

Discrete-time Security Markets

This chapter deals mostly with *finite markets* – that is, discrete-time models of financial markets in which all relevant quantities take a finite number of values. Essentially, we follow here the approach of Harrison and Pliska (1981); a more exhaustive analysis of finite markets can be found in Taqqu and Willinger (1987). An excellent introduction to discrete-time financial mathematics is given by Pliska (1997) and Shreve (2004). A monograph by Föllmer and Schied (2000) is the most comprehensive source in the area.

The detailed treatment of finite models of financial markets presented below is not motivated by their practical importance (except for binomial or multinomial models). The main motivation comes rather from the fact that the most important ideas and results of arbitrage pricing can be presented in a more transparent way by working first in a finite-dimensional framework.

Since the number of dates is finite, there is no loss of generality if we take the set of dates $\mathcal{T} = \{0, 1, \dots, T^*\}$. Let Ω be an arbitrary finite set, $\Omega = \{\omega_1, \omega_2, \dots, \omega_d\}$, and let $\mathcal{F} = \mathcal{F}_{T^*}$ be the σ -field of all subsets of Ω , i.e., $\mathcal{F} = 2^\Omega$. We consider a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^{T^*}$, where \mathbb{P} is an arbitrary probability measure on $(\Omega, \mathcal{F}_{T^*})$ such that $\mathbb{P}\{\omega_j\} > 0$ for every $j = 1, 2, \dots, d$. We assume throughout that the σ -field \mathcal{F}_0 is trivial; that is, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Since \mathcal{T} and Ω are both finite sets, all random variables and all stochastic processes are necessarily bounded. Thus they are integrable with respect to any probability measure \mathbb{P} considered in what follows. A vector of prices of k *primary securities* is modelled by means of an \mathbb{F} -adapted \mathbb{R}^k -valued, nonnegative stochastic process $Z = (Z^1, Z^2, \dots, Z^k)$. A k -dimensional process $Z = (Z^1, Z^2, \dots, Z^k)$ is said to be \mathbb{F} -adapted, if, for any $i = 1, 2, \dots, k$ and any $t \leq T^*$, the random variable Z_t^i is \mathcal{F}_t -measurable. For brevity, we shall say that a given process is adapted, instead of \mathbb{F} -adapted, if no confusion may arise. We assume throughout that $\mathcal{F}_t = \mathcal{F}_t^Z = \sigma(Z_0, Z_1, \dots, Z_t)$; that is, the filtration \mathbb{F} is generated by the observations of the price process Z . A *trading strategy* (a *dynamic portfolio*) is an \mathbb{R}^k -valued \mathbb{F} -adapted process $\phi = (\phi^1, \phi^2, \dots, \phi^k)$. At any date t , the i^{th} component, ϕ_t^i , of a portfolio ϕ determines the number of units of the i^{th} asset that are held in the portfolio at this date.

2.1 The Cox-Ross-Rubinstein Model

Before we start the analysis of a general finite market, we present a particular model, which is a direct extension of a one-period two-state model with two securities to the multi-period set-up. Let T be a positive integer, interpreted as the time to maturity of a derivative contract, expressed in some convenient units of time. A European call option written on one share of a stock S paying no dividends during the option's lifetime, is formally equivalent to the claim X whose payoff at time T is contingent on the stock price S_T , and equals

$$X = (S_T - K)^+ \stackrel{\text{def}}{=} \max\{S_T - K, 0\}. \quad (2.1)$$

The call option value (or price) at the expiry date T equals $C_T = (S_T - K)^+$. Our first aim is to put the price on the option at any instant $t = 0, \dots, T$, when the price of a risky asset (a stock) is modelled by the Cox et al. (1979a) multiplicative binomial lattice, commonly known as the Cox-Ross-Rubinstein (CRR, for short) model of a stock price. Since, for good reasons, models of this form are by far the most popular discrete-time financial models, it seems legitimate to refer to the CRR model as the *benchmark* discrete-time model.

2.1.1 Binomial Lattice for the Stock Price

We consider a discrete-time model of a financial market with the set of dates $0, 1, \dots, T^*$, and with two primary traded securities: a risky asset, referred to as a *stock*, and a risk-free investment, called a *savings account* (or a *bond*).

Let us first describe the savings account. We assume that it has a constant rate of return $r > -1$ over each time period $[t, t + 1]$, meaning that its price process $B = (B_t)_{t=0}^{T^*}$ equals (by convention $B_0 = 1$)

$$B_t = (1 + r)^t = \hat{r}^t, \quad \forall t = 0, 1, \dots, T^*, \quad (2.2)$$

where we set $\hat{r} = 1 + r$. We postulate that the stock price $S = (S_t)_{t=0}^{T^*}$ satisfies

$$S_{t+1}/S_t \in \{u, d\} \quad (2.3)$$

for $t = 0, 1, \dots, T^* - 1$, where $0 < d < u$ are real numbers and S_0 is a strictly positive constant. It will be essential to assume that, given the level S_t of the stock price at time t , both possible future states at time $t + 1$, that is, $S_{t+1}^u = uS_t$ and $S_{t+1}^d = dS_t$, have strictly positive probabilities of occurrence, also when we take into account all past observations S_0, S_1, \dots, S_t of the price process S . Formally, we postulate that

$$\mathbb{P}\{S_{t+1} = uS_t \mid S_0, S_1, \dots, S_t\} > 0, \quad \mathbb{P}\{S_{t+1} = dS_t \mid S_0, S_1, \dots, S_t\} > 0. \quad (2.4)$$

To put it more intuitively, given the full observation of the sample path of the stock price S up to time t , the investors should never be in a position to tell with certainty

whether the stock price will reach the upper level uS_t or the lower level dS_t at the end of the next time period. Since (2.4) states that at any date t the conditional distribution of S_{t+1} is non-degenerate under the actual (i.e., the real-world) probability \mathbb{P} , we shall refer to (2.4) as the *non-degeneracy condition*. We shall argue that the replicating strategies (and thus also arbitrage prices) of derivative assets are independent of the choice of the actual probability measure \mathbb{P} on the underlying probability space, provided that the non-degeneracy property (2.4) is valid under \mathbb{P} .

As a consequence, it is sufficient to focus on the simplest probabilistic model of the stock price with the desired features. To this end, we pick an arbitrary number $p \in (0, 1)$ and we introduce a sequence ξ_t , $t = 1, 2, \dots, T^*$ of mutually independent random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with identical probability distribution

$$\mathbb{P}\{\xi_t = u\} = p = 1 - \mathbb{P}\{\xi_t = d\}, \quad \forall t = 1, 2, \dots, T^*.$$

We now formally define the stock price process S by setting

$$S_t = S_0 \prod_{j=1}^t \xi_j, \quad \forall t = 0, 1, \dots, T^*. \quad (2.5)$$

The sequence of independent and identically distributed random variables ξ_t , $t = 1, 2, \dots, T^*$ plays the role of a *driving noise* in the stochastic dynamics of the stock price, which can be seen here as a *geometric random walk*. It is apparent that under the present assumptions we have

$$\begin{aligned} \mathbb{P}\{S_{t+1} = uS_t \mid S_0, S_1, \dots, S_t\} &= \mathbb{P}\{\xi_{t+1} = u\} = p > 0, \\ \mathbb{P}\{S_{t+1} = dS_t \mid S_0, S_1, \dots, S_t\} &= \mathbb{P}\{\xi_{t+1} = d\} = 1 - p > 0, \end{aligned}$$

and thus (2.4) holds. Equivalently, the value of the stock price at time t is given as

$$S_t = S_0 \exp\left(\sum_{j=1}^t \zeta_j\right), \quad \forall t = 0, 1, \dots, T^*, \quad (2.6)$$

where $\zeta_1, \zeta_2, \dots, \zeta_{T^*}$ are independent, identically distributed random variables:

$$\mathbb{P}\{\zeta_t = \ln u\} = p = 1 - \mathbb{P}\{\zeta_t = \ln d\}, \quad \forall t = 1, 2, \dots, T^*.$$

Due to representation (2.6), the process S given by (2.5) is frequently referred to as an *exponential random walk*. It is also clear that $\ln S_t = \ln S_{t-1} + \zeta_t$ for $t = 1, 2, \dots, T^*$. This means that the logarithm of the stock price (the *log-price*) follows an *arithmetic random walk*. Finally, let us introduce the sequence of logarithmic returns (the *log-returns*) $\mu_1, \mu_2, \dots, \mu_{T^*}$ on the stock by setting

$$\mu_t \stackrel{\text{def}}{=} \ln\left(\frac{S_t}{S_{t-1}}\right) = \zeta_t.$$

Obviously, the log-returns $\mu_1, \mu_2, \dots, \mu_{T^*}$ are independent and identically distributed under \mathbb{P} .

In view of (2.5), it is clear that a typical sample path of the stock price can be represented as a sequence $(s_0, s_1, \dots, s_{T^*})$ of real numbers such that $s_0 = S_0$ and the ratio s_{i+1}/s_i equals either u or d . If we take, for example, $T^* = 5$, then a sample path of S may look, for instance, as follows:

$$(S_0, uS_0, u^2S_0, u^2dS_0, u^2d^2S_0, u^3d^2S_0)$$

or

$$(S_0, dS_0, udS_0, ud^2S_0, ud^3S_0, u^2d^3S_0).$$

For mathematical convenience, we will later formally identify each sample path of the process S with a sequence of length T^* of zeroes and ones, by associating the number 1 to u , and the number 0 to d . Manifestly, for the two sample paths given above, the corresponding sequences of zeroes and ones are $(1, 1, 0, 0, 1)$ and $(0, 1, 0, 0, 1)$ respectively. The collection of all sequences of zeroes and ones will also play the role of the underlying probability space Ω , so that we formally set $\Omega = 2^{T^*}$. This also means that in the CRR model with T^* periods the space $\Omega = \{\omega_1, \omega_2, \dots, \omega_d\}$ comprises 2^{T^*} elementary events, each elementary event ω_k representing in fact a particular sample path of the stock price.

2.1.2 Recursive Pricing Procedure

Let us re-emphasize that the standing assumption that the random variables ξ_t , $t = 1, 2, \dots, T^*$ are mutually independent and identically distributed under the actual probability \mathbb{P} is not essential for our further purposes; we make this assumption, without loss of generality, for mathematical convenience.

As we will see in what follows, the arbitrage price of any European or American contingent claim in the binomial model of a financial market is independent of the choice of the probability of upward and downward movements of the stock price at any particular node. Indeed, it is uniquely determined by the assumed values of the stock price – that is, by the postulated form of sample paths of the stock price process.

Formally, the arbitrage price depends only on the specification of the payoff, the initial stock price S_0 and the value of parameters u , d and r . Hence the valuation results that we are going to derive will appear to be distribution-free, meaning that they do not depend on the specification of the actual probability measure \mathbb{P} .

Let us introduce some notation. For any $t \leq T$, we write α_t to denote the number of shares held during the period $[t, t + 1)$, while β_t stands for the dollar investment in the savings account during this period.

To determine the arbitrage price of a European call option we shall show, using backward induction, that by adjusting his or her dynamic portfolio $\phi_t = (\alpha_t, \beta_t)$, $t = 0, 1, \dots, T - 1$, at the beginning of each period, an investor is able to mimic the payoff of an option at time T for every state. We shall refer to this fact by saying that the contingent claim $X = (S_T - K)^+$ admits a unique, dynamic, replicating,

self-financing strategy. Replication and valuation of a European put option can be reduced, through the put-call parity relationship, to that of a European call.

For concreteness, we shall now focus on a European call option. It will soon become clear, however, that the backward induction method presented below can be directly applied to any European contingent claim X of the form $X = g(S_T)$. It is common to refer to such claims as *path-independent*, as opposed to *path-dependent* claims that have the form $X = h(S_0, S_1, \dots, S_T)$; that is, they depend on the whole sample path. In particular, we shall see that for an arbitrary path-independent European claim we need to compute only $t + 1$ values at any date $t = 0, 1, \dots, T^*$. By contrast, in the case of a path-dependent claim one has to deal with (at most) 2^t values at any date $t = 0, 1, \dots, T$. The backward induction method (as well as the risk-neutral valuation method presented in Sect. 2.2.2) can be applied to both cases with no essential changes, however.

First step. Given a fixed maturity date $1 \leq T \leq T^*$, we start our analysis by considering the last period before the expiry date, $[T - 1, T]$. We assume that a portfolio replicating the terminal payoff of a call option is established at time $T - 1$, and remains fixed until the expiry date T . We thus need to find a portfolio $\phi_{T-1} = (\alpha_{T-1}, \beta_{T-1})$ at the beginning of the last period for which the terminal wealth $V_T(\phi)$, which equals

$$V_T(\phi) = \alpha_{T-1}S_T + \beta_{T-1}\hat{r}, \quad (2.7)$$

replicates the option payoff C_T ; that is, we have $V_T(\phi) = C_T$. Combining (2.1) with (2.7), we get the following equality

$$\alpha_{T-1}S_T + \beta_{T-1}\hat{r} = (S_T - K)^+. \quad (2.8)$$

By virtue of our assumptions, we have $S_T = S_{T-1}\xi_{T-1}$; hence we may rewrite (2.8) in a more explicit form

$$\begin{cases} \alpha_{T-1}uS_{T-1} + \beta_{T-1}\hat{r} = (uS_{T-1} - K)^+, \\ \alpha_{T-1}dS_{T-1} + \beta_{T-1}\hat{r} = (dS_{T-1} - K)^+. \end{cases}$$

This simple system of two linear equations can be solved easily, yielding

$$\alpha_{T-1} = \frac{(uS_{T-1} - K)^+ - (dS_{T-1} - K)^+}{S_{T-1}(u - d)}, \quad (2.9)$$

and

$$\beta_{T-1} = \frac{u(dS_{T-1} - K)^+ - d(uS_{T-1} - K)^+}{\hat{r}(u - d)}. \quad (2.10)$$

Furthermore, the wealth $V_{T-1}(\phi)$ of this portfolio at time $T - 1$ equals

$$\begin{aligned} V_{T-1}(\phi) &= \alpha_{T-1}S_{T-1} + \beta_{T-1} \\ &= \hat{r}^{-1} \left(p_*(uS_{T-1} - K)^+ + (1 - p_*)(dS_{T-1} - K)^+ \right), \end{aligned}$$

where we write $p_* = (\hat{r} - d)/(u - d) = (1 + r - d)/(u - d)$.

As we shall show later in this section, the number p_* specifies the probability of the rise of the stock price over each period $[t, t + 1]$ under the martingale measure for the process $S^* = S/B$. We assume from now on that $d < 1 + r < u$ so that the number p_* belongs to the open interval $(0, 1)$. Assuming the absence of arbitrage in the market model,¹ the wealth $V_{T-1}(\phi)$ agrees with the value (that is, the *arbitrage price*) of a call option at time $T - 1$. Put another way, the equality $C_{T-1} = V_{T-1}(\phi)$ is valid.

Second step. We continue our analysis by considering the preceding period, $[T - 2, T - 1]$. In this step, we seek a portfolio $\phi_{T-2} = (\alpha_{T-2}, \beta_{T-2})$ created at time $T - 2$ in such a way that its wealth at time $T - 1$ replicates option value C_{T-1} ; that is

$$\alpha_{T-2}S_{T-1} + \beta_{T-2}\hat{r} = C_{T-1}. \quad (2.11)$$

Note that since $C_{T-1} = V_{T-1}(\phi)$, the dynamic trading strategy ϕ constructed in this way will possess the *self-financing property* at time $T - 1$

$$\alpha_{T-2}S_{T-1} + \beta_{T-2}\hat{r} = \alpha_{T-1}S_{T-1} + \beta_{T-1}. \quad (2.12)$$

Basically, the self-financing feature means that the portfolio is adjusted at time $T - 1$ (and more generally, at any trading date) in such a way that no withdrawals or inputs of funds take place. Since $S_{T-1} = S_{T-2}\xi_{T-2}$ and $\xi_{T-2} \in \{u, d\}$, we get the following equivalent form of equality (2.11)

$$\begin{cases} \alpha_{T-2}uS_{T-2} + \beta_{T-2}\hat{r} = C_{T-1}^u, \\ \alpha_{T-2}dS_{T-2} + \beta_{T-2}\hat{r} = C_{T-1}^d, \end{cases} \quad (2.13)$$

where we set

$$C_{T-1}^u = \frac{1}{\hat{r}} \left(p_*(u^2S_{T-2} - K)^+ + (1 - p_*)(udS_{T-2} - K)^+ \right)$$

and

$$C_{T-1}^d = \frac{1}{\hat{r}} \left(p_*(udS_{T-2} - K)^+ + (1 - p_*)(d^2S_{T-2} - K)^+ \right).$$

In view of (2.13), it is evident that

$$\alpha_{T-2} = \frac{C_{T-1}^u - C_{T-1}^d}{S_{T-2}(u - d)}, \quad \beta_{T-2} = \frac{uC_{T-1}^d - dC_{T-1}^u}{\hat{r}(u - d)}.$$

Consequently, the wealth $V_{T-2}(\phi)$ of the portfolio $\phi_{T-2} = (\alpha_{T-2}, \beta_{T-2})$ at time $T - 2$ equals

$$\begin{aligned} V_{T-2}(\phi) &= \alpha_{T-2}S_{T-2} + \beta_{T-2} = \frac{1}{\hat{r}} \left(p_*C_{T-1}^u + (1 - p_*)C_{T-1}^d \right) \\ &= \frac{1}{\hat{r}^2} \left(p_*^2(u^2S_{T-2} - K)^+ + 2p_*q_*(udS_{T-2} - K)^+ + q_*^2(d^2S_{T-2} - K)^+ \right). \end{aligned}$$

¹ We will return to this point later in Sect. 2.2. Let us only mention here that the necessary and sufficient condition for the absence of arbitrage has the same form as in the case of the one-period model; that is, it is exactly the condition $d < 1 + r < u$ we have just imposed.

Using the same arbitrage arguments as in the first step, we argue that the wealth $V_{T-2}(\phi)$ of the portfolio ϕ at time $T - 2$ gives the arbitrage price at time $T - 2$, i.e., $C_{T-2} = V_{T-2}(\phi)$.

General induction step. It is evident that by repeating the above procedure, one can completely determine the option price at any date $t \leq T$, as well as the (unique) trading strategy ϕ that replicates the option. Summarizing, the above reasoning provides a recursive procedure for finding the value of a call with any number of periods to go. It is worth noting that in order to value the option at a given date t and for a given level of the current stock price S_t , it is enough to consider a sub-lattice of the CRR binomial lattice starting from S_t and ranging over $T - t$ periods.

