Preface

Besides giving an introduction to Commutative Algebra – the theory of commutative rings – this book is devoted to the study of projective modules and the minimal number of generators of modules and ideals.

The notion of a module over a ring R is a generalization of that of a vector space over a field k. The axioms are identical. But whereas every vector space possesses a basis, a module need not always have one. Modules possessing a basis are called free. So a finitely generated free R-module is of the form R^n for some $n \in \mathbb{N}$, equipped with the usual operations. A module is called projective, iff it is a direct summand of a free one. Especially a finitely generated R-module P is projective iff there is an R-module Q with $P \oplus Q \cong R^n$ for some n. Remarkably enough there do exist nonfree projective modules. Even there are nonfree P such that $P \oplus R^m \cong R^n$ for some m and n. Modules P having the latter property are called stably free. On the other hand there are many rings, all of whose projective modules are free, e.g. local rings and principal ideal domains. (A commutative ring is called local iff it has exactly one maximal ideal.) For two decades it was a challenging problem whether every projective module over the polynomial ring $k[X_1, \dots, X_n]$ with a field k was free. It was known from the beginning that such a module had to be stably free. The statement that it should be actually free was called Serre's Conjecture. This was proved independently by D. Quillen and A. Suslin in 1976. We give several proofs of it.

Later we show how vector bundles over a compact Hausdorff space X (more generally vector bundles of special type over any topological space) can be interpreted as projective modules over the ring of (real, complex or quaternion) continuous functions on X. This gives the concept of projective modules an intuitive meaning. For instance it is no surprise that nontrivial vector bundles exist – at least once one has seen the Möbius band.

In the second half of the book we study the question what one can say about the minimal number of generators of certain ideals. This often – but not always – is connected with the theory of projective modules. We begin with dimension theory on commutative so named Noetherian rings, i.e. such whose ideals are finitely generated. (This property of rings was first identified and studied by E. Noether, a student of D. Hilbert.) Its fundamental theorem states the equality of two numbers: Let R be a local Noetherian ring with maximal ideal $\mathfrak{m}$. Then the minimal number n of elements $a_1, \dots, a_n$ such that $\mathfrak{m}$ is a minimal prime over-ideal of $(a_1, \dots, a_n)$ equals the number of steps of a maximal chain of prime ideals in R, i.e.

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n = \mathfrak{m}$$

with maximal n. This number n is called the (Krull) dimension of R. If $\mathfrak{m}$ itself can be generated by $n = \dim(R)$ elements then R is called a regular local ring. A not necessarily local ring whose ideals are finitely generated is called regular, if all its localizations are so. (To every commutative ring R and every prime ideal $\mathfrak{p}$ of it one can associate in a canonical way a local ring, the localization of R in $\mathfrak{p}$.) An example is $k[X_1, \dots, X_n]$ with a field k. All of its maximal ideals can be generated by n elements.

Other examples are the domains all of whose ideals are projective modules. Namely these are the regular Noetherian domains of dimension ≤ 1, i.e. whose nonzero prime ideals are maximal. These rings are called Dedekind rings. We give a complete classification of the finitely generated modules, especially of the projective ones over those rings. Also we prove the theorem of the finiteness of the class number, which says that over certain 'classical' Dedekind rings there are only finitely many projective modules upto 'stable isomorphy', i.e. upto 'adding free modules'.

The Forster-Swan Theorem gives upper bounds to the number of generators of modules if one knows these numbers for the localizations. As a consequence one sees that an ideal I of $R = k[X_1, \dots, X_n]$ can be generated by $n + 1$ generators if R/I is regular.

Improving an old theorem of Kronecker, in 1972 Eisenbud and Evans and independently Storch have shown that every prime ideal in $k[X_1, \dots, X_n]$ is a minimal prime overideal of an ideal generated by n elements. (In the book we will give a version of this theorem which is not restricted to prime ideals.) Geometrically this means that every set defined by polynomial equations can already be defined by n equations.

The last chapter is dedicated to the question: Under what hypotheses can one describe an algebraic curve in the affine n-space by $n - 1$ equations? i.e. let $\mathfrak{p}$ be a prime ideal of $R = k[X_1, \dots, X_n]$ with $\dim(R/\mathfrak{p}) = 1$. When is $\mathfrak{p}$ a minimal prime overideal of an ideal, generated by $n - 1$ elements? (Also here the good formulation does not restrict to prime ideals.) We show that the answer is positive in the following two cases:

1. $R/\mathfrak{p}$ is regular, i.e. a Dedekind ring. (Mohan Kumar)
2. k is of positive characteristic. (Cowsik and Nori)

The general answer is still not known. That is the reality today.

