7

Serre's Splitting Theorem and Lindel's Proof of Serre's Conjecture

In this chapter all rings are supposed to be *commutative*. We shall present a proof of the Quillen-Suslin theorem due to H. Lindel which only uses the concept of the Krull dimension of a noetherian ring, and its property that it increases by one for a polynomial extension in one variable, and that it decreases by one if we go modulo a non-zero divisor.

7.1 Serre's Splitting Theorem

In Section 1.1 we had proved the Prime Avoidance Lemma: *Let R be a ring, $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r \in \operatorname{Spec}(R)$, I an ideal of R and $x \in R$. If $x + I \subset \bigcup_{i=1}^r \mathfrak{p}_i$, then $(x, I) \subset \mathfrak{p}_{i_0}$ for some i_0.*

The above statement implies

Lemma 7.1.1 *Let $(a_1, \dots, a_n) \not\subset \mathfrak{p}_i$, for prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$. Then there exist $b_2, \dots, b_n \in R$ such that $c = a_1 + b_2 a_2 + \cdots + b_n a_n \notin \bigcup_{i=1}^r \mathfrak{p}_i$.*

Indeed, set $x = a_1$ and $I = (a_2, \dots, a_n)$. $\square$

Proposition 7.1.2 *Let A be a Noetherian ring and $I \subset A$ an ideal of height $\geq n$ generated by n elements $a_1, \dots, a_n \in A$. Then there is a matrix $\varepsilon \in \mathrm{E}_n(A)$ such that the n-tuple $(b_1, \dots, b_n) := (a_1, \dots, a_n)\varepsilon$ generates I, and satisfies*

(1) $b_i = a_i + \sum_{j>i} c_{ij} a_j$, *for all* $1 \leq i \leq n$, *for some* $c_{ij} \in A$,

(2) $\operatorname{ht}(b_1, \dots, b_i) = i$, *for* $1 \leq i \leq n$.

In particular, if $n \geq \dim(A) + 2$, then $(a_1, \dots, a_n)$ is completable to a matrix of $\mathrm{E}_n(A)$.

Proof. By the above lemma we can find $c_{12} \dots, c_{1n}$ in A, such that the element $b_1 = a_1 + c_{12}a_2 + \dots + c_{1n}a_n$ does not belong to any of the minimal prime ideals of A. Applying the above lemma again we can find $c_{21}, c_{23}, \dots, c_{2n}$ in A, such that the element $b_2' = a_2 + c_{21}b_1 + c_{23}a_3 + \dots + c_{2n}a_n$ does not belong to any of the minimal prime overideals of (b_1). But then $b_2 := b_2' - c_{21}b_1 = a_2 + c_{23}a_3 + \dots + c_{2n}a_n$ also does not belong to any of the minimal prime overideals of (b_1). In particular $\mathrm{ht}(b_1, b_2) = 2$. Proceeding as above we can obtain a set of generators as required.

Now to the last sentence. Clearly $(a_1, \dots, a_n) \sim_{\mathrm{E}_n(A)} (b_1, \dots, b_n)$. Let $d := \dim(A)$. Then $d + 1 < n$. Since $\mathrm{ht}(b_1, \dots, b_{d+1}) > \dim(A)$, already $(b_1, \dots, b_{d+1})$ is unimodular. So $(b_1, \dots, b_{d+1}, \dots, b_n)$ is 'elementarily' completable, since a proper subrow is unimodular. □

Remark 7.1.3 From Proposition 7.1.2 we can conclude that if $(a_1, \dots, a_r, s)$ is in $\mathrm{Um}_{r+1}(A)$ then there are elements $c_1, \dots, c_r \in A$ such that $\mathrm{ht}(a_1 + sc_1, \dots, a_r + sc_r) \geq r$.

Let us illustate this in the special case when $r = 2$; the general case is similarly done. By Proposition 7.1.2 there are $\lambda, \mu_1, \mu_2 \in A$ such that the ideal $(a_1 + \lambda a_2 + \mu_1 s,\ a_2 + \mu_2 s)$ has height ≥ 2. Note that we can write $a_1 + \lambda a_2 + \mu_1 s = a_1 + \lambda(a_2 + \mu_2 s) + \mu_1' s$. But then $\mathrm{ht}(a_1 + \mu_1' s,\ a_2 + \mu_2 s) \geq 2$.

