12. Market LIBOR Models

As was mentioned already, the acronym LIBOR stands for the *London Interbank Offered Rate*. It is the rate of interest offered by banks on deposits from other banks in eurocurrency markets. Also, it is the floating rate commonly used in interest rate swap agreements in international financial markets (in domestic financial markets as the reference interest rate for a floating rate loans it is customary to take a *prime rate* or a *base rate*). LIBOR is determined by trading between banks and changes continuously as economic conditions change. For more information on market conventions related to the LIBOR and Eurodollar futures, we refer to Sect. 9.3.4.

In this chapter, we present an overview of recently developed methodologies related to the arbitrage-free modelling of *market rates,* such as LIBORs. In contrast to more traditional aproaches, term structure models developed recently by, among others, Miltersen et al. (1997), Brace et al. (1997), Musiela and Rutkowski (1997), Jamshidian (1997a), Hunt et al. (1996, 2000), Hunt and Kennedy (1996, 1997, 1998a), and Andersen and Andreasen (2000b), are tailored to handle the most actively traded interest-rate options, such as caps and swaptions. For this reason, they typically enjoy a higher degree of tractability than the classical term structure models based on the diffusion-type behavior of instantaneous (spot or forward) rates.

Recall that the Heath-Jarrow-Morton methodology of term structure modelling is based on the arbitrage-free dynamics of instantaneous, continuously compounded forward rates. The assumption that instantaneous rates exist is not always convenient, since it requires a certain degree of smoothness with respect to the *tenor* (i.e., maturity) of bond prices and their volatilities. An alternative construction of an arbitrage-free family of bond prices, making no reference to the instantaneous, continuously compounded rates, is in some circumstances more suitable.

The first step in this direction was taken by Sandmann and Sondermann (1994), who focused on the effective annual interest rate (cf. Sect. 10.1.2). This idea was further developed by Goldys et al. (1994), Musiela (1994), Sandmann et al. (1995), Miltersen et al. (1997) and Brace et al. (1997).

It is worth pointing out that in all these papers, the HJM framework is adopted (at least implicitly). For instance, Goldys et al. (1994) introduce a HJM-type model based on the rate $j(t,T)$, which is related to the instantaneous forward rate through the formula $1 + j(t,T) = e^{f(t,T)}$. The model put forward in this paper assumes a deterministic volatility function for the process $j(t,T)$. A slightly more general case of nominal annual rates $q(t,T)$, which satisfy (δ representing the duration of each compounding period)

$$(1 + \delta q(t,T))^{1/\delta} = e^{f(t,T)},$$

was studied by Musiela (1994), who assumes the deterministic volatility $\gamma(t,T)$ of each nominal annual rate $q(t,T)$. This implies the following form of the coefficient σ in the dynamics of the instantaneous forward rate

$$\sigma(t,T) = \delta^{-1}\big(1 - e^{-\delta f(t,T)}\big)\gamma(t,T),$$

so that the model is indeed well-defined (that is, instantaneous forward rates, and thus also the nominal annual rates, do not explode). Unfortunately, these models do not give closed-form solutions for zero-coupon bond options, and thus a numerical approach to option pricing is required. Miltersen et al. (1997) focus on the actuarial (or effective) forward rates $a(t,T,U)$ satisfying

$$\big(1 + a(t,T,T+\delta)\big)^{\delta} = \exp\Big(\int_T^{T+\delta} f(t,u)\,du\Big).$$

They show that a closed-form solution for the bond option price is available when $\delta = 1$. More specifically, an *interest rate cap* is priced according to the market standard (see Sect. 12.4.1 for details). However, the model is not explicitly identified and its arbitrage-free features are not examined, thus leaving open the question of pricing other interest rate derivatives. These problems were addressed in part in a paper by Sandmann et al. (1995), where a lognormal-type model based on an add-on forward rate (*add-on yield*) $f_s(t,T,T+\delta)$, where

$$1 + \delta f_s(t,T,T+\delta) = \exp\Big(\int_T^{T+\delta} f(t,u)\,du\Big),$$

was analyzed. Finally, using a different approach, Brace et al. (1997) explicitly identify the dynamics of all rates $f_s(t,T,T+\delta)$ under the martingale measure $\mathbb{P}^*$ and analyze the properties of the model.

Let us summarize the content of this chapter. We start by describing in Sect. 12.1 forward and futures LIBORs. The properties of the LIBOR in the Gaussian HJM model are also dealt with in this section. Subsequently, in Sect. 12.4 we present various approaches to LIBOR market models. Further properties of these models are examined in Sect. 12.5. In Sect. 12.2 we describe interest rate cap and floor agreements. Next, we provide in Sect. 12.3 the valuation results for these contracts within the framework of the Gaussian HJM model. In Sect. 12.6, we deal with the valuation of contingent claims within the framework of the lognormal LIBOR market model. In the last section, we present briefly some extensions of this model.

12.1 Forward and Futures LIBORs

We shall frequently assume that we are given a prespecified collection of reset/settlement dates $0 \leq T_0 < T_1 < \cdots < T_n$, referred to as the *tenor structure.* Also, we shall write $\delta_j = T_j - T_{j-1}$ for $j = 1, \ldots, n$. As usual, $B(t,T)$ stands for the price at time t of a T-maturity zero-coupon bond, $\mathbb{P}^*$ is the spot martingale measure, while $\mathbb{P}_{T_j}$ (respectively, $\mathbb{P}_{T_{j+1}}$) is the forward martingale measure associated with the date T_j (respectively, T_{j+1}). The corresponding Brownian motions are denoted by W^* and W^{T_j} (respectively, $W^{T_{j+1}}$). Also, we write $F_B(t,T,U) = B(t,T)/B(t,U)$. Finally, $\pi_t(X)$ is the value (that is, the arbitrage price) at time t of a European claim X.

Our first task is to examine those properties of interest rate forward and futures contracts that are universal, in the sense that do not rely on specific assumptions imposed on a particular model of the term structure of interest rates. To this end, we fix an index j, and we consider various interest rates related to the period $[T_j, T_{j+1}]$.

12.1.1 One-period Swap Settled in Arrears

Let us first consider a one-period swap agreement settled in arrears; i.e., with the *reset date* T_j and the *settlement date* T_{j+1} (more realistic multi-period swap agreements are examined in Chap. 13). By the contractual features, the long party pays $\delta_{j+1}\kappa$ and receives $B^{-1}(T_j, T_{j+1}) - 1$ at time T_{j+1}. Equivalently, he pays an amount $Y_1 = 1 + \delta_{j+1}\kappa$ and receives $Y_2 = B^{-1}(T_j, T_{j+1})$ at this date. The values of these payoffs at time $t \leq T_j$ are

$$\pi_t(Y_1) = B(t, T_{j+1})\big(1 + \delta_{j+1}\kappa\big), \quad \pi_t(Y_2) = B(t, T_j).$$

The second equality above is trivial, since the payoff Y_2 is equivalent to the unit payoff at time T_j. Consequently, for any fixed $t \leq T_j$, the value of the *forward swap rate* that makes the contract worthless at time t can be found by solving for $\kappa = \kappa_t$ the following equation

$$\pi_t(Y_1) = B(t, T_{j+1})\big(1 + \delta_{j+1}\kappa_t\big) = B(t, T_j) = \pi_t(Y_2).$$

It is thus apparent that

$$\kappa_t = \frac{B(t, T_j) - B(t, T_{j+1})}{\delta_{j+1} B(t, T_{j+1})}, \quad \forall\, t \in [0, T_j].$$

Note that κ_t coincides with the *forward LIBOR* $L(t, T_j)$ which, by convention, is set to satisfy

$$1 + \delta_{j+1} L(t, T_j) \overset{\text{def}}{=} \frac{B(t, T_j)}{B(t, T_{j+1})}.$$

It is also useful to observe that

$$1 + \delta_{j+1} L(t, T_j) = F_B(t, T_j, T_{j+1}) = \mathbb{E}_{\mathbb{P}_{T_{j+1}}}\big(B^{-1}(T_j, T_{j+1}) \,|\, \mathcal{F}_t\big), \qquad (12.1)$$

where the last equality is a consequence of the definition of the forward measure $\mathbb{P}_{T_{j+1}}$. We conclude that in order to determine the forward LIBOR $L(\cdot, T_j)$, it is enough to find the forward price of the claim $B^{-1}(T_j, T_{j+1})$ for the settlement date T_{j+1}. Furthermore, it is evident that the process $L(\cdot, T_j)$ necessarily follows a martingale under the forward probability measure $\mathbb{P}_{T_{j+1}}$. Recall that in the HJM framework, we have

$$dF_B(t, T_j, T_{j+1}) = F_B(t, T_j, T_{j+1})\big(b(t, T_j) - b(t, T_{j+1})\big) \cdot dW_t^{T_{j+1}} \qquad (12.2)$$

under $\mathbb{P}_{T_{j+1}}$, where $b(\cdot, T)$ is the price volatility of the T-maturity zero-coupon bond. On the other hand, $L(\cdot, T_j)$ can be shown to admit the following representation

$$dL(t, T_j) = L(t, T_j)\lambda(t, T_j) \cdot dW_t^{T_{j+1}}$$

for a certain adapted process $\lambda(\cdot, T_j)$. Combining the last two formulas with (12.1), we arrive at the following fundamental relationship

$$\frac{\delta_{j+1} L(t, T_j)}{1 + \delta_{j+1} L(t, T_j)}\, \lambda(t, T_j) = b(t, T_j) - b(t, T_{j+1}), \quad \forall\, t \in [0, T_j]. \qquad (12.3)$$

It is worth stressing that equality (12.3) will play an essential role in the construction of the so-called *lognormal LIBOR market model*. For instance, in the construction based on the backward induction, relationship (12.3) will allow us to specify uniquely the forward measure for the date T_j, provided that $\mathbb{P}_{T_{j+1}}$, $W^{T_{j+1}}$ and the volatility $\lambda(t, T_j)$ are known (we may postulate, for instance, that $\lambda(\cdot, T_j)$ is a given deterministic function).

Recall that in the HJM framework the Radon-Nikodým density of $\mathbb{P}_{T_j}$ with respect to $\mathbb{P}_{T_{j+1}}$ is known to satisfy

$$\frac{d\mathbb{P}_{T_j}}{d\mathbb{P}_{T_{j+1}}} = \mathcal{E}_{T_j}\Big(\int_0^{\cdot} \big(b(t, T_j) - b(t, T_{j+1})\big) \cdot dW_t^{T_{j+1}}\Big). \qquad (12.4)$$

In view of (12.3), we thus have

$$\frac{d\mathbb{P}_{T_j}}{d\mathbb{P}_{T_{j+1}}} = \mathcal{E}_{T_j}\Big(\int_0^{\cdot} \frac{\delta_{j+1} L(t, T_j)}{1 + \delta_{j+1} L(t, T_j)}\, \lambda(t, T_j) \cdot dW_t^{T_{j+1}}\Big).$$

For our further purposes, it is also useful to observe that this density admits the following representation

$$\frac{d\mathbb{P}_{T_j}}{d\mathbb{P}_{T_{j+1}}} = cF_B(T_j, T_j, T_{j+1}) = c\big(1 + \delta_{j+1} L(T_j, T_j)\big), \quad \mathbb{P}_{T_{j+1}}\text{-a.s.}, \qquad (12.5)$$

where $c > 0$ is a normalizing constant, and thus we have that

$$\frac{d\mathbb{P}_{T_j}}{d\mathbb{P}_{T_{j+1}}}\,|\mathcal{F}_t = cF_B(t, T_j, T_{j+1}) = c\big(1 + \delta_{j+1} L(t, T_j)\big), \quad \mathbb{P}_{T_{j+1}}\text{-a.s.},$$

for any date $t \le T_j$.

Finally, the dynamics of the process $L(\cdot, T_j)$ under the probability measure $\mathbb{P}_{T_j}$ are given by a somewhat involved stochastic differential equation

$$dL(t,T_j) = L(t,T_j)\left(\frac{\delta_{j+1}L(t,T_j)|\lambda(t,T_j)|^2}{1+\delta_{j+1}L(t,T_j)}\,dt + \lambda(t,T_j)\cdot dW_t^{T_j}\right).$$

As we shall see in what follows, it is nevertheless not hard to determine the probability law of $L(\cdot, T_j)$ under the forward measure $\mathbb{P}_{T_j}$ – at least in the case of the deterministic volatility function $\lambda(\cdot, T_j)$.

12.1.2 One-period Swap Settled in Advance

Consider now a similar swap that is, however, settled in advance – that is, at time T_j. Our first goal is to determine the forward swap rate implied by such a contract. Note that under the present assumptions, the long party (formally) pays an amount $Y_1 = 1 + \delta_{j+1}\kappa$ and receives $Y_2 = B^{-1}(T_j, T_{j+1})$ at the settlement date T_j (which coincides here with the reset date). The values of these payoffs at time $t \le T_j$ admit the following representations

$$\pi_t(Y_1) = B(t,T_j)\big(1+\delta_{j+1}\kappa\big),$$

and

$$\pi_t(Y_2) = B(t,T_j)\mathbb{E}_{\mathbb{P}_{T_j}}(B^{-1}(T_j,T_{j+1})\,|\,\mathcal{F}_t).$$

The value $\kappa = \tilde{\kappa}_t$ of the *modified forward swap rate* that makes the swap agreement settled in advance worthless at time t can be found from the equality $\pi_t(Y_1) = \pi_t(Y_2)$, where

$$\pi_t(Y_1) = B(t,T_j)\big(1+\delta_{j+1}\kappa\big)$$

and

$$\pi_t(Y_2) = B(t,T_j)\mathbb{E}_{\mathbb{P}_{T_j}}(B^{-1}(T_j,T_{j+1})\,|\,\mathcal{F}_t).$$

It is clear that

$$\tilde{\kappa}_t = \delta_{j+1}^{-1}\big(\mathbb{E}_{\mathbb{P}_{T_j}}(B^{-1}(T_j,T_{j+1})\,|\,\mathcal{F}_t) - 1\big).$$

We are in a position to introduce the *modified forward LIBOR* $\tilde{L}(t, T_j)$ by setting

$$\tilde{L}(t,T_j) \stackrel{\text{def}}{=} \delta_{j+1}^{-1}\big(\mathbb{E}_{\mathbb{P}_{T_j}}(B^{-1}(T_j,T_{j+1})\,|\,\mathcal{F}_t) - 1\big), \quad \forall\, t \in [0,T_j].$$

Let us make two remarks. First, it is clear that finding the modified LIBOR $\tilde{L}(\cdot, T_j)$ is essentially equivalent to pricing the claim $B^{-1}(T_j, T_{j+1})$ at T_j (more precisely, we need to know the forward price of this claim for the date T_j). Second, it is useful to observe that

$$\tilde{L}(t,T_j) = \mathbb{E}_{\mathbb{P}_{T_j}}\left(\frac{1-B(T_j,T_{j+1})}{\delta_{j+1}B(T_j,T_{j+1})}\,\Big|\,\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}_{T_j}}(L(T_j,T_j)\,|\,\mathcal{F}_t).$$

In particular, it is evident that at the reset date T_j the two forward LIBORs introduced above coincide, since manifestly

$$\tilde{L}(T_j, T_j) = \frac{1 - B(T_j, T_{j+1})}{\delta_{j+1} B(T_j, T_{j+1})} = L(T_j, T_j).$$

To summarize, the standard forward LIBOR $L(\cdot, T_j)$ satisfies

$$L(t, T_j) = \mathbb{E}_{\mathbb{P}_{T_{j+1}}}(L(T_j, T_j) \,|\, \mathcal{F}_t), \quad \forall\, t \in [0, T_j],$$

with the initial condition

$$L(0, T_j) = \frac{B(0, T_j) - B(0, T_{j+1})}{\delta_{j+1} B(0, T_{j+1})},$$

while for the modified LIBOR $\tilde{L}(\cdot, T_j)$ we have

$$\tilde{L}(t, T_j) = \mathbb{E}_{\mathbb{P}_{T_j}}(\tilde{L}(T_j, T_j) \,|\, \mathcal{F}_t), \quad \forall\, t \in [0, T_j],$$

with the initial condition

$$\tilde{L}(0, T_j) = \delta_{j+1}^{-1} \big(\mathbb{E}_{\mathbb{P}_{T_j}}(B^{-1}(T_j, T_{j+1})) - 1 \big).$$

Note that the last condition depends not only on the initial term structure, but also on the volatilities of bond prices (see, e.g., formula (12.11) below).