The 'book' started as a M. Phil. project of the junior author with Selby Jose, in the course of which they found the expository article [106] of Valla. The initiative and determination of the senior author, and the urging of colleagues, led to *Selby's thesis* becoming into a book which could be used as a graduate course or for an intensive workshop.

The whole book tells the story of a philosophy of J-P. Serre and his vision of relating that philosophy to problems in Affine geometry. A thorough development of this subject till the end of 1980 is done in this book. The intermix of Classical Algebraic K-theory and Complete Intersection problems is emphasized in this text for the first time. The results of Eisenbud-Evans, Swan's connection between vector bundles and projective modules, Lindel's proof of Serre's conjecture appear for the first time in a student text form.

The book is almost self-contained, and serves as an introduction to basic Commutative algebra and its applications in problems of affine algebraic geometry. In a first reading, the student could skip Chapter 5; but a better understanding of the subject and its interconnection with other parts of Mathematics can only be had by eventually perusing this chapter. The reader could also skip the Eisenbud-Evans theorem in Chapter 9, §9.4; though he should know that this is the best achieved via *general position arguments*, the recurrent theme.

The material in this text has been crystallized from various books, research and expository articles. Earlier works on this subject are the notes of Geyer, Ohm, Badesču; survey articles by Valla, Lyubeznik, Murthy, Bass, Suslin; and the books of Lam, Szpiro, Kunz, Mandal respectively. Following the tradition in the books written by J-P. Serre, we have not let the exercises interrupt the flow of reading. We have placed them chapter-wise at the very end. The exercises are challenging; the student who wades through them will be very proficient to work in this active research area. This is because most of the exercises are culled from the research papers of experts.

Interesting developments, not found here, include- Effective methods of Sturmfels, etc.; Bose on Serre's conjecture via Groebner bases, and its relation to problems in Electrical Engineering; numerous applications of the Local-Global techniques due to Bass, Suslin, Asanuma, etc; generalizations of Serre's conjecture: the Bass conjectures, the Bass-Quillen conjecture by Lindel, Popescu, and the Anderson conjecture over monoidal rings due to Gubeladze; surprising developments in the orthogonal groups due to Parimala, Sridharan, Ojanguren; analogous results of Suslin and his students in the classical groups; G. Lyubeznik's work in higher dimension which uses the Cowsik-Nori Theorem as the inductive start; Murthy's results on complete intersection questions; and Mohan Kumar's on Eisenbud-Evans conjectures over affine algebras; later developments, inspired by Nori, by Bhatwadekar, Mandal, Raja Sridharan on complete interestion and Euler classes; parallel development on complete intersections over real algebras by Ojanguren, Ischebeck, etc.; Herzog on monomial curves; results of Warfield, Stafford, etc. on development of Serre's conjecture and the Eisenbud-Evans theorem in non-commutative noetherian rings; examples of Macaulay, Moh; projective varieties, and the corresponding problems there; connections with local cohomology started by Faltings, and developed in Lyubeznik's thesis; scheme theoretic and ideal theoretic questions studied by Hartshorne, Abhyankar amongst others; local algebra - 'symbolic primes' results of Cowsik, Huneke, etc.; motivations from algebraic topology; principal G-bundles; ...

We would like to thank many colleagues for their valuable suggestions which encouraged us, and have helped to improve the material presented. We thank our wives for their support. The senior author acknowledges the help from Deutsche Forschungsgemeinschaft (DFG) and T.I.F.R. which made it possible for him to interact with the junior author. The book was finally completed in the idyllic setting at the I.C.T.P. this summer, whom we thank for their hospitality. Finally, we look forward to hearing from you, to improve the material and its presentation.

September 13, 2004

Friedrich Ischebeck
Ravi A. Rao

http://www.springer.com/978-3-540-23032-8

Ideals and Reality
Projective Modules and Number of Generators of Ideals
Ischebeck, F.; Rao, R.A.
2005, XIV, 336 p., Hardcover
ISBN: 978-3-540-23032-8