Lemma 7.1.4 *Let $a_1, \dots, a_n$, $s \in A$. Then there are elements $c_i \in A$, $1 \leq i \leq n$ such that $\mathrm{ht}(a_1 + sc_1, \dots, a_i + sc_i)A_s \geq i$ for $1 \leq i \leq n$ in the ring A_s.*

(If I is an ideal of A, by IA_s we clearly denote the ideal of A_s which is generated by the image of I.)

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be those minimal prime ideals of A which do not contain s. Since $s \notin \mathfrak{p}_i$ for all i, we conclude that $a_1 + sA \not\subset \bigcup_{i=1}^m \mathfrak{p}_i$, i.e. there is a $c_1 \in A$ such that $(a_1 + sc_1)A_s$ has height ≥ 1.

Let $\mathfrak{p}_1', \dots, \mathfrak{p}_{m'}'$ be those minimal prime overideals of $(a_1 + sc_1)$ which do not contain s. Again, as above, $a_2 + sA \not\subset \bigcup \mathfrak{p}_i'$, i.e. there is a $c_2 \in A$ such that $a_2 + sc_2 \notin \bigcup \mathfrak{p}_i'$ whence $\mathrm{ht}(a_1 + sc_1, a_2 + sc_2)A_s \geq 2$.

Continuing thus we get $\mathrm{ht}(a_1 + sc_1, \dots, a_i + sc_i)A_s \geq i$ for all $1 \leq i \leq n$. □

We shall 'globalize' the above lemma next.

Definitions 7.1.5 *Let M be an A-module and $z \in M$.*

a) *z is called* **unimodular** *if Az is a direct summand of M and* $\mathrm{Ann}(z) = (0)$. *The set of all unimodular elements of M is denoted by* $\mathrm{Um}(M)$.

b) *The* **order ideal** *$\mathcal{O}_M(z)$ of z is defined by*

$$\mathcal{O}_M(z) := \{\varphi(z) \mid \varphi \in \mathrm{Hom}_A(M, A)\}.$$

Clearly $\mathcal{O}_M(z)$ is an ideal. And z is unimodular, if and only if $\mathcal{O}_M(z) = A$.

Remarks 7.1.6 a) If $S \subset A$ is multiplicative and M finitely presented, then $S^{-1}\mathcal{O}_M(z) = \mathcal{O}_{S^{-1}M}(z_S)$.

This follows from the fact that $S^{-1}\mathrm{Hom}_A(M, A) \cong \mathrm{Hom}_{A_S}(M_S, A_S)$ canonically, if M is finitely presented.

b) Let $f = (f_1, \dots, f_n) \in A^n =: F$. Then $\mathcal{O}_F(f) = \sum_{i=1}^n Af_i$.

Lemma 7.1.7 *Let P be a projective A-module of rank r and $p \in P$. Let further $s \in A$ such that P_s is free. Then there exists a $q \in P$ such that $\mathrm{ht}\mathcal{O}_P(p + sq)A_s \geq r$. In particular, if $\overline{p} \in \mathrm{Um}(P/sP)$ over $A/(s)$, then $\mathrm{ht}\mathcal{O}_P(p + sq) \geq r$.*

Proof. Let $p_1, \dots, p_r \in P$ such that their images in P_s make up a basis of P_s over A_s, and write

$$p_s = \sum_{i=1}^{r} \frac{a_i}{s^n} \cdot \frac{p_i}{1} \quad \text{with} \quad a_i \in A,\ n \in \mathbb{N}.$$

We may assume $n > 0$. By Lemma 7.1.4 there are elements $c_i \in A$ such that $\mathrm{ht}(a_1 + c_1 s^n, \dots, a_r + c_r s^n)A_s \geq r$. Let $q := s^{n-1} \sum c_i p_i$.

By the above remarks

$$\mathcal{O}_P(p + sq)A_s = \mathcal{O}_{P_s}((p + sq)_s) \supset (a_1 + c_1 s^n, \dots, a_r + c_r s^n)A_s,$$

which implies $\mathrm{ht}\mathcal{O}_P(p + sq)A_s \geq r$.