12.1.3 Eurodollar Futures

As was mentioned in Sect. 9.3, *Eurodollar futures contract* is a futures contract in which the LIBOR plays the role of an underlying asset. By convention, at the contract's maturity date T_j, the quoted Eurodollar futures price $E(T_j, T_j)$ is set to satisfy

$$E(T_j, T_j) \stackrel{\text{def}}{=} 1 - \delta_{j+1} L(T_j).$$

Equivalently, in terms of the price of a zero-coupon bond we have $E(T_j, T_j) = 2 - B^{-1}(T_j, T_{j+1})$. From the general properties of futures contracts, it follows that the Eurodollar futures price at time $t \leq T_j$ equals

$$E(t, T_j) \stackrel{\text{def}}{=} \mathbb{E}_{\mathbb{P}^*}(E(T_j, T_j)) = 1 - \delta_{j+1} \mathbb{E}_{\mathbb{P}^*} \big(L(T_j, T_j) \,|\, \mathcal{F}_t \big)$$

and thus

$$E(t, T_j) = 2 - \mathbb{E}_{\mathbb{P}^*} \big(B^{-1}(T_j, T_{j+1}) \,|\, \mathcal{F}_t \big). \tag{12.6}$$

Recall that the probability measure $\mathbb{P}^*$ represents the spot martingale measure in a given model of the term structure. It seems natural to introduce the concept of the *futures LIBOR,* associated with the Eurodollar futures contract, through the following definition.

Definition 12.1.1. Let $E(t,T_j)$ be the Eurodollar futures price at time t for the settlement date T_j. The implied *futures LIBOR* $L^f(t,T_j)$ satisfies

$$E(t,T_j) = 1 - \delta_{j+1} L^f(t,T_j), \quad \forall\, t \in [0,T_j]. \tag{12.7}$$

It follows immediately from (12.6)-(12.7) that the following equality is valid

$$1 + \delta_{j+1} L^f(t,T_j) = \mathbb{E}_{\mathbb{P}^*}\big(B^{-1}(T_j,T_{j+1}) \,\big|\, \mathcal{F}_t\big).$$

Equivalently, we have

$$L^f(t,T_j) = \delta_{j+1}^{-1}\Big(\mathbb{E}_{\mathbb{P}^*}\big(B^{-1}(T_j,T_{j+1}) \,\big|\, \mathcal{F}_t\big) - 1\Big) = \mathbb{E}_{\mathbb{P}^*}(L(T_j,T_j) \,|\, \mathcal{F}_t).$$

It is thus clear that the futures LIBOR follows a martingale under the spot martingale measure $\mathbb{P}^*$.

12.1.4 LIBOR in the Gaussian HJM Model

In this section, we make a standing assumption that the bond price volatilities $b(t,T_j)$ are deterministic functions, that is, we place ourselves within the Gaussian HJM framework. In this case, it is not hard to express forward and futures LIBORs in terms of bond prices and bond price volatilities. Furthermore, as soon as the dynamics of various rates under forward probability measures are known explicitly, it is straightforward to value interest-rate sensitive derivatives. Recall that in the HJM framework we have

$$dF_B(t,T_j,T_{j+1}) = F_B(t,T_j,T_{j+1})\big(b(s,T_j) - b(s,T_{j+1})\big) \cdot dW_t^{T_{j+1}}$$

with the terminal condition $F_B(T_j,T_j,T_{j+1}) = B^{-1}(T_j,T_{j+1})$. Also, the spot and forward Brownian motions are known to satisfy

$$dW_t^{T_j} = dW_t^{T_{j+1}} - \big(b(s,T_j) - b(s,T_{j+1})\big)\, dt, \tag{12.8}$$

and

$$dW_t^* = dW_t^{T_j} + b(s,T_j)\, dt. \tag{12.9}$$

In view of the relationships above, it is quite standard to establish the following proposition (see Flesaker (1993b) for related results). It is worth pointing out that in the present framework there are no ambiguities in the definition of the spot probability measure (this should be contrasted with the case of the discrete-tenor lognormal model of forward LIBORs, in which the spot measure is not uniquely defined).

For conciseness, we shall frequently write $F_B(t) = F_B(t,T_j,T_{j+1})$. Also, we write, as usual,

$$\gamma(t,T_j,T_{j+1}) = b(t,T_j) - b(t,T_{j+1})$$

to denote the volatility of the process $F_B(t,T_j,T_{j+1})$.

Proposition 12.1.1. *Assume the Gaussian HJM model of the term structure of interest rates. Then the following relationships are valid*

$$1 + \delta_{j+1} L(t, T_j) = F_B(t, T_j, T_{j+1}), \tag{12.10}$$

$$1 + \delta_{j+1} \tilde{L}(t, T_j) = F_B(t, T_j, T_{j+1}) \, e^{\int_t^{T_j} |\gamma(u, T_j, T_{j+1})|^2 \, du}, \tag{12.11}$$

$$1 + \delta_{j+1} L^f(t, T_j) = F_B(t, T_j, T_{j+1}) \, e^{-\int_t^{T_j} b(u, T_{j+1}) \cdot \gamma(u, T_j, T_{j+1}) \, du}. \tag{12.12}$$

Proof. For brevity, we shall write $F_B(t) = F_B(t, T_j, T_{j+1})$. The first formula is in fact universal (see (12.1)). For the second, note that (cf. (12.2))

$$dF_B(t) = F_B(t) \gamma(t, T_j, T_{j+1}) \cdot \big(dW_t^{T_j} + \gamma(t, T_j, T_{j+1}) \, dt \big).$$

Consequently,

$$F_B(T_j) = F_B(t) \exp \Big(\int_t^{T_j} \gamma_u \cdot \big(dW_u^{T_j} + \gamma_u \, du \big) - \frac{1}{2} \int_t^{T_j} |\gamma_u|^2 du \Big),$$

where we write $\gamma_u = \gamma(u, T_j, T_{j+1})$. Since $B^{-1}(T_j, T_{j+1}) = F_B(T_j)$, upon taking conditional expectation with respect to the σ-field $\mathcal{F}_t$, we obtain (12.11). Furthermore, we have

$$dF_B(t) = F_B(t) \gamma_t \cdot \big(dW_t^* - b(t, T_{j+1}) \, dt \big)$$

and thus

$$F_B(T_j) = F_B(t) \exp \Big(\int_t^{T_j} \gamma_u \cdot \big(dW_u^* - b(u, T_{j+1}) \, du \big) - \frac{1}{2} \int_t^{T_j} |\gamma_u|^2 du \Big).$$

This leads to equality (12.12). □

Dynamics of the forward LIBORs are also easy to find, as the following corollary shows.

Corollary 12.1.1. *We have*

$$dL(t, T_j) = \delta_{j+1}^{-1} \big(1 + \delta_{j+1} L(t, T_j) \big) \gamma(t, T_j, T_{j+1}) \cdot dW_t^{T_{j+1}}, \tag{12.13}$$

$$d\tilde{L}(t, T_j) = \delta_{j+1}^{-1} \big(1 + \delta_{j+1} \tilde{L}(t, T_j) \big) \gamma(t, T_j, T_{j+1}) \cdot dW_t^{T_j}, \tag{12.14}$$

$$dL^f(t, T_j) = \delta_{j+1}^{-1} \big(1 + \delta_{j+1} L^f(t, T_j) \big) \gamma(t, T_j, T_{j+1}) \cdot dW_t^*. \tag{12.15}$$

Proof. Formula (12.13) is an immediate consequence of (12.1) combined with (12.2). Expressions (12.14) and (12.15) can be derived by applying Itô's rule to equalities (12.11) and (12.12) respectively. □

From Corollary 12.1.1, it is rather clear that closed-form expressions for values of options written on forward or futures LIBORs are not available in the Gaussian HJM framework.

12.2 Interest Rate Caps and Floors

An *interest rate cap* (known also as a *ceiling rate agreement,* or briefly *CRA*) is a contractual arrangement where the grantor (seller) has an obligation to pay cash to the holder (buyer) if a particular interest rate exceeds a mutually agreed level at some future date or dates. Similarly, in an *interest rate floor,* the grantor has an obligation to pay cash to the holder if the interest rate is below a preassigned level. When cash is paid to the holder, the holder's net position is equivalent to borrowing (or depositing) at a rate fixed at that agreed level. This assumes that the holder of a cap (or floor) agreement also holds an underlying asset (such as a deposit) or an underlying liability (such as a loan). Finally, the holder is not affected by the agreement if the interest rate is ultimately more favorable to him than the agreed level. This feature of a cap (or floor) agreement makes it similar to an option.

Specifically, a *forward start cap* (or a *forward start floor*) is a strip of caplets (floorlets), each of which is a call (put) option on a forward rate respectively. Let us denote by κ and by δ_j the cap strike rate and the length of a caplet respectively. We shall check that an interest rate caplet (i.e., one leg of a cap) may also be seen as a put option with strike price 1 (per dollar of principal) that expires at the caplet start day on a discount bond with face value $1 + \kappa\delta_j$ maturing at the caplet end date. This property makes the valuation of a cap relatively simple; essentially, it can be reduced to the problem of option pricing on zero-coupon bonds.

Similarly to the swap agreements examined in the next chapter, interest rate caps and floors may be settled either *in arrears* or *in advance.* In a forward cap or floor with the notional principal N settled in arrears at dates T_j, $j = 1, \dots, n$, where $T_j - T_{j-1} = \delta_j$ the cash flows at times T_j are

$$N\big(L(T_{j-1}, T_{j-1}) - \kappa\big)^+ \delta_j$$

and

$$N\big(\kappa - L(T_{j-1}, T_{j-1})\big)^+ \delta_j$$

respectively, where the (spot) LIBOR $L(T_{j-1}, T_{j-1})$ is determined at the reset date T_{j-1}, and it formally satisfies

$$B(T_{j-1}, T_j)^{-1} = 1 + \delta_j L(T_{j-1}, T_{j-1}). \tag{12.16}$$

The arbitrage price at time $t \le T_0$ of a *forward cap,* denoted by $\mathbf{FC}_t$, is

$$\mathbf{FC}_t = \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}^*}\Big(\frac{B_t}{B_{T_j}}\big(L(T_{j-1}, T_{j-1}) - \kappa\big)^+ \delta_j \,\Big|\, \mathcal{F}_t\Big). \tag{12.17}$$

We have assumed here, without loss of generality, that the notional principal $N = 1$. This convention will be in force throughout the rest of this chapter. Let us consider a *caplet* (i.e., one leg of a cap) with reset date T_{j-1} and settlement date $T_j = T_{j-1} + \delta_j$.

The value at time t of a caplet equals (for simplicity, we write $\tilde{\delta}_j = 1+\kappa\delta_j$)

$$
\begin{aligned}
\mathbf{Cpl}_t &= \mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_{T_j}} \Big(\delta_j^{-1}\big(B(T_{j-1},T_j)^{-1}-1\big)-\kappa\Big)^+ \delta_j \,\Big|\, \mathcal{F}_t \Big\} \\
&= \mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_{T_j}} \Big(\frac{1}{B(T_{j-1},T_j)} - \tilde{\delta}_j\Big)^+ \,\Big|\, \mathcal{F}_t \Big\} \\
&= \mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_{T_{j-1}}} \Big(\frac{1}{B(T_{j-1},T_j)} - \tilde{\delta}_j\Big)^+ \mathbb{E}_{\mathbb{P}^*}\Big(\frac{B_{T_{j-1}}}{B_{T_j}} \,\Big|\, \mathcal{F}_{T_{j-1}}\Big) \,\Big|\, \mathcal{F}_t \Big\} \\
&= \mathbb{E}_{\mathbb{P}^*}\Big\{ \frac{B_t}{B_{T_{j-1}}} \Big(1 - \tilde{\delta}_j B(T_{j-1},T_j)\Big)^+ \,\Big|\, \mathcal{F}_t \Big\} \\
&= B(t,T_{j-1})\,\mathbb{E}_{\mathbb{P}_{T_{j-1}}}\Big\{ \big(1 - \tilde{\delta}_j B(T_{j-1},T_j)\big)^+ \,\Big|\, \mathcal{F}_t \Big\},
\end{aligned}
$$

where the last equality was deduced from Lemma 9.6.3. It is apparent that a caplet is essentially equivalent to a put option on a zero-coupon bond; it may also be seen as an option on a one-period forward swap.

Since the cash flow of the j^{th} caplet at time T_j is a $\mathcal{F}_{T_{j-1}}$-measurable random variable, we may also use Corollary 9.6.1 to express the value of the cap in terms of expectations under forward measures. Indeed, from (9.38) we have

$$\mathbf{FC}_t = B(t,T_{j-1}) \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}_{T_{j-1}}}\Big(B(T_{j-1},T_j)\big(L(T_{j-1},T_{j-1})-\kappa\big)^+\delta_j \,\Big|\, \mathcal{F}_t\Big).$$

Consequently, using (12.16) we get once again the equality

$$\mathbf{FC}_t = B(t,T_{j-1}) \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}_{T_{j-1}}}\Big(\big(1-\tilde{\delta}_j B(T_{j-1},T_j)\big)^+ \,\Big|\, \mathcal{F}_t\Big),$$

which is valid for every $t \in [0, T_{j-1}]$.

Finally, the equivalence of a cap and a put option on a zero-coupon bond can be explained in an intuitive way. For this purpose, it is enough to examine two basic features of both contracts: the exercise set and the payoff value. Let us consider the j^{th} caplet. A caplet is exercised at time T_{j-1} if and only if $L(T_{j-1}) - \kappa > 0$, or equivalently, if

$$B(T_{j-1},T_j)^{-1} = 1 + L(T_{j-1},T_{j-1})(T_j - T_{j-1}) > 1 + \kappa\delta_j = \tilde{\delta}_j.$$

The last inequality holds whenever $B(T_{j-1},T_j) < \tilde{\delta}_j^{-1}$. This shows that both of the considered options are exercised in the same circumstances. If exercised, the caplet pays $\delta_j(L(T_{j-1},T_{j-1})-\kappa)$ at time T_j, or equivalently,

$$\delta_j B(T_{j-1},T_j)\big(L(T_{j-1},T_{j-1})-\kappa\big) = \tilde{\delta}_j\,\big(\tilde{\delta}_j^{-1} - B(T_{j-1},T_j)\big)$$

at time T_{j-1}. This shows once again that the j^{th} caplet, with strike level κ and nominal value 1, is essentially equivalent to a put option with strike price $(1+\kappa\delta_j)^{-1}$ and nominal value $(1+\kappa\delta_j)$ written on the corresponding zero-coupon bond with maturity T_j.

The price of a *forward floor* at time $t \in [0, T]$ equals

$$\mathbf{FF}_t = \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}^*}\Big(\frac{B_t}{B_{T_j}}\big(\kappa - L(T_{j-1}, T_{j-1})\big)^+ \delta_j \,\Big|\, \mathcal{F}_t\Big). \tag{12.18}$$

Using a trivial equality

$$\big(\kappa - L(T_{j-1}, T_{j-1})\big)^+ \delta_j = \big(L(T_{j-1}, T_{j-1}) - \kappa\big)^+ \delta_j - \big(L(T_{j-1}, T_{j-1}) - \kappa\big)\delta_j,$$

we find that the following cap-floor parity relationship is satisfied at any time $t \in [0, T]$ (the three contracts are assumed to have the same payment dates)

$$\textit{Forward Cap}\,(t) \;-\; \textit{Forward Floor}\,(t) \;=\; \textit{Forward Swap}\,(t),$$

For a description of a (multi-period) forward swap, we refer to the next chapter. This relationship can also be verified by a straightforward comparison of the corresponding cash flows of both portfolios. Let us finally mention that by a *cap* (respectively, *floor*), we mean a forward cap (respectively, forward floor) with $t = T$.

12.3 Valuation in the Gaussian HJM Model

We assume that the bond price volatility is a deterministic function, that is, we place ourselves within the Gaussian HJM framework. Recall that for any two maturity dates U, T we write $F_B(t, T, U) = B(t, T)/B(t, U)$, so that the function $\gamma(t, T, U) = b(t, T) - b(t, U)$ represents the volatility of $F_B(t, T, U)$.

12.3.1 Plain-vanilla Caps and Floors

The following lemma is an immediate consequence of Proposition 11.3.1 and the equivalence of a caplet and a specific put option on a zero-coupon bond.

Lemma 12.3.1. *Consider a caplet with settlement date T, accrual period δ, and strike level κ, that pays at time $T + \delta$ the amount $(L(T, T) - \kappa)^+\delta$. Its arbitrage price at time $t \in [0, T]$ in the Gaussian HJM set-up equals*

$$\mathbf{Cpl}_t = B(t, T)\Big(N\big(e_1(t, T)\big) - \tilde{\delta} F_B(t, T + \delta, T) N\big(e_2(t, T)\big)\Big),$$

where $\tilde{\delta} = 1 + \kappa\delta$ and

$$e_{1,2}(t, T) = \frac{\ln F_B(t, T, T + \delta) - \ln \tilde{\delta} \pm \frac{1}{2} v^2(t, T)}{v(t, T)}$$

with

$$v^2(t, T) = \int_t^T |\gamma(u, T, T + \delta)|^2 \, du.$$

The next result, which is an almost immediate consequence of Lemma 12.3.1, provides a generic pricing formula for a forward cap in the Gaussian HJM set-up.

