open sets* are naturally defined in M ; briefly, they are the closed sets and the open sets in M .

The following theorem holds:

A set M is connected if and only if M never splits into two nonempty parts which are both closed and open in M .

Indeed, suppose that M is disconnected, i.e., M splits into two parts M_1 and M_2 neither of which contains adherent points of the other. Then M_1 and M_2 themselves contain the points that are adherent to them in M . Hence, M_1 and M_2 are closed. Since they are complementary to each other, they are also open in M .

Now, suppose that M splits into two parts M' and M'' closed in M . This means that M' and M'' themselves contain their adherent points in M , i.e., M' and M'' do not adhere to one another, and M is therefore disconnected. Since M' and M'' are complementary to each other with respect to M , the fact that M' is closed implies that M'' is open, and M' is open for a similar reason. Consequently, splitting into two closed sets is also splitting into open sets. The converse is proved in exactly the same manner: splitting into open sets is also splitting into closed sets. This completes the proof of the theorem.

A subset M_1 of a set M is a *connected component* of M if M_1 is connected and is not contained in any connected subset of M but for M_1 itself.

Any set is connected or is the union of disjoint connected components.

The proof is based on the following theorem:

If connected sets M_ξ have at least one common point a , then their union is a connected set.

First, we prove the latter assertion. Denote the union of the sets M_ξ by M . Were M disconnected, it would split into parts M' and M'' which do not adhere to one another.

Since the union of M' and M'' is M , each set M_ξ either lies entirely in one of these sets M' and M'' or has common points with both the sets.

The second possibility is excluded. Indeed, if some M_ξ contains two pieces M'_ξ and M''_ξ which belong to M' and M'' , the pieces M'_ξ and M''_ξ cannot adhere to one another since M' and M'' do not. As a result, M_ξ would be disconnected, contradicting the assumption.

Since the second possibility is excluded, each M_ξ lies entirely in M' or M'' . Because all the sets M_ξ have a common point, M' and M'' would also have a common point, i.e., would adhere to one another. This contradiction proves that M never splits into parts M' and M'' which do not adhere to one another. Hence, M is connected.

Now, we prove the former assertion on the decomposition into connected components. A one point set is connected, because it never splits at all.

Take some point x of a given set M and consider the union of all connected sets containing x and contained in M . There are such sets, for the point x itself is one of them. This union, denote it by $M(x)$, is connected by virtue of the theorem just proved above. By construction, $M(x)$ is not a proper subset of any connected set; hence, it is a connected component.

If $M(y)$ is a set constructed similarly for another point y , then $M(y)$ is connected also. If $M(y)$ has common points with $M(x)$, then the union $M(x) \cup M(y)$ is connected by the same theorem. This union contains x and

therefore must be contained in $M(x)$, since by construction $M(x)$ includes all connected sets that contain the point x . The same holds for $M(y)$ as well. Therefore, if the sets $M(x)$ and $M(y)$ have common points, then they coincide.

Defining the sets $M(x)$ for all points x in M , we see that every two of these sets either coincide or are disjoint. The set M thus splits into disjoint connected components $M(x)$, as claimed.

We also indicate the following two propositions:

1. *If, for every two points in a set M , there is a connected subset of M containing the points, then the set M itself is connected.*

2. *Every two points in a connected open set of Euclidean space (or in n -dimensional arithmetical space) can be joined by a polygonal line contained in the set.*

These propositions will not be used in the sequel and we leave their proofs to the reader.

2.8.5 Mappings. Let X and Y be two sets in the same space or in two different spaces. The assignment to each point x in X of some point y in Y is a *mapping* from X into Y . Denoting the mapping by φ , we write $y = \varphi(x)$. To each part X' of X there corresponds a part Y' of Y , i.e., $Y' = \varphi(X')$. The set Y' is the *image* of X' and X' is a *preimage* of Y' .

A function $y = f(x)$ of a real variable x , defined on an interval X , is nothing but a mapping from the interval into the real axis. Similarly, a function of n variables is a mapping from a set of n -dimensional space into the real axis.