If $\overline{p} \in \mathrm{Um}(P/sP)$ then $1 + sx' \in \mathcal{O}_P(p)$ for some $x' \in A$, whence $1 + sx \in \mathcal{O}_P(p + sq)$ for some $x \in A$. Therefore, if a prime ideal $\mathfrak{p} \supset \mathcal{O}_P(p + sq)$ then $s \notin \mathfrak{p}$. Hence $\mathrm{ht}\mathcal{O}_P(p + sq) = \mathrm{ht}\mathcal{O}_P(p + sq)A_s \geq r$. □

Theorem 7.1.8 (Serre's Splitting Theorem) *Let A be a commutative Noetherian ring of finite Krull dimension. Let P be a finitely generated projective A-module of rank $>$ dim A. Then P has a unimodular element. Moreover, if $s \in A$ such that P_s is free and $(p, s) \in \mathrm{Um}(P \oplus A)$ then there is a $q \in P$ such that $p + sq \in \mathrm{Um}(P)$.*

Proof 1. (H. Lindel): We may assume that A is a reduced ring with connected spectrum, whence rk P is constant. We prove the result by induction on dim A. If dim $A = 0$, then A is a finite direct product of fields and P is free. The second assertion is then easily verified.

If S is the multiplicatively closed subset of all non-zero-divisors of A then by Corollary 6.4.6 the ring $S^{-1}A$ is a finite direct product of fields. Hence $S^{-1}P$ is free; so P_s is free for some $s \in S$. Since $\dim(A/(s)) < \dim\ A$, by the induction hypothesis $\mathrm{Um}(P/sP) \neq \emptyset$. Let $\overline{p} \in \mathrm{Um}(P/sP)$ where $\overline{p}$ is the residue class of $p \in P$. By Lemma 7.1.7 there is a $q \in P$ such that $\mathrm{ht}\ \mathcal{O}_P(p + sq) \geq \mathrm{rk}\ P$. By $\mathrm{rk}\ P > \dim\ A$ we see $p + sq \in \mathrm{Um}(P)$. □

Proof 2. (R. A. Rao): We sketch the idea of the proof. As before we may assume that the ring A is reduced. View the ring A as a fibre product

$$A = A_s \times_{A_{sT}} A_T = \{(x, y) \in A_s \times A_T \ \mid \ x_T = y_s\},$$

where $T = 1 + sA$. (It is an easy check that the fibre product is equal to A.) Similarly, the projective module P is the fibre product

$$P_s \times_{A_{sT}} P_T = \{(m, n) \in P_s \times P_T \mid m_T = n_s\}.$$

We use induction on the dimension of A.

We choose a non-zero-divisor s so that P_s is free. Let $p_1 \in \mathrm{Um}(P_s)$, and let $p_2 \in \mathrm{Um}(P_T)$ which will exist by induction. Suppose that there is an $\alpha \in \mathrm{Aut}(P_{sT})$ such that

1. $(p_1)_T\alpha = (p_2)_s$,
2. $\alpha = (\alpha_1)_T(\alpha_2)_s$, $\alpha_1 \in \mathrm{Aut}(P_s)$, $\alpha_2 \in \mathrm{Aut}(P_T)$.

Then $(p_1\alpha_1)_T = (p_2\alpha_1)_s$, and so $p = (p_1\alpha_1, p_2\alpha_2^{-1}) \in P_s \times_{I_{sT}} P_T = P$. Since p is "locally unimodular" it is unimodular.

To check the existence of an α with the desired properties one observes that by the general position arguments there is an elementary matrix $\varepsilon \in \mathrm{E}_r(A_{sT})$ such that $(p_1)_T\varepsilon = (p_2)_s$. So we only have to verify that the second property holds.

It is well known that the elementary matrices have the desired 'splitting property'. More generally, one can show that

Proposition 7.1.9 *Let $s, t \in R$ with $(s, t) = 1$. Let P be a projective R-module such that P_{st} is free of rank n, and let $\varepsilon \in \mathrm{E}_n(R_{st})$ (regarded as a subset of $\mathrm{Aut}(P_{st}) \cong \mathrm{Aut}(R^n_{st}) = \mathrm{GL}_n(R_{st})$.) Then $\varepsilon = (\varepsilon_1)_t(\varepsilon_2)_s$, for some $\varepsilon_1 \in \mathrm{Aut}(P_s)$, $\varepsilon_2 \in \mathrm{Aut}(P_t)$.* □

This follows from the fact that elementary matrices are homotopic to the identity. For then one can apply Quillen's Splitting Lemma 4.3.8 to a homotopy.