Proposition 12.3.1. *Assume the Gaussian HJM framework, so that the volatilities $\gamma(t, T_{j-1}, T_j)$ are deterministic. Then the arbitrage price at time $t \le T_0$ of an interest rate cap with strike level κ, settled in arrears at times T_j, $j = 1, \dots, n$, equals*

$$\mathbf{FC}_t = \sum_{j=1}^{n} B(t, T_{j-1}) \Big(N\big(e_1(t, T_{j-1})\big) - \tilde{\delta}_j F_B(t, T_j, T_{j-1}) N\big(e_2^j(t, T_{j-1})\big) \Big)$$

where $\tilde{\delta}_j = 1 + \kappa \delta_j$ *and*

$$e_{1,2}(t, T_{j-1}) = \frac{\ln F_B(t, T_{j-1}, T_j) - \ln \tilde{\delta}_j \pm \frac{1}{2} v^2(t, T_{j-1})}{v(t, T_{j-1})}$$

with

$$v^2(t, T_{j-1}) = \int_t^{T_{j-1}} |\gamma(u, T_{j-1}, T_j)|^2 \, du.$$

Proof. We represent the price of a forward cap in the following way

$$\begin{aligned} \mathbf{FC}_t &= \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}^*} \Big\{ \frac{B_t}{B_{T_j}} \left(L(T_{j-1}, T_{j-1}) - \kappa \right)^+ \delta_j \, \Big| \, \mathcal{F}_t \Big\} \\ &= \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}^*} \Big\{ \frac{B_t}{B_{T_j}} \Big(\big(B(T_{j-1}, T_j)^{-1} - 1\big) \delta_j^{-1} - \kappa \Big)^+ \delta_j \, \Big| \, \mathcal{F}_t \Big\} \\ &= \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}^*} \Big\{ \frac{B_t}{B_T} \Big(1 - \tilde{\delta}_j B(T, T_j) \Big)^+ \, \Big| \, \mathcal{F}_t \Big\} = \sum_{j=1}^{n} \mathbf{Cpl}_t^j, \end{aligned}$$

where $\mathbf{Cpl}_t^j$ stands for the price at time t of the j^{th} caplet. The assertion now follows from Lemma 12.3.1. □

To derive the valuation formula for a floor, it is enough to make use of the cap-floor parity, that is, the universal relationship $\mathbf{FC}_t - \mathbf{FF}_t = \mathbf{FS}_t$. By combining the valuation formulas for caps and swaps, we find easily that, under the assumptions of Proposition 12.3.1, the arbitrage price of a floor is given by the expression

$$\mathbf{FF}_t = \sum_{j=1}^{n} \Big(\tilde{\delta}_j B(t, T_j) N\big(- e_2^j(t)\big) - B(t, T_{j-1}) N\big(- e_1^j(t)\big) \Big).$$

In the derivation of the last formula we have used, in particular, the universal (i.e., model independent) valuation formula (13.2) for swaps, which will be established in Sect. 13.1.1 below.

12.3.2 Exotic Caps

A large variety of *exotic caps* is offered to institutional clients of financial institutions. In this section, we develop pricing formulas for some of them within the Gaussian HJM set-up.

Dual-strike caps. The *dual-strike cap* (known also as a *N-cap*) is an interest rate cap that has a lower strike κ_1, an upper strike κ_2 (with $\kappa_1 \leq \kappa_2$), and a trigger, say l. So long as the floating rate L is below the level l, the N-cap owner enjoys protection at the lower strike κ_1. For periods when L is at or above the level l, the N-cap owner has protection at the upper strike level κ_2. Let us consider an N-cap on principal 1 settled in arrears at times $T_j, j = 1, \ldots, n$, where $T_j - T_{j-1} = \delta_j$ and $T_0 = T$. It is clear that the cash flow of the N-cap at time T_j equals

$$c_j = \big(L(T_{j-1}) - \kappa_1\big)^+ \delta_j \, \mathbb{1}_{\{L(T_{j-1}) < l\}} + \big(L(T_{j-1}) - \kappa_2\big)^+ \delta_j \mathbb{1}_{\{L(T_{j-1}) \geq l\}}.$$

where $L(T_{j-1}) = L(T_{j-1}, T_{j-1})$. It is not hard to check that the N-cap price at time $t \in [0, T]$ is

$$\begin{aligned} \mathbf{NC}_t \;=\; & \sum_{j=0}^{n-1} B(t,T_j)\Big(N\big(h_2^j(t,l)\big) - N\big(h_2^j(t,\kappa_1 \wedge l)\big) + N\big(-h_2^j(t,\kappa_2 \vee l)\big)\Big) \\ & - \sum_{j=0}^{n-1} (1+\kappa_1\delta_{j+1}) B(t,T_{j+1})\Big(N\big(h_1^j(t,l)\big) - N\big(h_1^j(t,\kappa_1 \wedge l)\big)\Big) \\ & - \sum_{j=0}^{n-1} (1+\kappa_2\delta_{j+1}) B(t,T_{j+1}) N\big(-h_1^j(t,\kappa_2 \vee l)\big), \end{aligned}$$

where

$$h_{1,2}^j(t,\kappa) = \frac{\ln(1+\kappa\delta_{j+1}) - \ln F_B(t,T_j,T_{j+1}) \pm \frac{1}{2}v^2(t,T_j)}{v(t,T_j)}$$

and $v^2(t,T_j)$ is given in Proposition 12.3.1.

Bounded caps. A *bounded cap* (or a *B-cap*) consists of a sequence of caplets in which the difference between the fixed and floating levels is paid only if the total payments to date are less than some prescribed level b (let us stress that other kinds of B-caps exist). Let us first consider a particular *B-caplet* maturing at a reset date T_{j-1}. The corresponding cash flow will be paid in arrears at time T_j only if the accumulated cash flows at time T_{j-1}, due to resets at times T_k and cash flows of the B-cap paid at times $T_{k+1}, k = 0, \ldots, j-2$, are still less than b. More formally, the cash flow of a B-caplet maturing at T_{j-1} equals

$$c_j(\kappa, b) = (L(T_{j-1}) - \kappa)^+ \, \delta_j \, \mathbb{1}_D,$$

where D stands for the following set

$$D = \Big\{ \sum_{k=1}^{j} \big(L(T_{k-1}) - \kappa\big)^{+} \delta_k \, \le \, b \Big\} = \Big\{ \sum_{k=1}^{j} \big(B(T_{k-1}, T_k)^{-1} - \tilde{\delta}_k\big)^{+} \, \le \, b \Big\},$$

where, as usual, $\tilde{\delta}_k = 1 + \kappa \delta_k$. The amount $c_j(\kappa, b)$ is paid at time T_j. The arbitrage price of a B-caplet at time $t \le T$ therefore equals

$$\mathbf{BCpl}_{\,t} = \mathbb{E}_{\mathbb{P}^*} \Big\{ \frac{B_t}{B_{T_j}} \Big(L(T_{j-1}) - \kappa \Big)^{+} \delta_j \, \mathbb{1}_D \, \Big| \, \mathcal{F}_t \Big\},$$

or equivalently,

$$\mathbf{BCpl}_{\,t} = \mathbb{E}_{\mathbb{P}^*} \Big\{ \frac{B_t}{B_{T_{j-1}}} \Big(1 - \tilde{\delta}_j B(T_{j-1}, T_j) \Big)^{+} \mathbb{1}_D \, \Big| \, \mathcal{F}_t \Big\}.$$

Using the standard forward measure method, the last equality can be given the following form

$$\mathbf{BCpl}_{\,t} = B(t, T_{j-1}) \mathbb{E}_{\mathbb{P}_{T_{j-1}}} \Big\{ \Big(1 - \tilde{\delta}_j B(T_{j-1}, T_j) \Big)^{+} \mathbb{1}_D \, \Big| \, \mathcal{F}_t \Big\},$$

where $\mathbb{E}_{\mathbb{P}_{T_{j-1}}}$ stands for the expectation under the forward measure $\mathbb{P}_{T_{j-1}}$. Furthermore,

$$B(T_{k-1}, T_k) = \frac{B(t, T_k)}{B(t, T_{k-1})} \exp \Big(- \int_t^{T_{k-1}} \gamma_k(u) \cdot dW_u^{T_{j-1}}$$

$$\frac{1}{2} \int_t^{T_{k-1}} |\gamma_k(u)|^2 \, du - \int_t^{T_{k-1}} \gamma_k(u) \cdot \gamma(u, T_{k-1}, T_{j-1}) \, du \Big),$$

where $\gamma_k(u) = \gamma(u, T_{k-1}, T_k)$. The random variable $(\xi_1(t), \ldots, \xi_j(t))$, where

$$\xi_k(t) = \int_t^{T_{k-1}} \gamma_k(u) \cdot dW_u^{T_{j-1}}$$

for $k = 1, \ldots, j$, is independent of the σ-field $\mathcal{F}_t$ under the forward measure $\mathbb{P}_{T_{j-1}}$. Furthermore, its probability law under $\mathbb{P}_{T_{j-1}}$ is Gaussian $N(0, \Gamma)$, where the entries of the matrix Γ are (notice that $\gamma_{kk}(t) = v_k^2(t)$)

$$\gamma_{kl}(t) = \int_t^{T_{k-1} \wedge T_{l-1}} \gamma_k(u) \cdot \gamma_l(u) \, du.$$

It is apparent that

$$\mathbf{BCpl}_{\,t} = \mathbb{E}_{\mathbb{P}_{T_{j-1}}} \Big\{ \Big(B(t, T_{j-1}) - \tilde{\delta}_j B(t, T_j) \, e^{-\xi_{j-1}(t) - v_{j-1}^2(t)/2} \Big)^{+} \mathbb{1}_{D_{j-1}} \, \Big| \, \mathcal{F}_t \Big\},$$

where D_{j-1} stands for the set

$$D_{j-1} = \Big\{ \sum_{k=1}^{j} \Big(\frac{B(t, T_{k-1})}{B(t, T_k)} \, e^{\xi_k(t) + \alpha_{kj}(t) + v_k^2(t)/2} - \tilde{\delta}_j \Big)^{+} \le b \Big\},$$

and

$$\alpha_{kj}(t) = \int_t^{T_{k-1}} \gamma_k(u) \cdot \gamma(u, T_{k-1}, T_{j-1})\, ds.$$

Denoting

$$A_{j-1} = \Big\{ \xi_{j-1} \geq \ln \frac{\tilde{\delta}_j B(t,T_j)}{B(t,T_{j-1})} + \frac{1}{2} v^2_{j-1}(t) \Big\},$$

we arrive at the following expression

$$\begin{aligned} \mathbf{BCpl}_t &= B(t,T_{j-1}) \mathbb{P}_{T_{j-1}} \{A_{j-1} \cap D_{j-1}\} \\ &\tilde{\delta}_j B(t,T_j)\, \mathbb{E}_{\mathbb{P}_{T_{j-1}}} \Big(\exp\big(-\xi_{j-1}(t) - v^2_{j-1}(t)/2\big) \mathbb{1}_{A_{j-1} \cap D_{j-1}} \Big). \end{aligned}$$

12.3.3 Captions

Since a caplet is essentially a put option on a zero-coupon bond, a European call option on a caplet is an example of a compound option. More exactly, it is a call option on a put option with a zero-coupon bond as the underlying asset of the put option. Hence, the valuation of a call option on a caplet can be done similarly as in Chap. 6 (provided, of course, that the model of a zero-coupon bond price has sufficiently nice properties). A call option on a cap, or a *caption,* is thus a call on a portfolio of put options. To price a caption observe that its payoff at expiry is

$$\mathbf{CC}_T = \Big(\sum_{j=1}^{n} \mathbf{Cpl}^{\,j}_T - K \Big)^+,$$

where as usual $\mathbf{Cpl}^{\,j}_T$ stands for the price at time T of the j^{th} caplet of the cap, T is the call option's expiry date and K is its strike price. Suppose that we place ourselves within the framework of the spot rate models of Chap. 9.5 – for instance, the Hull and White model. Typically, the caplet price is an increasing function of the current value of the spot rate r_t. Let r^* be the critical level of interest rate, which is implicitly determined by the equality $\sum_{j=1}^n \mathbf{Cpl}^{\,j}_T(r^*) = K$. It is clear that the option is exercised when the rate r_T is greater than r^*. Let us introduce numbers K_j by setting $K_j = \mathbf{Cpl}^{\,j}_T(r^*)$ for $j = 1, \dots, n$. It is easily seen that the caption's payoff is equal to the sum of the payoffs of n call options on particular caplets, with K_j being the corresponding strike prices. Consequently, the caption's price $\mathbf{CC}_t$ at time $t \leq T_1$ is given by the formula $\mathbf{CC}_t = \sum_{j=1}^n C_t(\mathbf{Cpl}^{\,j}, T, K_j)$, where $C_t(\mathbf{Cpl}^{\,j}, T, K_j)$ is the price at time t of a call option with expiry date T and strike level K_j written on the j^{th} caplet (see Hull and White (1994)). An option on a cap (or floor) can also be studied within the Gaussian HJM framework (see Brace and Musiela (1997)). However, results concerning caption valuation within this framework are less explicit than in the case of the Hull and White model.

12.4 LIBOR Market Models

The goal of this section is to present various approaches to the direct modelling of forward LIBORs. We focus here on the model's construction, its basic properties, and the valuation of the most typical derivatives. For further details, the interested reader is referred to the papers by Musiela and Sondermann (1993), Sandmann and Sondermann (1993), Goldys et al. (1994), Sandmann et al. (1995), Brace et al. (1997), Jamshidian (1997a), Miltersen et al. (1997), Musiela and Rutkowski (1997), Rady (1997), Sandmann and Sondermann (1997), Rutkowski (1998b, 1999a), Yasuoka (2001), Galluccio and Hunter (2003, 2004), and Glasserman and Kou (2003).

The issues related to the model's implementation, including model calibration and the valuation of exotic LIBOR and swap rate derivatives, are treated in Brace (1996), Brace et al. (1998, 2001a), Hull and White (1999), Schlögl (1999), Uratani and Utsunomiya (1999), Lotz and Schlögl (1999), Schoenmakers and Coffey (1999), Andersen (2000), Andersen and Andreasen (2000b), Brace and Womersley (2000), Dun et al. (2000), Hull and White (2000), Glasserman and Zhao (2000), Sidenius (2000), Andersen and Brotherton-Ratcliffe (2001), De Yong et al. (2001a, 2001b), Pelsser et al. (2002), Wu (2002), Wu and Zhang (2002), d'Aspremont (2003), Glasserman and Merener (2003), Galluccio et al. (2003a), Jäckel and Rebonato (2003), Kawai (2003), Pelsser and Pietersz (2003), and Piterbarg (2003a, 2003c).