A mapping is *continuous* if it does not violate the adherence relation (every violation is a discontinuity). In other words, a mapping φ is continuous if it preserves the adherence of points, i.e., if x is an adherent point of X' , then $\varphi(x)$ is an adherent point of $\varphi(X')$.

In an arithmetical space, a point x adheres to X' if in X' there are points x' whose coordinates x'_1, \dots, x'_n are arbitrarily close to the coordinates of the point x . This is equivalent to the fact that in X' there is a sequence of points $x^{(1)}, x^{(2)}, \dots$ converging to x in the sense that the differences $x_i^{(k)} - x_i$ (between the coordinates of the points $x^{(k)}$ and of the point x) tend to zero. Therefore, a mapping φ from one arithmetical space into another is continuous if and only if $x^{(k)} \rightarrow x$ implies that $\varphi(x^{(k)}) \rightarrow \varphi(x)$, which is in perfect agreement with the usual definition of the continuity of a function.

A mapping φ is said to be *one-to-one*, or *injective*, if not only to each point x φ assigns a unique point $y = \varphi(x)$, but also to different x 's it assigns different y 's. In other words, to each y there may correspond only one x . In this case, the inverse mapping φ^{-1} associating x 's with y 's is single-valued.

If a mapping φ and its inverse φ^{-1} are continuous, then φ is said to be *continuous in both directions*.

A one-to-one mapping continuous in both directions is called a *homeomorphism*. If a set X admits a homeomorphism onto a set Y , then X is *homeomorphic* to Y . Since any homeomorphism is one-to-one and continuous in both directions, both sets X and Y play the same role here, and so we say that they are *homeomorphic to each other*.

The one-to-one property of a mapping means that points under such a mapping neither split nor merge together. Continuity in both directions means that adherences neither disappear nor appear. Hence, from the standpoint of adherences alone (abstracting from other properties of elements and sets) two homeomorphic sets have the same structure. They are absolutely the same topologically. All topological conclusions, i.e., conclusions relying only upon the adherence relation and valid for one of the sets are also valid for the other. This is quite similar to the fact that two congruent figures (i.e., figures admitting a distance-preserving mapping between their points) are geometrically the same. The geometric properties based on the concept of length are the same for congruent figures; similarly, the topological properties based on the concept of adherence are the same for homeomorphic sets.

2.8.6 Manifolds. By an *n -dimensional manifold* we mean a topological space in which each point has a neighborhood homeomorphic to the interior of an n -dimensional cube and in which any two points have disjoint neighborhoods.⁵¹ In other words, a manifold is a space each of whose points has a neighborhood into which coordinates x_1, \dots, x_n that vary in some interval may be introduced, say $a - \varepsilon < x_i < a + \varepsilon$ ($i = 1, 2, \dots, n$), and the correspondence between the points of the neighborhood and their coordinates is one-to-one, onto, and continuous in both directions. We also assume that any two points possess nonintersecting neighborhoods.

Every open set of n -dimensional arithmetical space is an n -dimensional manifold. Indeed, coordinates have already been introduced in such a space. An open set is characterized by the requirement that none of its points is an adherent point of the complement. This means that all points whose coordinates are close to the coordinates of a point a of the set are also contained in the set. That is, along with any point $a(a_1, \dots, a_n)$ the set contains all points x with coordinates x_1, x_2, \dots, x_n such that $|a_i - x_i| < \varepsilon$, or $a - \varepsilon < x_i < a + \varepsilon$ ($i = 1, \dots, n$). By definition, this means that the set is an n -dimensional manifold.

An arbitrary development without boundary provides an example of a two-dimensional manifold. Each point of the development has a neighborhood homeomorphic to a square or, which is equivalent, to a disk. This is obvious for points inside the polygons of the development. A neighborhood of a point on an edge is the union of two “half-neighborhoods” on polygons that

⁵¹It is connectedness and the existence of a covering of the space by at most countably many neighborhoods each homeomorphic to the interior of a cube (as well as partitioning into simplices) which is usually included in the notion of manifold. We do not require these conditions.

are to be glued together along the edge. Finally, a neighborhood of a vertex is the union of angular sectors touching at the vertex in accordance to the “rules for gluing” of the polygons.