Path-independent European claims. As already mentioned, the recursive valuation procedure applies with virtually no changes to any European claim of the form $X = g(S_T)$. Indeed, a specific formula for the option's payoff was used in the first step only. If $X = g(S_T)$, we replace (2.8) by

$$\alpha_{T-1}S_T + \beta_{T-1}\hat{r} = g(S_T), \quad (2.14)$$

so that we now need to solve the following pair of linear equations:

$$\begin{cases} \alpha_{T-1}uS_{T-1} + \beta_{T-1}\hat{r} = g(uS_{T-1}), \\ \alpha_{T-1}dS_{T-1} + \beta_{T-1}\hat{r} = g(dS_{T-1}). \end{cases}$$

The unique solution $\alpha_{T-1}, \beta_{T-1}$ depends on S_{T-1} , but not on the values of the stock price at times $0, 1, \dots, T - 2$. Hence the replicating portfolio can be written as $\alpha_{T-1} = f_{T-1}(S_{T-1})$ and $\beta_{T-1} = h_{T-1}(S_{T-1})$ for some functions $f_{T-1}, h_{T-1} : \mathbb{R}_+ \rightarrow \mathbb{R}$. Consequently, the *arbitrage price* $\pi_{T-1}(X) = V_{T-1}(\phi)$ of X at time $T - 1$ can be represented as $\pi_{T-1}(X) = g_{T-1}(S_{T-1})$ for some function $g_{T-1} : \mathbb{R}_+ \rightarrow \mathbb{R}$.

In the next step, we consider the period $[T - 2, T - 1]$ and a path-independent claim $g_{T-1}(S_{T-1})$ that settles at time $T - 1$. A similar reasoning as above shows that $\pi_{T-2}(X) = g_{T-2}(S_{T-2})$ for some function $g_{T-2} : \mathbb{R}_+ \rightarrow \mathbb{R}$.

We conclude that there exists a sequence of functions $g_t : \mathbb{R}_+ \rightarrow \mathbb{R}$ (with $g_T = g$) such that the arbitrage price process $\pi(X)$ satisfies $\pi_t(X) = g_t(S_t)$ for $t = 0, 1, \dots, T$. As we shall see in what follows, this *Markovian feature* of the replicating strategy and the arbitrage price of a path-independent European contingent claim can also be deduced from the Markov property of the stock price under the martingale measure.

Path-dependent European claims. In the case of a general (that is, possibly path-dependent) European contingent claim, formula (2.14) becomes

$$\alpha_{T-1}S_T + \beta_{T-1}\hat{r} = g(S_0, S_1, \dots, S_T). \quad (2.15)$$

It is essential to make clear that α_{T-1} and β_{T-1} depend not only on the level of the stock price at time $T - 1$ (i.e., on the choice of the node at time $T - 1$), but on the entire sample path that connects the initial state $S_0 = s_0$ with a given node at time $T - 1$.

Let $(s_0, s_1, \dots, s_{T-1})$ be a particular sample path of the stock price process S that connects s_0 with a generic value s_{T-1} at time $T - 1$. In order to find the replicating portfolio for this particular sample path, we need to solve for the unknowns $\alpha_{T-1}(s_0, s_1, \dots, s_{T-1})$ and $\beta_{T-1}(s_0, s_1, \dots, s_{T-1})$ the following pair of linear equations

$$\begin{cases} \alpha_{T-1}(s_0, s_1, \dots, s_{T-1})us_{T-1} + \beta_{T-1}(s_0, s_1, \dots, s_{T-1})\hat{r} = g^u, \\ \alpha_{T-1}(s_0, s_1, \dots, s_{T-1})ds_{T-1} + \beta_{T-1}(s_0, s_1, \dots, s_{T-1})\hat{r} = g^d, \end{cases}$$

where $g^u = g(s_0, s_1, \dots, s_{T-1}, us_{T-1})$ and $g^d = g(s_0, s_1, \dots, s_{T-1}, ds_{T-1})$.

Of course, the number of sample paths connecting s_0 with the current level of the stock price at time $T - 1$ depends on the choice of a node. It is also clear that the total number of sample paths of length $T - 1$ is 2^{T-1} . Thus the arbitrage price of X at time $T - 1$ will have at most 2^{T-1} different values. As soon as the arbitrage price

$$\pi_{T-1}(X) = V_{T-1}(\phi) = \alpha_{T-1}(s_0, s_1, \dots, s_{T-1})s_{T-1} + \beta_{T-1}(s_0, s_1, \dots, s_{T-1})$$

is found at each node s_{T-1} and for each sample path connecting s_0 with this node, we are in a position to apply the same procedure as above to the (path-dependent) claim $\pi_{T-1}(X)$ settling at $T - 1$. In this way, we are able to find the sequence of prices $\pi_{T-2}(X), \pi_{T-3}(X), \dots, \pi_1(X), \pi_0(X)$, as well as replicating strategy $\phi_t = (\alpha_t, \beta_t)$, $t = 0, 1, \dots, T$ for a claim X . Since this procedure has a unique solution for any X , it is fairly clear that any European contingent claim can be replicated, and thus it is *attainable* in the CRR model; we refer to this property as the *completeness* of the CRR model.

Let \mathcal{F}_t^S stand for the σ -field of all events of \mathcal{F} generated by the observations of the stock price S up to the date t ; formally $\mathcal{F}_t^S = \sigma(S_0, S_1, \dots, S_t)$, where $\sigma(S_0, S_1, \dots, S_t)$ denotes the least σ -field with respect to which the random variables S_0, S_1, \dots, S_t are measurable. We write \mathbb{F}^S to denote the filtration² generated by the stock price, so that $\mathbb{F}^S = (\mathcal{F}_t^S)_{t \leq T^*}$.

By construction of a replicating strategy, it is evident that, for any fixed t , the random variables α_t, β_t defining the portfolio at time t , as well as the wealth $V_t(\phi)$ of this portfolio, are measurable with respect to the σ -field \mathcal{F}_t^S . Hence the processes $\phi = (\alpha, \beta)$ and $V(\phi)$ are *adapted* to the filtration \mathbb{F}^S generated by the stock price, or briefly, \mathbb{F}^S -adapted.

We conclude that the unique replicating strategy for an arbitrary European contingent claim in the CRR model follows an \mathbb{F}^S -adapted stochastic process and is *self-financing*, in the sense that the equality

$$\alpha_{t-1}S_t + \beta_{t-1}\hat{r} = \alpha_t S_t + \beta_t \quad (2.16)$$

holds for every $t = 1, \dots, T - 1$. From the considerations above, we conclude that the following result is valid within the set-up of the CRR model.

² Recall that a *filtration* is simply an increasing family of σ -fields.

Proposition 2.1.1 Any European claim X is attainable in the CRR model. Its replicating strategy ϕ and arbitrage price $\pi(X)$ are \mathbb{F}^S -adapted processes. If, in addition, a claim X is path-independent, the arbitrage price at time $t = 0, 1, \dots, T$ is a function of the current level S_t of the stock price, so that, for any $t = 0, 1, \dots, T$, we have $\pi_t(X) = g_t(S_t)$ for some function $g_t : \mathbb{R}_+ \rightarrow \mathbb{R}$.

2.1.3 CRR Option Pricing Formula

It appears that the recursive pricing procedure leads to an explicit formula for the arbitrage price of a European call (or put) option in the CRR model. Before we state this result, we find it convenient to introduce some notation.

For any fixed natural number m , let the function $a_m : \mathbb{R}_+ \rightarrow \mathbb{N}^*$ be given by the formula (\mathbb{N}^* stands hereafter for the set of all nonnegative integers)

$$a_m(x) = \inf \{ j \in \mathbb{N}^* \mid xu^j d^{m-j} > K \},$$

where, by convention, $\inf \emptyset = \infty$.

Let us set $a^d = a_m(dx)$ and $a^u = a_m(ux)$. It is not difficult to check that for any $x > 0$, we have either $a^d = a^u$ or $a^d = a^u + 1$. For ease of notation, we write

$$\Delta_m(x, j) = \binom{m}{j} p_*^j (1 - p_*)^{m-j} (u^j d^{m-j} x - K).$$

Proposition 2.1.2 For every $m = 1, 2, \dots, T$, the arbitrage price of a European call option at time $t = T - m$ is given by the Cox-Ross-Rubinstein valuation formula

$$C_{T-m} = S_{T-m} \sum_{j=a}^m \binom{m}{j} \bar{p}^j (1 - \bar{p})^{m-j} - \frac{K}{\hat{r}^m} \sum_{j=a}^m \binom{m}{j} p_*^j (1 - p_*)^{m-j}, \quad (2.17)$$

where $a = a_m(S_{T-m})$, $p_* = (\hat{r} - d)/(u - d)$ and $\bar{p} = p_* u / \hat{r}$. At time $t = T - m - 1$, the unique replicating strategy $\phi_{T-m-1} = (\alpha_{T-m-1}, \beta_{T-m-1})$ equals

$$\begin{aligned} \alpha_{T-m-1} &= \sum_{j=a^d}^m \binom{m}{j} \bar{p}^j (1 - \bar{p})^{m-j} + \frac{\delta \Delta_m(u S_{T-m-1}, a^u)}{S_{T-m-1}(u - d)}, \\ \beta_{T-m-1} &= -\frac{K}{\hat{r}^{m+1}} \sum_{j=a^d}^m \binom{m}{j} p_*^j (1 - p_*)^{m-j} - \frac{\delta d \Delta_m(u S_{T-m-1}, a^u)}{\hat{r}(u - d)}, \end{aligned}$$

where $a^d = a_m(d S_{T-m-1})$, $a^u = a_m(u S_{T-m-1})$ and $\delta = a^d - a^u$.

Proof. Straightforward calculations yield $1 - \bar{p} = d(1 - p_*)/\hat{r}$, and thus

$$\bar{p}^j (1 - \bar{p})^{m-j} = p_*^j (1 - p_*)^{m-j} u^j d^{m-j} / \hat{r}^m.$$

Formula (2.17) is therefore equivalent to the following

$$\begin{aligned}
C_{T-m} &= \frac{1}{\hat{r}^m} \sum_{j=a}^m \binom{m}{j} p_*^j (1-p_*)^{m-j} \left(u^j d^{m-j} S_{T-m} - K \right) \\
&= \frac{1}{\hat{r}^m} \sum_{j=0}^m \binom{m}{j} p_*^j (1-p_*)^{m-j} \left(u^j d^{m-j} S_{T-m} - K \right)^+.
\end{aligned}$$

We will now proceed by induction with respect to m . For $m = 0$, the above formula is manifestly true, since it reduces to $C_T = (S_T - K)^+$. Assume now that C_{T-m} is the arbitrage price of a European call option at time $T - m$. We have to select a portfolio $\phi_{T-m-1} = (\alpha_{T-m-1}, \beta_{T-m-1})$ for the period $[T-m-1, T-m]$ (that is, established at time $T-m-1$ at each node of the binomial lattice) in such a way that the portfolio's wealth at time $T - m$ replicates the value C_{T-m} of the option. Formally, the wealth of the portfolio $(\alpha_{T-m-1}, \beta_{T-m-1})$ needs to satisfy the relationship

$$\alpha_{T-m-1} S_{T-m} + \beta_{T-m-1} \hat{r} = C_{T-m}, \quad (2.18)$$

which in turn is equivalent to the following pair of equations

$$\begin{cases} \alpha_{T-m-1} u S_{T-m-1} + \beta_{T-m-1} \hat{r} = C_{T-m}^u, \\ \alpha_{T-m-1} d S_{T-m-1} + \beta_{T-m-1} \hat{r} = C_{T-m}^d, \end{cases}$$

where

$$\begin{aligned}
C_{T-m}^u &= \frac{1}{\hat{r}^m} \sum_{j=0}^m \binom{m}{j} p_*^j (1-p_*)^{m-j} \left(u^{j+1} d^{m-j} S_{T-m-1} - K \right)^+ \\
&= \frac{1}{\hat{r}^m} \sum_{j=a^u}^m \binom{m}{j} p_*^j (1-p_*)^{m-j} \left(u^{j+1} d^{m-j} S_{T-m-1} - K \right) \\
C_{T-m}^d &= \frac{1}{\hat{r}^m} \sum_{j=0}^m \binom{m}{j} p_*^j (1-p_*)^{m-j} \left(u^j d^{m-j+1} S_{T-m-1} - K \right)^+ \\
&= \frac{1}{\hat{r}^m} \sum_{j=a^d}^m \binom{m}{j} p_*^j (1-p_*)^{m-j} \left(u^j d^{m-j+1} S_{T-m-1} - K \right).
\end{aligned}$$

Consequently, we have (we write $q_* = 1 - p_*$)

$$\begin{aligned}
\alpha_{T-m-1} &= \frac{C_{T-m}^u - C_{T-m}^d}{S_{T-m-1}(u - d)} \\
&= \frac{1}{\hat{r}^m(u - d)} \sum_{j=a^d}^m \binom{m}{j} p_*^j q_*^{m-j} (u^{j+1} d^{m-j} - u^j d^{m-j+1}) \\
&\quad + \frac{\delta \Delta_m(u S_{T-m-1}, a^u)}{S_{T-m-1}(u - d)} \\
&= \sum_{j=a^d}^m \binom{m}{j} \bar{p}^j (1 - \bar{p})^{m-j} + \frac{\delta \Delta_m(u S_{T-m-1}, a^u)}{S_{T-m-1}(u - d)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
 \beta_{T-m-1} &= \frac{uC_{T-m}^d - dC_{T-m}^u}{\hat{r}(u-d)} \\
 &= \frac{1}{\hat{r}^{m+1}(u-d)} \sum_{j=a^d}^m \binom{m}{j} p_*^j (1-p_*)^{m-j} (dK - uK) \\
 &\quad - \frac{\delta d \Delta_m(uS_{T-m-1}, a^u)}{\hat{r}(u-d)} \\
 &= -\frac{K}{\hat{r}^{m+1}} \sum_{j=a^d}^m \binom{m}{j} p_*^j (1-p_*)^{m-j} - \frac{\delta d \Delta_m(uS_{T-m-1}, a^u)}{\hat{r}(u-d)}.
 \end{aligned}$$

The wealth of this portfolio at time $T - m - 1$ equals (note that just established explicit formulas for the replicating portfolio are not employed here)

$$\begin{aligned}
 C_{T-m-1} &= \alpha_{T-m-1} S_{T-m-1} + \beta_{T-m-1} \\
 &= (u-d)^{-1} (C_{T-m}^u - C_{T-m}^d + \hat{r}^{-1} (uC_{T-m}^d - dC_{T-m}^u)) \\
 &= \hat{r}^{-1} (p_* C_{T-m}^u + (1-p_*) C_{T-m}^d) \\
 &= \frac{1}{\hat{r}^{m+1}} \left\{ \sum_{j=0}^m \binom{m}{j} p_*^{j+1} q_*^{m-j} (u^{j+1} d^{m-j} S_{T-m-1} - K)^+ \right. \\
 &\quad \left. + \sum_{j=0}^m \binom{m}{j} p_*^j q_*^{m+1-j} (u^j d^{m+1-j} S_{T-m-1} - K)^+ \right\} \\
 &= \frac{1}{\hat{r}^{m+1}} \left\{ \sum_{j=1}^{m+1} \binom{m}{j-1} p_*^j q_*^{m+1-j} (u^j d^{m+1-j} S_{T-m-1} - K)^+ \right. \\
 &\quad \left. + \sum_{j=0}^m \binom{m}{j} p_*^j q_*^{m+1-j} (u^j d^{m+1-j} S_{T-m-1} - K)^+ \right\}.
 \end{aligned}$$

Using the last formula and the equality $\binom{m}{j-1} + \binom{m}{j} = \binom{m+1}{j}$, we obtain

$$\begin{aligned}
 C_{T-m-1} &= \frac{1}{\hat{r}^{m+1}} \left\{ p_*^{m+1} (u^{m+1} S_{T-m-1} - K)^+ \right. \\
 &\quad + \sum_{j=1}^m \binom{m}{j} p_*^j q_*^{m+1-j} (u^j d^{m+1-j} S_{T-m-1} - K)^+ \\
 &\quad + \sum_{j=1}^m \binom{m}{j-1} p_*^j q_*^{m+1-j} (u^j d^{m+1-j} S_{T-m-1} - K)^+ \\
 &\quad \left. + q_*^{m+1} (d^{m+1} S_{T-m-1} - K)^+ \right\} \\
 &= \frac{1}{\hat{r}^{m+1}} \sum_{j=0}^{m+1} \binom{m+1}{j} p_*^j q_*^{m+1-j} (u^j d^{m+1-j} S_{T-m-1} - K)^+. \quad \square
 \end{aligned}$$

Note that the CRR valuation formula (2.17) makes no reference to the subjective (or actual) probability p . Intuitively, the pricing formula does not depend on the investor's attitudes toward risk. The only assumption made with regard to the behavior of an individual is that all investors prefer more wealth to less wealth, and thus have an incentive to take advantage of risk-free profitable investments. Consequently, if arbitrage opportunities were present in the market, no market equilibrium would be possible. This feature of arbitrage-free markets explains the term *partial equilibrium approach*, frequently used in economic literature in relation to arbitrage pricing of derivative securities.

2.2 Martingale Properties of the CRR Model

Our next goal is to analyze the no-arbitrage features of the CRR model. As mentioned already, it is convenient to work on Ω related to the *canonical space* of the process $S = (S_t)_{t=0}^{T^*}$. We start by introducing a finite probability space Ω ; namely, for a fixed natural number T^* , we consider

$$\Omega = \{\omega = (a_1, a_2, \dots, a_{T^*}) \mid a_j = 1 \text{ or } a_j = 0\}. \quad (2.19)$$

In the present context, it will be sufficient to consider a specific class \mathcal{P} of probability measures on the measurable space (Ω, \mathcal{F}) , where \mathcal{F} is the σ -field of all subsets of Ω , i.e., $\mathcal{F} = 2^\Omega$. For a generic elementary event $\omega = (a_1, a_2, \dots, a_{T^*})$, we define its probability $\mathbb{P}\{\omega\}$ by setting, for a fixed $p \in (0, 1)$,

$$\mathbb{P}\{\omega\} = p^{\sum_{j=1}^{T^*} a_j} (1-p)^{T^* - \sum_{j=1}^{T^*} a_j}. \quad (2.20)$$

Definition 2.2.1 We denote by \mathcal{P} the class of all probability measures on the canonical space (Ω, \mathcal{F}) given by (2.19) that have the form (2.20).

It is clear that any element $\mathbb{P} \in \mathcal{P}$ is uniquely determined by the value of the parameter p . For any $j = 1, 2, \dots, T^*$, we denote by A_j the event $A_j = \{\omega \in \Omega \mid a_j = 1\}$. It is not difficult to check that the events A_j , $j = 1, 2, \dots, T^*$ are mutually independent; moreover, $\mathbb{P}\{A_j\} = p$ for every $j = 1, 2, \dots, T^*$. We are now in a position to define a sequence of random variables ξ_j , $j = 1, 2, \dots, T^*$ by setting

$$\xi_j(\omega) = ua_j + d(1 - a_j), \quad \forall \omega \in \Omega. \quad (2.21)$$

The random variables ξ_j are easily seen to be independent and identically distributed, with the following probability distribution under \mathbb{P}

$$\mathbb{P}\{\xi_j = u\} = p = 1 - \mathbb{P}\{\xi_j = d\}, \quad \forall t \leq T^*. \quad (2.22)$$

We shall show that the unique martingale measure for the process $S^* = S/B$ belongs to the class \mathcal{P} . The assumption that the real-world probability is selected from this class is not essential, though. It is enough to assume that \mathbb{P} is such that any potential sample path of the price process S has a strictly positive probability, or equivalently, that the non-degeneracy condition (2.4) is satisfied.