The main motivation for the introduction of the lognormal LIBOR model was the market practice of pricing caps (and swaptions) by means of Black-Scholes-like formulas. For this reason, we shall first describe how market practitioners value caps. The formulas commonly used by practitioners assume that the underlying instrument follows a geometric Brownian motion under some probability measure, $\mathbb{Q}$ say. Since the formal definition of this probability measure is not available, we shall informally refer to $\mathbb{Q}$ as the *market probability.*

12.4.1 Black's Formula for Caps

Let us consider an interest rate cap with expiry date T and fixed strike level κ. Market practice is to price the option assuming that the underlying forward interest rate process is lognormally distributed with zero drift. Let us first consider a caplet – that is, one leg of a cap. Assume that the forward LIBOR $L(t,T)$, $t \in [0,T]$, for the accrual period of length δ follows a geometric Brownian motion under the 'market probability', $\mathbb{Q}$ say. More specifically,

$$dL(t,T) = L(t,T)\sigma\, dW_t, \tag{12.19}$$

where W follows a one-dimensional standard Brownian motion under $\mathbb{Q}$, and σ is a strictly positive constant. The unique solution of (12.19) is

$$L(t,T) = L(0,T)\exp\left(\sigma W_t - \tfrac{1}{2}\sigma^2 t^2\right), \quad \forall\, t \in [0,T].$$

The 'market price' at time t of a caplet with expiry date T and strike level κ is now found from the formula

$$\mathbf{Cpl}_t = \delta B(t, T+\delta)\, \mathbb{E}_{\mathbb{Q}}\big((L(T,T)-\kappa)^+ \,\big|\, \mathcal{F}_t\big).$$

More explicitly, for any $t \in [0,T]$ we have

$$\mathbf{Cpl}_t = \delta B(t, T+\delta)\Big(L(t,T) N\big(\hat{e}_1(t,T)\big) - \kappa N\big(\hat{e}_1(t,T)\big)\Big),$$

where

$$\hat{e}_{1,2}(t,T) = \frac{\ln(L(t,T)/\kappa) \pm \frac{1}{2}\, \hat{v}_0^2(t,T)}{\hat{v}_0(t,T)}$$

and $\hat{v}_0^2(t,T) = \sigma^2(T-t)$. This means that market practitioners price caplets using Black's formula, with discount from the settlement date $T+\delta$. A cap settled in arrears at times T_j, $j = 1, \dots, n$, where $T_j - T_{j-1} = \delta_j$, $T_0 = T$, is priced by the formula

$$\mathbf{FC}_t = \sum_{j=1}^{n} \delta_j B(t, T_j)\Big(L(t, T_{j-1}) N\big(\hat{e}_1(t, T_{j-1})\big) - \kappa N\big(\hat{e}_2(t, T_{j-1})\big)\Big),$$

where for every $j = 0, \dots, n-1$

$$\hat{e}_{1,2}(t, T_{j-1}) = \frac{\ln(L(t, T_{j-1})/\kappa) \pm \frac{1}{2}\, \hat{v}^2(t, T_{j-1})}{\hat{v}(t, T_{j-1})}$$

and $\hat{v}^2(t, T_{j-1}) = \sigma_j^2(T_{j-1} - t)$ for some constants σ_j, $j = 1, \dots, n$. The constant σ_j is referred to as the *implied volatility* of the j^{th} caplet. Thus, for a fixed strike κ we obtain in this way the term structure of caplet volatilities. Since the implied caplet volatilities usually depend on the strike level, we also observe the volatility smile in the caplets market. In practice, caps are quoted in terms of implied volatilities, assuming a flat term structure for underlying caplets. The term structure of caplets volatilities can be stripped from market prices of caps.

The market convention described above implicitly assumes (at least in the case of flat caplet volatilities) that for any maturity T_j the corresponding forward LIBOR has a lognormal probability law under the 'market probability'. As we shall see in what follows, the valuation formula obtained for caps (and floors) in the lognormal LIBOR market model agrees with market practice.

Recall that in the general framework of stochastic interest rates, the price of a *forward cap* equals (see formula (12.17))

$$\mathbf{FC}_t = \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}^*}\Big(\frac{B_t}{B_{T_j}}\big(L(T_{j-1}) - \kappa\big)^+ \delta_j \,\Big|\, \mathcal{F}_t\Big) = \sum_{j=1}^{n} \mathbf{Cpl}_t^j, \tag{12.20}$$

where

$$\mathbf{Cpl}_t^j = B(t, T_j)\, \mathbb{E}_{\mathbb{P}_{T_j}}\Big(\big(L(T_{j-1}) - \kappa\big)^+ \delta_j \,\Big|\, \mathcal{F}_t\Big) \tag{12.21}$$

for every $j = 1, \dots, n$.

12.4.2 Miltersen, Sandmann and Sondermann Approach

The first attempt to provide a rigorous construction a lognormal model of forward LIBORs was done by Miltersen, Sandmann and Sondermann in their paper published in 1997 (see also Musiela and Sondermann (1993), Goldys et al. (1994), and Sandmann et al. (1995) for related studies). As a starting point of their analysis, Miltersen et al. (1997) postulate that the forward LIBOR process $L(t,T)$ satisfies

$$dL(t,T) = \mu(t,T)\,dt + L(t,T)\lambda(t,T)\cdot dW_t^*,$$

with a deterministic volatility function $\lambda(t,T+\delta)$. It is not difficult to deduce from the last formula that the forward price of a zero-coupon bond satisfies

$$dF(t,T+\delta,T) = -F(t,T+\delta,T)\big(1-F(t,T+\delta,T)\big)\lambda(t,T)\cdot dW_t^T.$$

Subsequently, they focus on the partial differential equation satisfied by the function $v = v(t,x)$ that expresses the forward price of the bond option in terms of the forward bond price. The PDE for the option's price is

$$\frac{\partial v}{\partial t} + \frac{1}{2}|\lambda(t,T)|^2 x^2 (1-x)^2 \frac{\partial^2 v}{\partial x^2} = 0 \tag{12.22}$$

with the terminal condition, $v(T,x) = (K-x)^+$.

It is interesting to note that the PDE (12.22) was previously solved by Rady and Sandmann (1994) who worked within a different framework, however. In fact, they were concerned with the valuation of a bond option for the Bühler and Käsler (1989) model.

By solving the PDE (12.22), Miltersen et al. (1997) derived not only the closed-form solution for the price of a bond option (this goal was already achieved in Rady and Sandmann (1994)), but also the "market formula" for the caplet's price. It should be stressed, however, that the existence of a lognormal family of LIBORs $L(t,T)$ with different maturities T was not formally established in a definitive manner by Miltersen et al. (1997) (although some partial results were provided). The positive answer to the problem of existence of such a model was given by Brace et al. (1997), who also start from the continuous-time HJM framework.

12.4.3 Brace, Gątarek and Musiela Approach

To introduce formally the notion of a *forward LIBOR,* we assume that we are given a family $B(t,T)$ of bond prices, and thus also the collection $F_B(t,T,U)$ of forward processes. Let us fix a horizon date T^*. In contrast to the previous section, we shall now assume that a strictly positive real number $\delta < T^*$ representing the length of the accrual period, is fixed throughout. By definition, the forward δ-LIBOR rate $L(t,T)$ for the future date $T \le T^* - \delta$ prevailing at time t is given by the conventional market formula

$$1 + \delta L(t,T) = F_B(t,T,T+\delta), \quad \forall\, t \in [0,T].$$

Comparing this formula with (9.40), we find that $L(t,T) = f_s(t,T,T+\delta)$, so that the forward LIBOR $L(t,T)$ represents in fact the add-on rate prevailing at time t over the future time period $[T,T+\delta]$. We can also re-express $L(t,T)$ directly in terms of bond prices, as for any $T \in [0,T^* - \delta]$ we have

$$1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)}, \quad \forall\, t \in [0,T]. \tag{12.23}$$

In particular, the initial term structure of forward LIBORs satisfies

$$L(0,T) = \frac{B(0,T) - B(0,T+\delta)}{\delta B(0,T+\delta)}. \tag{12.24}$$

Assume that we are given a family $F_B(t,T,T^*)$ of forward processes satisfying

$$dF_B(t,T,T^*) = F_B(t,T,T^*)\gamma(t,T,T^*) \cdot dW_t^{T^*}.$$

on $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbb{P}_{T^*})$, where W^{T^*} is a standard Brownian motion under $\mathbb{P}_{T^*}$. Then it is not hard to derive the dynamics of the associated family of forward LIBORs. For instance, one finds that under the forward measure $\mathbb{P}_{T+\delta}$ we have

$$dL(t,T) = \delta^{-1} F_B(t,T,T+\delta)\, \gamma(t,T,T+\delta) \cdot dW_t^{T+\delta},$$

where $W_t^{T+\delta}$ and $\mathbb{P}_{T+\delta}$ are defined by (see (9.42))

$$W_t^{T+\delta} = W_t^{T^*} - \int_0^t \gamma(u,T+\delta,T^*)\, du.$$

The process $W^{T+\delta}$ is a standard Brownian motion with respect the probability measure $\mathbb{P}_{T+\delta} \sim \mathbb{P}_{T^*}$ defined on $(\Omega, \mathcal{F}_T)$ by means of the Radon-Nikodým density (see (9.43)

$$\frac{d\mathbb{P}_{T+\delta}}{d\mathbb{P}_{T^*}} = \mathcal{E}_{T+\delta}\Big(\int_0^{\cdot} \gamma(u,T+\delta,T^*) \cdot dW_u^{T^*} \Big) \quad \mathbb{P}\text{-a.s.}$$

This means that $L(t,T)$ solves the equation

$$dL(t,T) = \delta^{-1}(1 + \delta L(t,T))\, \gamma(t,T,T+\delta) \cdot dW_t^{T+\delta} \tag{12.25}$$

subject to the initial condition (12.24). Suppose that forward LIBORs $L(t,T)$ are strictly positive. Then formula (12.25) can be rewritten as follows

$$dL(t,T) = L(t,T)\, \lambda(t,T) \cdot dW_t^{T+\delta}, \tag{12.26}$$

where for every $t \in [0,T]$ we have

$$\lambda(t,T) = \frac{1 + \delta L(t,T)}{\delta L(t,T)}\, \gamma(t,T,T+\delta). \tag{12.27}$$

We thus see that the collection of forward processes uniquely specifies the family of forward LIBORs.

The construction of a model of forward LIBORs relies on the following set of assumptions.

(LR.1) For any maturity $T \leq T^* - \delta$, we are given a $\mathbb{R}^d$-valued, bounded $\mathbb{F}$-adapted process $\lambda(t,T)$ representing the volatility of the forward LIBOR process $L(t,T)$.

(LR.2) We assume a strictly decreasing and strictly positive initial term structure $B(0,T), T \in [0,T^*]$, and thus an initial term structure $L(0,T)$ of forward LIBORs

$$L(0,T) = \frac{B(0,T) - B(0,T+\delta)}{\delta B(0,T+\delta)}, \quad \forall\, T \in [0,T^* - \delta].$$

Note that the volatility λ is a stochastic process, in general. In the special case when $\lambda(t,T)$ is a (bounded) deterministic function, a model we are going to construct is termed *lognormal LIBOR market model* for a fixed accrual period.

Remarks. Needless to say that the boundedness of λ can be weakened substantially. In fact, we shall frequently postulate, to simplify the exposition, that a volatility process (or function) is bounded in order but it is clear that a suitable integrability conditions are sufficient.

To construct a model satisfying (LR.1)-(LR.2), Brace et al. (1997) place themselves in the HJM set-up and they assume that for every $T \in [0,T^*]$, the volatility $b(t,T)$ vanishes for every $t \in [(T-\delta) \vee 0, T]$.

The construction presented in Brace et al. (1997) relies on forward induction, as opposed to the backward induction, which will be used in what follows. They start by postulating that the dynamics of $L(t,T)$ under the martingale measure $\mathbb{P}^*$ are governed by the following SDE

$$dL(t,T) = \mu(t,T)\,dt + L(t,T)\lambda(t,T) \cdot dW_t^*,$$

where λ is known, but the drift coefficient μ is unspecified. Recall that the arbitrage-free dynamics of the instantaneous forward rate $f(t,T)$ are

$$df(t,T) = \sigma(t,T) \cdot \sigma^*(t,T)\,dt + \sigma(t,T) \cdot dW_t^*.$$

In addition, we have the following relationship (cf. (12.23))

$$1 + \delta L(t,T) = \exp\Big(\int_T^{T+\delta} f(t,u)\,du\Big). \tag{12.28}$$

Applying Itô's formula to both sides of (12.28), and comparing the diffusion terms, we find that

$$\sigma^*(t,T+\delta) - \sigma^*(t,T) = \int_T^{T+\delta} \sigma(t,u)\,du = \frac{\delta L(t,T)}{1+\delta L(t,T)}\,\lambda(t,T).$$

To solve the last equation for σ^* in terms of L, it is necessary to impose some kind of 'initial condition' on the process σ^*, or, equivalently, on the coefficient σ in the dynamics of $f(t,T)$.

For instance, by setting $\sigma(t,T)=0$ for $0 \le t \le T \le t+\delta$ (this choice was postulated in Brace et al. (1997)), we obtain the following relationship

$$b(t,T) = -\sigma^*(t,T) = -\sum_{k=1}^{[\delta^{-1}T]} \frac{\delta L(t,T-k\delta)}{1+\delta L(t,T-k\delta)}\,\lambda(t,T-k\delta). \qquad (12.29)$$

The existence and uniqueness of solutions to the SDEs that govern the instantaneous forward rate $f(t,T)$ and the forward LIBOR $L(t,T)$ for σ^* given by (12.29) can be shown rather easily, using the forward induction. Taking this result for granted, we conclude that the process $L(t,T)$ satisfies, under the spot martingale measure $\mathbb{P}^*$,

$$dL(t,T) = L(t,T)\sigma^*(t,T)\cdot\lambda(t,T)\,dt + L(t,T)\lambda(t,T)\cdot dW_t^*,$$

or equivalently,

$$dL(t,T) = L(t,T)\lambda(t,T)\cdot dW_t^{T+\delta}$$

under the forward measure $\mathbb{P}_{T+\delta}$. In this way, Brace et al. (1997) were able to specify completely their generic model of forward LIBORs. In particular, in the case of deterministic volatilities $\lambda(t,T)$, we obtain the lognormal model of forward LIBORs, that is, the model in which the process $L(t,T)$ is lognormal under $\mathbb{P}_{T+\delta}$ for any maturity $T>0$ and for a fixed δ.

Let us note that this model is sometimes referred to as the LLM model (that is, Lognormal LIBOR Market model) model or as the BGM model (that is, Brace-Gątarek-Musiela model).

12.4.4 Musiela and Rutkowski Approach

As an alternative to forward induction, we describe the backward induction approach to the modelling of forward LIBORs. We shall now focus on the modelling of a finite family of forward LIBORs that are associated with a prespecified collection $T_0 < \cdots < T_n$ of reset/settlement dates. The construction presented below is based on the one given by Musiela and Rutkowski (1997).

Let us start by recalling the notation. We assume that we are given a predetermined collection of reset/settlement dates $0 \le T_0 < T_1 < \cdots < T_n$ referred to as the *tenor structure*. Let us write $\delta_j = T_j - T_{j-1}$ for $j=1,\dots,n$, so that $T_j = T_0 + \sum_{i=1}^{j}\delta_i$ for every $j=0,\dots,n$.

Since δ_j is not necessarily constant, the assumption of a fixed accrual period δ is now relaxed, and thus a model will be more suitable for practical purposes. Indeed, in most LIBOR derivatives the accrual period (day-count fraction) varies over time, and thus it is essential to have a model of LIBORs that is capable of mimicking this important real-life feature.

We find it convenient to set $T^* = T - n$ and

$$T^*_m = T^* - \sum_{j=n-m+1}^{n} \delta_j = T_{n-m}, \quad \forall\, m = 0, \dots, n.$$

For any $j = 0, \dots, n-1$, we define the forward LIBOR $L(t, T_j)$ by setting

$$L(t, T_j) = \frac{B(t, T_j) - B(t, T_{j+1})}{\delta_{j+1} B(t, T_{j+1})}, \quad \forall\, t \in [0, T_j].$$

Let us introduce the notion of a martingale probability associated with the forward LIBOR $L(t, T_{j-1})$.

Definition 12.4.1. Let us fix $j = 1, \dots, n$. A probability measure $\mathbb{P}_{T_j}$ on $(\Omega, \mathcal{F}_{T_j})$, equivalent to $\mathbb{P}$, is said to be the *forward LIBOR measure* for the date T_j if, for every $k = 1, \dots, n$, the relative bond price

$$U_{n-j+1}(t, T_k) \stackrel{\mathrm{def}}{=} \frac{B(t, T_k)}{B(t, T_j)}, \quad \forall\, t \in [0, T_k \wedge T_j],$$

follows a local martingale under $\mathbb{P}_{T_j}$.

It is clear that the notion of forward LIBOR measure is formally identical with that of a forward martingale measure for a given date. The slight modification of our previous terminology emphasizes our intention to make a clear distinction between various kinds of forward probabilities, which we are going to study in the sequel. Also, it is trivial to observe that the forward LIBOR $L(t, T_j)$ necessarily follows a local martingale under the forward LIBOR measure for the date T_{j+1}. If, in addition, it is a strictly positive process and the underlying filtration is generated by a Brownian motion, the existence of the associated volatility process can be justified easily.

In our further development, we shall go the other way around; that is, we will assume that for any date T_j, the volatility $\lambda(t, T_j)$ of the forward LIBOR $L(t, T_j)$ is exogenously given. Basically, it can be a deterministic $\mathbb{R}^d$-valued function of time, an $\mathbb{R}^d$-valued function of the underlying forward LIBORs, or a d-dimensional stochastic process adapted to a Brownian filtration. For simplicity, we assume that the volatilities of forward LIBORs are bounded (of course, this assumption can be relaxed).