By gluing, i.e., identifying, we can build three-dimensional manifolds. For example, take a cube and identify the symmetric points of its opposite faces. Some three-dimensional manifold results. This method applies to constructing n -dimensional manifolds: they originate by gluing from “developments” formed by n -dimensional polyhedra. The topological structure of such a manifold is determined by the law for gluing of the development. Compare this with Euler’s Theorem expressing the characteristic property of the law for gluing when the sphere is glued together from a development.

In our book, manifolds will not be defined by means of such a construction, but in accordance with the general definition, by introducing coordinates in a neighborhood of each element (point of the manifold). For example, all closed convex polyhedra with v vertices form a $3v$ -dimensional manifold. Each vertex is determined by three coordinates and all v vertices, by $3v$ coordinates. A closed convex polyhedron is determined by its vertices, and hence by the same $3v$ coordinates.

Given a polyhedron P_0 , sufficiently small shifts of its vertices never take one of the vertices into the convex hull of the others. Therefore, shifted vertices also span a polyhedron P with v vertices. Consequently, the coordinates of vertices may be changed in a neighborhood of the point corresponding to the polyhedron P_0 , and thus the “neighborhood” of the polyhedron is mapped onto a $3v$ -dimensional cube.

The adherence relation in this manifold of polyhedra is defined in the natural way. A polyhedron P_0 “adheres to” a set of polyhedra P if, among the latter, there are polyhedra whose vertices are arbitrarily close to the vertices of P_0 .

Manifolds defined in a similar manner will be used in the proofs of existence theorems in Chapters 4, 7, and 9.

2.9 The Domain Invariance Theorem

2.9.1 We prove the following theorem, on which the Mapping Lemma is based:

Under a homeomorphic mapping of an n -dimensional manifold R_1 onto a subset of an n -dimensional manifold R_2 , the image of every open set in R_1 is an open set in R_2 .⁵²

⁵²This theorem might seem trivial, since a homeomorphic mapping of one space onto another always takes open sets into open sets by the definition of homeomorphism. However, what was said above implies only that the image of an open set is open relative to the image of the space R_1 in R_2 . For example, under the identical

An open set is characterized by the requirement that each of its points is interior, i.e., may be surrounded by a neighborhood homeomorphic to an n -dimensional ball and contained in the set under consideration. For such a neighborhood, we can take a convex n -dimensional polyhedron with the least possible number of vertices. Such a polyhedron is called an n -dimensional *simplex*. It has $n + 1$ vertices. For $n = 0, 1, 2, 3$ simplices are respectively the point, the segment, the triangle, and the tetrahedron. Each $(n - 1)$ -dimensional face of an n -dimensional simplex is obviously an $(n - 1)$ -dimensional simplex.

Obviously, it suffices to prove that:

Under a homeomorphic mapping φ of an n -dimensional simplex T onto a subset of n -dimensional Euclidean space E , every interior point p of T goes to an interior point of the set $\Phi = \varphi(T)$ in E .

The proof consists in characterizing the interior points of a simplex by a property which is shown to be invariant under homeomorphisms.

First of all, we introduce the notion of a closed covering of a closed set: a finite collection of closed sets A_1, A_2, \dots, A_s is a *closed covering* of a closed set Φ if the union of the sets A_i is the set Φ .

A number k is the *multiplicity* of a covering if there exists a point of the set Φ which simultaneously belongs to k sets A_i and, at the same time, none of the points of Φ simultaneously belongs to a greater number of sets A_i . If the diameters⁵³ of all sets A_i are all less than a number $\varepsilon > 0$, then we are dealing with an ε -covering.