2.2.1 Martingale Measures

Let us return to the multiplicative binomial lattice modelling the stock price. In the present framework, the process S is determined by the initial stock price S_0 and the sequence ξ_j , $j = 1, 2, \dots, T^*$ of independent random variables given by (2.21). More precisely, the sequence $S = (S_t)_{t=0}^{T^*}$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by means of (2.5), or equivalently, by the relation

$$S_{t+1} = \xi_{t+1} S_t, \quad \forall t < T^*, \quad (2.23)$$

with $S_0 > 0$. Let us introduce the process S^* of the *discounted* stock price by setting

$$S_t^* = S_t / B_t = S_t / \hat{r}^t, \quad \forall t \leq T^*. \quad (2.24)$$

Let \mathcal{D}_t^S be the family of decompositions of Ω generated by random variables S_u , $u \leq t$; that is, $\mathcal{D}_t^S = \mathcal{D}(S_0, S_1, \dots, S_t)$ for every $t \leq T^*$. It is clear that the family \mathcal{D}_t^S , $t \leq T^*$, of decompositions is an increasing family of σ -fields, meaning that $\mathcal{D}_t^S \subset \mathcal{D}_{t+1}^S$ for every $t \leq T^* - 1$. Note that the family \mathcal{D}_t^S is also generated by the family $\xi_1, \xi_2, \dots, \xi_{T^*}$ of random variables, more precisely,

$$\mathcal{D}_t^S = \mathcal{D}(\xi_0, \xi_1, \dots, \xi_t), \quad \forall t \leq T^*,$$

where by convention $\xi_0 = 1$. The family \mathcal{D}_t^S , $t \leq T^*$, models a discrete-time flow of information generated by the observations of stock prices. In financial interpretation, the decomposition \mathcal{D}_t^S represents the market information available to all investors at time t . Let us set $\mathcal{F}_t^S = \sigma(\mathcal{D}_t^S)$ for every $t \leq T^*$, where $\sigma(\mathcal{D}_t^S)$ is the σ -field generated by the decomposition \mathcal{D}_t^S . It is clear that for $t \leq T^*$

$$\mathcal{F}_t^S = \sigma(S_0, S_1, \dots, S_t) = \sigma(S_0^*, S_1^*, \dots, S_t^*) = \mathcal{F}_t^{S^*}.$$

Recall that we write $\mathbb{F}^S = (\mathcal{F}_t^S)_{t \leq T^*}$ to denote the family of natural σ -fields of the process S , or briefly, the *natural filtration* of the process S .

Let \mathbb{P} be a probability measure satisfying (2.4) – that is, such that $\mathbb{P}\{\omega\} > 0$ for any elementary event $\omega = (a_1, a_2, \dots, a_{T^*})$. Note that all probability measures satisfying (2.4) are mutually equivalent.

Definition 2.2.2 A probability measure \mathbb{P}^* equivalent to \mathbb{P} is called a *martingale measure* for the discounted stock price process $S^* = (S_t^*)_{t=0}^{T^*}$ if

$$\mathbb{E}_{\mathbb{P}^*}(S_{t+1}^* | \mathcal{F}_t^S) = S_t^*, \quad \forall t \leq T^* - 1, \quad (2.25)$$

that is, if the process S^* follows a *martingale* under \mathbb{P}^* with respect to the filtration \mathbb{F}^S . In this case, we say that the discounted stock price S^* is a $(\mathbb{P}^*, \mathbb{F}^S)$ -*martingale*, or briefly, a \mathbb{P}^* -*martingale*, if no confusion may arise.

Since a martingale measure \mathbb{P}^* corresponds to the choice of the savings account as a numeraire asset, it is also referred to as a *spot martingale measure*. The problem of existence and uniqueness of a spot martingale measure in the CRR model can be solved completely, as the following result shows.

Proposition 2.2.1 *A martingale measure \mathbb{P}^* for the discounted stock price S^* exists if and only if $d < 1 + r < u$. In this case, the martingale measure \mathbb{P}^* is the unique element from the class \mathcal{P} that corresponds to $p = p_* = (1 + r - d)/(u - d)$.*

Proof. Using (2.23)–(2.24), we may re-express equality (2.25) in the following way

$$\mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-(t+1)} \xi_{t+1} S_t | \mathcal{F}_t^S) = \hat{r}^{-t} S_t, \quad \forall t \leq T^* - 1, \quad (2.26)$$

or equivalently

$$\hat{r}^{-(t+1)} S_t \mathbb{E}_{\mathbb{P}^*}(\xi_{t+1} | \mathcal{F}_t^S) = \hat{r}^{-t} S_t, \quad \forall t \leq T^* - 1.$$

Since $S_t > 0$, for the last equality to hold it is necessary and sufficient that

$$\mathbb{E}_{\mathbb{P}^*}(\xi_{t+1} | \mathcal{F}_t^S) = 1 + r. \quad (2.27)$$

Since the right-hand side in (2.27) is non-random, we conclude that

$$\mathbb{E}_{\mathbb{P}^*}(\xi_{t+1} | \mathcal{F}_t^S) = \mathbb{E}_{\mathbb{P}^*}(\xi_{t+1}).$$

In view of Lemma 2.2.1 below, this means that ξ_{t+1} is independent of \mathcal{F}_t^S under \mathbb{P}^* , and thus independent of the random variables $\xi_1, \xi_2, \dots, \xi_t$ under \mathbb{P}^* . Since (2.27) holds for any t , it is easy to deduce by induction that the random variables ξ_1, \dots, ξ_{T^*} are necessarily independent under \mathbb{P}^* . It thus suffices to find the distribution of ξ_t under \mathbb{P}^* . We have, for any $t = 1, 2, \dots, T^*$

$$\mathbb{E}_{\mathbb{P}^*}(\xi_t) = u\mathbb{P}^*\{\xi_t = u\} + d(1 - \mathbb{P}^*\{\xi_t = u\}) = 1 + r.$$

By solving this equation, we find that $\mathbb{P}^*\{\xi_t = u\} = p_* = (1 + r - d)/(u - d)$ for every t . Note that the last equality defines a probability measure \mathbb{P}^* if and only if $d \leq 1 + r \leq u$. Moreover, \mathbb{P}^* belongs to the class \mathcal{P} and is equivalent to \mathbb{P} if and only if $d < 1 + r < u$. \square

Lemma 2.2.1 *Let ξ be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}\{\xi = a\} + \mathbb{P}\{\xi = b\} = 1$ for some real numbers a, b . Let \mathcal{G} be a sub- σ -field of \mathcal{F} . If $\mathbb{E}_{\mathbb{P}}(\xi | \mathcal{G}) = \mathbb{E}_{\mathbb{P}}\xi$ then ξ is independent of \mathcal{G} .*

Proof. Equality $\mathbb{E}_{\mathbb{P}}(\xi | \mathcal{G}) = \mathbb{E}_{\mathbb{P}}\xi$ implies that for any event $B \in \mathcal{G}$ we have

$$\int_B \xi d\mathbb{P} = \int_B \mathbb{E}_{\mathbb{P}}\xi d\mathbb{P}.$$

Let us write $A = \{\xi = a\}$. Then we obtain (we denote by A^c the complement of A in Ω , so that $A^c = \Omega \setminus A$), for any event $B \in \mathcal{G}$,

$$\int_B (a\mathbb{1}_A + b\mathbb{1}_{A^c}) d\mathbb{P} = a\mathbb{P}\{A \cap B\} + b\mathbb{P}\{A^c \cap B\} = \mathbb{P}\{B\}(a\mathbb{P}\{A\} + b\mathbb{P}\{A^c\}).$$

The last equality implies that $\mathbb{P}\{A \cap B\} = \mathbb{P}\{A\}\mathbb{P}\{B\}$, as expected. \square

Note that the stock price follows under the unique martingale measure \mathbb{P}^* , an exponential random walk (cf. formula (2.6)), with the probability of upward movement equal to p_* . This feature explains why it was possible, with no loss of generality, to restrict attention to the special class of probability measures on the underlying canonical space Ω . It is important to point out that the martingale measure \mathbb{P}^* is not exogenously introduced in order to model the observed real-world fluctuations of stock prices. Unlike the actual probability \mathbb{P} , the martingale measure \mathbb{P}^* should be seen as a technical tool that proves to be useful in the arbitrage valuation of derivative securities.

In view of relationship (2.23) it is clear that we have $S_{t+1}^* = \hat{r}^{-1} \xi_{t+1} S_t^*$. The following corollary is thus an immediate consequence of the independence of the random variables $\xi_1, \xi_2, \dots, \xi_{T^*}$ under \mathbb{P}^* .

Corollary 2.2.1 *The stock price S and the discounted stock price S^* are Markov processes under the spot martingale measure \mathbb{P}^* with respect to the filtration $\mathbb{F}^S = \mathbb{F}^{S^*}$.*

Let Φ stand for the class of all self-financing (see (2.16)) trading strategies in the CRR model. As in the previous chapter, by an *arbitrage opportunity* we mean a strategy $\phi \in \Phi$ with zero initial wealth, nonnegative terminal wealth, and such that the terminal wealth is strictly positive with positive probability (the formal definition is given in Sect. 2.6.3 below). We say that a model is *arbitrage-free* if such a trading strategy does not exist.

Corollary 2.2.2 *The CRR binomial model (S, B, Φ) is arbitrage-free if and only if $d < 1 + r < u$.*

Proof. Suppose first that the condition $d < 1 + r < u$ does not hold. Then it is possible to produce an example of an arbitrage opportunity by considering, for instance, the first time-period and proceeding as in the case of a single-period model (buy stock if $1 + r \leq d$ and short stock if $1 + r \geq u$). We conclude that the condition $d < 1 + r < u$ is necessary for the arbitrage-free property of the model. The proof of sufficiency relies on the existence of a martingale measure established in Proposition 2.2.1. It is enough to show that the discounted wealth process of any self-financing strategy follows a martingale under \mathbb{P}^* . We do not go into details here, since this part of the proposition follows immediately from Proposition 2.6.2. \square

The notion of a martingale measure (or a *risk-neutral probability*) is not universal, in the sense that it is relative to a specific choice of a numeraire asset. It can be verified, for instance, that the unique martingale measure for the relative bond price $\bar{B} = B/S$ is the unique element $\bar{\mathbb{P}}$ from the class \mathcal{P} that corresponds to the following value of p (cf. (1.16))

$$p = \bar{p} = \left(\frac{1}{d} - \frac{1}{\hat{r}} \right) \frac{ud}{u - d} = \frac{p_* u}{\hat{r}}.$$

It is easily seen that \bar{p} is in the open interval $(0, 1)$ if and only if $d < 1 + r < u$ (that is, if and only if p_* belongs to this interval). Hence the following result.

Proposition 2.2.2 *A martingale measure $\bar{\mathbb{P}}$ for the process $\bar{B} = B/S$ exists if and only if $d < 1 + r < u$. In this case, the martingale measure $\bar{\mathbb{P}}$ is the unique element from the class \mathcal{P} that corresponds to $p = \bar{p}$.*

2.2.2 Risk-neutral Valuation Formula

We shall show that the CRR option pricing formula of Proposition 2.1.2 may be alternatively derived by the direct evaluation of the conditional expectation, under the martingale measure \mathbb{P}^* , of the discounted option's payoff.

Proposition 2.2.3 *Consider a European call option, with expiry date T and strike price K , written on one share of a common stock whose price S is assumed to follow the CRR multiplicative binomial process (2.5). Then for any $m = 0, 1, \dots, T$ the arbitrage price C_{T-m} , given by formula (2.17), coincides with the conditional expectation*

$$C_{T-m}^* = \mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-m}(S_T - K)^+ \mid \mathcal{F}_{T-m}^S). \quad (2.28)$$

Proof. It is enough to find the conditional expectation in (2.28) explicitly. Recall that we have

$$S_T = S_{T-m} \xi_{T-m+1} \xi_{T-m+2} \dots \xi_T = S_{T-m} \eta_m,$$

where we write $\eta_m = \xi_{T-m+1} \xi_{T-m+2} \dots \xi_T$. Note that the stock price S_{T-m} is manifestly an \mathcal{F}_{T-m}^S -measurable random variable, whereas the random variable η_m is independent of the σ -field \mathcal{F}_{T-m}^S .

Hence, by applying the well-known property of conditional expectations (see Lemma A.1.1) to the random variables $\psi = S_{T-m}$, $\eta = \eta_m$ and to the function $h(x, y) = \hat{r}^{-m}(xy - K)^+$, one finds that

$$C_{T-m}^* = \mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-m}(S_T - K)^+ \mid \mathcal{F}_{T-m}^S) = H(S_{T-m}),$$

where the function $H : \mathbb{R} \rightarrow \mathbb{R}$ equals

$$H(x) = \mathbb{E}_{\mathbb{P}^*}(h(x, \eta_m)) = \mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-m}(x\eta_m - K)^+), \quad \forall x \in \mathbb{R}.$$

The random variables $\xi_{T-m+1}, \xi_{T-m+2}, \dots, \xi_T$ are also mutually independent and identically distributed under \mathbb{P}^* , with $\mathbb{P}^*\{\xi_j = u\} = p_* = 1 - \mathbb{P}^*\{\xi_j = d\}$. It is thus clear that

$$H(x) = \hat{r}^{-m} \sum_{j=0}^m \binom{m}{j} p_*^j (1 - p_*)^{m-j} (xu^j d^{m-j} - K)^+.$$

Using equalities $\bar{p} = p_* u / \hat{r}$ and $1 - \bar{p} = (1 - p_*) d / \hat{r}$, we conclude that

$$C_{T-m}^* = \sum_{j=a}^m \binom{m}{j} \left(S_{T-m} \bar{p}^j (1 - \bar{p})^{m-j} - K \hat{r}^{-m} p_*^j (1 - p_*)^{m-j} \right),$$

where $a = a_m(S_{T-m}) = \inf \{j \in \mathbb{N}^* \mid S_{T-m} u^j d^{m-j} > K\}$. □

One might wonder if the risk-neutral valuation formula (2.28) remains in force for a larger class of financial models and European contingent claims. Generally speaking, the answer to this question is positive, even if the interest rate is assumed to follow a stochastic process, that is, for any claim X maturing at T and every $t = 0, 1, \dots, T$, the arbitrage price $\pi_t(X)$ equals

$$\pi_t(X) = B_t \mathbb{E}_{\mathbb{P}^*}(B_T^{-1} X \mid \mathcal{F}_t^S), \quad \forall t \leq T. \quad (2.29)$$

For the CRR model, the last formula follows, for instance, from the recursive pricing procedure described in Sect. 2.1.2. A general result for a discrete-time finite model is established in Sect. 2.6.5.

The risk-neutral valuation formula (2.29) makes it clear, that the valuation is a linear map from the space of contingent claim that settle at time T (that is, the linear space of all \mathcal{F}_T -measurable random variables) to \mathbb{R} . The *put-call parity* relationship

$$C_t - P_t = S_t - K \hat{r}^{-(T-t)}$$

can now be easily established by applying the risk-neutral valuation formula to the payoff $X = C_T - P_T = S_T - K$.

2.2.3 Change of a Numeraire

The following variant of the risk-neutral valuation formula, based on the martingale measure $\tilde{\mathbb{P}}$ of Proposition 2.2.2, is also valid for any contingent claim X

$$\pi_t(X) = S_t \mathbb{E}_{\tilde{\mathbb{P}}}(S_T^{-1} X \mid \mathcal{F}_t^S), \quad \forall t \leq T. \quad (2.30)$$

The last formula can be proved using (2.29), the Bayes formula, and the expression for the Radon-Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to \mathbb{P}^* , which is given by the following result.

Proposition 2.2.4 *The Radon-Nikodým derivative process η of $\tilde{\mathbb{P}}$ with respect to \mathbb{P}^* equals, for every $t = 0, 1, \dots, T^*$,*

$$\eta_t = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} \Big|_{\mathcal{F}_t^S} = \frac{B_0 S_t}{S_0 B_t}. \quad (2.31)$$

Proof. It is clear that \mathbb{P}^* and $\tilde{\mathbb{P}}$ are equivalent on $(\Omega, \mathcal{F}_{T^*})$ and thus the Radon-Nikodým derivative $\eta_{T^*} = d\tilde{\mathbb{P}}/d\mathbb{P}^*$ exists. Since $S^* = S/B$ is a martingale under \mathbb{P}^* and, by virtue of general properties of the Radon-Nikodým derivative, we have that $\eta_t = \mathbb{E}_{\mathbb{P}^*}(\eta_{T^*} \mid \mathcal{F}_t^S)$, it suffices to show that

$$\eta_{T^*} = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} = \frac{B_0 S_{T^*}}{S_0 B_{T^*}}. \quad (2.32)$$

It is easily seen that the last equality defines an equivalent probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_{T^*})$ since $\eta_{T^*} > 0$ and $\mathbb{E}_{\mathbb{P}^*} \eta_{T^*} = 1$. It suffices to verify that the martingale property of the process $S^* = S/B$ under \mathbb{P}^* implies that the process B/S is a martingale under $\tilde{\mathbb{P}}$. We thus assume that, for $u \leq t$,

$$\mathbb{E}_{\mathbb{P}^*}(S_t/B_t \mid \mathcal{F}_u^S) = S_u/B_u.$$

Using the Bayes formula (see Lemma A.1.4), we obtain

$$\begin{aligned} \mathbb{E}_{\bar{\mathbb{P}}}(B_t/S_t \mid \mathcal{F}_u^S) &= \frac{\mathbb{E}_{\mathbb{P}^*}(\eta_T^* B_t/S_t \mid \mathcal{F}_u^S)}{\mathbb{E}_{\mathbb{P}^*}(\eta_T^* \mid \mathcal{F}_u^S)} \\ &= \frac{\mathbb{E}_{\mathbb{P}^*}(\mathbb{E}_{\mathbb{P}^*}(\eta_T^* \mid \mathcal{F}_t^S) B_t/S_t \mid \mathcal{F}_u^S)}{\mathbb{E}_{\mathbb{P}^*}(\eta_T^* \mid \mathcal{F}_u^S)} \\ &= \frac{\mathbb{E}_{\mathbb{P}^*}(\eta_t B_t/S_t \mid \mathcal{F}_u^S)}{\eta_u} \\ &= \frac{\mathbb{E}_{\mathbb{P}^*}((S_t/B_t)(B_t/S_t) \mid \mathcal{F}_u^S)}{S_u/B_u} \\ &= B_u/S_u. \end{aligned}$$

We conclude that the probability measure $\bar{\mathbb{P}}$ given by (2.32) is indeed the unique martingale measure for the process $\bar{B} = B/S$. \square

Let us check that the right-hand sides in (2.29) and (2.30) coincide for any contingent claim X . Since X and S_T are \mathcal{F}_T^S -measurable random variables, the Bayes formula yields

$$\begin{aligned} S_t \mathbb{E}_{\bar{\mathbb{P}}}(S_T^{-1} X \mid \mathcal{F}_t^S) &= S_t \frac{\mathbb{E}_{\mathbb{P}^*}(\eta_T S_T^{-1} X \mid \mathcal{F}_t^S)}{\mathbb{E}_{\mathbb{P}^*}(\eta_T \mid \mathcal{F}_t^S)} \\ &= S_t \frac{\mathbb{E}_{\mathbb{P}^*}(\eta_T S_T^{-1} X \mid \mathcal{F}_t^S)}{\eta_t} \\ &= S_t \frac{\mathbb{E}_{\mathbb{P}^*}(S_T B_T^{-1} S_T^{-1} X \mid \mathcal{F}_t^S)}{S_t B_t^{-1}} \\ &= B_t \mathbb{E}_{\mathbb{P}^*}(B_T^{-1} X \mid \mathcal{F}_t^S). \end{aligned}$$

We are in a position to state the following corollary.