Our aim is to construct a family $L(t, T_j)$, $j = 0, \dots, n-1$ of forward LIBORs, a collection of mutually equivalent probability measures $\mathbb{P}_{T_j}$, $j = 1, \dots, n$, and a family W^{T_j}, $j = 0, \dots, n-1$ of processes in such a way that: (i) for any $j = 1, \dots, n$ the process W^{T_j} is a d-dimensional standard Brownian motion under the probability measure $\mathbb{P}_{T_j}$, (ii) for any $j = 1, \dots, n-1$, the forward LIBOR $L(t, T_j)$ satisfies the SDE

$$dL(t, T_j) = L(t, T_j)\, \lambda(t, T_j) \cdot dW_t^{T_{j+1}}, \quad \forall\, t \in [0, T_j],$$

with the initial condition

$$L(0,T_j) = \frac{B(0,T_j) - B(0,T_{j+1})}{\delta_{j+1} B(0,T_{j+1})}.$$

As was mentioned already, the construction of the model is based on backward induction. We start by defining the forward LIBOR with the longest maturity, T_{n-1}. We postulate that $L(t,T_{n-1}) = L(t,T_1^*)$ is governed under the underlying probability measure $\mathbb{P}$ by the following SDE (note that, for simplicity, we have chosen the underlying probability measure $\mathbb{P}$ to play the role of the forward LIBOR measure for the date T^*)

$$dL(t,T_1^*) = L(t,T_1^*)\,\lambda(t,T_1^*) \cdot dW_t,$$

with the initial condition

$$L(0,T_1^*) = \frac{B(0,T_1^*) - B(0,T^*)}{\delta_n B(0,T^*)}.$$

Put another way, we have

$$L(t,T_1^*) = \frac{B(0,T_1^*) - B(0,T^*)}{\delta_n B(0,T^*)}\, \mathcal{E}_t\Big(\int_0^{\cdot} \lambda(u,T_1^*) \cdot dW_u\Big)$$

for $t \in [0,T_1^*]$. Since $B(0,T_1^*) > B(0,T^*)$, it is clear that the $L(t,T_1^*)$ follows a strictly positive martingale under $\mathbb{P}_{T^*} = \mathbb{P}$.

The next step is to define the forward LIBOR for the date T_2^*. For this purpose, we need to introduce first the forward martingale measure for the date T_1^*. By definition, it is a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, and such that processes

$$U_2(t,T_k^*) = \frac{B(t,T_k^*)}{B(t,T_1^*)}$$

are $\mathbb{Q}$-local martingales. It is important to observe that the process $U_2(t,T_k^*)$ admits the following representation

$$U_2(t,T_k^*) = \frac{U_1(t,T_k^*)}{\delta_n L(t,T_1^*) + 1}.$$

The following auxiliary result is a straightforward consequence of Itô's rule.

Lemma 12.4.1. *Let G and H be real-valued adapted processes, such that*

$$dG_t = \alpha_t \cdot dW_t, \quad dH_t = \beta_t \cdot dW_t.$$

Assume, in addition, that $H_t > -1$ for every t and write $Y_t = (1 + H_t)^{-1}$. Then

$$d(Y_t G_t) = Y_t\big(\alpha_t - Y_t G_t \beta_t\big) \cdot \big(dW_t - Y_t \beta_t\, dt\big).$$

It follows immediately from Lemma 12.4.1 that

$$dU_2(t,T_k^*) = \eta_t^k \cdot \Big(dW_t - \frac{\delta_n L(t,T_1^*)}{1 + \delta_n L(t,T_1^*)}\, \lambda(t,T_1^*)\, dt\Big)$$

for a certain process η^k.

It is therefore enough to find a probability measure under which the process

$$W_t^{T_1^*} \stackrel{\text{def}}{=} W_t - \int_0^t \frac{\delta_n L(u, T_1^*)}{1 + \delta_n L(u, T_1^*)} \lambda(u, T_1^*) \, du = W_t - \int_0^t \gamma(u, T_1^*) \, du,$$

where $t \in [0, T_1^*]$, follows a standard Brownian motion (the definition of $\gamma(t, T_1^*)$ is clear from the context). This can easily be achieved using Girsanov's theorem, as we may put

$$\frac{d\mathbb{P}_{T_1^*}}{d\mathbb{P}} = \mathcal{E}_{T_1^*}\Big(\int_0^{\cdot} \gamma(u, T_1^*) \cdot dW_u \Big), \quad \mathbb{P}\text{-a.s.}$$

We are in a position to specify the dynamics of the forward LIBOR for the date T_2^* under $\mathbb{P}_{T_1^*}$, namely we postulate that

$$dL(t, T_2^*) = L(t, T_2^*) \, \lambda(t, T_2^*) \cdot dW_t^{T_1^*},$$

with the initial condition

$$L(0, T_2^*) = \frac{B(0, T_2^*) - B(0, T_1^*)}{\delta_{n-1} B(0, T_1^*)}.$$

Let us now assume that we have found processes $L(t, T_1^*), \ldots, L(t, T_m^*)$. In particular, the forward LIBOR measure $\mathbb{P}_{T_{m-1}^*}$ and the associated Brownian motion $W^{T_{m-1}^*}$ are already specified. Our aim is to determine the forward LIBOR measure $\mathbb{P}_{T_m^*}$. It is easy to check that

$$U_{m+1}(t, T_k^*) = \frac{U_m(t, T_k^*)}{\delta_{n-m} L(t, T_m^*) + 1}.$$

Using Lemma 12.4.1, we obtain the following relationship

$$W_t^{T_m^*} = W_t^{T_{m-1}^*} - \int_0^t \frac{\delta_{n-m} L(u, T_m^*)}{1 + \delta_{n-m} L(u, T_m^*)} \lambda(u, T_m^*) \, du$$

for $t \in [0, T_m^*]$. The forward LIBOR measure $\mathbb{P}_{T_m^*}$ can thus be found easily using Girsanov's theorem. Finally, we define the process $L(t, T_{m+1}^*)$ as the solution to the SDE

$$dL(t, T_{m+1}^*) = L(t, T_{m+1}^*) \, \lambda(t, T_{m+1}^*) \cdot dW_t^{T_m^*},$$

with the initial condition

$$L(0, T_{m+1}^*) = \frac{B(0, T_{m+1}^*) - B(0, T_m^*)}{\delta_{n-m} B(0, T_m^*)}.$$

Remarks. If the volatility coefficient $\lambda(t, T_m) : [0, T_m] \to \mathbb{R}^d$ is deterministic, then, for each date $t \in [0, T_m]$, the random variable $L(t, T_m)$ has a lognormal probability law under the forward martingale measure $\mathbb{P}_{T_{m+1}}$. In this case, the model is referred to as the lognormal LIBOR model.

12.4.5 Jamshidian's Approach

The backward induction approach to modelling of forward LIBORs presented in the preceding section was re-examined and modified by Jamshidian (1997a). In this section, we present briefly his alternative approach to the modelling of forward LIBORs.

As was made apparent in the previous section, in the direct modelling of LIBORs, no explicit reference is made to the bond price processes, which are used to define formally a forward LIBOR through equality (12.23). Nevertheless, to explain the idea that underpins Jamshidian's approach, we shall temporarily assume that we are given a family of bond prices $B(t, T_j)$ for the future dates T_j, $j = 0, \dots, n$. By definition, the *spot LIBOR measure* is that probability measure equivalent to $\mathbb{P}$, under which all relative bond prices are local martingales, when the price process obtained by rolling over one-period bonds, is taken as a numeraire. The existence of such a measure can be either postulated, or derived from other conditions. Let us define

$$G_t = B(t, T_{m(t)}) \prod_{j=1}^{m(t)} B^{-1}(T_{j-1}, T_j), \tag{12.30}$$

for every $t \in [0, T^*]$, where we set

$$m(t) = \inf \{k \in \mathbf{N} \,|\, T_0 + \sum_{i=1}^{k} \delta_i \geq t\} = \inf \{k \in \mathbf{N} \,|\, T_k \geq t\}.$$

It is easily seen that G_t represents the wealth at time t of a portfolio that starts at time 0 with one unit of cash invested in a zero-coupon bond of maturity T_0, and whose wealth is then reinvested at each date T_j, $j = 0, \dots, n-1$, in zero-coupon bonds maturing at the next date; that is, at time T_{j+1}.

Definition 12.4.2. A *spot LIBOR measure* $\mathbb{P}^L$ is any probability measure on $(\Omega, \mathcal{F}_{T^*})$ equivalent to a reference probability $\mathbb{P}$, and such that the relative prices $B(t, T_j)/G_t$, $j = 1, \dots, n$, are local martingales under $\mathbb{P}^L$.

Note that

$$\frac{B(t, T_{k+1})}{G_t} = \prod_{j=1}^{m(t)} \big(1 + \delta_j L(T_{j-1}, T_{j-1})\big)^{-1} \prod_{j=m(t)+1}^{k} \big(1 + \delta_j L(t, T_{j-1})\big)^{-1},$$

so that all relative bond prices $B(t, T_j)/G_t$, $j = 1, \dots, n$ are uniquely determined by a collection of forward LIBORs. In this sense, the *rolling bond* G is the correct choice of the numeraire asset in the present set-up. We shall now concentrate on the derivation of the dynamics under $\mathbb{P}^L$ of forward LIBOR processes $L(t, T_j)$, $j = 1, \dots, n$. Our aim is to show that the joint dynamics of forward LIBORs involve only the volatilities of these processes (as opposed to volatilities of bond prices or some other processes).

Put differently, we shall show that it is possible to define the whole family of forward LIBORs simultaneously under a single probability measure (of course, this feature can also be deduced from the previously examined construction). To facilitate the derivation of the dynamics of $L(t, T_j)$, we postulate temporarily that bond prices $B(t, T_j)$ follow Itô processes under the underlying probability measure $\mathbb{P}$, more explicitly

$$dB(t, T_j) = B(t, T_j)\big(a(t, T_j)\, dt + b(t, T_j) \cdot dW_t\big) \tag{12.31}$$

for every $j = 1, \ldots, n$, where, as before, W is a d-dimensional standard Brownian motion under an underlying probability measure $\mathbb{P}$ (it should be stressed, however, that we do not assume here that $\mathbb{P}$ is a forward (or spot) martingale measure). Combining (12.30) with (12.31), we obtain

$$dG_t = G_t\big(a(t, T_{m(t)})\, dt + b(t, T_{m(t)}) \cdot dW_t\big).$$

Furthermore, by applying Itô's rule to equality

$$1 + \delta_{j+1} L(t, T_j) = \frac{B(t, T_j)}{B(t, T_{j+1})}, \tag{12.32}$$

we find that

$$dL(t, T_j) = \mu(t, T_j)\, dt + \zeta(t, T_j) \cdot dW_t,$$

where

$$\mu(t, T_j) = \frac{B(t, T_j)}{\delta_{j+1} B(t, T_{j+1})}\big(a(t, T_j) - a(t, T_{j+1})\big) - \zeta(t, T_j) b(t, T_{j+1})$$

and

$$\zeta(t, T_j) = \frac{B(t, T_j)}{\delta_{j+1} B(t, T_{j+1})}\big(b(t, T_j) - b(t, T_{j+1})\big). \tag{12.33}$$

Using (12.32) and (12.33), we arrive at the following relationship

$$b(t, T_{m(t)}) - b(t, T_{j+1}) = \sum_{k=m(t)}^{j} \frac{\delta_{k+1} \zeta(t, T_k)}{1 + \delta_{k+1} L(t, T_k)}. \tag{12.34}$$

By the definition of a spot LIBOR measure $\mathbb{P}^L$, each relative price process $B(t, T_j)/G_t$ follows a local martingale under $\mathbb{P}^L$. Since, in addition, $\mathbb{P}^L$ is assumed to be equivalent to $\mathbb{P}$, it is clear (because of Girsanov's theorem) that it is given by the Doléans exponential, that is,

$$\frac{d\mathbb{P}^L}{d\mathbb{P}} = \mathcal{E}_{T^*}\Big(\int_0^{\cdot} h_u \cdot dW_u\Big), \quad \mathbb{P}\text{-a.s.}$$

for some adapted process h. It it not hard to check, using Itô's rule, that h needs to satisfy, for every $t \in [0, T_j]$ and every $j = 1, \ldots, n$,

$$a(t, T_j) - a(t, T_{m(t)}) = \big(b(t, T_{m(t)}) - h_t\big) \cdot \big(b(t, T_j) - b(t, T_{m(t)})\big).$$

Combining (12.33) with the formula above, we obtain

$$\frac{B(t,T_j)}{\delta_{j+1}B(t,T_{j+1})}\big(a(t,T_j)-a(t,T_{j+1})\big)=\zeta(t,T_j)\cdot\big(b(t,T_{m(t)})-h_t\big),$$

and this in turn yields

$$dL(t,T_j)=\zeta(t,T_j)\cdot\Big(\big(b(t,T_{m(t)})-b(t,T_{j+1})-h_t\big)\,dt+dW_t\Big).$$

Using the last formula, (12.34) and Girsanov's theorem, we arrive at the following result, due to Jamshidian (1997a).

Proposition 12.4.1. *For any $j=0,\dots,n-1$, the process $L(t,T_j)$ satisfies*

$$dL(t,T_j)=\sum_{k=m(t)}^{j}\frac{\delta_{k+1}\zeta(t,T_k)\cdot\zeta(t,T_j)}{1+\delta_{k+1}L(t,T_k)}\,dt+\zeta(t,T_j)\cdot dW_t^L,$$

where the process $W_t^L=W_t-\int_0^t h_u\,du$ is a d-dimensional standard Brownian motion under the spot LIBOR measure $\mathbb{P}^L$.

To further specify the model, we postulate that the processes $\zeta(t,T_j)$, $j=1,\dots,n$ are exogenously given. Specifically, let

$$\zeta(t,T_j)=\lambda_j\big(t,L(t,T_j),L(t,T_{j+1}),\dots,L(t,T_n)\big),\quad\forall\,t\in[0,T_j],$$

where $\lambda_j:[0,T_j]\times\mathbb{R}^{n-j+1}\to\mathbb{R}^d$ are given functions. This leads to the following system of SDEs

$$dL(t,T_j)=\sum_{k=m(t)}^{j}\frac{\delta_{k+1}\lambda_k(t,L_k(t))\cdot\lambda_j(t,L_j(t))}{1+\delta_{k+1}L(t,T_k)}\,dt+\lambda_j(t,L_j(t))\cdot dW_t^L,$$

where, for brevity, we write

$$L_j(t)=(L(t,T_j),L(t,T_{j+1}),\dots,L(t,T_n)).$$

Under standard regularity assumptions imposed on the set of coefficients λ_j, this system of SDEs can be solved recursively, starting from the SDE for the process $L(t,T_{n-1})$. In this way, one can produce a large variety of alternative versions of a forward LIBOR model, including the CEV LIBOR model and a simple version of displaced-diffusion model (these stochastic volatility versions of a LIBOR model are presented briefly in Sect. 12.7).

Let us finally observe that the lognormal LIBOR market model corresponds to the choice of $\zeta(t,T_j)=\lambda(t,T_j)L(t,T_j)$, where $\lambda(t,T_j):[0,T_j]\to\mathbb{R}^d$ is a deterministic function for every $j=0,\dots n-1$. In this case we deal with the following system of SDEs

$$\frac{dL(t,T_j)}{L(t,T_j)}=\sum_{k=m(t)}^{j}\frac{\delta_{k+1}L(t,T_k)\lambda(t,T_k)\cdot\lambda(t,T_j)}{1+\delta_{k+1}L(t,T_k)}\,dt+\lambda(t,T_j)\cdot dW_t^L.$$

If we decide to use the probability measure $\mathbb{P}^L$ to value a given contingent claim X, its arbitrage price will be expressed in units of the rolling bond G.

12.5 Properties of the Lognormal LIBOR Model

We make the standing assumptions the volatilities of forward LIBORs $L(t, T_j)$ for $j = 0, \dots, n-1$ are deterministic. In other words, we place ourselves within the framework of the lognormal LIBOR model. It is interesting to note that in all approaches, there is a uniquely determined correspondence between forward measures (and forward Brownian motions) associated with different dates (it is based on relationships (12.3) and (12.8)). On the other hand, however, there is a considerable degree of ambiguity in the way in which the spot martingale measure is specified (in some instances, it is not introduced at all). Consequently, the futures LIBOR $L^f(t, T_j)$, which equals (cf. Sect. 12.1.3)

$$L^f(t, T_j) = \mathbb{E}_{\mathbb{P}^*}(L(T_j, T_j) \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}(\tilde{L}(T_j, T_j) \,|\, \mathcal{F}_t), \tag{12.35}$$

is not necessarily specified in the same way in various approaches to the LIBOR market model.