By using these notions, we give a topologically invariant characteristic for interior points of a closed set in n -dimensional Euclidean space:

For a point p of a closed set Φ in n -dimensional Euclidean space to be an interior point of Φ , it is necessary and sufficient that there exist a closed covering $\alpha = \{A_1, \dots, A_s\}$ of Φ of multiplicity $n + 1$ such that: (1) p is the only point of Φ belonging to $n + 1$ sets A_i ; (2) every covering α' of Φ differing from α only in a sufficiently small neighborhood $U(p)$ of p ⁵⁴ is of multiplicity

mapping of a straight line into a plane, the image of the straight line obviously is not an open set relative to the plane, although it is open relative to itself. The assertion that we are going to prove consists in the fact that under the conditions of the theorem the image of every open set is an open set open relative to the whole space R_2 . It is easy to verify that this assertion may be rephrased in the following simpler form: the homeomorphic image of a manifold R_1 in a manifold R_2 has no boundary.

⁵³The diameter of a set is the least upper bound of the distance between two points of the set. The distance between two points is understood, as usual, to be the square root of the sum of squares of the differences between the coordinates of the points.

⁵⁴Let a covering α consist of the sets A_1, \dots, A_s and a covering α' , of the sets A'_1, \dots, A'_t , with $s \neq t$ in the general case. Denote by $\overline{A_i}$ ($\overline{A'_i}$) the part of the set A_i (A'_i) which has no common points with $U(p)$. We say that α' differs from α only in $U(p)$, provided that each $\overline{A_i}$ coincides with some $\overline{A'_i}$ and vice versa.

no less than $n + 1$ (here by $U(p)$ we mean a neighborhood of p relative to Φ , i.e., the intersection of a neighborhood of p in n -dimensional Euclidean space with the set Φ).

The assertion is equivalent to the theorem that we are proving. To verify this, it suffices to demonstrate that all the above conditions characterizing interior points of an n -dimensional simplex are invariant under homeomorphisms. It is readily seen that the notions of multiplicity of a covering and of a “sufficiently small” neighborhood $U(p)$ are topologically invariant. It remains to show that any homeomorphism takes a closed covering of a simplex T to a closed covering of the image $\Phi = \varphi(T)$ of T and vice versa. Since the elements of a closed covering of a simplex, as well as the simplex itself, are closed and bounded sets of Euclidean space, we need only prove the following assertion: the homeomorphic image of a closed and bounded set in n -dimensional Euclidean space is closed and bounded. The proof is very simple: the boundedness of the image is immediate from the Weierstrass Theorem claiming that a continuous function given on a closed and bounded set is bounded above and below. The fact that the image is closed follows from the Bolzano–Weierstrass Theorem. (Let m be an accumulation point⁵⁵ of the image. To a sequence of points converging to m in the image corresponds an infinite sequence of points in the preimage, and this sequence has an accumulation point by the Bolzano–Weierstrass Theorem. By the continuity of homeomorphisms, this accumulation point is mapped into the point m , and therefore the latter belongs to the image. Thus, the image contains all its accumulation points and hence is closed.)

2.9.2 Returning to the property of interior points stated above, we first prove that each noninterior point does not to possess this property. To this end, we need the following lemma:

Lemma 1. *For an arbitrarily small $\varepsilon > 0$, there exists a closed ε -covering of the boundary \tilde{T} of T of multiplicity not exceeding n .*

We begin the proof by indicating an ε -covering of multiplicity $n + 1$ (with ε arbitrarily small) for n -dimensional space E . For $n = 1$, the required covering is constructed by partitioning the real axis into equal intervals. For $n > 1$, the required covering is constructed by induction: n -dimensional space is partitioned into sufficiently thin parallel layers by $(n - 1)$ -dimensional planes. Let us partition one of these planes into equal $(n - 1)$ -dimensional cubes in the manner provided by the induction hypothesis and orthogonally project this partition onto the other $(n - 1)$ -dimensional planes. Then each layer is partitioned into n -dimensional parallelepipeds. Shifting neighboring layers in parallel, we reach a position in which each vertex of every parallelepiped lies inside an $(n - 1)$ -dimensional face of a parallelepiped of the neighboring layer

⁵⁵ A point m is an *accumulation point* of a set M if each neighborhood of m contains infinitely many points of M other than m .