Corollary 7.1.10 *Let R be a one-dimensional Noetherian ring and P a finitely generated projective R-module of rank r. Then $P \cong \bigwedge^r P \oplus R^{r-1}$.*

Proof. By repeatedly using Serre's Splitting Theorem we get $P \cong Q \oplus R^{r-1}$ with Q projective of rank 1. Taking the r-th exterior power will now give us $\bigwedge^r P \cong Q$. $\square$

Let P be a projective R-module. We first describe some automorphisms of $P \oplus R$ called **Flips** which correspond to the elementary transformations of a free module. Let $p, q \in P$, $\varphi \in P^* = \mathrm{Hom}_R(P, R)$, $a \in R$. The following automorphisms of $P \oplus R$ are called Flips:

$$(p, a) \mapsto (p + aq, a),$$

$$(p, a) \mapsto (p, a + \varphi(p)).$$

These have the following nice property: If I is an ideal of A, then every Flip of $(P/IP) \oplus (R/I)$ over the ring R/I can be lifted to one of $P \oplus R$.

As a consequence we now derive the famous Cancellation Theorem of Hyman Bass, which was proved in the early sixties:

Theorem 7.1.11 (Bass' Cancellation Theorem) *Let R be a Noetherian ring of dimension d and P a finitely generated projective R-module of rank $> d$. Then P is "cancellative", i.e. $P \oplus Q \cong P' \oplus Q$, for some finitely generated projective module Q implies that $P \cong P'$. In fact, if $(p, a) \in \mathrm{Um}(P \oplus R)$ then there is a product τ of Flips such that $(p, a)\tau = (0, 1)$.*

Proof. We may assume that R is a reduced ring.

Let $Q \oplus Q' = F$ be free; then $P \oplus F \cong P' \oplus F$. So we may assume Q is free above. Therefore, it will suffice to show that $P \oplus R \cong P' \oplus R$ implies that $P \cong P'$. Let $(p, a) \in P \oplus R$ denote the image of $(0, 1) \in P' \oplus R$. Then $(p, a) \in \mathrm{Um}(P \oplus R)$; i.e. $\mathcal{O}_P(p) + Ra = R$. To show that P is cancellative, it suffices to show that there is an automorphism $\tau \in \mathrm{Aut}(P \oplus R)$ such that $(p, a)\tau = (0, 1)$. For then $P' \cong \mathrm{coker}(p, a) \cong \mathrm{coker}(0, 1) = P$. We show that there is a flip τ satisfying this, by induction on d. If $d = 0$ then R is a finite direct product of fields, and the assertion is easy.

To perform the induction step, let us first recall that by the Splitting Theorem there is a $p_0 \in \mathrm{Um}(P)$. Further there is a non-zero-divisor $s \in R$ such that P_s is free. Since $\dim(R/(s)) < \dim R$, by induction hypothesis by applying a product of flips we can map $(\overline{p}, \overline{a})$ to $(\overline{0}, \overline{1})$. By further flips we map $(\overline{0}, \overline{1})$ first to $(\overline{p_0}, \overline{1})$ and finally to $(\overline{p_0}, \overline{0})$. (The 'overline' denotes 'modulo (s)'.) Then we lift the composition of these flips to get an automorphism τ' such that, if $(p, a)\tau' = (p', a')$ then $\overline{p'} \in \mathrm{Um}(\overline{P})$, and $a' = sr \in sR$.

By Lemma 7.1.7 there are $q \in P$, $n \in \mathbb{N}$ such that $\mathrm{ht}(\mathcal{O}_{P_{sr}}(p' + (sr)^n q) \geq d+1$. Note that if a prime ideal $\mathfrak{p} \supset \mathcal{O}_P(p' + (sr)^n q)$ then $sr \notin \mathfrak{p}$. This is due to the unimodularity of $(p' + (sr)^n q, sr)$! Therefore, any minimal prime overideal of

$\mathcal{O}_P(p'+(sr)^n q)$ does not contain sr. Consequently, $\mathrm{ht}(\mathcal{O}_P(p'+(sr)^n q) \geq d+1$; whence $p'+(sr)^n q \in \mathrm{Um}(P)$.