For a given function $g : \mathbb{R} \to \mathbb{R}$ and a fixed date $u \leq T_j$, we are interested in the payoff of the form $X = g\big(L(u, T_j)\big)$ that settles at time T_j. Particular cases of such payoffs are

$$X_1 = g\big(B^{-1}(T_j, T_{j+1})\big), \ X_2 = g\big(B(T_j, T_{j+1})\big), \ X_3 = g\big(F_B(u, T_{j+1}, T_j)\big).$$

Recall that

$$B^{-1}(T_j, T_{j+1}) = 1 + \delta_{j+1} L(T_j) = 1 + \delta_{j+1} \tilde{L}(T_j) = 1 + \delta_{j+1} L^f(T_j).$$

The choice of the “pricing measure” is thus largely matter of convenience. Similarly, we have

$$B(T_j, T_{j+1}) = \frac{1}{1 + \delta_{j+1} L(T_j, T_j)} = F_B(T_j, T_{j+1}, T_j). \tag{12.36}$$

More generally, the forward price of a T_{j+1}-maturity bond for the settlement date T_j equals

$$F_B(u, T_{j+1}, T_j) = \frac{B(u, T_{j+1})}{B(u, T_j)} = \frac{1}{1 + \delta_{j+1} L(u, T_j)}. \tag{12.37}$$

To value a European contigent claim $X = g(L(u, T_j)) = \tilde{g}(F_B(u, T_{j+1}, T_j))$ settling at time T_j we may use the risk-neutral valuation formula

$$\pi_t(X) = B(t, T_j) \, \mathbb{E}_{\mathbb{P}_{T_j}}(X \,|\, \mathcal{F}_t), \quad \forall \, t \in [0, T_j].$$

It is thus clear that to value a claim in the case $u \leq T_j$, it is enough to know the dynamics of either $L(t, T_j)$ or $F_B(t, T_{j+1}, T_j)$ under the forward martingale measure $\mathbb{P}_{T_j}$. When $u = T_j$, we may equally well use the dynamics under $\mathbb{P}_{T_j}$ of either the process $\tilde{L}(t, T_j)$, or the $L^f(t, T_j)$.

For instance,

$$\pi_t(X_1) = B(t,T_j)\mathbb{E}_{\mathbb{P}_{T_j}}(B^{-1}(T_j,T_{j+1}) \,|\, \mathcal{F}_t),$$

or equivalently,

$$\pi_t(X_1) = B(t,T_j)\mathbb{E}_{\mathbb{P}_{T_j}}(F_B^{-1}(T_j,T_{j+1},T_j) \,|\, \mathcal{F}_t),$$

but also

$$\pi_t(X_1) = B(t,T_j)\big(1+\delta_{j+1}\mathbb{E}_{\mathbb{P}_{T_j}}(Z(T_j) \,|\, \mathcal{F}_t)\big),$$

where $Z(T_j) = L(T_j) = \tilde{L}(T_j) = L^f(T_j)$.

12.5.1 Transition Density of the LIBOR

We shall now derive the transition probability density function of the process $L(t,T_j)$ under the forward martingale measure $\mathbb{P}_{T_j}$. Let us first prove the following related result that is of independent interest (it was first established by Jamshidian (1993)).

Proposition 12.5.1. *Let $t \le u \le T_j$. Then we have*

$$\mathbb{E}_{\mathbb{P}_{T_j}}\big(L(u,T_j) \,|\, \mathcal{F}_t\big) = L(t,T_j) + \frac{\delta_{j+1}\mathrm{Var}_{\mathbb{P}_{T_{j+1}}}\big(L(u,T_j) \,|\, \mathcal{F}_t\big)}{1+\delta_{j+1}L(t,T_j)}.$$

In the case of the lognormal model of forward LIBORs, we have

$$\mathbb{E}_{\mathbb{P}_{T_j}}\big(L(u,T_j) \,|\, \mathcal{F}_t\big) = L(t,T_j)\left(1+\frac{\delta_{j+1}L(t,T_j)\big(e^{v_j^2(t,u)}-1\big)}{1+\delta_{j+1}L(t,T_j)}\right),$$

where

$$v_j^2(t,u) = \mathrm{Var}_{\mathbb{P}_{T_{j+1}}}\Big(\int_t^u \lambda(s,T_j)\cdot dW_s^{T_{j+1}}\Big) = \int_t^u |\lambda(s,T_j)|^2\,ds.$$

Proof. Combining (12.5) with the martingale property of $L(t,T_j)$ under $\mathbb{P}_{T_{j+1}}$, we obtain

$$\mathbb{E}_{\mathbb{P}_{T_j}}\big(L(u,T_j) \,|\, \mathcal{F}_t\big) = \frac{\mathbb{E}_{\mathbb{P}_{T_{j+1}}}\big((1+\delta_{j+1}L(u,T_j))L(u,T_j) \,|\, \mathcal{F}_t\big)}{1+\delta_{j+1}L(t,T_j)},$$

so that

$$\mathbb{E}_{\mathbb{P}_{T_j}}\big(L(u,T_j) \,|\, \mathcal{F}_t\big) = L(t,T_j) + \frac{\delta_{j+1}\,\mathbb{E}_{\mathbb{P}_{T_{j+1}}}\big((L(u,T_j)-L(t,T_j))^2 \,|\, \mathcal{F}_t\big)}{1+\delta_{j+1}L(t,T_j)}.$$

In the case of the lognormal model, we have

$$L(u,T_j) = L(t,T_j)\,e^{\eta(t,u)-\frac{1}{2}v_j^2(t,u)},$$

where

$$\eta(t,u) = \int_t^u \lambda(s,T_j)\,dW_s^{T_{j+1}}. \tag{12.38}$$

Consequently,

$$\mathbb{E}_{\mathbb{P}_{T_{j+1}}}\big((L(u,T_j) - L(t,T_j))^2 \,|\, \mathcal{F}_t\big) = L^2(t,T_j)\big(e^{v_j^2(t,u)} - 1\big).$$

This gives the desired equality. □

To derive the transition probability density function of the process $L(t,T_j)$, note that for any $t \le u \le T_j$ and any bounded Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ we have

$$\mathbb{E}_{\mathbb{P}_{T_j}}\big(g(L(u,T_j)) \,|\, \mathcal{F}_t\big) = \frac{\mathbb{E}_{\mathbb{P}_{T_{j+1}}}\Big(g(L(u,T_j))\,\big(1 + \delta_{j+1}L(u,T_j)\big) \,\Big|\, \mathcal{F}_t\Big)}{1 + \delta_{j+1}L(t,T_j)}.$$

The following simple lemma appears to be useful in what follows.

Lemma 12.5.1. *Let ζ be a nonnegative random variable on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with the probability density function $f_{\mathbb{P}}$. Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$. Suppose that for any bounded Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ we have*

$$\mathbb{E}_{\mathbb{P}}(g(\zeta)) = \mathbb{E}_{\mathbb{Q}}\big((1+\zeta)g(\zeta)\big).$$

Then the probability density function $f_{\mathbb{Q}}$ of ζ under $\mathbb{Q}$ satisfies $f_{\mathbb{P}}(y) = (1 + y)f_{\mathbb{Q}}(y)$.

Proof. The assertion is in fact trivial since, by assumption,

$$\int_{-\infty}^{\infty} g(y)f_{\mathbb{P}}(y)\,dy = \int_{-\infty}^{\infty} g(y)(1+y)f_{\mathbb{Q}}(y)\,dy$$

for any bounded Borel measurable function $g : \mathbb{R} \to \mathbb{R}$. □

Assume the lognormal LIBOR model, and fix $x \in \mathbb{R}$. Recall that for any $t \ge u$ we have

$$L(u,T_j) = L(t,T_j)\, e^{\eta(t,u) - \frac{1}{2}\mathrm{Var}_{\mathbb{P}_{T_{j+1}}}(\eta(t,u))},$$

where $\eta(t,u)$ is given by (12.38) (so that it is independent of the σ-field $\mathcal{F}_t$). The Markovian property of $L(t,T_j)$ under the forward measure $\mathbb{P}_{T_{j+1}}$ is thus apparent. Denote by $p_L(t,x,u,y)$ the transition probability density function under $\mathbb{P}_{T_{j+1}}$ of the process $L(t,T_j)$. Elementary calculations involving Gaussian densities yield

$$p_L(t,x,u,y) = \frac{1}{\sqrt{2\pi}v_j(t,u)y}\,\exp\left\{-\frac{\big(\ln(y/x) + \frac{1}{2}v_j^2(t,u)\big)^2}{2v_j^2(t,u)}\right\}$$

for any $x, y > 0$ and arbitrary $t < u$, where

$$p_L(t,x,u,y) = \mathbb{P}_{T_{j+1}}\{L(u,T_j) = y \,|\, L(t,T_j) = x\}.$$

Taking into account Lemma 12.5.1, we conclude that the transition probability density function of the process[1] $L(t,T_j)$, under the forward martingale measure $\mathbb{P}_{T_j}$, satisfies

$$\tilde{p}_L(t,x,u,y) = \mathbb{P}_{T_j}\{L(u,T_j) = y \,|\, L(t,T_j) = x\} = \frac{1+\delta_{j+1}y}{1+\delta_{j+1}x}\, p_L(t,x,u,y).$$

We are now in a position to state the following result.

Corollary 12.5.1. *The transition probability density function under $\mathbb{P}_{T_j}$ of the forward LIBOR $L(t,T_j)$ equals*

$$\tilde{p}_L(t,x,u,y) = \frac{1+\delta_{j+1}y}{\sqrt{2\pi}v_j(t,u)\, y(1+\delta_{j+1}x)} \exp\left\{ -\frac{\big(\ln(y/x) + \frac{1}{2}v_j^2(t,u)\big)^2}{2v_j^2(t,u)} \right\}$$

for any $t < u$ and arbitrary $x, y > 0$.

12.5.2 Transition Density of the Forward Bond Price

Observe that the forward bond price $F_B(t,T_{j+1},T_j)$ satisfies

$$F_B(t,T_{j+1},T_j) = \frac{B(t,T_{j+1})}{B(t,T_j)} = \frac{1}{1+\delta_{j+1}L(t,T_j)}. \tag{12.39}$$

First, this implies that in the lognormal LIBOR model, the dynamics of the forward bond price $F_B(t,T_{j+1},T_j)$ are governed by the following stochastic differential equation, under $\mathbb{P}_{T_j}$,

$$dF_B(t) = -F_B(t)\big(1 - F_B(t)\big)\lambda(t,T_j)\cdot dW_t^{T_j}, \tag{12.40}$$

where we write $F_B(t) = F_B(t,T_{j+1},T_j)$. If the initial condition in (12.40) satisfies $0 < F_B(0) < 1$, then this equation can be shown to admit a unique strong solution (it satisfies $0 < F_B(t) < 1$ for every $t > 0$). This makes it clear that the forward bond price $F_B(t,T_{j+1},T_j)$, and thus also the LIBOR $L(t,T_j)$, are Markovian under $\mathbb{P}_{T_j}$.

Using the formula established in Corollary 12.5.1 and relationship (12.39), one can find the transition probability density function of the Markov process $F_B(t,T_{j+1},T_j)$ under $\mathbb{P}_{T_j}$; that is,

$$p_B(t,x,u,y) = \mathbb{P}_{T_j}\{F_B(u,T_{j+1},T_j) = x \,|\, F_B(t,T_{j+1},T_j) = y\}.$$

We have the following result (see Rady and Sandmann (1994), Miltersen et al. (1997) and Jamshidian (1997a)).

[1] The Markov property of $L(t,T_j)$ under $\mathbb{P}_{T_j}$ follows from the properties of the forward price, which will be established in Sect. 12.5.2.

Corollary 12.5.2. *The transition probability density function under $\mathbb{P}_{T_j}$ of the forward bond price $F_B(t, T_{j+1}, T_j)$ equals*

$$p_B(t,x,u,y) = \frac{x}{\sqrt{2\pi}v_j(t,u)y^2(1-y)} \exp\left\{ -\frac{\left(\ln \frac{x(1-y)}{y(1-x)} + \frac{1}{2}v_j^2(t,u)\right)^2}{2v_j^2(t,u)} \right\}$$

for any $t < u$ and arbitrary $0 < x, y < 1$.

Proof. Let us fix $x \in (0,1)$. Using (12.39), it is easy to show that

$$p_B(t,x,u,y) = y^{-2}\, \tilde{p}_L\Big(t, \frac{1-x}{\delta x}, u, \frac{1-y}{\delta y}\Big),$$

where $\delta = \delta_{j+1}$. The formula now follows from Corollary 12.5.1. □

Goldys (1997) established the following result (we find it convenient to defer the proof of equality (12.42) to Sect. 12.6.3).

Proposition 12.5.2. *Let X be a solution of the following stochastic differential equation*

$$dX_t = -X_t\big(1 - X_t\big)\lambda(t) \cdot dW_t, \quad X_0 = x, \tag{12.41}$$

where W follows a standard Brownian motion under $\mathbb{P}$, and $\lambda : \mathbb{R}_+ \to \mathbb{R}$ is a locally square integrable function. Then for any nonnegative Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ and any $u > 0$ we have

$$\mathbb{E}_{\mathbb{P}}(g(X_u)) = (1-x)\mathbb{E}_{\mathbb{Q}}\big(g(h_1(\zeta))\big) + x\mathbb{E}_{\mathbb{Q}}\big(g(h_2(\zeta))\big), \tag{12.42}$$

where ζ has, under $\mathbb{Q}$, a Gaussian law with zero mean value and variance

$$\mathrm{Var}_{\mathbb{Q}}(\zeta) = v^2(0,u) = \int_0^u |\lambda(s)|^2\, ds.$$

Furthermore,

$$\frac{1}{h_{1,2}(y)} = 1 + \exp\Big(y + \ln\frac{1-x}{x} \pm \frac{1}{2}v^2(0,u)\Big), \quad \forall\, y \in \mathbb{R}.$$

Before we end this chapter, we shall check that formula (12.39), which gives the transition probability density function of the forward bond price, can be re-derived using formula (12.42). According to (12.42), we have $\mathbb{E}_{\mathbb{P}_{T_j}}(g(X_u)) = I_1 + I_2$, where

$$I_1 = \frac{1-x}{\sqrt{2\pi}v(0,u)} \int_{-\infty}^{\infty} g(h_1(z))\, e^{-z^2/2v^2(0,u)}\, dz$$

and

$$I_2 = \frac{x}{\sqrt{2\pi}v(0,u)} \int_{-\infty}^{\infty} g(h_2(z))\, e^{-z^2/2v^2(0,u)}\, dz.$$

First, let us set $y = h_1(z)$ in I_1, so that $dy = y(1-y)dz$. Then we obtain

$$I_1 = \frac{1-x}{\sqrt{2\pi}v(0,u)} \int_0^1 \frac{g(y)}{y(1-y)}\, k(x,y)\, dy$$

where we set

$$k(x,y) = \exp\left\{ -\frac{\left(\ln \frac{x(1-y)}{y(1-x)} - \frac{1}{2}v^2(0,u)\right)^2}{2v^2(0,u)} \right\}.$$

Equivalently, we have

$$I_1 = \frac{1-x}{\sqrt{2\pi}v(0,u)} \int_0^1 \frac{xg(y)}{(1-x)y^2}\, \tilde{k}(x,y)\, dy.$$

where

$$\tilde{k}(x,y) = \exp\left\{ -\frac{\left(\ln \frac{x(1-y)}{y(1-x)} + \frac{1}{2}v^2(0,u)\right)^2}{2v^2(0,u)} \right\}.$$

Similarly, the change of variable $y = h_2(z)$ in I_2 (so that once again $dy = y(1-y)\,dz$) leads to the following equality

$$I_2 = \frac{x}{\sqrt{2\pi}v(0,u)} \int_0^1 \frac{g(y)}{y(1-y)}\, \tilde{k}(x,y)\, dy.$$

Simple algebra now yields (recall that $\mathbb{E}_{\mathbb{P}_{T_j}}(g(X_u)) = I_1 + I_2$)

$$I_1 + I_2 = \frac{1}{\sqrt{2\pi}v(0,u)} \int_0^1 \frac{xg(y)}{y^2(1-y)} \exp\left\{ -\frac{\left(\ln \frac{x(1-y)}{y(1-x)} + \frac{1}{2}v^2(0,u)\right)^2}{2v^2(0,u)} \right\} dy.$$

It is interesting to note that the formula above generalizes easily to the case $0 \le t < u$. Indeed, it is enough to consider the stochastic differential equation (12.41) with the initial condition $X_t = x$ at time t. As a result, we obtain the following formula for the conditional expectation

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}_{T_j}}(g(X_u) \,|\, X_t = x) \\ &\quad = \frac{1}{\sqrt{2\pi}v(t,u)} \int_0^1 \frac{xg(y)}{y^2(1-y)} \exp\left\{ -\frac{\left(\ln \frac{x(1-y)}{y(1-x)} + \frac{1}{2}v^2(t,u)\right)^2}{2v^2(t,u)} \right\} dy. \end{aligned}$$

It is worth observing that an application of the last formula to the process $X_t = F_B(t, T_{j+1}, T_j)$ leads to an alternative derivation of the formula established in Corollary 12.5.2.