(the case $n = 2$ is shown in Fig. 72). Then, obviously, the multiplicity of the covering of the space by such parallelepipeds is greater by 1 than the multiplicity of the covering of the $(n - 1)$ -dimensional plane and thus equals $n + 1$.

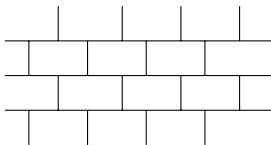


Fig. 72

Now, let T be an arbitrary n -dimensional simplex in E . Consider the constructed ε -covering of E . It is seen from its construction that the distances between the points of E simultaneously belonging to $n + 1$ cubes of the covering are greater than some positive number δ .

Suppose that there are such points on the boundary \tilde{T} of T . Choose a direction in E which is not parallel to any of the $(n - 1)$ -dimensional faces of T and perform a small parallel shift of T in this direction so that none of the points of E belonging to $n + 1$ cubes of the partition and not contained in \tilde{T} is not shifted into \tilde{T} . (Points not lying on \tilde{T} while belonging to $n + 1$ cubes of the covering cannot be arbitrarily close to \tilde{T} ; otherwise, they would have an accumulation point and hence, among them, there would exist points arbitrarily close to each other.) Moreover, since the direction of the shift is chosen so that each point that belonged to \tilde{T} is shifted outside \tilde{T} , after the shift there are no points on \tilde{T} that can possibly belong to $n + 1$ cubes of the covering of E . But then the intersections of the cubes of the covering with \tilde{T} , being closed sets (because the intersection of any number of closed sets is itself closed) of diameter $< \varepsilon$, form the required covering of the boundary \tilde{T} , completing the proof of the lemma.

Suppose that p is not an interior point of Φ . Consider a closed covering $\alpha = \{A_1, \dots, A_s\}$ of multiplicity $n + 1$ of Φ . Assume that p is the only point of Φ belonging to $n + 1$ sets A_i ; let us show that we can construct another closed covering α' of multiplicity n which coincides with α everywhere except in an arbitrarily small neighborhood $U(p)$ of the point p (here $U(p)$ is a neighborhood relative to Φ).

Construct an n -dimensional simplex T whose interior contains the point p and whose intersection with Φ is contained in $U(p)$. Since p is the only point of Φ belonging to $n + 1$ sets A_i , none of the points of the boundary \tilde{T} of T belongs to more than n sets A_i . It follows that, for a sufficiently small $\varepsilon > 0$, each set M of diameter $< \varepsilon$ which lies in \tilde{T} has common points with at most n sets A_i . Indeed, if this is not so, then we can choose a sequence of positive numbers ε_j ($j = 1, 2, \dots$) converging to zero and, for each ε_j ,

find a set M_j on \tilde{T} , of diameter $< \varepsilon_j$, having common points with $n + 1$ sets $A_{i_1}, \dots, A_{i_{n+1}}$. Since finitely many sets A_i may form only finitely many different combinations of $n + 1$ sets, we can choose $n + 1$ sets A^1, \dots, A^{n+1} for which there are arbitrarily large indices j such that all the sets A^1, \dots, A^{n+1} have common points with the sets M_j , and consequently for each such index j there exists a point b_j that belongs to \tilde{T} and whose distance from each of the sets A^1, \dots, A^{n+1} does not exceed ε_j .⁵⁶ Since all the points b_j lie in a bounded part of the space (namely, in \tilde{T}), the Bolzano-Weierstrass Theorem implies that the set of points b_j has an accumulation point b in \tilde{T} . In an arbitrarily small neighborhood of b , there are points b_j having arbitrarily large indices and therefore arbitrarily close to each of the sets A^1, \dots, A^{n+1} . Hence, b adheres to each of the sets A^1, \dots, A^{n+1} . Since these sets are closed, b belongs simultaneously to all of them, which is excluded by assumption.