Now we can perform the following sequence of flips

$$(p', sr) \mapsto (p' + (sr)^n q, sr) \mapsto (p' + (sr)^n q, 1) \mapsto (0, 1).$$

□

7.2 Lindel's Proof of Serre's Conjecture

Theorem 7.2.1 *Let R be a Noetherian ring and P a finitely genrated projective module over the polynomial ring $A := R[X_1, \dots, X_n]$ of rank $>$ dim R. Then P possesses a unimodular element. (cf. Definition 7.1.5.)*

This means that P admits a free direct summand of rank 1. By induction on rk P this implies that P is free, if R is a field. If dim R=1 and Spec(R) is connected, then P splits into a direct sum of a free module and a rank-1-projective one. So it is free, if R is a principal domain. (It is extended from R, if R is a Dedekind ring, more generally, if R is a so called seminormal Noetherian ring of dimension 1. We will not go into this.)

It was asked in the early seventies by H. Bass, whether Theorem 7.2.1 might be true. It was established by S. M. Bhatwadekar and A. Roy in [6], where they used the Quillen-Suslin Theorem to start the induction process on dim(R). Later H. Lindel could do without this, and so gave a new proof of Serre's Conjecture. Here we outline a variant of his argument.

Remarks 7.2.2 a) Let P be a projective A-module and I an ideal of A. Consider $\overline{P} := P/IP$ as an A/I-module and let $a \in \mathcal{O}_{\overline{P}}(\overline{z})$, where $\overline{z}$ denotes the residue class of z. Then there is a $b \in \mathcal{O}_P(z)$ whose residue class in A/I is a.

Namely, since P is projective, one can lift every homomorphism $P/IP \longrightarrow A/I$ to a homomorphism $P \to A$.

b) Recall that, if $f = (f_1, \dots, f_n) \in A^n =: F$, then $\mathcal{O}_F(f) = \sum_{i=1}^n Af_i$. Especially if $\eta : A \to A$ is a ring endomorphism, f as above and $f_\eta := (\eta(f_1), \dots, \eta(f_n))$. Then $\mathcal{O}_F(f_\eta) = \eta(\mathcal{O}_F(f))A$, where we denote by the latter term the ideal of A, generated by $\eta(\mathcal{O}_F(f))$.

Let us apply Remark b) to prove

Lemma 7.2.3 *Let $f(X) = (f_1(X), \dots, f_n(X)) \in R[X]^n$, $a \in R \cap \mathcal{O}_F(f)$. If there is a $g(X) \in R[X]$ with $1 + Xg(X) \in \mathcal{O}_F(f)$, then $f(abX) := (f_1(abX), \dots, f_n(abX))$ is unimodular in $F := R[X]^n$ for every $b \in R$.*

Proof. Clearly the map $\eta : R[X] \to R[X],\ h(X) \mapsto h(abX)$ is an endomorphism of the ring $R[X]$. So by Remark b) we get

$1{+}abXg(abX) = \eta(1{+}Xg(X)) \in \mathcal{O}_F(f(abX)$ and also $a = \eta(a) \in \mathcal{O}_F(f(abX)$.

But a and $1 + abXg(abX)$ generate the unit ideal of $R[X]$. $\square$

Proposition 7.2.4 *Let M be a finitely presented $A[X]$-module, $m \in M$, $s \in A$ such that*

(1) *M_s is free over $A_s[X]$,*

(2) *$(1 + XA[X]) \cap \mathcal{O}_M(m) \neq \emptyset$,*

(3) *$(1 + sA) \cap \mathcal{O}_M(m) \neq \emptyset$.*

Then there is a unimodular $m' \in M$ with $m' \equiv m \pmod{sXM}$.

Proof. By (3) there is an $r \in A$ such that $1 - sr \in \mathcal{O}_M(m)$. We may rename sr by s. Namely, if (1) holds for s it holds for rs; and if the assertion holds for rs it holds for s. So we may assume

$$1 - s \in \mathcal{O}_M(m) \tag{7.1}$$

Fix an identification $M_s = A_s[X]^n$ and let $m_s = (f_1, \dots, f_n)$ with $f_i \in A_s[X]$. For $N \in \mathbb{N}$, consider the endomorphism

$$g(X) \mapsto g((1 - s^N)X) = g(X - s^N X)$$

of $A_s[X]$. We can write

$$\Big(f_1((1 - s^N)X), \dots, f_n((1 - s^N)X)\Big) = (f_1, \dots, f_n) + s^N Xv$$

with some $v \in M_s$. We choose N big enough such that there is a $w \in M$ with $s^N Xv = sXw_s$. We set $m' := m + sXw$. Then

$$m \equiv m' \pmod{sXM} \ \text{ and } \ m'_s = \Big(f_1((1 - s^N)X), \dots, f_n((1 - s^N)X)\Big)$$