12.6 Valuation in the Lognormal LIBOR Model

We start by considering the valuation of plain-vanilla caps and floors. Subsequently, we shall study the case of a generic path-independent claim.

12.6.1 Pricing of Caps and Floors

We shall now examine the valuation of caps within the lognormal LIBOR model of Sect. 12.4.4. To this end, we formally assume that $k \le n$. Dynamics of the forward LIBOR process $L(t, T_{j-1})$ under the forward martingale measure $\mathbb{P}_{T_j}$ are known to be

$$dL(t, T_{j-1}) = L(t, T_{j-1})\, \lambda(t, T_{j-1}) \cdot dW_t^{T_j}, \tag{12.43}$$

where W^{T_j} is a d-dimensional Brownian motion under the forward measure $\mathbb{P}_{T_j}$, and $\lambda(t, T_{j-1}) : [0, T_{j-1}] \to \mathbb{R}^d$ is a deterministic function. Consequently, for every $t \in [0, T_{j-1}]$ we have

$$L(t, T_{j-1}) = L(0, T_{j-1})\, \mathcal{E}_t\Big(\int_0^{\cdot} \lambda(u, T_{j-1}) \cdot dW_u^{T_j} \Big).$$

To the best of our knowledge, the cap valuation formula (12.44) was first established in a rigorous way by Miltersen et al. (1997), who focused on the dynamics of the forward LIBOR for a given date. Equality (12.44) was subsequently re-derived independently in Goldys (1997) and Rady (1997), who used probabilistic methods (though they dealt with a European bond option, their results are essentially equivalent to equality (12.44)). Finally, the same was established by means of the forward measure approach in Brace et al. (1997), where an arbitrage-free continuous-time model of all forward LIBORs was presented. It is instructive to compare the valuation formula (12.44) with the formula of Proposition 12.3.1, which holds for a Gaussian HJM case. The following proposition is an immediate consequence of formulas (12.20)-(12.21), combined with dynamics (12.43). Since the proof of the next result is rather standard, it is provided for the sake of completeness.

Proposition 12.6.1. *Consider an interest rate cap with strike level κ, settled in arrears at times T_j, $j = 1, \dots, k$. Assuming the lognormal LIBOR model, the price of a cap at time $t \in [0, T]$ equals*

$$\mathbf{FC}_t = \sum_{j=1}^{k} \delta_j B(t, T_j) \Big(L(t, T_{j-1}) N\big(\tilde{e}_1^j(t)\big) - \kappa\, N\big(\tilde{e}_2^j(t)\big)\Big), \tag{12.44}$$

where

$$\tilde{e}_{1,2}^j(t) = \frac{\ln(L(t, T_{j-1})/\kappa) \pm \frac{1}{2}\, \tilde{v}_j^2(t)}{\tilde{v}_j(t)}$$

and

$$\tilde{v}_j^2(t) = \int_t^{T_{j-1}} |\lambda(u, T_{j-1})|^2\, du.$$

Proof. We fix j and we consider the j^{th} caplet, with the payoff at time T_j

$$\mathbf{Cpl}^{\,j}_{T_j} = \delta_j (L(T_{j-1}) - \kappa)^+ = \delta_j L(T_{j-1})\, \mathbb{1}_D - \delta_j \kappa\, \mathbb{1}_D, \tag{12.45}$$

where $D = \{L(T_{j-1}) > \kappa\}$ is the exercise set. Since the caplet settles at time T_j, it is convenient to use the forward measure $\mathbb{P}_{T_j}$ to find its arbitrage price. We have

$$\mathbf{Cpl}^{\,j}_t = B(t,T_j)\, \mathbb{E}_{\mathbb{P}_{T_j}}\big(\mathbf{Cpl}^{\,j}_{T_j} \,|\, \mathcal{F}_t\big), \quad \forall\, t \in [0,T_j].$$

Obviously, it is enough to find the value of a caplet for $t \in [0, T_{j-1}]$. In view of (12.45), it suffices to compute the following conditional expectations

$$\begin{aligned}\mathbf{Cpl}^{\,j}_t &= \delta_j B(t,T_j)\, \mathbb{E}_{\mathbb{P}_{T_j}}\big(L(T_{j-1})\, \mathbb{1}_D \,\big|\, \mathcal{F}_t\big) - \kappa\delta_j B(t,T_j)\, \mathbb{P}_{T_j}(D \,|\, \mathcal{F}_t)\\ &= \delta_j B(t,T_j)(I_1 - I_2),\end{aligned}$$

where the meaning of I_1 and I_2 is clear from the context. Recall that the spot LIBOR $L(T_{j-1}) = L(T_{j-1}, T_{j-1})$ is given by the formula

$$L(T_{j-1}) = L(t,T_{j-1}) \exp\Big(\int_t^{T_{j-1}} \lambda^{j-1}_u \cdot dW^{T_j}_u - \frac{1}{2}\int_t^{T_{j-1}} |\lambda^{j-1}_u|^2\, du \Big),$$

where we set $\lambda^{j-1}_u = \lambda(u, T_{j-1})$. Since λ^{j-1} is a deterministic function, the probability law under $\mathbb{P}_{T_j}$ of the Itô integral

$$\zeta(t,T_{j-1}) = \int_t^{T_{j-1}} \lambda^{j-1}_u \cdot dW^{T_j}_u$$

is Gaussian, with zero mean and the variance

$$\mathrm{Var}_{\mathbb{P}_{T_j}}\,(\zeta(t,T_{j-1})) = \int_t^{T_{j-1}} |\lambda^{j-1}_u|^2\, du.$$

It is thus straightforward to check that

$$I_2 = \kappa\, N\left(\frac{\ln(L(t,T_{j-1}) - \ln\kappa - \frac{1}{2}v_j^2(t)}{v_j(t)} \right).$$

To evaluate I_1, we introduce an auxiliary probability measure $\hat{\mathbb{P}}_{T_j}$, equivalent to $\mathbb{P}_{T_j}$ on $(\Omega, \mathcal{F}_{T_{j-1}})$, by setting

$$\frac{d\hat{\mathbb{P}}_{T_j}}{d\mathbb{P}_{T_j}} = \mathcal{E}_{T_{j-1}}\Big(\int_0^{\cdot} \lambda^{j-1}_u \cdot dW^{T_j}_u \Big).$$

Then the process $\hat{W}^{T_j}$, given by the formula

$$\hat{W}^{T_j}_t = W^{T_j}_t - \int_0^t \lambda^{j-1}_u\, du, \quad \forall\, t \in [0, T_{j-1}],$$

is the d-dimensional standard Brownian motion under $\hat{\mathbb{P}}_{T_j}$.

Furthermore, the forward price $L(T_{j-1})$ admits the following representation under $\hat{\mathbb{P}}_{T_j}$, for $t \in [0, T_{j-1}]$,

$$L(T_{j-1}) = L(t, T_{j-1}) \exp\Big(\int_t^{T_{j-1}} \lambda_u^{j-1} \cdot d\hat{W}_u^{T_j} + \frac{1}{2} \int_t^{T_{j-1}} |\lambda_u^{j-1}|^2 du \Big).$$

Since

$$I_1 = L(t, T_{j-1}) \mathbb{E}_{\mathbb{P}_{T_j}} \Big(\mathbb{1}_D \exp\Big(\int_t^{T_{j-1}} \lambda_u^{j-1} \cdot dW_u^{T_j} - \frac{1}{2} \int_t^{T_{j-1}} |\lambda_u^{j-1}|^2 du \Big) \Big| \mathcal{F}_t \Big)$$

from the abstract Bayes rule we get $I_1 = L(t, T_{j-1})\, \hat{\mathbb{P}}_{T_j}(D \,|\, \mathcal{F}_t)$. Arguing in much the same way as for I_2, we thus obtain

$$I_1 = L(t, T_{j-1})\, N \left(\frac{\ln L(t, T_{j-1}) - \ln \kappa + \frac{1}{2} v_j^2(t)}{v_j(t)} \right).$$

This completes the proof of the proposition. □

As before, to derive the valuation formula for a floor, it is enough to make use of the cap-floor parity.

12.6.2 Hedging of Caps and Floors

It is clear the replicating strategy for a cap is a simple sum of replicating strategies for caplets. It is therefore enough to focus on a particular caplet. Let us denote by $F_C(t, T_j)$ the forward price of the j^{th} caplet for the settlement date T_j. From (12.44), it is clear that

$$F_C(t, T_j) = \delta_j L(t, T_{j-1}) N\big(\tilde{e}_1^j(t)\big) - \kappa\, N\big(\tilde{e}_2^j(t)\big),$$

so that an application of Itô's formula yields (the calculations here are essentially the same as in the classical Black-Scholes model)

$$dF_C(t, T_j) = \delta_j N\big(\tilde{e}_1^j(t)\big)\, dL(t, T_{j-1}).$$

Let us consider the following self-financing trading strategy in the T_j-forward market, that is, with all values expressed in units of T_j-maturity zero-coupon bonds. We start our trade at time 0 with $F_C(0, T_j)$ units of zero-coupon bonds; we need thus to invest $\mathbf{Cpl}_0^j = F_C(0, T_j) B(0, T_j)$ of cash at time 0. At any time $t \le T_{j-1}$, we take $\psi_t^j = N\big(\tilde{e}_1^j(t)\big)$ positions in one-period forward swaps over the period $[T_{j-1}, T_j]$. The associated gains process $\tilde{G}$, in the T_j forward market, satisfies $\tilde{G}_0 = 0$ and[2]

$$d\tilde{G}_t = \delta_j \psi_t^j\, dL(t, T_{j-1}) = \delta_j N\big(\tilde{e}_1^j(t)\big)\, dL(t, T_{j-1}) = dF_C(t, T).$$

[2] To get a more intuitive insight into this formula, it is convenient to consider first a discretized version of ψ.

Consequently,

$$F_C(T_{j-1},T_j) = F_C(0,T_j) + \int_0^{T_{j-1}} \delta_j \psi_t^j \, dL(t,T_{j-1}) = F_C(0,T_j) + \tilde{G}_{T_{j-1}}.$$

It should be stressed that dynamic trading is restricted to the interval $[0,T_{j-1}]$ only. The gains/losses (involving the initial investment) are incurred at time T_j, however. All quantities in the last formula are expressed in units of T_j-maturity zero-coupon bonds. Also, the caplet's payoff is known already at time T_{j-1}, so that it is completely specified by its forward price $F_C(T_{j-1},T_j) = \mathbf{Cpl}^j_{T_{j-1}}/B(T_{j-1},T_j)$. It is thus clear that the strategy ψ introduced above replicates the j^{th} caplet.

Formally, the replicating strategy also has the second component, η_t^j say, representing the number of forward contracts, with the settlement date T_j, on T_j-maturity bond. Note that $F_B(t,T_j,T_j)=1$ for every $t \le T_j$, and thus $dF_B(t,T_j,T_j)=0$. Hence, for the T_j-forward value of our strategy, we get

$$\tilde{V}_t(\psi^j,\eta^j) = \eta_t^j = F_C(t,T_j)$$

and

$$d\tilde{V}_t(\psi^j,\eta^j) = \psi_t^j \delta_j \, dL(t,T_{j-1}) + \eta_t^j \, dF_B(t,T_j,T_j) = \delta_j N\big(\tilde{e}_1^j(t)\big)\, dL(t,T_{j-1}).$$

It should be stressed that, with the exception for the initial investment at time 0 in T_j-maturity bonds, no trading in bonds is required for replication of a caplet. In practical terms, the hedging of a cap within the framework of the lognormal LIBOR model in done exclusively through dynamic trading in the underlying one-period forward swaps. In this interpretation, the component η^j represents simply the future (i.e., as of time T_{j-1}) effects of continuous trading in forward contracts. The same remarks (and similar calculations) apply also to floors.

Alternatively, replication of a caplet can be done in the spot (that is, cash) market, using two simple portfolios of bonds. Indeed, it is easily seen that for the process

$$V_t(\psi^j,\eta^j) = B(t,T_{j-1})\tilde{V}_t(\psi^j,\eta^j) = \mathbf{Cpl}^j_t$$

we have

$$V_t(\psi^j,\eta^j) = \psi_t^j\big(B(t,T_{j-1}) - B(t,T_j)\big) + \eta_t^j \, dF_B(t,T_j,T_j)$$

and

$$\begin{aligned} dV_t(\psi^j,\eta^j) &= \psi_t^j \, d\big(B(t,T_{j-1}) - B(t,T_j)\big) + \eta_t^j \, dB(t,T_j) \\ &= N\big(\tilde{e}_1^j(t)\big)\, d\big(B(t,T_{j-1}) - B(t,T_j)\big) + \eta_t^j \, dB(t,T_j). \end{aligned}$$

In this interpretation, the components ψ^j and η^j represent the number of units of portfolios $B(t,T_{j-1}) - B(t,T_j)$ and $B(t,T_j)$ that are held at time t.

12.6.3 Valuation of European Claims

We follow here the approach due to Goldys (1997). Let X be a solution to the stochastic differential equation

$$\begin{cases} dX_t = -X_t(1-X_t)\lambda(t)\cdot dW_t, \\ X_0 = x, \end{cases} \tag{12.46}$$

where we assume that the function $\lambda : [0,T^*] \to \mathbb{R}^d$ is bounded and measurable and, as usual, W is a standard Brownian motion defined on a filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$. For any $x \in (0,1)$, the existence of a unique global solution to (12.46) can be deduced easily from the general theory of stochastic differential equations. However, in Lemma 12.6.1 below we provide a direct proof by means of a simple transformation which is also crucial for the further calculations. Consider the following stochastic differential equation

$$dZ_t = \frac{1}{2}\frac{1-e^{-Z_t}}{1+e^{-Z_t}}\,|\lambda(t)|^2 dt - \lambda(t)\cdot dW_t \tag{12.47}$$

with the initial condition $Z_0 = z$. Since the drift term in this equation is represented by a bounded and globally Lipschitz function, equation (12.47) is known to have a unique strong non-exploding solution for any initial condition $z \in \mathbb{R}$ (see, e.g., Theorem 5.2.9 in Karatzas and Shreve (1998a)).

Lemma 12.6.1. *For any $x \in (0,1)$, the process*

$$X_t = \big(1+e^{-Z_t}\big)^{-1}, \quad \forall\, t \in [0,T^*], \tag{12.48}$$

where z satisfies

$$Z_0 = z = \ln\frac{x}{1-x}$$

is the unique strong and non-exploding solution to equation (12.46)*. Moreover, $0 < X_t < 1$ for every $t \in [0,T^*]$.*

Proof. It is easy to see that equation (12.47) can be rewritten in the form

$$dZ_t = \big(X_t - \tfrac{1}{2}\big)|\lambda(t)|^2\, dt - \lambda(t)\cdot dW_t.$$

Hence, applying the Itô formula to the process X given by formula (12.48), we find that

$$dX_t = -X_t(1-X_t)\lambda(t)\cdot dW_t.$$

Hence, the process X is indeed a solution to (12.46). Conversely, if X is any local (weak) solution to (12.46) then it is in fact a strong solution because the diffusion coefficient in (12.46) is locally Lipschitz. Moreover, using the Itô formula again, one can check that the process

$$Z_t = \ln\frac{X_t}{1-X_t}, \quad \forall\, t \in [0,T^*],$$

is a strong solution of (12.47). Thus, it can be continued to a global one which is unique. The last part of the lemma follows from the definition of the process X, and the uniqueness of solutions to (12.46). □

The next result provides a representation of the expected value $\mathbb{E}_{\mathbb{P}}(g(X_t))$ in terms of the integral with respect to the Gaussian probability law.