Thus, we can choose an $\varepsilon > 0$ such that the subsets of \tilde{T} of diameter less than ε cannot simultaneously intersect more than n sets A_i . Using Lemma 1, let us construct an $\frac{\varepsilon}{2}$ -covering $\beta = \{B_1, \dots, B_m\}$ of the boundary \tilde{T} of the simplex T . Since p is not an interior point of Φ , there is a point O inside T which does not belong to Φ . Consider the join of the set B_j ($j = 1, \dots, m$) with the point O , i.e., the set B'_j which contains the corresponding set B_j and all the straight line segments (with their endpoints included) that connect the points of the set B_j to the point O .

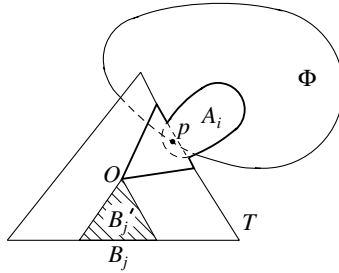


Fig. 73

Now, we construct a closed covering α_β of the set Φ and the simplex T as follows: first, we replace each set A_i by the part of A_i not containing interior points of the simplex T (if A_i lies entirely inside T , then we simply exclude it); obviously, each A_i is thus replaced by some closed set A''_i . We replace the part of the covering inside the simplex T by the covering of T by the “pyramids” B'_j (Fig. 73). Next, we “glue” the pyramids B'_j to the sets A_i according to the following rule: if B'_j has common points with some sets A''_i , then we glue B'_j to one and only one of these sets; if B'_j has common points

⁵⁶The distance from a point to a set is the greatest lower bound for the distances from the point to the points of the set.

with none of the sets A_i'' , then we leave B_j' without changes. Afterwards, we define the covering α_β as follows: the elements of the covering are the sets obtained from the sets A_j'' by taking the union of A_j'' and all the sets B_j' glued to A_j'' (if nothing is glued to A_j'' , then A_j'' is included in the covering α_β without changes) and also the sets B_j' that are not glued at all (Fig. 73).

We assert that each point other than O belongs to at most n sets of the covering α_β . Indeed, an arbitrary point lying beyond T can belong to $n+1$ sets in α_β only if it belongs to $n+1$ corresponding sets in α , which is impossible since the point p lies inside T . Indeed if such a point b existed inside T , its projection from O to \tilde{T} would simultaneously belong to $n+1$ sets B_j , which cannot possibly occur. It remains to consider the points of \tilde{T} . Let a be such a point. Since the diameter of each set B_j is less than $\varepsilon/2$, the diameter of the union of all sets B_j containing the point a (by construction, there are at most n such sets) is less than ε . Therefore, this union has common points with at most n sets A_i'' . Thus, the number of pyramids B_j' containing the point a and the number of sets A_j'' to which the pyramids can be glued in constructing the covering α_β is less than n . But since each pyramid B_j' is glued to at most one set A_j'' , the number of sets of the covering α_β which contain the point a is also at most n .

Now, replacing each set of the covering α_β by its intersection with Φ , we obtain a covering α' of Φ which differs from the original covering α only inside $U(p)$ and is of multiplicity not exceeding n (since the point O lies outside of Φ and is excluded from our considerations when we pass from α_β to α').

We have thus established that our test for interior points is sufficient, since it obviously fails for points that are not interior. It remains to prove that the test is necessary, i.e., it holds for every interior point of the set Φ .

2.9.3 So, let p be an interior point. Take a simplex T in Φ whose interior contains the point p . Draw the $(n-1)$ -dimensional planes through p parallel to the $(n-1)$ -dimensional faces of T .

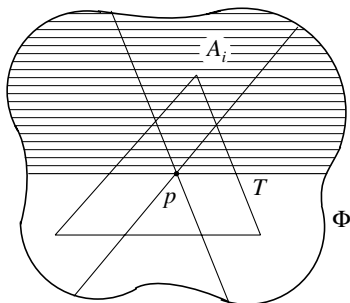


Fig. 74

Each such plane divides the space, and the set Φ as well, into two parts. One of these parts contains exactly one vertex of the simplex T , while the other contains all the remaining vertices. Denote by A_1, \dots, A_{n+1} those parts which contain exactly one vertex of the simplex. These sets obviously form a closed covering α of the set Φ and p is the only point of Φ which belongs to $n+1$ sets of the covering (Fig. 74). Let us prove that if a neighborhood $U(p)$ of p is so small that it is disjoint from the boundary of the simplex T , then it is impossible to obtain a covering of lesser multiplicity by modifying α in $U(p)$.