Corollary 2.2.3 *For any European contingent claim X settling at time T the arbitrage price satisfies, for every $t = 0, 1, \dots, T$,*

$$\pi_t(X) = B_t \mathbb{E}_{\mathbb{P}^*}(B_T^{-1} X \mid \mathcal{F}_t^S) = S_t \mathbb{E}_{\bar{\mathbb{P}}}(S_T^{-1} X \mid \mathcal{F}_t^S). \quad (2.33)$$

The following result furnishes an alternative representation for the arbitrage price of European call and put options.

Proposition 2.2.5 *The arbitrage price at time t of a European call option settling at time T equals*

$$C_t = S_t \bar{\mathbb{P}}\{S_T > K \mid \mathcal{F}_t^S\} - K \hat{r}^{-(T-t)} \mathbb{P}^*\{S_T > K \mid \mathcal{F}_t^S\}, \quad (2.34)$$

where $\bar{\mathbb{P}}$ and \mathbb{P}^* are the unique martingale measures for the processes $\bar{B} = B/S$ and $S^* = S/B$ respectively. The arbitrage price at time t of a European put option is given by

$$C_t = K \hat{r}^{-(T-t)} \mathbb{P}^* \{S_T < K \mid \mathcal{F}_t^S\} - S_t \bar{\mathbb{P}} \{S_T < K \mid \mathcal{F}_t^S\}. \quad (2.35)$$

Proof. To derive (2.34), let us denote $D = \{S_T > K\}$ and let us observe that

$$C_T = (S_T - K)^+ = S_T \mathbb{1}_D - K \mathbb{1}_D.$$

Consequently, using the linearity of the arbitrage price and Corollary 2.2.3, we obtain

$$C_t = \pi_t(S_T \mathbb{1}_D) - \pi_t(K \mathbb{1}_D) = S_t \mathbb{E}_{\bar{\mathbb{P}}} (S_T^{-1} S_T \mathbb{1}_D \mid \mathcal{F}_t^S) - K B_t \mathbb{E}_{\mathbb{P}^*} (B_T^{-1} \mathbb{1}_D \mid \mathcal{F}_t^S).$$

It is easily seen that the last formula yields (2.34). The proof of (2.35) is similar.

□

2.3 The Black-Scholes Option Pricing Formula

We will now show that the classical Black-Scholes option valuation formula (2.40) can be obtained from the CRR option valuation result (2.17) by an asymptotic procedure, using a properly chosen sequence of binomial models. To this end, we need to examine the asymptotic properties of the CRR model when the number of steps goes to infinity and, simultaneously, the size of time and space steps tends to zero in an appropriate way.

In contrast to the previous section, $T > 0$ is a fixed, but arbitrary, real number. For any n of the form $n = 2^k$, we divide the interval $[0, T]$ into n equal subintervals I_j of length $\Delta_n = T/n$, namely, $I_j = [j\Delta_n, (j+1)\Delta_n]$ for $j = 0, 1, \dots, n-1$. Note that n corresponds, in a sense, to the natural number T^* in the preceding section.

Let us first introduce the modified accumulation factor. We write r_n to denote the risk-free rate of return over each interval $I_j = [j\Delta_n, (j+1)\Delta_n]$. Hence the price B^n of the risk-free asset equals

$$B_{j\Delta_n}^n = (1 + r_n)^j, \quad \forall j = 0, 1, \dots, n.$$

We thus deal here with a sequence B^n of savings accounts. The same remark applies to the sequence S^n of binomial lattices describing the evolution of the stock price.

For any n , we assume that the stock price S^n can appreciate over the period I_j by u_n or decline by d_n . Specifically, we set $S_{(j+1)\Delta_n}^n = \xi_{j+1}^n S_{j\Delta_n}^n$ for $j = 0, 1, \dots, n-1$, where for any fixed n and j , ξ_j^n is a random variable with values in the two-element set $\{u_n, d_n\}$.

In view of Proposition 2.1.2, we may assume, without loss of generality, that for any n the random variables ξ_j^n , $j = 1, 2, \dots, n$ are defined on a common probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$, are mutually independent, and

$$\mathbb{P}_n \{\xi_j^n = u_n\} = p_n = 1 - \mathbb{P}_n \{\xi_j^n = d_n\}, \quad \forall j = 1, \dots, n,$$

for some number $p_n \in (0, 1)$. Recall that the choice of the parameter $p_n \in (0, 1)$ is not relevant from the viewpoint of arbitrage pricing.

To ensure the convergence of the CRR option valuation formula to the Black-Scholes one, we need to impose, in addition, specific restrictions on the asymptotic behavior of the quantities r_n , u_n and d_n . Let us put

$$1 + r_n = e^{r\Delta_n}, \quad u_n = e^{\sigma\sqrt{\Delta_n}}, \quad d_n = u_n^{-1}, \quad (2.36)$$

where $r \geq 0$ and $\sigma > 0$ are given real numbers. It is worth noting that for every $n > r^2\sigma^{-2}T$ we have $d_n = u_n^{-1} < 1 + r_n < u_n$ and thus the CRR model (S^n, B^n, Φ^n) is arbitrage-free for n sufficiently large. Also, using elementary arguments, we obtain

$$\lim_{n \rightarrow +\infty} p_{*,n} = \lim_{n \rightarrow +\infty} \frac{1 + r_n - d_n}{u_n - d_n} = \lim_{n \rightarrow +\infty} \frac{e^{r\Delta_n} - e^{-\sigma\sqrt{\Delta_n}}}{e^{\sigma\sqrt{\Delta_n}} - e^{-\sigma\sqrt{\Delta_n}}} = 1/2$$

and

$$\lim_{n \rightarrow +\infty} \bar{p}_n = \lim_{n \rightarrow +\infty} \frac{p_{*,n}u_n}{1 + r_n} = 1/2.$$

As was mentioned earlier, we seek the asymptotic value of the call option price when the number of time periods tends to infinity. Assume that $t = jT/2^k$ for some natural j and k ; that is, t is an arbitrary dyadic number from the interval $[0, T]$. Given any such number, we introduce the sequence $m_n(t)$ by setting

$$m_n(t) = n(T - t)/T, \quad \forall n \in \mathbb{N}. \quad (2.37)$$

Then the sequence $m_n(t)$ has natural values for $n = 2^l$ sufficiently large. Furthermore, $T - t = m_n(t)\Delta_n$ so that $m_n(t)$ represents the number of trading periods in the interval $[t, T]$ (at least for large l). Note also that

$$\lim_{n \rightarrow +\infty} (1 + r_n)^{-m_n(t)} = \lim_{n \rightarrow +\infty} e^{-r\Delta_n m_n(t)} = e^{-r(T-t)}. \quad (2.38)$$

For a generic value of stock price at time t , $S_t = S_{T-m_n(t)\Delta_n} = s$, we define

$$b_n(t) = \inf\{j \in \mathbb{N}^* \mid su_n^j d_n^{m_n(t)-j} > K\}. \quad (2.39)$$

The next proposition provides the derivation of the classical Black-Scholes option valuation formula by means of an asymptotic procedure. A direct analysis of the continuous-time Black-Scholes option pricing model, based on the Itô stochastic calculus, is presented in Sect. 3.1.

Proposition 2.3.1 *For any dyadic $t \in [0, T]$, the following convergence holds*

$$\lim_{n \rightarrow +\infty} \sum_{j=b_n(t)}^{m_n(t)} \binom{m_n(t)}{j} \{S_t \bar{p}_n^j \bar{q}_n^{m_n(t)-j} - K \hat{r}_n^{-m_n(t)} p_{*,n}^j q_{*,n}^{m_n(t)-j}\} = C_t,$$

where $q_{*,n} = 1 - p_{*,n}$, $\bar{q}_n = 1 - \bar{p}_n$, and C_t is given by the Black-Scholes formula

$$C_t = S_t N(d_1(S_t, T - t)) - K e^{-r(T-t)} N(d_2(S_t, T - t)), \quad (2.40)$$

where

$$d_1(s, t) = \frac{\ln(s/K) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}, \quad (2.41)$$

$$d_2(s, t) = d_1(s, t) - \sigma\sqrt{t} = \frac{\ln(s/K) + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}, \quad (2.42)$$

and N stands for the standard Gaussian cumulative distribution function,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du, \quad \forall x \in \mathbb{R}.$$

Before proceeding to the proof of the proposition, let us make a few comments. We wish to make clear that the limit of the CRR option price depends essentially on the choice of sequences u_n and d_n . For the choice of u_n s and d_n s that we have made here, the asymptotic dynamic of the stock price is that of the *geometric Brownian motion* (known also as the *geometric Wiener process*). This means, in particular, that the asymptotic evolution of the stock price under the martingale measure may be described by a stochastic process whose sample paths almost all follow continuous functions; furthermore, the risk-neutral probability distribution of the continuous-time stock price at any time t is lognormal.

Proof of Proposition 2.3.1. Let $S_t = s$ be the generic value of the stock price at time t . Our first goal is to check that

$$\lim_{n \rightarrow +\infty} \sum_{j=b_n(t)}^{m_n(t)} \binom{m_n(t)}{j} \bar{p}_n^j (1 - \bar{p}_n)^{m_n(t)-j} = N(d_1(s, T - t)). \quad (2.43)$$

It is essential to observe that

$$\sum_{j=b_n(t)}^{m_n(t)} \binom{m_n(t)}{j} \bar{p}_n^j (1 - \bar{p}_n)^{m_n(t)-j} = \mathbb{Q}\{b_n(t) \leq \gamma_n \leq m_n(t)\},$$

where for every n the random variable γ_n has, under \mathbb{Q} , the binomial distribution with parameters $m_n(t)$ and \bar{p}_n .

Let us write $\sigma_n = \sqrt{\bar{p}_n(1 - \bar{p}_n)}$ and let us introduce the normalized sequence $\tilde{\gamma}_n$ by setting

$$\tilde{\gamma}_n = \frac{\gamma_n - \mathbb{E}_{\mathbb{Q}}(\gamma_n)}{\sqrt{\text{Var}_{\mathbb{Q}}(\gamma_n)}} = \frac{\gamma_n - m_n(t)\bar{p}_n}{\sqrt{m_n(t)\bar{p}_n(1 - \bar{p}_n)}} = \frac{\sum_{j=1}^{m_n(t)} (\zeta_j^n - \bar{p}_n)}{\sigma_n \sqrt{m_n(t)}},$$

where ζ_j^n , $j = 1, \dots, n$ are independent, identically distributed random variables with Bernoulli distribution and the parameter \bar{p}_n

$$\mathbb{Q}\{\zeta_j^n = 1\} = \bar{p}_n = 1 - \mathbb{Q}\{\zeta_j^n = 0\}.$$

We wish first to check that the sequence $\tilde{\gamma}_n$ of random variables converges in distribution to the standard Gaussian distribution. To this end, let us denote by \tilde{f}_n the characteristic function of the random variable $\tilde{\zeta}_j^n = \zeta_j^n - \bar{p}_n$. We have

$$\tilde{f}_n(z) = \mathbb{E}_{\mathbb{Q}}(e^{iz\tilde{\zeta}_j^n}) = e^{-iz\bar{p}_n} \mathbb{E}_{\mathbb{Q}}(e^{iz\zeta_j^n}) = e^{-iz\bar{p}_n} (\bar{p}_n e^{iz} + 1 - \bar{p}_n).$$

It is not difficult to check that

$$\tilde{f}_n(z) = 1 - \frac{\sigma_n^2 z^2}{2} + o(z^2),$$

where the symbol $o(z^2)$ represents a quantity satisfying $\lim_{z \rightarrow 0} o(z^2)/z^2 = 0$. Consequently, by virtue of independence of the random variables $\tilde{\zeta}_j^n$, $j = 1, \dots, n$, the characteristic function $g_n(z) = \mathbb{E}_{\mathbb{Q}}(e^{iz\tilde{\gamma}_n})$ of the random variable $\tilde{\gamma}_n$ can be represented as follows:

$$g_n(z) = \left\{ \tilde{f}_n \left(\frac{z}{\sigma_n \sqrt{m_n(t)}} \right) \right\}^{m_n(t)} = \left\{ 1 - \frac{z^2}{2m_n(t)} + o \left(\frac{z^2}{m_n(t)} \right) \right\}^{m_n(t)}.$$

From the last expression, it is thus apparent that the following point-wise convergence is valid:

$$\lim_{n \rightarrow +\infty} g_n(z) = e^{-z^2/2}, \quad \forall z \in \mathbb{R}. \quad (2.44)$$

Recall the well-known fact that the function $e^{-z^2/2}$ is the characteristic function of the standard Gaussian distribution $N(0, 1)$.

The convergence in distribution of the normalized sequence $\tilde{\gamma}_n$ to the standard Gaussian distribution now follows from the point-wise convergence (2.44) of the corresponding sequence g_n of characteristic functions to the characteristic function of the standard Gaussian distribution (see, for instance, Theorem III.3.1 in Shiryaev (1984)).

Furthermore, it is clear that

$$\lim_{n \rightarrow +\infty} \frac{m_n(t) - m_n(t)\bar{p}_n}{\sqrt{m_n(t)\bar{p}_n(1 - \bar{p}_n)}} = \lim_{n \rightarrow +\infty} \sqrt{\bar{m}_n(t)\bar{p}_n^{-1}(1 - \bar{p}_n)} = +\infty.$$

Hence, using (2.39), we obtain

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{b_n(t) - m_n(t)\bar{p}_n}{\sqrt{m_n(t)\bar{p}_n(1 - \bar{p}_n)}} \\ &= \lim_{n \rightarrow +\infty} \frac{\frac{\ln(K/s) + m_n(t)\sigma\sqrt{\Delta_n}}{2\sigma\sqrt{\Delta_n}} - m_n(t)\bar{p}_n}{\sqrt{m_n(t)\bar{p}_n(1 - \bar{p}_n)}} \\ &= \lim_{n \rightarrow +\infty} \frac{\ln(K/s) + \sigma m_n(t)\sqrt{\Delta_n}(1 - 2\bar{p}_n)}{2\sigma\sqrt{m_n(t)\Delta_n}\bar{p}_n(1 - \bar{p}_n)} \\ &= \frac{\ln(K/s) - (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} = -d_1(s, T - t) \end{aligned}$$

since

$$\lim_{n \rightarrow +\infty} m_n(t) \sqrt{\Delta_n} (1 - 2\bar{p}_n) = -(T - t) \left(\frac{r}{\sigma} + \frac{\sigma}{2} \right).$$

The convergence in distribution of the sequence γ_n to the standard Gaussian distribution $N(0, 1)$ and the equality $1 - N(x) = N(-x)$ for $x \in \mathbb{R}$ yield

$$\lim_{n \rightarrow +\infty} \mathbb{Q}\{b_n(t) \leq \gamma_n \leq m_n(t)\} = 1 - N(-d_1(s, T - t)) = N(d_1(s, T - t)).$$

This completes the proof of equality (2.43). Reasoning in a similar manner, we will now check that

$$\lim_{n \rightarrow +\infty} \sum_{j=b_n(t)}^{m_n(t)} \binom{m_n(t)}{j} p_{*,n}^j (1 - p_{*,n})^{m_n(t)-j} = N(d_2(s, T - t)). \quad (2.45)$$

The sum in the left-hand side of formula (2.45) is equal to the probability $\mathbb{Q}\{b_n(t) \leq \gamma_n^* \leq m_n(t)\}$, where for every n the random variable γ_n^* has, under \mathbb{Q} , the binomial distribution with parameters $m_n(t)$ and $p_{*,n}$. Moreover, we have

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{b_n(t) - m_n(t)p_{*,n}}{\sqrt{m_n(t)p_{*,n}(1 - p_{*,n})}} \\ &= \lim_{n \rightarrow +\infty} \frac{\frac{\ln(K/s) + m_n(t)\sigma\sqrt{\Delta_n}}{2\sigma\sqrt{\Delta_n}} - m_n(t)p_{*,n}}{\sqrt{m_n(t)p_{*,n}(1 - p_{*,n})}} \\ &= \lim_{n \rightarrow +\infty} \frac{\ln(K/s) + \sigma m_n(t)\sqrt{\Delta_n}(1 - 2p_{*,n})}{2\sigma\sqrt{m_n(t)\Delta_n p_{*,n}(1 - p_{*,n})}} \\ &= \frac{\ln(K/s) - (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} = -d_2(s, T - t) \end{aligned}$$

since

$$\lim_{n \rightarrow +\infty} m_n(t) \sqrt{\Delta_n} (1 - 2p_{*,n}) = (T - t) \left(\frac{\sigma}{2} - \frac{r}{\sigma} \right).$$

Therefore

$$\lim_{n \rightarrow +\infty} \mathbb{Q}\{b_n(t) \leq \gamma_n^* \leq m_n(t)\} = 1 - N(-d_2(s, T - t)) = N(d_2(s, T - t)).$$

The proposition now follows by combining (2.38), (2.43), and (2.45). \square

For a different choice of the sequences u_n and d_n , the stock price may asymptotically follow a stochastic process with discontinuous sample paths. For instance, if we put $u_n = u$ and $d_n = e^{ct/n}$ then the stock process will follow asymptotically a log-Poisson process, examined by Cox and Ross (1975). This noticeable feature is related to the fact that we deal with a triangular array of random variables and thus the class of asymptotic probability distributions is larger than in the case of the classical central limit theorem. More advanced problems related to the convergence of discrete-time financial models to continuous-time counterparts

were studied by, among others, Madan et al. (1989), He (1990), Amin (1991), Willinger and Taqqu (1991), Duffie and Protter (1992), Cutland et al. (1993b), Amin and Khanna (1994), Rachev and Rüschendorf (1994), Hubalek and Schachermayer (1998), Leisen (1998), Broadie et al. (1998), Lesne et al. (2000) and Lesne and Prigent (2001). The recent monograph by Prigent (2003) is an excellent source of expertise in the area of weak convergence of financial models.

2.4 Valuation of American Options

In this section, we are concerned with the arbitrage valuation of American options written on a stock S within the framework of the CRR binomial model of a stock price. Due to the possibility of an early exercise of an American option, the problem of pricing and hedging of such claims cannot be reduced to a simple replication of the terminal payoff. Nevertheless, the valuation of American options will still be based on the no-arbitrage arguments.