Since M is finitely presented, we see $(1 - s)_s \in \mathcal{O}_{M_s}(m_s)$ by Remark 7.1.6 and Relation (7.1). So, using (ii), the lemma, applied to the ring A_s, to $a = (1 - s)_s$ and $ab = (1 - s^N)_s$, gives that m'_s is unimodular in M_s over $A_s[X]$. But this means that $s^r \in \mathcal{O}_M(m')$ for some $r \in \mathbb{N}$.

Finally by (7.1) there is a linear map $\alpha : M \to A$ with $\alpha(m) = 1 - s$. Therefore $\alpha(m') = \alpha(m - sXw) \in 1 + sA$. Together with $s^r \in \mathcal{O}_M(m')$ this implies $1 \in \mathcal{O}_M(m')$, i.e. that m' is unimodular. $\square$

Definition 7.2.5 *Let I be an ideal of a polynomial ring $A[X]$ (in one indeterminate). By $l(I)$ we denote the set consisting of 0 and all leading coefficients of $f \in I \setminus \{0\}$. Obviously $l(I)$ is an ideal of A.*

Lemma 7.2.6 *(H. Bass, A. Suslin)*
Let A be a Noetherian ring and I an ideal of $A[X]$. Then $\mathrm{ht}_A l(I) \geq \mathrm{ht}_{A[X]} I$.

Proof. Let first I be a prime ideal and $\mathfrak{p} = I \cap A$. If $I = \mathfrak{p}[X]$, then clearly $l(I) = \mathfrak{p}$. With $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}[X])$ the lemma follows in this case. If on the other hand $I \neq \mathfrak{p}[X]$, i.e. $I \supsetneq \mathfrak{p}[X]$, let $g \in I \setminus \mathfrak{p}[X]$. Then there is an $h \in \mathfrak{p}[X]$, such that the leading coefficient of $f = g - h$ does not belong to $\mathfrak{p}$. Since $f \in I$ and $\mathfrak{p} \subset l(I)$, we have $\mathrm{ht}\, l(I) \geq \mathrm{ht}(\mathfrak{p}) + 1 = \mathrm{ht}\, I$. (The latter equation is Theorem 6.9.3 b)).

Now let I be arbitrary and $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ the prime ideals of $A[X]$ which are minimal over I. (Their number is finite, since $A[X]$ is Noetherian.) The case $I = A[X]$ being obvious, we may assume $r > 0$. Then $(\prod_i \mathfrak{q}_i)^N \subset I$ for some $N \in \mathbb{N}$. Since apparently $l(I) \cdot l(J) \subset l(IJ)$, one derives $\prod l(\mathfrak{q}_i)^N \subset l(I)$.

Let $\mathfrak{p} \supset l(I)$ be a prime ideal of A with $\mathrm{ht}\, \mathfrak{p} = \mathrm{ht}\, l(I)$. Then $l(\mathfrak{q}_i) \subset \mathfrak{p}$ for some I, and so

$$\mathrm{ht}\ I \leq \mathrm{ht}\, \mathfrak{q}_i \leq \mathrm{ht}\, \mathfrak{p} = \mathrm{ht}\, l(I).$$

□

Proposition 7.2.7 *Let I be an ideal of a polynomial ring $R[X_1, \ldots, X_n]$ with* ht $I >$ dim R. *Then there exists a so called Nagata transformation of variables, fixing X_n, and sending $X_i \mapsto X_i' = X_i + X_n^{r_i}$, for suitable $r_i \in \mathbb{N}$, $1 \leq i \leq n-1$, such that I contains a polynomial that is monic in X_n with coefficients in $R[X_1', \ldots, X_{n-1}']$.*

Proof. Induction on n. The case $n = 1$ follows directly from the Lemma 7.2.6. So assume $n > 1$.