Proposition 12.6.2. *Let $g : [0,1] \to \mathbb{R}$ be a nonnegative Borel function such that the random variable $g(X_t)$ is $\mathbb{P}$-integrable for some initial condition $x \in (0,1)$ and some $t > 0$. Then the expected value $\mathbb{E}_{\mathbb{P}}(g(X_t))$ is given by the formula*

$$\mathbb{E}_{\mathbb{P}}(g(X_t)) = \sqrt{x(1-x)}e^{-\frac{v^2(0,t)}{8}}\,\mathbb{E}_{\mathbb{Q}}\left\{\big(\eta^{-1}+\eta\big)g\big((1+\eta^2)^{-1}\big)\right\},$$

where

$$\eta = \exp\Big(-\frac{z+\zeta}{2}\Big), \quad z = \ln\frac{x}{1-x},$$

and the random variable ζ has under $\mathbb{Q}$ a Gaussian law with zero mean value and the variance

$$v^2(0,t) = \int_0^t |\lambda(u)|^2 du.$$

Proof. The proof of the proposition, due to Goldys (1997), is based on a simple idea that for any $\mathbb{P}$-integrable random variable, U say, we have

$$\mathbb{E}_{\mathbb{P}}U = \mathbb{E}_{\mathbb{P}}(X_tU) + \mathbb{E}_{\mathbb{P}}(Y_tU), \tag{12.49}$$

where $Y_t = 1 - X_t$. Next, it is essential to observe that

$$X_t = x\,\mathcal{E}_t\Big(-\int_0^{\cdot} Y_u\lambda(u)\cdot dW_u\Big) \tag{12.50}$$

and

$$Y_t = (1-x)\,\mathcal{E}_t\Big(\int_0^{\cdot} X_u\lambda(u)\cdot dW_u\Big). \tag{12.51}$$

Furthermore, from (12.47) it follows that

$$dZ_t = -\lambda(t)\cdot\Big(dW_t - \big(X_t - \tfrac{1}{2}\big)\lambda(t)\,dt\Big),$$

or equivalently,

$$dZ_t = -\lambda(t)\cdot\Big(dW_t - \big(Y_t + \tfrac{1}{2}\big)\lambda(t)\,dt\Big).$$

Let us introduce the auxiliary probability measures $\tilde{\mathbb{P}}$ and $\hat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_t)$ by setting

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_t\Big(\int_0^{\cdot} \big(X_u - \tfrac{1}{2}\big)\lambda(u)\cdot dW_u\Big) \stackrel{\text{def}}{=} \tilde{\eta}_t$$

and

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_t\Big(\int_0^{\cdot} \big(Y_u + \tfrac{1}{2}\big)\lambda(u)\cdot dW_u\Big) \stackrel{\text{def}}{=} \hat{\eta}_t.$$

Note that

$$Z_t = z - \int_0^t \lambda(u)\cdot d\tilde{W}_u = z - \int_0^t \lambda(u)\cdot d\hat{W}_u,$$

where the processes

$$\tilde{W}_t = W_t - \int_0^t \big(X_u - \tfrac{1}{2}\big)\lambda(u)\,du$$

$$\hat{W}_t = W_t - \int_0^t \big(Y_u + \tfrac{1}{2}\big)\lambda(u)\,du$$

are known to be standard Brownian motions under $\tilde{\mathbb{P}}$ and $\hat{\mathbb{P}}$ respectively. Simple manipulations show that

$$\tilde{\eta}_t = \mathcal{E}_t\Big(\int_0^{\cdot} X_u\lambda(u)\cdot dW_u\Big)\,\exp\Big(-\frac{1}{2}\int_0^t \lambda(u)\cdot d\tilde{W}_u + \frac{1}{8}\int_0^t |\lambda(u)|^2du\Big)$$

and thus, using (12.51), we obtain

$$Y_t = (1-x)\,e^{-\frac{v^2(0,t)}{8}}\,\tilde{\eta}_t \exp\Big(\frac{1}{2}\int_0^t \lambda(u)\cdot d\tilde{W}_u\Big). \tag{12.52}$$

Similarly, we have

$$\hat{\eta}_t = \mathcal{E}_t\Big(\int_0^{\cdot} Y_u\lambda(u)\cdot dW_u\Big)\,\exp\Big(\frac{1}{2}\int_0^t \lambda(u)\cdot d\hat{W}_u + \frac{1}{8}\int_0^t |\lambda(u)|^2du\Big).$$

Hence, in view of (12.50), we get

$$X_t = x\,e^{-\frac{v^2(0,t)}{8}}\,\hat{\eta}_t \exp\Big(-\frac{1}{2}\int_0^t \lambda(u)\cdot d\hat{W}_u\Big).$$

The last equality yields

$$\mathbb{E}_{\mathbb{P}}(X_t g(X_t)) = x\,e^{-\frac{v^2(0,t)}{8}}\,\mathbb{E}_{\hat{\mathbb{P}}}\left\{g(X_t)\exp\Big(-\frac{1}{2}\int_0^t \lambda(u)\cdot d\hat{W}_u\Big)\right\},$$

and (12.52) gives

$$\mathbb{E}_{\mathbb{P}}(Y_t g(X_t)) = (1-x)\,e^{-\frac{v^2(0,t)}{8}}\,\mathbb{E}_{\tilde{\mathbb{P}}}\left\{g(X_t)\exp\Big(\frac{1}{2}\int_0^t \lambda(u)\cdot d\tilde{W}_u\Big)\right\}.$$

Using (12.48) and (12.49), we conclude that

$$\mathbb{E}_{\mathbb{P}}(g(X_t)) = e^{-\frac{v^2(0,t)}{8}}\,\mathbb{E}_{\mathbb{Q}}\left\{g\left(\frac{1}{1+e^{-(z+\zeta)}}\right)\,\Big(xe^{\frac{1}{2}\zeta} + (1-x)e^{-\frac{1}{2}\zeta}\Big)\right\},$$

where ζ is, under $\mathbb{Q}$, a Gaussian random variable with zero mean value and variance $v^2(0,t)$. The last formula is equivalent to the formula in the statement of the proposition. $\square$

To establish directly formula (12.42), a slightly different change of the underlying probability measure $\mathbb{P}$ is convenient. Namely, we put

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_t\Big(\int_0^{\cdot} X_u \lambda(u) \cdot dW_u\Big)$$

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_t\Big(\int_0^{\cdot} Y_u \lambda(u) \cdot dW_u\Big).$$

Now, under $\tilde{\mathbb{P}}$ and $\hat{\mathbb{P}}$ we have

$$dZ_t = -\tfrac{1}{2}\lambda(t)\,dt - \lambda(t)\cdot d\tilde{W}_t$$

$$dZ_t = \tfrac{1}{2}\lambda(t)\,dt - \lambda(t)\cdot d\hat{W}_t$$

respectively, where

$$\tilde{W}_t = W_t - \int_0^t X_u \lambda(u)\,du, \quad \hat{W}_t = W_t - \int_0^t Y_u \lambda(u)\,du,$$

$t \in [0, T^*]$, are standard Brownian motions under the corresponding probabilities. Formula (12.42) can thus be derived easily from representation (12.49) combined with equalities (12.50)-(12.51).

Corollary 12.6.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative Borel function. Then for every $x \in (0,1)$ and any $0 < t < T$ the conditional expectation $\mathbb{E}_{\mathbb{P}}(g(X_T)\,|\,\mathcal{F}_t)$ equals $\mathbb{E}_{\mathbb{P}}(g(X_T)\,|\,\mathcal{F}_t) = k(X_t)$, where the function $k : (0,1) \to \mathbb{R}$ is given by the formula*

$$k(x) = \sqrt{x(1-x)}e^{-\frac{v^2(t,T)}{8}}\,\mathbb{E}_{\mathbb{Q}}\left\{g\left(\frac{1}{1+e^{-(z+\zeta)}}\right)\left(e^{(z+\zeta)/2} + e^{-(z+\zeta)/2}\right)\right\}$$

with $z = \ln x/(1-x)$, and the random variable ζ has under $\mathbb{Q}$ a Gaussian law with zero mean value and variance $v^2(t,T) = \int_t^T |\lambda(u)|^2 du$.

12.6.4 Bond Options

Our next goal is to establish the bond option valuation formula within the framework of the lognormal LIBOR model. It is interesting to notice that an identical formula was previously established by Rady and Sandmann (1994), who adopted the PDE approach, and who worked with a different model, however. In this section, in which we follow Goldys (1997), we present a probabilistic approach to the bond option valuation. His derivation of the bond option price is based on Corollary 12.6.1. Note that this result can be used to the valuation of an arbitrary European claim.

Proposition 12.6.3. *The price C_t at time $t \le T_{j-1}$ of a European call option, with expiration date T_{j-1} and strike price $0 < K < 1$, written on a zero-coupon bond maturing at $T_j = T_{j-1} + \delta_j$, equals*

$$C_t = (1-K)B(t,T_j)N\big(l_1^j(t)\big) - K(B(t,T_{j-1}) - B(t,T_j))N\big(l_2^j(t)\big),$$

where

$$l_{1,2}^j(t) = \frac{\ln\left((1-K)B(t,T_j)\right) - \ln\left(K\big(B(t,T_{j-1}) - B(t,T_j)\big)\right) \pm \frac{1}{2}\tilde{v}_j(t)}{\tilde{v}_j(t)}$$

and

$$\tilde{v}_j^2(t) = \int_t^{T_{j-1}} |\lambda(u,T_{j-1})|^2 du.$$

Proof. Let us write $T = T_{j-1}$ and $T + \delta = T_j$. In view of Corollary 12.6.1, it is clear that

$$C_t = B(t,T)\,\mathbb{E}_{\mathbb{P}_T}\big((F(T,T+\delta,T) - K)^+ \,|\, \mathcal{F}_t\big) = B(t,T)k(x),$$

where $x = F(t,T+\delta,T)$ and $g(y) = (y-K)^+$. Using the notation $\tilde{v} = \tilde{v}_j(t)$ and

$$l = \tilde{v}^{-1}\Big(\ln\frac{x(1-K)}{K(1-x)}\Big),$$

we obtain

$$k(x) = \sqrt{x(1-x)}\,e^{-\frac{1}{8}\tilde{v}^2}\int_{-l}^{\infty}\left(\frac{1}{1+e^{-\tilde{y}}} - K\right)\left(e^{-\frac{1}{2}\tilde{y}} + e^{\frac{1}{2}\tilde{y}}\right)n(y)\,dy,$$

where $\tilde{y} = z + \tilde{v}y$, and n stands for the standard normal density. Let us set

$$h(y) = e^{-\frac{1}{2}(z+\tilde{v}y)} + e^{\frac{1}{2}(z+\tilde{v}y)}.$$

Then

$$k(x) = \sqrt{x(1-x)}\,e^{-\frac{1}{8}\tilde{v}^2}\left(\int_{-l}^{\infty}\frac{h(y)n(y)}{1+e^{-(z+\tilde{v}y)}}\,dy - K\int_{-l}^{\infty}h(y)n(y)\,dy\right).$$

Equivalently,

$$k(x) = \sqrt{x(1-x)}\,e^{-\frac{1}{8}\tilde{v}^2}\,(I_1 - KI_2),$$

where

$$I_1 = \int_{-l}^{\infty} e^{\frac{1}{2}(z+\tilde{v}y)}n(y)\,dy = e^{\frac{1}{8}\tilde{v}^2+\frac{1}{2}z}\big(1 - N(-l - \tfrac{1}{2}\tilde{v})\big)$$

and

$$I_2 = I_1 + e^{\frac{1}{8}\tilde{v}^2-\frac{1}{2}z}\big(1 - N(-l + \tfrac{1}{2}\tilde{v})\big).$$

Consequently, we find that

$$\begin{aligned} k(x) &= \sqrt{x(1-x)}(1-K)e^{\frac{1}{2}z}\big(1 - N(-l - \tfrac{1}{2}\tilde{v})\big) \\ &\quad - K\sqrt{x(1-x)}e^{-\frac{1}{2}z}\big(1 - N(-l + \tfrac{1}{2}\tilde{v})\big), \end{aligned}$$

or, after simplification,

$$k(x) = x(1-K)N(l + \tfrac{1}{2}\tilde{v}) - K(1-x)N(l - \tfrac{1}{2}\tilde{v}).$$

Since

$$x = F(t, T+\delta, T) = B(t, T+\delta)/B(t,T),$$

the proof of the proposition is completed. □

Using the put-call parity relationship for options written on a zero-coupon bond,

$$C_t - P_t = B(t, T_j) - KB(t, T_{j-1}),$$

it is easy to check that the price of the corresponding put option equals

$$P_t = (K-1)B(t,T_j)N\big(-l_1(t)\big) - K\big(B(t,T_j) - B(t,T_{j-1})\big)N\big(-l_2(t)\big).$$

Recall that the j^{th} caplet is equivalent to the put option written on a zero-coupon bond, with expiry date T_{j-1} and strike price $K = \tilde{\delta}_j^{-1} = (1+\kappa d_j)^{-1}$. More precisely, the option's payoff should be multiplied by the nominal value $\tilde{\delta}_j = 1 + \kappa d_j$. Hence, using the last formula, we obtain

$$\mathbf{Cpl}_t^j = \big(B(t,T_{j-1}) - B(t,T_j)\big)N\big(-l_2^j(t)\big) - \kappa\delta_j B(t,T_j)N\big(-l_1^j(t)\big),$$

since clearly $K - 1 = -\kappa\delta_j\tilde{\delta}_j^{-1}$. To show that the last formula coincides with (12.44), it is enough to check that if $K = \tilde{d}_j^{-1}$, then the terms $-l_2^j(t)$ and $-l_1^j(t)$ coincide with the terms $\tilde{e}_1^j(t)$ and $\tilde{e}_2^j(t)$ of Proposition 12.6.1. In this way, we obtain an alternative probabilistic derivation of the cap valuation formula within the lognormal LIBOR market model.

Remarks. Proposition 12.6.3 shows that the replication of the bond option using the underlying bonds of maturity T_{j-1} and T_j is not straightforward. This should be contrasted with the case of the Gaussian HJM set-up in which hedging of bond options with the use of the underlying bonds is done in a standard way. This illustrates our general observation that each particular model of the term structure should be tailored to a specific class of derivatives and hedging instruments.

12.7 Extensions of the LLM Model

Let us emphasize that the constructions of the LIBOR market model presented in Sect. 12.4.4-12.4.5 do not require that the volatilities of LIBORs be deterministic functions. We have seen that they may be adapted stochastic processes, or some (deterministic or random) functions of the underlying forward LIBORs. The most popular choice of deterministic volatilities for forward LIBORs leads inevitably to the so-called lognormal LIBOR market model (also known as the LLM model or the BGM model).

Empirical studies have shown that the implied volatilities of market prices of caplets (and swaptions) tend to be decreasing functions of the strike level. Hence, it is of practical interest to develop stochastic volatility versions of the LIBOR market model capable of matching the observed volatility smiles of caplets.

CEV LIBOR model. A straightforward generalization of the lognormal LIBOR market model was examined by Andersen and Andreasen (2000b). In their approach, the assumption that the volatilities are deterministic functions was replaced by a suitable functional form of the volatility coefficient. The main emphasis in Andersen and Andreasen (2000b) is put on the use of the CEV process[3] as a model of a forward LIBOR. To be more specific, they postulate that, for every $t \in [0, T_j]$,

$$dL(t, T_j) = L^{\beta}(t, T_j)\,\lambda(t, T_j) \cdot dW_t^{T_{j+1}},$$

where $\beta > 0$ is a strictly positive constant. Under this specification of the dynamics of forward LIBORs with the exponent $\beta \neq 1$, they derive closed-form solutions for caplet prices in terms of the cumulative distribution function of a non-central χ^2 probability distribution. They show also that, depending on the choice of the parameter β, the implied Black volatilities for caplets, when considered as a function of the strike level $\kappa > 0$, exhibit downward- or upward-sloping skew.

Stochastic volatility LIBOR model. In a recent paper by Joshi and Rebonato (2003), the authors examine a stochastic volatility displaced-diffusion extension of a LIBOR market model. Recall that the stochastic volatility displaced-diffusion approach to the modelling of stochastic volatility was presented in Sect. 7.2.2. Basically, Joshi and Rebonato (2003 postulate that

$$d(f(t, T_i) + \alpha) = (f(t, T_i) + \alpha)\big(\mu^{\alpha}(t, T_i)\,dt + \sigma^{\alpha}(t, T_i)\,dW_t\big),$$

where the drift term can be found explicitly, and where by assumption the volatility $\sigma^{\alpha}(t, T_i)$ is given by the expression:

$$\sigma^{\alpha}(t, T_i) = \big(a_t + b_t(T_i - t)\big)e^{-c_t(T_i - t)} + d_t$$

for some mean-reverting diffusion processes $a_t, b_t, \ln c_t$ and $\ln d_t$. They argue that such a model is sufficiently flexible to be capable of describing in a realistic way not only the today's implied volatility surface, but also the observed changes in the market term structure of volatilities.

For other examples of stochastic volatility extensions of a LIBOR market model, the interested reader is referred to Rebonato and Joshi (2001), Gątarek (2003), and Piterbarg (2003a).

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[3] In the context of equity options, the CEV (*constant elasticity of variance*) process was introduced by Cox and Ross (1976). For more details, see Sect. 7.2.

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