2.4.1 American Call Options

Let us first consider the case of the American call option – that is, the option to buy a specified number of shares, which may be exercised at any time before the option expiry date T , or on that date. The exercise policy of the option holder is necessarily based on the information accumulated to date and not on the future prices of the stock. As in the previous chapter, we will write C_t^a to denote the arbitrage price at time t of an American call option written on one share of a stock. By arbitrage price of the American call we mean such price process C_t^a , $t \leq T$, that an extended financial market model – that is, a market with trading in risk-free bonds, stocks and American call options – remains arbitrage-free. Our first goal is to show that the price of an American call option in the CRR arbitrage-free market model coincides with the arbitrage price of a European call option with the same expiry date and strike price. For this purpose, it is sufficient to show that the American call option should never be exercised before maturity, since otherwise the option writer would be able to make risk-free profit. It is worth noting that, by convention, when an American call is exercised at time t , its payoff equals $(S_t - K)^+$, rather than $S_t - K$ (a similar convention applies to an American put). Due to this convention, we may postulate that an American call or put option should be exercised by its holder either prior to or at maturity date.

The argument hinges on the following simple inequality

$$C_t \geq (S_t - K)^+, \quad \forall t \leq T, \quad (2.46)$$

which can be justified in several ways. For instance, one may use the explicit formula (2.17), or apply the risk-neutral valuation formula (2.28). In the latter method, the argument is based on Jensen's conditional inequality applied to the convex function $f(x) = (x - K)^+$. In fact, we have (recall that $t = T - m$)

$$\mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-m}(S_T - K)^+ | \mathcal{F}_t^S) \geq (\mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-m}S_T | \mathcal{F}_t^S) - \hat{r}^{-m}K)^+ \geq (S_t - K)^+,$$

where the first inequality is the Jensen conditional inequality, and the second follows from the trivial inequality $K/\hat{r}^m \leq K$ (the assumption that $r \geq 0$ is essential here).

A more intuitive way of deriving (2.46) is based on no-arbitrage arguments. Note that since the option's price C_t is always nonnegative, it is sufficient to consider the case when the current stock price is greater than the exercise price – that is, when $S_t - K > 0$. Suppose, on the contrary, that $C_t < S_t - K$ for some t , i.e., $S_t - C_t > K$. Then it would be possible, with zero net initial investment, to buy at time t a call option, short a stock, and invest the sum $S_t - C_t$ in a savings account. By holding this portfolio unchanged up to the maturity date T , we would be able to lock in a risk-free profit. Indeed, the value of our portfolio at time T would satisfy (recall that $r \geq 0$)

$$\hat{r}^{T-t}(S_t - C_t) + C_T - S_T > \hat{r}^{T-t}K + (S_T - K)^+ - S_T \geq 0.$$

We conclude once again that inequality (2.46) is necessary for the absence of arbitrage opportunities.

Taking (2.46) for granted, we may deduce the property $C_t^a = C_t$ by simple no-arbitrage arguments. Suppose, on the contrary, that the writer of an American call is able to sell the option at time 0 at the price $C_0^a > C_0$ (it is evident that, at any time, an American option is worth at least as much as a European option with the same contractual features; in particular, $C_0^a \geq C_0$). In order to profit from this transaction, the option writer establishes a dynamic portfolio replicating the value process of the European call, and invests the remaining funds in risk-free bonds.

Suppose that the holder of the option decides to exercise it at instant t before the expiry date T . Then the option's writer locks in a risk-free profit, since the value of portfolio satisfies

$$C_t - (S_t - K)^+ + \hat{r}^t(C_0^a - C_0) > 0, \quad \forall t \leq T.$$

The above reasoning implies that the European and American call options are equivalent from the point of view of arbitrage pricing theory; that is, both options have the same price, and an American call should never be exercised by its holder before expiry.

The last statement means also that a risk-neutral investor who is long an American call should be indifferent between selling it before, and holding it to, the option's expiry date (provided that the market is efficient – that is, options are neither underpriced nor overpriced).

Let us show by still another intuitive reasoning that a holder of an American call should never exercise the option before its expiry date. Consider an investor who contemplates exercising an American call option at a certain date $t < T$. A better solution is to short one share of stock and to hold the option until its expiry date. Actually, exercising the option yields $S_t - K$ of cash at time t – that is, $\hat{r}^{T-t}(S_t - K)$ of cash at the option's expiry. The second trading strategy gives the payoff $\hat{r}^{T-t}S_t -$

$S_T + (S_T - K)^+$ at time T ; that is, either $\hat{r}^{T-t} S_t - K$ (if $S_T \geq K$) or $\hat{r}^{T-t} S_t - S_T$ (if $S_T < K$).

It is thus evident that in all circumstances the second portfolio outperforms the first if $r \geq 0$. It is interesting to observe that this argument can be easily extended to the case of uncertain future interest rates.

2.4.2 American Put Options

Since the early exercise feature of American put options was examined in Sect. 1.7, we will focus on the justification of the valuation formula. Let us denote by \mathcal{T} the class of all *stopping times* defined on the filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, where $\mathcal{F}_t = \mathcal{F}_t^S$ for every $t = 0, 1, \dots, T^*$.

By a *stopping time* we mean an arbitrary mapping $\tau : \Omega \rightarrow \{0, 1, \dots, T^*\}$ such that for any $t = 0, 1, \dots, T^*$, a random event $\{\omega \in \Omega \mid \tau(\omega) = t\}$ belongs to the σ -field \mathcal{F}_t . Intuitively, this property means that the decision whether to stop a process at time t (that is, whether to exercise an option at time t or not) depends on the stock price fluctuations up to time t only. Also, let $\mathcal{T}_{[t, T]}$ stand for the subclass stopping times τ satisfying $t \leq \tau \leq T$. Recall that, by convention, if an American put is exercised at time t , the payoff to its holder equals $(K - S_t)^+$, rather than $K - S_t$.

Corollary 1.7.1 and the preceding discussion suggest the following result.

Proposition 2.4.1 *The arbitrage price P_t^a of an American put option equals*

$$P_t^a = \max_{\tau \in \mathcal{T}_{[t, T]}} \mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-(\tau-t)}(K - S_\tau)^+ \mid \mathcal{F}_t), \quad \forall t \leq T. \quad (2.47)$$

Moreover, for any $t \leq T$ the stopping time τ_t^* that realizes the maximum in (2.47) is given by the expression (by convention $\min \emptyset = T$)

$$\tau_t^* = \min \{u \in \{t, t+1, \dots, T\} \mid (K - S_u)^+ \geq P_u^a\}. \quad (2.48)$$

Proof. The proof of the proposition is left to the reader as an exercise. Alternatively, we refer to Sect. 2.8 for a detailed study of valuation and hedging of American claims in a finite market model, which covers also the case of the CRR model. \square

The stopping time τ_t^* will be referred to as the *rational exercise time* of an American put option that is still alive at time t . Let us emphasize that the stopping time τ_t^* does not solve the optimal stopping problem for any individual, but only for those investors who are risk-neutral. Also, we do not claim that the rational stopping time after t is unique, in general. In fact, the stopping time τ_t^* given by (2.48) is the minimal rational exercise time after t . For more information on this issue, see Sect. 2.8.2.

Recall that the stock price S is Markovian under \mathbb{P}^* (see Corollary 2.2.1). An application of the *Bellman principle*³ reduces the optimal stopping problem (2.47) to an explicit recursive procedure that allows us to find the value function V^P . These observations lead to the following corollary to Proposition 2.4.1.

³ For an exposition of the stochastic optimal control, see, e.g., Bertsekas and Shreve (1978) or Zabczyk (1996).

Corollary 2.4.1 *Let the nonnegative adapted process U_t , $t \leq T$, be defined recursively by setting $U_T = (K - S_T)^+$ and, for $t \leq T - 1$,*

$$U_t = \max \left\{ K - S_t, \mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-1} U_{t+1} | \mathcal{F}_t) \right\}. \quad (2.49)$$

Then the arbitrage price P_t^a of an American put option at time t equals U_t . Moreover, the rational exercise time as of time t equals

$$\tau_t^* = \min \left\{ u \in \{t, t+1, \dots, T\} \mid K - S_u \geq U_u \right\}. \quad (2.50)$$

It is also possible to go the other way around – that is, to first show directly that the price P_t^a needs to satisfy the recursive relation, for $t \leq T - 1$,

$$P_t^a = \max \left\{ K - S_t, \mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-1} P_{t+1}^a | \mathcal{F}_t) \right\} \quad (2.51)$$

subject to the terminal condition $P_T^a = (K - S_T)^+$, and subsequently derive the equivalent representation (2.47). In the case of the CRR model (indeed, in the case of any discrete-time security pricing model), the latter approach appears to be the simplest way to value American options. The main reason for this is that an apparently difficult valuation problem is thus reduced to the simple one-period case. To show this we shall argue, as usual, by contradiction. To start with, we assume that (2.51) fails to hold for $t = T - 1$. If this is the case, by reasoning along the same lines as in Sect. 1.7, one may easily construct at time $T - 1$ a portfolio producing risk-free profit at time T . We thus conclude that necessarily

$$P_{T-1}^a = \max \left\{ K - S_{T-1}, \mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-1} (K - S_T)^+ | \mathcal{F}_T) \right\}.$$

The next step is to consider the time period $[T - 2, T - 1]$, with $T - 1$ now playing the role of the terminal date, and P_{T-1}^a being the terminal payoff. This procedure may be repeated as many times as needed.

Summarizing, in the case of the CRR model, the arbitrage pricing of an American put reduces to the following simple recursive recipe, which is valid for every $t = 0, 1, \dots, T - 1$,

$$P_t^a = \max \left\{ K - S_t, \hat{r}^{-1} (p_* P_{t+1}^{au} + (1 - p_*) P_{t+1}^{ad}) \right\} \quad (2.52)$$

with $P_T^a = (K - S_T)^+$. Note that P_{t+1}^{au} and P_{t+1}^{ad} represent the value of the American put in the next step corresponding to the upward and downward movement of the stock price starting from a given node on the lattice – that is, to the values uS_t and dS_t of the stock price at time $t + 1$, respectively.

2.4.3 American Claims

The above results may be easily extended to the case of an arbitrary claim of American style. We shall assume that an American claim does not produce any payoff unless it is exercised, so that it is not the most general definition one may envisage.

Definition 2.4.1 An American contingent claim $X^a = (X, \mathcal{T}_{[0,T]})$ expiring at time T consists of a sequence of payoffs $(X_t)_{t=0}^T$, where X_t is an \mathcal{F}_t -measurable random variable for $t = 0, 1, \dots, T$, and of the set $\mathcal{T}_{[0,T]}$ of admissible exercise policies.

We interpret X_t as the payoff received by the holder of the claim X^a upon exercising it at time t . Note that the set of admissible exercise policies is restricted to the class $\mathcal{T}_{[0,T]}$ of all stopping times of the filtration \mathbb{F}^S with values in $\{0, 1, \dots, T\}$. Let $h : \mathbb{R} \times \{0, 1, \dots, T\} \rightarrow \mathbb{R}$ be an arbitrary function. We say that X^a is an American contingent claim associated with the *reward function* h if the equality $X_t = h(S_t, t)$ holds for every $t = 0, 1, \dots, T$.

An American contingent claim is said to be *path-independent* when its generic payoffs X_t do not depend on the whole sample path up to time t , but only on the current value, S_t , of the stock price. It is clear that a claim is path-independent if and only if it is associated with a certain reward function h .

Arbitrage valuation of any American claim in a discrete-time model is based on a simple recursive procedure. In order to price a path-independent American claim in the case of the CRR model, it is sufficient to move backward in time along the binomial lattice. If an American contingent claim is path-dependent, such a simple recipe is no longer applicable (for examples of efficient numerical procedures for valuing path-dependent options, we refer to Hull and White (1993c)). We have, however, the following result, whose proof is omitted. Once again, we refer to Sect. 2.8 for a detailed study of arbitrage pricing of American claims in a finite market model.

Proposition 2.4.2 For every $t \leq T$, the arbitrage price $\pi(X^a)$ of an arbitrary American claim X^a in the CRR model equals

$$\pi_t(X^a) = \max_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-(\tau-t)} X_\tau \mid \mathcal{F}_t).$$

The price process $\pi(X^a)$ can be determined using the following recurrence relation, for every $t \leq T - 1$,

$$\pi_t(X^a) = \max \left\{ X_t, \mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-1} \pi_{t+1}(X^a) \mid \mathcal{F}_t) \right\} \quad (2.53)$$

subject to the terminal condition $\pi_T(X^a) = X_T$. In the case of a path-independent American claim X^a with the reward function h we have that, for every $t \leq T - 1$,

$$\pi_t(X^a) = \max \left\{ h(S_t, t), \hat{r}^{-1} (p_* \pi_{t+1}^u(X^a) + (1 - p_*) \pi_{t+1}^d(X^a)) \right\}, \quad (2.54)$$

where, for a generic stock price value S_t at time t , we write $\pi_{t+1}^u(X^a)$ and $\pi_{t+1}^d(X^a)$ to denote the values of the price process $\pi(X^a)$ at time $t + 1$ in the nodes that correspond to the upward and downward movements of the stock price during the time-period $[t, t + 1]$ – that is, for the values uS_t and dS_t of the stock price at time $t + 1$, respectively.

By a slight abuse of notation, we will sometimes write $X_t^a = \pi_t(X^a)$ to denote the arbitrage price at time t of an American claim X^a . Hence (2.53) becomes (cf. (2.51))

$$X_t^a = \max \left\{ X_t, \mathbb{E}_{\mathbb{P}^*}(\hat{r}^{-1} X_{t+1}^a | \mathcal{F}_t) \right\} \quad (2.55)$$

with the terminal condition $X_T^a = X_T$. Similarly, formula (2.54) takes a more concise form (cf. (2.52))

$$X_t^a = \max \left\{ h(S_t, t), \hat{r}^{-1} (p_* X_{t+1}^{au} + (1 - p_*) X_{t+1}^{ad}) \right\}. \quad (2.56)$$

2.5 Options on a Dividend-paying Stock

So far we have assumed that a stock pays no dividend during an option's lifetime. Suppose now that the stock pays dividends, and the dividend policy is of the following specific form: the stock maintains a constant yield, κ , on each ex-dividend date. We shall restrict ourselves to the last period before the option's expiry. However, the analysis we present below carries over to the more general case of multi-period trading. We assume that the option's expiry date T is an ex-dividend date. This means that the shareholder will receive at that time a dividend payment d_T that amounts to $\kappa u S_{T-1}$ or $\kappa d S_{T-1}$, according to the actual stock price fluctuation. On the other hand, we postulate that the ex-dividend stock price at the end of the period will be either $u(1 - \kappa) S_{T-1}$ or $d(1 - \kappa) S_{T-1}$. This corresponds to the traditional assumption that the stock price declines on the ex-dividend date by exactly the dividend amount. Therefore, the option's payoff C_T^κ at expiry is either

$$C_T^u = (u(1 - \kappa) S_{T-1} - K)^+ \quad \text{or} \quad C_T^d = (d(1 - \kappa) S_{T-1} - K)^+,$$

depending on the stock price fluctuation during the last period. If someone is long a stock, he or she receives the dividend at the end of the period; a party in a short position has to make restitution for the dividend to the party from whom the stock was borrowed. Under these assumptions, the replicating strategy of a call option is determined by the following system of equations (independently of the sign of α_{T-1} ; that is, whether the position is long or short)

$$\begin{cases} \alpha_{T-1} u S_{T-1} + \beta_{T-1} \hat{r} = (u(1 - \kappa) S_{T-1} - K)^+, \\ \alpha_{T-1} d S_{T-1} + \beta_{T-1} \hat{r} = (d(1 - \kappa) S_{T-1} - K)^+. \end{cases}$$

Note that, in contrast to the option payoff, the terminal value of the portfolio $(\alpha_{T-1}, \beta_{T-1})$ is not influenced by the fact that T is the ex-dividend date. This nice feature of portfolio's wealth depends essentially on our assumption that the ex-dividend drop of the stock price coincides with the dividend payment. Solving the above equations for α_{T-1} and β_{T-1} , we find

$$\alpha_{T-1} = \frac{(u_\kappa S_{T-1} - K)^+ - (d_\kappa S_{T-1} - K)^+}{S_{T-1}(u - d)} = \frac{C_T^u - C_T^d}{S_T^u - S_T^d}$$

and

$$\beta_{T-1} = \frac{u(d_\kappa S_{T-1} - K)^+ - d(u_\kappa S_{T-1} - K)^+}{\hat{r}(u - d)} = \frac{uC_T^d - dC_T^u}{\hat{r}(u - d)},$$

where $u_\kappa = (1 - \kappa)u$ and $d_\kappa = (1 - \kappa)d$. By standard arguments, we conclude that the price C_{T-1}^κ of the option at the beginning of the period equals

$$C_{T-1}^\kappa = \alpha_{T-1} S_{T-1} + \beta_{T-1} = \hat{r}^{-1}(p_* C_T^u + (1 - p_*) C_T^d),$$

or explicitly

$$C_{T-1}^\kappa = \hat{r}^{-1} \left(p_*(u_\kappa S_{T-1} - K)^+ + (1 - p_*)(d_\kappa S_{T-1} - K)^+ \right), \quad (2.57)$$

where $p_* = (\hat{r} - d)/(u - d)$. Working backwards in time from the expiry date, one finds the general formula for the arbitrage price of a European call option, provided that the ex-dividend dates and the dividend ratio $\kappa \in (0, 1)$ are known in advance. If we price a put option, the corresponding hedging portfolio at time $T - 1$ satisfies

$$\begin{cases} \alpha_{T-1} u S_{T-1} + \beta_{T-1} \hat{r} = (K - u_\kappa S_{T-1})^+, \\ \alpha_{T-1} d S_{T-1} + \beta_{T-1} \hat{r} = (K - d_\kappa S_{T-1})^+. \end{cases}$$

This yields the following expression for the arbitrage price of a put option at time $T - 1$

$$P_{T-1}^\kappa = \hat{r}^{-1}(p_* P_T^u + (1 - p_*) P_T^d).$$

Once again, for any set of ex-dividend dates known in advance, the price of a European put option at time t can be derived easily by backward induction. Generally speaking, it is clear that the price of a call option is a decreasing function of the dividend yield κ (cf. formula (2.57)). Similarly, the price of a put option increases when κ increases. Both above relationships are rather intuitive, as the dividend payments during the option's lifetime make the underlying stock less valuable at an option's expiry than it would be if no dividends were paid. Also, one can easily extend the above analysis to include dividend policies in which the amount paid on any ex-dividend date depends on the stock price at that time in a more general way (we refer to Sect. 3.2 for more details).

Before we end this section, let us summarize the basic features of American options. We have argued that in the CRR model of a financial market, European and American call options on a stock paying no dividends during the option's lifetime are equivalent (this holds, indeed, in any arbitrage-free market model). This means, in particular, that an American call option should never be exercised before its expiry date. If the underlying stock pays dividends during the option's lifetime, it may be rational to exercise an American call before expiry (but only on a pre-dividend day – that is, one period before the next dividend payment). It is important to notice that the arbitrage valuation of an American call option written on a dividend-paying stock can be done, as usual, by means of backward induction.