Set $B := R[X_1, \ldots, X_{n-1}]$ and view A as a polynomial ring in one indeterminate X_n over B. Then $l(I) \subset B$ is of height $>$ dim R. By induction hypothesis we may assume, that there is a $g \in l(I)$ which is monic in X_1, i.e. of the form

$$g = X_1^T + g_{T-1} X_1^{T-1} + \cdots + g_0 \quad \text{with} \quad g_i \in R[X_2, \ldots, X_{n-1}].$$

By the definition of $l(I)$, in I there is a polynomial $f \in A$ of the form

$$f = g \cdot X_n^N + b_{N-1} X_n^{N-1} + \cdots + b_0 \quad \text{with} \quad b_i \in B.$$

Let M be the highest power of X_1 occurring in the b_i and let $K \in \mathbb{N}$ be specified later. Then set

$$Y_i := X_i \quad \text{for} \quad i < n \quad \text{and} \quad Y_n := X_n - X_1^K.$$

Now $g \cdot X_n^N = (Y_1^T + g_{T-1} Y_1^{T-1} + \cdots + g_0)(Y_n + Y_1^K)^N$ is monic of degree $T + KN$ in Y_1, whereas Y_1 occurs in $b_{N-1} X_n^{N-1} + \cdots + b_0$ with exponent $\leq M + K(N-1)$. If we choose K sufficiently large, $T + KN > M + K(N-1)$, and so f is monic in Y_1. □

Proof of the theorem: Induction on on n (the number of variables). If $n = 0$, this is Serre's Splitting Theorem 7.1.8. We may assume that R hence $R[X_1, \dots, X_n]$ is a reduced ring with connected spectrum.

Assume first that $R = k$ is a field. Then $R[X_1, \dots, X_n] = k[X_1][X_2, \dots, X_n]$. Then by the induction hypothesis $P \cong L \oplus R[X_1, \dots, X_n]^{r-1}$ where L is a projective $k[X_1, \dots, X_n]$-module of rank 1. Since $k[X_1, \dots, X_n]$ is a factorial, L, and hence P is free.

For general R let S be the set of all non-zero-divisors in R. Since R is reduced and Noetherian, $S^{-1}R$ is a finite direct product of fields. By the case, handled above, $S^{-1}P$ is free. Since P is finitely generated. there is a non-zero-divisor $s \in R$ such that P_s is free.

Consider P/sX_nP over $\Big(R[X_n]/(sX_n)\Big)[X_1, \dots, X_{n-1}]$. And note that P_{sX_n} is free over $R[X_1, \dots, X_n]_{sX_n}$. Since the rank of P/sX_nP over $R[X_1, \dots, X_n]/(sX_n)$ equals that of P and so is bigger than $\dim\, R \geq \dim\, R[X]/(sX_n)$ (*cf.* Theorem 6.9.3 c)), by induction hypothesis there is a $p \in P$, such that its residue class $\overline{p}$ is unimodular in P/sX_nP over $R[X_1, \dots, X_n]/(sX_n)$. Therefore for every $q \in P$ there is an $h \in R[X_1, \dots, X_n]$ with

$$1 + sX_nh \in \mathcal{O}_P(p + sX_nq). \tag{7.2}$$

Now we write $t := sX_n$. By Lemma 7.1.7 there is a $q \in P$ such that $\mathrm{ht}(\mathcal{O}_{P_t}(p + tq)_t) \geq \mathrm{rk}(P)$.

But $\overline{p + tq} = \overline{p}$, and so again by Lemma 7.1.7 we have $\mathrm{ht}(\mathcal{O}_P(p + tq)) \geq \mathrm{rk}(P) > \dim(R)$. Applying Proposition 7.2.7, we see that $\mathcal{O}_P(p + tq)$ contains a monic polynomial $f(X_n) \in R[X'_1, \dots, X'_{n-1}][X_n]$ with coefficients in $A := R[X'_1, \dots, X'_{n-1}]$, for some suitable variables $X'_1, \dots, X'_{n-1}$.

This implies that $A[X_n]/\mathcal{O}_P(p+tq)$ is integral over $A/A \cap \mathcal{O}_P(p+tq)$. So, since s by the relation (7.2) is invertible in the first ring, it is invertible in the second one. This means $(1 + sA) \cap \mathcal{O}_P(p + tq) \neq \emptyset$. So the hypotheses of Proposition 7.2.4 are fulfilled for $M = P$ and $m = p + tq$. Therefore $\mathrm{Um}(P) \neq \emptyset$. $\square$

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