Suppose that the covering α has been changed in $U(p)$ somehow. In the general case, the change consists in replacing the sets A_i by other sets coinciding with the former outside $U(p)$ and, possibly, in adding new sets that lie entirely in $U(p)$. Adjoin each of these “new” sets to one of the deformed sets A_i and denote the sets obtained in this way by \tilde{A}_i . We now show that there is a point simultaneously belonging to all the sets \tilde{A}_i . To this end, we make use of the following lemma:

The Sperner Lemma. *Let K be an arbitrary triangulation⁵⁷ of an n -dimensional simplex T with vertices e_1, e_2, \dots, e_{n+1} . Suppose that to each vertex e'_k of the triangulation K we assign a vertex $S(e'_k)$ of the simplex T which lies on a face of T containing e'_k and having the least possible dimension (the simplex itself is its own n -dimensional face; therefore, to an interior point e'_k there may correspond an arbitrary vertex of the simplex). Then there is an n -dimensional simplex T_i of K whose vertices are mapped to different (and hence all) vertices of the simplex T .*

A simplex T_i whose vertices are mapped into different vertices of the simplex T is said to be *normal*. The lemma will be proved, if we establish that the number of normal simplices is odd. We prove this by induction on n . For $n=0$ the assertion is trivial. Assume it valid for $n-1$. Take the face $|e_1 \dots e_n|$ of T with vertices e_1, \dots, e_n . Given a simplex T_i , an $(n-1)$ -dimensional face of T_i whose vertices are mapped into e_1, \dots, e_n will be called *distinguished*. Obviously, if a simplex T_i has distinguished faces but is not normal, then T_i has exactly two such faces. For this reason, were the number of normal simplices even, the total number of distinguished faces of all simplices T_i would be even as well. Since each distinguished face lying inside T belongs to two simplices, the total number of such faces is also even. It follows that the number of distinguished faces lying on $(n-1)$ -dimensional faces of the simplex T would be even, too. By the definition of the mapping S , such faces may only lie on faces with vertices e_1, \dots, e_n . However, this is impossible by the induction hypothesis, since the distinguished faces lying on the face

⁵⁷A *triangulation* of a simplex T is a partition of T into simplices T_i such that the intersection of two simplices T_i and T_j is a face (of any dimension) of each of them, or the empty set. The vertices of the simplices T_i are called vertices of the triangulation.

$|e_1, \dots, e_n|$ play the role of normal simplices under the restriction of the mapping S to the face $|e_1, \dots, e_n|$. This completes the proof of the lemma.

We now prove that the sets \tilde{A}_i have a common point. To this end, take an arbitrary triangulation K of our simplex T . Define some mapping from the vertices of the triangulation K to the vertices of the simplex T , satisfying the condition that a vertex of T and the corresponding vertex of the triangulation belong to the same set \tilde{A}_i . By construction, the set \tilde{A}_i containing a given vertex has no points in common with the opposite face; therefore, the mapping S satisfies the conditions of the Sperner Lemma and consequently there is a simplex whose vertices, being mapped into different vertices of T , belong to all the sets $\tilde{A}_1, \dots, \tilde{A}_{n+1}$ (because each vertex of T belongs to exactly one set \tilde{A}_i). Since the triangulation K can be arbitrarily fine, from the fact that the sets \tilde{A}_i are closed we easily infer that they have a common point. However, by passing to the sets \tilde{A}_i , we can only diminish the multiplicity of the covering. Therefore, the original covering obtained from α by an arbitrary change in the neighborhood $U(p)$ has multiplicity $\geq n + 1$. This concludes the proof of the theorem.



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