On the other hand, we know that the properties of American and European put options with the same contractual features are distinct, in general, as in some circum-

stances the holder of an American put written on a non-dividend-paying stock should exercise her right to sell the stock before the option's expiry date. If the underlying stock pays dividends during a put option's lifetime, the probability of early exercise declines, and thus the arbitrage price of an American put becomes closer to the price of the otherwise identical put option of European style.

2.6 Security Markets in Discrete Time

This section deals mostly with *finite markets* – that is, discrete-time models of financial markets in which all relevant quantities take a finite number of values. The case of discrete-time models with infinite state space is treated very briefly. Essentially, we follow here the approach of Harrison and Pliska (1981); a more exhaustive analysis of finite markets can be found in Taqqu and Willinger (1987). An excellent introduction to discrete-time financial mathematics is given in a monograph by Pliska (1997). A monograph by Föllmer and Schied (2000) is the most comprehensive source in the area.

The detailed treatment of finite models of financial markets presented below is not motivated by their practical importance (except for binomial or multinomial models). The main motivation comes rather from the fact that the most important ideas and results of arbitrage pricing can be presented in a more transparent way by working first in a finite-dimensional framework.

We need first to introduce some notation. Since the number of dates is assumed to be a finite ordered set, there is no loss of generality if we take the set of dates $T = \{0, 1, \dots, T^*\}$. Let Ω be an arbitrary finite set, $\Omega = \{\omega_1, \omega_2, \dots, \omega_d\}$ say, and let $\mathcal{F} = \mathcal{F}_{T^*}$ be the σ -field of all subsets of Ω , i.e., $\mathcal{F} = 2^\Omega$. We consider a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T^*}$, where \mathbb{P} is an arbitrary probability measure on $(\Omega, \mathcal{F}_{T^*})$, such that $\mathbb{P}\{\omega_j\} > 0$ for every $j = 1, 2, \dots, d$. We assume throughout that the σ -field \mathcal{F}_0 is trivial; that is, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. A vector of prices of k primary securities is modelled by means of an \mathbb{F} -adapted, \mathbb{R}^k -valued, nonnegative stochastic process $Z = (Z^1, Z^2, \dots, Z^k)$. Recall that a k -dimensional process $Z = (Z^1, Z^2, \dots, Z^k)$ is said to be \mathbb{F} -adapted, if for any $i = 1, 2, \dots, k$, and any $t \leq T^*$, the random variable Z_t^i is \mathcal{F}_t -measurable.

Since the underlying probability space and the set of dates are both finite sets, all random variables and all stochastic processes considered in finite markets are necessarily bounded. For brevity, we shall say that a given process is adapted, instead of \mathbb{F} -adapted, if no confusion may arise. A *trading strategy* (also called a *dynamic portfolio*) is an arbitrary \mathbb{F} -adapted, \mathbb{R}^k -valued process $\phi = (\phi^1, \phi^2, \dots, \phi^k)$. At any date t , the i^{th} component, ϕ_t^i , of a portfolio ϕ determines the number of units of the i^{th} asset that are held in the portfolio at this date.

We do not postulate, in general, that $\mathcal{F}_t = \mathcal{F}_t^Z = \sigma(Z_0, Z_1, \dots, Z_t)$, that is, that the underlying filtration \mathbb{F} is generated by the observations of the price process Z .

2.6.1 Finite Spot Markets

Unless explicitly stated otherwise, we assume throughout that all assets are perfectly divisible and the market is frictionless, i.e., no restrictions on the short-selling of assets, nor transaction costs or taxes, are present.

In this section, the security prices Z^1, Z^2, \dots, Z^k are interpreted as *spot prices* (or *cash prices*) of certain financial assets. In order to avoid any confusion with the case of *futures markets*, which will be studied in the subsequent section, we shall denote hereafter the price process Z by $S = (S^1, S^2, \dots, S^k)$. In some places, it will be essential to assume that the price process of at least one asset follows a strictly positive process (such a process will play the role of a *numeraire* in what follows). Therefore, we assume, without loss of generality, that $S_t^k > 0$ for every $t \leq T^*$. To emphasize the special role of this particular asset, we will sometimes write B instead of S^k .

As mentioned above, the component ϕ_t^i of a trading strategy ϕ stands for the number of units of the i^{th} security held by an investor at time t . This implies that $\phi_t^i S_t^i$ represents the amount of funds invested in the i^{th} security at time t . The term “funds” is used here for the sake of terminological convenience only. In fact, we assume only that the prices of all primary securities are expressed in units of a certain common asset, which is thus used as a benchmark. The benchmark asset should have monotone appeal, meaning that either (a) all individuals prefer more units of this asset to less, or (b) all individuals prefer less units of this asset to more (we prefer to assume that (a) holds). Thus, the value of any contingent claim will also be expressed in units of the benchmark asset.

In our further development, we will sometimes express the original prices of all traded assets in terms of a fixed primary security; referred to as a *numeraire*. The modified processes will be referred to as *relative prices* (or *discounted prices*, if the numeraire corresponds to a bond price). The original prices of primary securities may be seen as relative prices with respect to the benchmark asset, which is not explicitly specified, however.

2.6.2 Self-financing Trading Strategies

In view of our conventions, the following definition of the wealth of a spot trading strategy ϕ is self-explanatory.

Definition 2.6.1 The *wealth process* $V(\phi)$ of a spot trading strategy ϕ is given by the equality (the dot “ \cdot ” stands for the usual inner product in \mathbb{R}^k)

$$V_t(\phi) = \phi_t \cdot S_t = \sum_{i=1}^k \phi_t^i S_t^i, \quad \forall t \leq T^*.$$

The initial wealth $V_0(\phi) = \phi_0 S_0$ is also referred to as the *initial investment* of the trading strategy ϕ . Since both S_0 and ϕ_0 are \mathcal{F}_0 -measurable random variables, they may be identified with some vectors in \mathbb{R}^k , therefore the initial wealth $V_0(\phi)$

of any portfolio is a real number. Subsequently, at any instant $t = 1, 2, \dots, T^*$, the portfolio ϕ may be rebalanced in such a way that there are no infusions of external funds, and no funds are withdrawn (in particular, the definition of a self-financing strategy assumes that no intertemporal consumption takes place). In the discrete-time spot market setup, these natural assumptions are easily formalized by means of the following definition.

Definition 2.6.2 A spot trading strategy ϕ is said to be *self-financing* if it satisfies the following condition, for every $t = 0, 1, \dots, T^* - 1$,

$$\phi_t \cdot S_{t+1} = \phi_{t+1} \cdot S_{t+1}. \quad (2.58)$$

Intuitively, after a portfolio ϕ_0 is set up at time 0, its revisions are allowed at times $1, 2, \dots, T^*$ only. In other words, it is held fixed over each time period $(t, t + 1)$ for $t = 0, 1, \dots, T^* - 1$. Notice that the rebalancing of a portfolio ϕ at the terminal date T^* is also allowed. If a trading strategy ϕ is self-financing, its revision at time T^* does not affect the terminal wealth $V_{T^*}(\phi)$, however. In fact, by virtue of (2.58), the terminal wealth $V_{T^*}(\phi)$ is uniquely determined by the form ϕ_{T^*-1} of the portfolio at time $T^* - 1$ and the vector S_{T^*} of terminal prices of primary securities. Summarizing, when dealing with replication of contingent claims, we may assume that $T^* - 1$ is the last date when a portfolio may be rebalanced. No wonder that the notion of a gains process $G(\phi)$, which is assumed to represent the *capital gains* earned by the holder of the dynamic portfolio ϕ , does not take into account the random variable ϕ_{T^*} .

We denote by Φ the class of all self-financing spot trading strategies. Let us observe that the class Φ is a vector space. Indeed, it is easily seen that for every $\phi, \psi \in \Phi$ and arbitrary real numbers c, d , the linear combination $c\phi + d\psi$ also represents a self-financing trading strategy.

Definition 2.6.3 The *gains process* $G(\phi)$ of an arbitrary spot trading strategy ϕ equals, for every $t = 0, 1, \dots, T^*$,

$$G_t(\phi) = \sum_{u=0}^{t-1} \phi_u \cdot (S_{u+1} - S_u). \quad (2.59)$$

In view of (2.59), it is clear that we consider here primary securities which do not pay intertemporal cash flows to their holders (such as dividends earned by a stockholder, or coupons received by a bond-holder from the issuer of a bond). The following useful lemma relates the gains process $G(\phi)$ of a self-financing strategy ϕ to its wealth process $V(\phi)$.

Lemma 2.6.1 A spot trading strategy ϕ is self-financing if and only if we have that, for every $t = 0, 1, \dots, T^*$,

$$V_t(\phi) = V_0(\phi) + G_t(\phi). \quad (2.60)$$

Proof. Assume first that ϕ is self-financing. Then, taking into account formulas (2.58)–(2.59), we obtain

$$\begin{aligned}
V_t(\phi) &= \phi_0 \cdot S_0 + \sum_{u=0}^{t-1} (\phi_{u+1} \cdot S_{u+1} - \phi_u \cdot S_u) \\
&= \phi_0 \cdot S_0 + \sum_{u=0}^{t-1} \phi_u \cdot (S_{u+1} - S_u) = V_0(\phi) + G_t(\phi),
\end{aligned}$$

so that (2.60) holds. The inverse implication is also easy to establish. \square

All definitions and results above can be easily extended to the case of a trading strategy ϕ over the time set $\{0, 1, \dots, T\}$ for some $T < T^*$. Also, any self-financing trading strategy on $\{0, 1, \dots, T\}$ can be extended to a self-financing trading strategy on $\{0, 1, \dots, T^*\}$ by postulating that the total wealth is invested at time T in the last asset, B . In that case, the terminal wealth of ϕ at time T^* satisfies $V_{T^*}(\phi) = B_{T^*} B_T^{-1} V_T(\phi)$. We will use this convention in what follows, without explicit mentioning.

2.6.3 Replication and Arbitrage Opportunities

We pursue an analysis of the spot market model $\mathcal{M} = (S, \Phi)$, where S is a k -dimensional, \mathbb{F} -adapted stochastic process and Φ stands for the class of all self-financing (spot) trading strategies.

By a *European contingent claim* X which settles at time T we mean an arbitrary \mathcal{F}_T -measurable random variable. Unless explicitly stated otherwise, we shall deal with European contingent claims, and we shall refer to them as *contingent claims* or simply *claims*. Since the space Ω is assumed to be a finite set with d elements, any claim X has the representation $X = (X(\omega_1), X(\omega_2), \dots, X(\omega_d)) \in \mathbb{R}^d$, and thus the class \mathcal{X} of all contingent claims which settle at some date $T \leq T^*$ may be identified with the linear space \mathbb{R}^d .

Definition 2.6.4 A *replicating strategy* for the contingent claim X , which settles at time T , is a self-financing trading strategy ϕ such that $V_T(\phi) = X$. Given a claim X , we denote by Φ_X the class of all trading strategies which replicate X .

The wealth process $V_t(\phi)$, $t \leq T$, of an arbitrary strategy ϕ from Φ_X is called a *replicating process* of X in \mathcal{M} . Finally, we say that a claim X is *attainable* in \mathcal{M} if it admits at least one replicating strategy. We denote by \mathcal{A} the class of all attainable claims.

Definition 2.6.5 A market model \mathcal{M} is called *complete* if every claim $X \in \mathcal{X}$ is attainable in \mathcal{M} , or equivalently, if for every \mathcal{F}_T -measurable random variable X there exists at least one trading strategy $\phi \in \Phi$ such that $V_T(\phi) = X$. In other words, a market model \mathcal{M} is complete whenever $\mathcal{X} = \mathcal{A}$.

Generally speaking, the completeness of a particular model of a financial market is a highly desirable property. We will show that under market completeness, any European contingent claim can be priced by arbitrage, and its price process can be mimicked by means of a self-financing dynamic portfolio. We need first to examine the arbitrage-free property of \mathcal{M} , however.

Definition 2.6.6 A trading strategy $\phi \in \Phi$ is called an *arbitrage opportunity* if $V_0(\phi) = 0$ and the terminal wealth of ϕ satisfies

$$\mathbb{P}\{V_{T^*}(\phi) \geq 0\} = 1 \quad \text{and} \quad \mathbb{P}\{V_{T^*}(\phi) > 0\} > 0.$$

We say that a spot market $\mathcal{M} = (S, \Phi)$ is *arbitrage-free* if there are no arbitrage opportunities in the class Φ of all self-financing trading strategies.

2.6.4 Arbitrage Price

In this section, X is an arbitrary attainable claim which settles at time T .

Definition 2.6.7 We say that X is *uniquely replicated* in \mathcal{M} if it admits a unique replicating process in \mathcal{M} ; that is, if the equality

$$V_t(\phi) = V_t(\psi), \quad \forall t \leq T,$$

holds for arbitrary trading strategies ϕ, ψ belonging to Φ_X . In this case, the process $V(\phi)$ is termed the *wealth process* of X in \mathcal{M} .

Proposition 2.6.1 Suppose that the market model \mathcal{M} is arbitrage-free. Then any attainable contingent claim X is uniquely replicated in \mathcal{M} .

Proof. Suppose, on the contrary, that there exists a time T attainable contingent claim X which admits two replicating strategies, say ϕ and ψ , such that for some $t < T$ we have: $V_u(\phi) = V_u(\psi)$ for every $u < t$, and $V_t(\phi) \neq V_t(\psi)$.

Assume first that $t = 0$ so that $V_0(\phi) > V_0(\psi)$ for some replicating strategies ϕ and ψ . Consider a strategy ζ which equals (recall that $B = S^k$)

$$\zeta_u = \psi_u - \phi_u + (0, \dots, 0, v_0 B_0^{-1}) \mathbb{1}_A,$$

where $v_0 = V_0(\phi) - V_0(\psi) > 0$. Then $V_0(\zeta) = 0$ and $V_{T^*}(\zeta) = v_0 B_0^{-1} B_{T^*} > 0$ for every ω , so that ζ is an arbitrage opportunity.

Let us now consider the case $t > 0$. We may assume, without loss of generality, that $\mathbb{P}\{A\} > 0$, where A stands for the event $\{V_t(\phi) > V_t(\psi)\}$. Denote by ξ the random variable $v_t = V_t(\phi) - V_t(\psi)$, and consider the following trading strategy η

$$\eta_u = \phi_u - \psi_u, \quad \forall u < t,$$

and

$$\eta_u = (\phi_u - \psi_u) \mathbb{1}_{A^c} + (0, \dots, 0, v_t B_t^{-1}) \mathbb{1}_A, \quad \forall u \geq t,$$

where A^c denotes the complement of the set A (i.e., if the event A occurs, both portfolio's are liquidated and the proceeds are invested in the k^{th} asset). It is apparent that the strategy η is self-financing and $V_0(\eta) = 0$. Moreover, its terminal wealth $V_{T^*}(\eta)$ equals

$$V_{T^*}(\eta) = v_t B_t^{-1} B_{T^*} \mathbb{1}_A.$$

Hence it is clear that $V_{T^*}(\eta) \geq 0$ and $\mathbb{P}\{V_{T^*}(\eta) > 0\} = \mathbb{P}\{A\} > 0$. We conclude that η is an arbitrage opportunity. This contradicts our assumption that the market \mathcal{M} is arbitrage-free. \square

The converse implication is not valid; that is, the uniqueness of the wealth process of any attainable contingent claim does not imply the arbitrage-free property of a market, in general. Therefore, the existence and uniqueness of the wealth process associated with any attainable claim is insufficient to justify the term *arbitrage price*. Indeed, it is trivial to construct a finite market in which all claims are uniquely replicated, but there exists a strictly positive claim, say Y , which admits a replicating strategy with negative initial investment (with negative manufacturing cost, using the terminology of Chap. 1). Suppose now that for every claim X , its price at time 0, $\pi_0(X)$, is defined as the initial investment of a strategy which replicates X . It is important to point out that the price functional π_0 , on the space \mathcal{X} of contingent claims, would not be supported by any kind of intertemporal equilibrium. In fact, any individual would tend to take an infinite position in any such claim Y (recall that we assume that all individuals are assumed to prefer more wealth to less).

We thus find it natural to formally introduce the notion of an arbitrage price in the following way.

Definition 2.6.8 Suppose that the market model \mathcal{M} is arbitrage-free. Then the wealth process of an attainable claim X is called the *arbitrage price process* (or simply, the *arbitrage price*) of X in \mathcal{M} . We denote it by $\pi_t(X)$, $t \leq T$.

2.6.5 Risk-neutral Valuation Formula

As mentioned earlier, the martingale approach to arbitrage pricing was first elaborated by Cox and Ross (1976) (although the idea of “risk-neutral” probabilities goes back to Arrow (1964, 1970)). In financial terminology, they showed that in a world with one stock and one bond, it is possible to construct preferences through a risk-neutral individual who gives the value of those claims which are priced by arbitrage. In this regard, let us mention that the *martingale measures*, which we are now going to introduce, are sometimes referred to as *risk-neutral probabilities*.

For the sake of notational simplicity, we write as usual $S^k = B$. This convention does not imply, however, that S^k should necessarily be interpreted as the price process of a risk-free bond. Recall, however, that we have assumed that S^k follows a strictly positive process. Let us denote by S^* the process of *relative* (or *discounted*) prices, which equals, for every $t = 0, 1, \dots, T^*$,

$$S_t^* = (S_t^1 B_t^{-1}, S_t^2 B_t^{-1}, \dots, S_t^k B_t^{-1}) = (S_t^{*1}, S_t^{*2}, \dots, S_t^{*(k-1)}, 1),$$

where we denote $S^{*i} = S^i B^{-1}$. Recall that the probability measures \mathbb{P} and \mathbb{Q} on (Ω, \mathcal{F}) are said to be *equivalent* if, for any event $A \in \mathcal{F}$, the equality $\mathbb{P}\{A\} = 0$ holds if and only if $\mathbb{Q}\{A\} = 0$. Similarly, \mathbb{Q} is said to be *absolutely continuous* with respect to \mathbb{P} if, for any event $A \in \mathcal{F}$, the equality $\mathbb{P}\{A\} = 0$ implies that $\mathbb{Q}\{A\} = 0$.

Definition 2.6.9 A probability measure \mathbb{P}^* on $(\Omega, \mathcal{F}_{T^*})$ equivalent to \mathbb{P} (absolutely continuous with respect to \mathbb{P} , respectively) is called a *martingale measure for S^** (a *generalized martingale measure for S^** , respectively) if the relative price S^* is a \mathbb{P}^* -martingale with respect to the filtration \mathbb{F} .

Recall that a k -dimensional process $S^* = (S^{*1}, S^{*2}, \dots, S^{*k})$ is a \mathbb{P}^* -martingale with respect to a filtration \mathbb{F} if $\mathbb{E}_{\mathbb{P}^*} |S_t^{*i}| < \infty$ for every i and $t = 0, 1, \dots, T^*$, and the equality

$$\mathbb{E}_{\mathbb{P}^*}(S_{t+1}^{*i} | \mathcal{F}_t) = S_t^{*i}$$

is valid for every i and $t = 0, 1, \dots, T^* - 1$. Of course, in the case of a finite space Ω , the integrability condition $\mathbb{E}_{\mathbb{P}^*} |S_t^{*i}| < \infty$ is always satisfied. The martingale property of S^* will be simply represented as $\mathbb{E}_{\mathbb{P}^*}(S_{t+1}^* | \mathcal{F}_t) = S_t^*$ for $t = 0, 1, \dots, T^* - 1$.

We denote by $\mathcal{P}(S^*)$ and $\bar{\mathcal{P}}(S^*)$ the class of all martingale measures for S^* and the class of all generalized martingale measures for S^* , respectively. It is easily seen that the inclusion $\mathcal{P}(S^*) \subseteq \bar{\mathcal{P}}(S^*)$ is valid. Moreover, it is not difficult to provide an example in which the class $\mathcal{P}(S^*)$ is empty, whereas the class $\bar{\mathcal{P}}(S^*)$ is not. Observe also that the notion of a martingale measure essentially depends on the choice of the numeraire – recall that we have chosen $S^k = B$ as a numeraire throughout. In the next step, we introduce the concept of a martingale measure for a market model \mathcal{M} .

Definition 2.6.10 A probability measure \mathbb{P}^* on $(\Omega, \mathcal{F}_{T^*})$ equivalent to \mathbb{P} (absolutely continuous with respect to \mathbb{P} , respectively) is called a *martingale measure for $\mathcal{M} = (S, \Phi)$* (a *generalized martingale measure for $\mathcal{M} = (S, \Phi)$* , respectively) if for every trading strategy $\phi \in \Phi$ the relative wealth process $V^*(\phi) = V(\phi)B^{-1}$ follows a \mathbb{P}^* -martingale with respect to the filtration \mathbb{F} .

We write $\mathcal{P}(\mathcal{M})$ ($\bar{\mathcal{P}}(\mathcal{M})$ respectively) to denote the class of all martingale measures (of all generalized martingale measures, respectively) for \mathcal{M} . Our goal is now to show that $\mathcal{P}(S^*) = \mathcal{P}(\mathcal{M})$ and $\bar{\mathcal{P}}(S^*) = \bar{\mathcal{P}}(\mathcal{M})$.

Lemma 2.6.2 *A trading strategy ϕ is self-financing if and only if the relative wealth process $V^*(\phi) = V(\phi)B^{-1}$ satisfies, for every $t = 0, 1, \dots, T^*$,*

$$V_t^*(\phi) = V_0^*(\phi) + \sum_{u=0}^{t-1} \phi_u \cdot \Delta_u S^*, \quad (2.61)$$

where $\Delta_u S^* = S_{u+1}^* - S_u^*$. If ϕ belongs to Φ then, for any (generalized) martingale measure \mathbb{P}^* , the relative wealth $V^*(\phi)$ is a \mathbb{P}^* -martingale with respect to the filtration \mathbb{F} .

Proof. Let us write $V = V(\phi)$ and $V^* = V^*(\phi)$. Then (2.61) is equivalent to

$$\Delta_t V^* = V_{t+1}^* - V_t^* = \phi_t \cdot \Delta_t S^*, \quad \forall t \leq T^* - 1. \quad (2.62)$$

But, on the one hand, we have that

$$V_{t+1}^* - V_t^* = \frac{V_{t+1}}{B_{t+1}} - \frac{V_t}{B_t} = \frac{\phi_{t+1} \cdot S_{t+1}}{B_{t+1}} - \frac{\phi_t \cdot S_t}{B_t},$$

and, on the other hand, we also see that

$$\phi_t \cdot \Delta_t S^* = \frac{\phi_t \cdot S_{t+1}}{B_{t+1}} - \frac{\phi_t \cdot S_t}{B_t}.$$

We thus conclude that condition (2.62) is indeed equivalent to the self-financing property: $\phi_t \cdot S_{t+1} = \phi_{t+1} \cdot S_{t+1}$ for $t = 0, 1, \dots, T^* - 1$.

For the second statement, it is enough to check that, for every $t = 0, 1, \dots, T^* - 1$,

$$\mathbb{E}_{\mathbb{P}^*}(V_{t+1}^* - V_t^* | \mathcal{F}_t) = 0.$$

Using (2.62), we obtain

$$\mathbb{E}_{\mathbb{P}^*}(V_{t+1}^* - V_t^* | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}(\phi_t \cdot (S_{t+1}^* - S_t^*) | \mathcal{F}_t) = \phi_t \cdot \mathbb{E}_{\mathbb{P}^*}(S_{t+1}^* - S_t^* | \mathcal{F}_t) = 0,$$

where the last equality follows from the martingale property of the relative price process S^* under \mathbb{P}^* . \square

In view of the last lemma, the following corollary is easy to prove.

Corollary 2.6.1 *A probability measure \mathbb{P}^* on $(\Omega, \mathcal{F}_{T^*})$ is a (generalized) martingale measure for the spot market model \mathcal{M} if and only if it is a (generalized) martingale measure for the relative price process S^* , i.e., $\mathcal{P}(S^*) = \mathcal{P}(\mathcal{M})$ and $\bar{\mathcal{P}}(S^*) = \bar{\mathcal{P}}(\mathcal{M})$.*

The next result shows that the existence of a martingale measure for \mathcal{M} is sufficient for the no-arbitrage property of \mathcal{M} . Recall that trivially $\mathcal{P}(\mathcal{M}) \subseteq \bar{\mathcal{P}}(\mathcal{M})$ so that the class $\bar{\mathcal{P}}(\mathcal{M})$ is manifestly non-empty if $\mathcal{P}(\mathcal{M})$ is so.

Proposition 2.6.2 *Assume that the class $\mathcal{P}(\mathcal{M})$ is non-empty. Then the spot market \mathcal{M} is arbitrage-free. Moreover, the arbitrage price process of any attainable contingent claim X , which settles at time T , is given by the risk-neutral valuation formula*

$$\pi_t(X) = B_t \mathbb{E}_{\mathbb{P}^*}(X B_T^{-1} | \mathcal{F}_t), \quad \forall t \leq T, \quad (2.63)$$

where \mathbb{P}^* is any (generalized) martingale measure for the market model \mathcal{M} associated with the choice of B as a numeraire.

Proof. Let \mathbb{P}^* be some martingale measure for \mathcal{M} . We know already that the relative wealth process $V^*(\phi)$ of any strategy $\phi \in \Phi$ follows a \mathbb{P}^* -martingale, and thus

$$V_t(\phi) = B_t V_t^*(\phi) = B_t \mathbb{E}_{\mathbb{P}^*}(V_T^*(\phi) | \mathcal{F}_t) = B_t \mathbb{E}_{\mathbb{P}^*}(V_T(\phi) B_T^{-1} | \mathcal{F}_t)$$

for every t . Since \mathbb{P}^* is equivalent to \mathbb{P} , it is clear that there are no arbitrage opportunities in the class Φ of self-financing trading strategies, hence the market \mathcal{M} is arbitrage-free.

Moreover, for any attainable contingent claim X which settles at time T , and any trading strategy $\phi \in \Phi_X$, we have

$$\pi_t(X) = V_t(\phi) = B_t \mathbb{E}_{\mathbb{P}^*}(V_T^*(\phi) | \mathcal{F}_t) = B_t \mathbb{E}_{\mathbb{P}^*}(X B_T^{-1} | \mathcal{F}_t). \quad (2.64)$$

This completes the proof in the case of a martingale measure \mathbb{P}^* . When \mathbb{P}^* is a generalized martingale measure for \mathcal{M} , all equalities in (2.64) remain valid (let us stress that the existence of a generalized martingale measure does not imply the absence of arbitrage, however). \square

Remarks. In a more general setting (e.g., in a continuous-time framework), a generalized martingale measure no longer plays the role of a pricing measure – that is, equality (2.63) may fail to hold, in general, if a martingale measure \mathbb{P}^* is merely absolutely continuous with respect to an underlying probability measure \mathbb{P} . The reason is that the Itô stochastic integral (as opposed to a finite sum) is not invariant with respect to an absolutely continuous change of a probability measure.

2.6.6 Existence of a Martingale Measure

This section addresses a basic question: is the existence of a martingale measure a necessary condition for absence of arbitrage in a finite model of a financial market? Results of this type are sometimes referred to as fundamental theorems of asset pricing. In the case of a finite market model, it was proved by Harrison and Pliska (1981); for a purely probabilistic approach we refer to Taqqu and Willinger (1987), who examine the case of a finite market, and to the papers of Dalang et al. (1990) and Schachermayer (1992), who deal with a discrete-time model with infinite state space (see also Harrison and Kreps (1979) for related results).

Recall that since $\Omega = \{\omega_1, \omega_2, \dots, \omega_d\}$, the space \mathcal{X} of all contingent claims that settle at time T^* may be identified with the finite-dimensional linear space \mathbb{R}^d . For any claim $X \in \mathcal{X}$, we write $X = (X(\omega_1), X(\omega_2), \dots, X(\omega_d)) = (x^1, x^2, \dots, x^d) \in \mathbb{R}^d$.

From Proposition 2.6.2, we know already that if the set of martingale measures is non-empty, then the market model \mathcal{M} is arbitrage-free. Our next goal is to show that this condition is also necessary for the no-arbitrage property of the market model \mathcal{M} .

Let us first recall the following definition.

Definition 2.6.11 A random variable $\tau : \Omega \rightarrow \{0, 1, \dots, T^*\}$ is said to be a *stopping time* with respect to the filtration \mathbb{F} if the event $\{\tau = t\}$ belongs to \mathcal{F}_t for every $t = 0, 1, \dots, T^*$.

Proposition 2.6.3 Suppose that the spot market model \mathcal{M} is arbitrage-free. Then the class $\mathcal{P}(\mathcal{M})$ of martingale measures for \mathcal{M} is non-empty.

Proof. We consider the following closed and bounded (i.e., compact) and convex subset of \mathbb{R}^d

$$\mathcal{X}_+ = \{X \in \mathbb{R}^d \mid X \geq 0 \text{ and } \mathbb{E}_{\mathbb{P}} X = 1\}.$$

Also, let \mathcal{A}_0^* stand for the class of all discounted claims that are attainable by trading strategies with zero initial wealth, that is,

$$\mathcal{A}_0^* = \{Y \in \mathbb{R}^d \mid Y = V_{T^*}^*(\phi) \text{ for some } \phi \in \Phi \text{ with } V_0(\phi) = 0\}.$$

Equivalently, noting that $V_{T^*}(\phi) = G_{T^*}(\phi)$ if and only if $V_0(\phi) = 0$,

$$\mathcal{A}_0^* = \{Y \in \mathbb{R}^d \mid Y = G_{T^*}^*(\phi) \text{ for some } \phi \in \Phi\}.$$

It is easily seen that \mathcal{A}_0^* is a linear subspace of \mathbb{R}^d . The crucial observation is that the assumption that \mathcal{M} is arbitrage-free implies that $\mathcal{X}_+ \cap \mathcal{A}_0^* = \emptyset$.

From a *separating hyperplane theorem* (see Corollary 2.6.3), we deduce that there exists a vector $Z = (z^1, z^2, \dots, z^d)$ such that $Z \cdot X > 0$ for every $X \in \mathcal{X}_+$ and $Z \cdot Y = \sum_{j=1}^d z^j y^j = 0$ for every $Y \in \mathcal{A}_0^*$. Let us denote $p_j = \mathbb{P}\{\omega_j\}$. Since the vector $\tilde{e}_j = (0, \dots, p_j^{-1}, \dots, 0) \in \mathbb{R}^d$ belongs to \mathcal{X}_+ for any $j = 1, 2, \dots, d$, we see that necessarily $z^j > 0$ for every $j = 1, 2, \dots, d$.

We define the probability measure \mathbb{P}^* on $(\Omega, \mathcal{F}_{T^*})$ by setting, for every $j = 1, 2, \dots, d$,

$$\mathbb{P}^*\{\omega_j\} = \frac{z^j}{\sum_{m=1}^d z^m} = cz^j.$$

Since $z^j > 0$ for every $j = 1, 2, \dots, d$, it is clear that \mathbb{P}^* is equivalent to \mathbb{P} . In view of Corollary 2.6.1, it thus remains to show that the relative price process S^* follows a \mathbb{P}^* -martingale. To this end, we observe that, by construction, the equality $\mathbb{E}_{\mathbb{P}^*} Y = cZ \cdot Y = 0$ holds for every $Y \in \mathcal{A}_0^*$.

Let τ be an arbitrary stopping time with respect to the filtration \mathbb{F} (see Definition 2.6.11). For any fixed $l \in \{1, 2, \dots, k-1\}$, we consider a trading strategy $\phi \in \Phi$ defined by the formula

$$\phi_t^i = \begin{cases} 0 & \text{for } i \neq l, i \neq k, \\ \mathbb{1}_{[0, \tau]}(t) & \text{for } i = l, \\ -S_0^l + S_t^l B_t^{-1} \mathbb{1}_{[\tau, T^*]}(t) & \text{for } i = k. \end{cases}$$

Note that ϕ is self-financing and $V_0(\phi) = 0$. Hence the discounted terminal wealth $V_{T^*}^*(\phi)$ belongs to the space \mathcal{A}_0^* , and thus

$$\mathbb{E}_{\mathbb{P}^*}(V_{T^*}^*(\phi)) = cZ \cdot V_{T^*}^*(\phi) = 0.$$

Since manifestly $V_{T^*}^*(\phi) = -S_0^l + S_\tau^l B_\tau^{-1}$, the formula above yields

$$\mathbb{E}_{\mathbb{P}^*}(S_\tau^l B_\tau^{-1}) = S_0^l B_0^{-1}.$$

We conclude that the equality $\mathbb{E}_{\mathbb{P}^*} S_\tau^{*l} = S_0^{*l}$ holds for an arbitrary stopping time τ . By considering stopping times of the form $\tau = t \mathbb{1}_A + T^* \mathbb{1}_{A^c}$, where A is an arbitrary event from \mathcal{F}_t , one can easily deduce that the process S^{*l} is a \mathbb{P}^* -martingale. Since l was an arbitrary number from the set $\{1, 2, \dots, k-1\}$, we conclude that the probability measure \mathbb{P}^* is a martingale measure for the k -dimensional process S^* . \square

The following result, which is an immediate consequence of Propositions 2.6.2 and 2.6.3, is a version of the so-called *First Fundamental Theorem of Asset Pricing*.

Theorem 2.6.1 *A finite spot market \mathcal{M} is arbitrage-free if and only if the class $\mathcal{P}(\mathcal{M})$ is non-empty.*

2.6.7 Completeness of a Finite Market

The next result links the completeness of a finite market to the uniqueness of a martingale measure. Any result of this kind is commonly referred to as the *Second Fundamental Theorem of Asset Pricing*.

Theorem 2.6.2 *Assume that a spot market model \mathcal{M} is arbitrage-free so that the class $\mathcal{P}(\mathcal{M})$ is non-empty. Then \mathcal{M} is complete if and only if the uniqueness of a martingale measure for \mathcal{M} holds.*

Proof. (\Rightarrow) Assume that \mathcal{M} is an arbitrage-free complete market. Then for every \mathcal{F}_{T^*} -measurable random variable X there exists at least one trading strategy $\phi \in \Phi$ such that $V_{T^*}(\phi) = X$. By virtue of Propositions 2.6.2 and 2.6.3, we have, for any \mathcal{F}_{T^*} -measurable random variable X ,

$$\pi_0(X) = B_0 \mathbb{E}_{\mathbb{P}^*}(X B_{T^*}^{-1}), \quad \forall X \in \mathcal{X},$$

where \mathbb{P}^* is any martingale measure from the non-empty class $\mathcal{P}(\mathcal{M})$. Therefore, for any two martingale measures \mathbb{P}_1^* and \mathbb{P}_2^* from the class $\mathcal{P}(\mathcal{M})$, we have that, for any \mathcal{F}_{T^*} -measurable random variable X ,

$$\mathbb{E}_{\mathbb{P}_1^*}(X B_{T^*}^{-1}) = \mathbb{E}_{\mathbb{P}_2^*}(X B_{T^*}^{-1}). \quad (2.65)$$

Let $A \in \mathcal{F}_{T^*}$ be any event and let us take $X = B_{T^*}^{-1} \mathbb{1}_A$. Then (2.65) yields the equality $\mathbb{P}_1^*\{A\} = \mathbb{P}_2^*\{A\}$, and thus we conclude that \mathbb{P}_1^* and \mathbb{P}_2^* coincide. This ends the proof of the “only if” clause.

(\Leftarrow) Assume now that there exists a unique element \mathbb{P}^* in $\mathcal{P}(\mathcal{M})$. Clearly, the market \mathcal{M} is then arbitrage-free; it rests to establish its completeness. Suppose, on the contrary, that the model \mathcal{M} is not complete, so that there exists a contingent claim which is not attainable in \mathcal{M} , that is, $\mathcal{A} \neq \mathcal{X}$.

We will show that in that case uniqueness of a martingale measure does not hold; specifically, we will construct a martingale measure \mathbb{Q} for \mathcal{M} different from \mathbb{P}^* . We denote by \mathcal{A}^* the class of all discounted claims attainable by some trading strategy, that is,

$$\mathcal{A}^* = \{ Y \in \mathbb{R}^d \mid Y = V_0(\phi) + G_T^*(\phi) \text{ for some } \phi \in \Phi \}.$$

Let us define the inner product in \mathbb{R}^d by the formula

$$\langle X, Y \rangle_{\mathbb{P}^*} := \mathbb{E}_{\mathbb{P}^*}(XY) = \sum_{j=1}^d x^j y^j \mathbb{P}^*\{\omega_j\}. \quad (2.66)$$

Observe that \mathcal{A}^* is a linear subspace of \mathbb{R}^d and $\mathcal{A}^* \neq \mathcal{X} = \mathbb{R}^d$. Hence there exists a non-zero random variable Z orthogonal to \mathcal{A}^* , in the sense that $\mathbb{E}_{\mathbb{P}^*}(ZY) = 0$ for every $Y \in \mathcal{A}^*$. Moreover, since the constant random variable $Y = 1$ belongs to \mathcal{A}^* (indeed, it simply corresponds to the buy-and-hold strategy in one unit of the asset $B = S^k$), we obtain that $\mathbb{E}_{\mathbb{P}^*} Z = 0$.

Let us define, for every $\omega_j \in \Omega$,

$$\mathbb{Q}\{\omega_j\} = \left(1 + \frac{Z(\omega_j)}{2\|Z\|_\infty}\right) \mathbb{P}^*\{\omega_j\},$$

where $\|Z\|_\infty = \max_{m=1,2,\dots,d} |Z(\omega_m)|$. We will show that \mathbb{Q} is a martingale measure, that is, $\mathbb{Q} \in \mathcal{P}(\mathcal{M})$. First, it is rather obvious that \mathbb{Q} is a probability measure equivalent to \mathbb{P}^* and thus it is equivalent to \mathbb{P} as well. Second, we will show that for any \mathbb{F} -adapted process ϕ we have that

$$\mathbb{E}_{\mathbb{Q}}\left(\sum_{u=0}^{T-1} \phi_u \cdot \Delta_u S^*\right) = 0. \quad (2.67)$$

We now argue that last equality implies that S^* is an \mathbb{F} -martingale under \mathbb{Q} . To check this claim, we fix t and i , and we take $\phi_t^i = \mathbb{1}_A$ for an arbitrary (but fixed) \mathcal{F}_t -measurable event A , and $\phi_u^i = 0$ for $t \neq u$ as well as $\phi_u^k = 0$ for $k \neq i$. We obtain the required equality $\mathbb{E}_{\mathbb{Q}}(S_{t+1}^{*i} - S_t^{*i} | \mathcal{F}_t) = 0$ for any $t = 0, 1, \dots, T-1$ and every $i = 1, \dots, k-1$ (for $i = k$, this condition is trivially satisfied, since $S^{*k} = 1$). We conclude that \mathbb{Q} belongs to $\mathcal{P}(S^*)$ and thus it belongs to $\mathcal{P}(\mathcal{M})$ as well. It is also rather obvious that $\mathbb{Q} \neq \mathbb{P}^*$.

To complete the proof, it remains to show that (2.67) is valid. To this end, we observe that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left(\sum_{u=0}^{T-1} \phi_u \cdot \Delta_u S^*\right) &= \mathbb{E}_{\mathbb{P}^*}\left(\sum_{u=0}^{T-1} \phi_u \cdot \Delta_u S^*\right) \\ &+ \frac{1}{2\|Z\|_\infty} \mathbb{E}_{\mathbb{P}^*}\left(Z \sum_{u=0}^{T-1} \phi_u \cdot \Delta_u S^*\right) =: J_1 + J_2. \end{aligned}$$

We first note that $J_1 = 0$ since the probability measure \mathbb{P}^* belongs to $\mathcal{P}(\mathcal{M})$. Furthermore, equality $J_2 = 0$ also holds, since it follows from the assumption that Z is orthogonal to \mathcal{A}^* and the observation that the sum $\sum_{u=0}^{T-1} \phi_u \cdot \Delta_u S^*$ belongs to \mathcal{A}^* . We conclude that equality (2.67) is indeed satisfied.

In that way, we have established the existence of two distinct martingale measures under non-completeness of a model \mathcal{M} . Therefore, the proof of the “if” clause is completed. \square

Corollary 2.6.2 *A contingent claim $X \in \mathcal{X}$ is attainable if and only if the map $\mathbb{P}^* \mapsto \mathbb{E}_{\mathbb{P}^*}(X B_T^{-1})$ from $\mathcal{P}(\mathcal{M})$ to \mathbb{R} is constant.*

Proof. We consider the case of $T = T^*$; the arguments for any $T \leq T^*$ are analogous. Suppose first that X is attainable. Then its arbitrage price at time 0 is well defined and it is unique. The risk-neutral valuation formula thus shows that the map $\mathbb{P}^* \mapsto \mathbb{E}_{\mathbb{P}^*}(X B_T^{-1})$ is constant.

Suppose now that the map is constant for a given non-zero claim X (of course, the zero claim is always attainable). We wish to show that X is attainable. Assume,

on the contrary, that X is not attainable and denote $X^* = XB_T^{-1}$. Let \mathbb{P}^* be some martingale measure from $\mathcal{P}(\mathcal{M})$ and let $Z = X^* - X^\perp$ where X^\perp is the orthogonal projection under \mathbb{P}^* of X^* on the class \mathcal{A}^* , in the sense of the inner product introduced in the proof of Theorem 2.6.2 (see formula (2.66)).

Since X^* does not belong to \mathcal{A}^* , the vector Z is non-zero and we may define the associated martingale measure $\mathbb{Q} \in \mathcal{P}(\mathcal{M})$ (once again, see the proof of Theorem 2.6.2). Finally, we note that $\mathbb{E}_{\mathbb{P}^*}(X^*) \neq \mathbb{E}_{\mathbb{Q}}(X^*)$, since

$$\mathbb{E}_{\mathbb{Q}}(X^*) = \mathbb{E}_{\mathbb{P}^*}(X^*) + \frac{1}{2\|Z\|_\infty} \mathbb{E}_{\mathbb{P}^*}(ZX^*)$$

and

$$\mathbb{E}_{\mathbb{P}^*}(ZX^*) = \mathbb{E}_{\mathbb{P}^*}((X^* - X^\perp)X^*) = \mathbb{E}_{\mathbb{P}^*}(X^*)^2 \neq 0.$$

This completes the proof.

□

2.6.8 Separating Hyperplane Theorem

We take for granted the following version of a *strong separating hyperplane theorem* (see, e.g., Rockafellar (1970) or Luenberger (1984)).

Theorem 2.6.3 *Let U be a closed, convex subset of \mathbb{R}^d such that $0 \notin U$. Then there exists a linear functional $y \in \mathbb{R}^d$ and a real number $\alpha > 0$ such that $y \cdot u \geq \alpha$ for every $u \in U$.*

In the proof of Proposition 2.6.3, we made use of the following corollary to Theorem 2.6.3.

Corollary 2.6.3 *Let W be a linear subspace of \mathbb{R}^d and let K be a compact, convex subset of \mathbb{R}^d such that $K \cap W = \emptyset$. Then there exists a linear functional $y \in \mathbb{R}^d$ and a real number $\alpha > 0$ such that $y \cdot w = 0$ for every $w \in W$ and $y \cdot k \geq \alpha$ for every $k \in K$.*

Proof. Let U stand for the following set

$$U = K - W = \{x \in \mathbb{R}^d \mid x = k - w, k \in K, w \in W\}.$$

It is easy to check that U is a convex, closed set. Since K and W are disjoint, we manifestly have that $0 \notin U$. From Theorem 2.6.3, we deduce the existence of $y \in \mathbb{R}^d$ and $\alpha > 0$, such that $y \cdot u \geq \alpha$ for every $u \in U$. This in turn implies that, for every $k \in K$ and $w \in W$,

$$y \cdot k - y \cdot w \geq \alpha. \quad (2.68)$$

Let us fix $k \in K$ and let us take λw instead of w , where λ is an arbitrary real number (recall that W is a linear space). Since λ is arbitrary and (2.68) holds, it is clear that $y \cdot w = 0$ for every $w \in W$. We conclude that $y \cdot k \geq \alpha$ for every $k \in K$. □

2.6.9 Change of a Numeraire

Assume that the price of the l^{th} asset is also strictly positive for some $l < k$ (recall the standing assumption that $S_t^k > 0$ for every t). Then it is possible to use S^l , rather than $B = S^k$, to express prices of all assets – that is, as a *numeraire asset*. It is clear that all general results established in the previous sections will remain valid if we use S^l instead of S^k . For instance, the model is arbitrage-free if and only if there exists a martingale measure, denoted by $\bar{\mathbb{P}}$, associated with the choice of S^l as a numeraire. Moreover, the uniqueness of a martingale measure $\bar{\mathbb{P}}$ associated with S^l is equivalent to the market completeness. Assume that a model is arbitrage-free and complete. By modifying the notation in the statement and the proof of Proposition 2.6.2, we deduce that the arbitrage price process of any European contingent claim X settling at time T is given by the risk-neutral valuation formula under $\bar{\mathbb{P}}$

$$\pi_t(X) = S_t^l \mathbb{E}_{\bar{\mathbb{P}}}((S_T^l)^{-1} X \mid \mathcal{F}_t), \quad \forall t \leq T. \quad (2.69)$$

The last formula can also be proved using directly (2.63) and the expression for the Radon-Nikodým derivative of $\bar{\mathbb{P}}$ with respect to \mathbb{P}^* , which is given by the following result, which extends Proposition 2.2.4.

Proposition 2.6.4 *The Radon-Nikodým derivative process η of $\bar{\mathbb{P}}$ with respect to \mathbb{P}^* equals, for every $t = 0, 1, \dots, T^*$,*

$$\eta_t = \frac{d\bar{\mathbb{P}}}{d\mathbb{P}^*} \Big|_{\mathcal{F}_t} = \frac{S_0^k S_t^l}{S_0^l S_t^k}. \quad (2.70)$$

Proof. It is clear that \mathbb{P}^* and $\bar{\mathbb{P}}$ are equivalent on $(\Omega, \mathcal{F}_{T^*})$ and thus the Radon-Nikodým derivative $\eta_{T^*} = d\bar{\mathbb{P}}/d\mathbb{P}^*$ exists. Note that S^l/S^k is a martingale under \mathbb{P}^* and, by virtue of general properties of the Radon-Nikodým derivative, we have that $\eta_t = \mathbb{E}_{\mathbb{P}^*}(\eta_{T^*} \mid \mathcal{F}_t)$. It thus suffices to show that

$$\eta_{T^*} = \frac{d\bar{\mathbb{P}}}{d\mathbb{P}^*} = \frac{S_0^k S_{T^*}^l}{S_0^l S_{T^*}^k}. \quad (2.71)$$

It is easily seen that the last equality defines an equivalent probability measure $\bar{\mathbb{P}}$ on $(\Omega, \mathcal{F}_{T^*})$ since $\eta_{T^*} > 0$ and $\mathbb{E}_{\mathbb{P}^*} \eta_{T^*} = 1$. Hence it remains to verify that if a process S^i/S^k is a martingale under \mathbb{P}^* then the process S^i/S^l is a martingale under $\bar{\mathbb{P}}$. We thus assume that, for every $u \leq t$,

$$\mathbb{E}_{\mathbb{P}^*}(S_t^i/S_t^k \mid \mathcal{F}_u) = S_u^i/S_u^k.$$

Using the Bayes formula (see Lemma A.1.4), we obtain

$$\begin{aligned}
\mathbb{E}_{\bar{\mathbb{P}}}(S_t^i/S_t^l \mid \mathcal{F}_u) &= \frac{\mathbb{E}_{\mathbb{P}^*}(\eta_{T^*} S_t^i/S_t^l \mid \mathcal{F}_u)}{\mathbb{E}_{\mathbb{P}^*}(\eta_{T^*} \mid \mathcal{F}_u)} \\
&= \frac{\mathbb{E}_{\mathbb{P}^*}(\mathbb{E}_{\mathbb{P}^*}(\eta_{T^*} \mid \mathcal{F}_t) S_t^i/S_t^l \mid \mathcal{F}_u)}{\mathbb{E}_{\mathbb{P}^*}(\eta_{T^*} \mid \mathcal{F}_u)} \\
&= \frac{\mathbb{E}_{\mathbb{P}^*}(\eta_t S_t^i/S_t^l \mid \mathcal{F}_u)}{\eta_u} \\
&= \frac{\mathbb{E}_{\mathbb{P}^*}((S_t^l/S_t^k)(S_t^i/S_t^l) \mid \mathcal{F}_u)}{S_u^l/S_u^k} \\
&= \frac{\mathbb{E}_{\mathbb{P}^*}(S_t^i/S_t^k \mid \mathcal{F}_u)}{S_u^l/S_u^k} \\
&= \frac{S_u^i/S_u^k}{S_u^l/S_u^k} = S_u^i/S_u^l.
\end{aligned}$$

We conclude that the probability measure $\bar{\mathbb{P}}$ given by (2.71) is indeed the unique martingale measure associated with the numeraire asset S^l . \square

2.6.10 Discrete-time Models with Infinite State Space

In this short subsection, we relax the standing assumption that the underlying probability space is finite. We assume instead that we are given a finite family of real-valued, discrete-time, stochastic processes S^1, S^2, \dots, S^k , defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T^*}$. We note that all definitions and results of Sect. 2.6.2–2.6.5 remain valid within the present set-up. Moreover, as in the finite case, it is possible to give a probabilistic characterization of discrete-time models of financial markets. The following result, which is a version of the First Fundamental Theorem of Asset Pricing, was first established by Dalang et al. (1990), and later re-examined by other authors.

Theorem 2.6.4 *A discrete-time spot market model $\mathcal{M} = (S, \Phi)$ is arbitrage-free if and only if there exists a martingale measure \mathbb{P}^* for S^* ; that is, the class $\mathcal{P}(S^*)$ is non-empty.*

To proof of sufficiency is based on similar arguments as in Sect. 2.6.5. One needs, however, to introduce the notion of a *generalized martingale* (see Jacod and Shiryaev (1998)), and thus we do not present the details here. The proof of necessity is more complicated. It is based, among others, on the following lemma (for its proof, see Rogers (1994)) and the *optional decomposition* of a generalized supermartingale (see Föllmer and Kabanov (1998)).

Lemma 2.6.3 *Let $X_t, t \leq T^*$, be a sequence of \mathbb{R}^k -valued random variables, defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, such that X_t is \mathcal{F}_t -measurable for*

every t , and let γ_t , $t \leq T^*$, be a sequence of \mathbb{R}^k -valued random variables such that γ_t is \mathcal{F}_{t-1} -measurable for every t and $|\gamma_t| < c$ for some constant c . Suppose that we have, for $t = 1, 2, \dots, T^*$,

$$\mathbb{P}\{\gamma_t \cdot X_t > 0 \mid \mathcal{F}_{t-1}\} > 0$$

and

$$\mathbb{P}\{\gamma_t \cdot X_t < 0 \mid \mathcal{F}_{t-1}\} > 0.$$

Then there exist a probability measure \mathbb{Q} equivalent to \mathbb{P} on $(\Omega, \mathcal{F}_{T^*})$ and such that the process X_t , $t \leq T^*$, is a \mathbb{Q} -martingale.

For the proof of the following version of the Second Fundamental Theorem of Asset Pricing, we refer to the paper by Jacod and Shiryaev (1998).

Theorem 2.6.5 *An arbitrage-free discrete-time spot market model $\mathcal{M} = (S, \Phi)$ is complete if and only if a martingale measure \mathbb{P}^* is unique.*

For further results on hedging in discrete-time incomplete models, one may consult the papers by Föllmer and Leukert (1999) (quantile hedging), Föllmer and Leukert (2000) (shortfall hedging), Frittelli (2000) (entropy-based approach), Rouge and El Karoui (2000) (utility-based approach), Carassus et al. (2002) (super-replication). Let us mention that results obtained in these papers are rarely explicit; in most cases only the existence of respective hedging strategies is established.

2.7 Finite Futures Markets

We continue working under the assumption that the underlying probability space is finite. Throughout this section, the coordinates Z^1, Z^2, \dots, Z^{k-1} of an \mathbb{R}^k -valued process Z are interpreted as *futures prices* of some either traded or non-traded assets. More precisely, $Z_t^i = f_{S^i}(t, T^i)$ represents the value of the futures price at time t of the i^{th} underlying security, in a futures contract with expiry date T^i . We assume here that $T^i \geq T^*$ for every $i = 1, 2, \dots, k-1$. For the ease of notation, we shall write briefly $f_t^i = f_{S^i}(t, T^i)$. Also, we write f to represent the random vector $(f^1, f^2, \dots, f^{k-1})$.

Furthermore, the last coordinate, $Z^k = S^k$, is assumed to model the spot price of a certain security. The process S^k may be taken to model the price of an arbitrary risky asset; however, we shall typically assume that S^k represents the price process of a risk-free asset – that is, of a savings account. The rationale for this assumption lies in the fact that we prefer to express spot prices of derivative securities, such as futures options, in terms of a spot security. In other words, the price process S^k will play the role of a benchmark for the prices of derivative securities. It is convenient to assume that the price process S^k is strictly positive.

2.7.1 Self-financing Futures Strategies

Our first goal is to introduce a class of self-financing futures strategies, along the similar lines as in Sect. 2.6.1. By a *futures trading strategy* we mean an arbitrary \mathbb{R}^k -valued, \mathbb{F} -adapted process ϕ_t , $t \leq T^*$. The coordinates ϕ_t^i , $i = 1, 2, \dots, k-1$, represent the number of long or short positions in a given futures contract assumed by an individual at time t , whereas $\phi_t^k S_t^k$ stands, as usual, for the cash investment in the spot security S^k at time t . As in the case of spot markets, we write $B = S^k$. In view of specific features of futures contracts, namely, the fact that it cost nothing to enter such a contract (cf. Sect. 1.2), the definition of the wealth process of a trading strategy is adjusted as follows.

Definition 2.7.1 The *wealth process* $V^f(\phi)$ of a futures trading strategy ϕ is an adapted stochastic process given by the equality

$$V_t^f(\phi) = \phi_t^k B_t, \quad \forall t \leq T^*. \quad (2.72)$$

In particular, the *initial investment* $V_0^f(\phi)$ of any futures portfolio ϕ equals $V_0^f(\phi) = \phi_0^k B_0$. Let us denote by $\phi^f = (\phi^1, \phi^2, \dots, \phi^{k-1})$ the futures position. Though ϕ^f is not present in formula (2.72), which defines the wealth process, it appears explicitly in the self-financing condition (2.73) as well as in expression (2.74) which describes the gains process (this in turn reflects the marking to market feature of a futures contract).

Definition 2.7.2 A futures trading strategy ϕ is said to be *self-financing* if and only if the condition

$$\phi_{t+1}^f \cdot (f_{t+1} - f_t) + \phi_t^k B_{t+1} = \phi_{t+1}^k B_{t+1} \quad (2.73)$$

is satisfied for every $t = 0, 1, \dots, T^* - 1$.

The *gains process* $G^f(\phi)$ of any futures trading strategy ϕ is given by the equality, for every $t = 0, 1, \dots, T^*$,

$$G_t^f(\phi) = \sum_{u=0}^{t-1} \phi_u^f \cdot (f_{u+1} - f_u) + \sum_{u=0}^{t-1} \phi_u^k (B_{u+1} - B_u). \quad (2.74)$$

Let us denote by Φ^f the vector space of all self-financing futures trading strategies. The following result is a counterpart of Lemma 2.6.1.

Lemma 2.7.1 A futures trading strategy ϕ is self-financing if and only if we have that, for every $t = 0, 1, \dots, T^*$,

$$V_t^f(\phi) = V_0^f(\phi) + G_t^f(\phi).$$

Proof. Taking into account (2.73)–(2.74), for any self-financing futures strategy ϕ we obtain

$$\begin{aligned}
V_t^f(\phi) &= V_0^f(\phi) + \sum_{u=0}^{t-1} (V_{u+1}^f - V_u^f) \\
&= V_0^f(\phi) + \sum_{u=0}^{t-1} (\phi_{u+1}^k B_{u+1} - \phi_u^k B_u) \\
&= V_0^f(\phi) + \sum_{u=0}^{t-1} (\phi_u^f \cdot (f_{u+1} - f_u) + \phi_u^k B_{u+1} - \phi_u^k B_u) \\
&= V_0^f(\phi) + \sum_{u=0}^{t-1} \phi_u^f \cdot (f_{u+1} - f_u) + \sum_{u=0}^{t-1} \phi_u^k (B_{u+1} - B_u) \\
&= V_0^f(\phi) + G_t^f(\phi)
\end{aligned}$$

for every $t = 0, 1, \dots, T^*$. This proves the “only if” clause. The proof of the “if” clause is left to the reader. \square

We say that a futures trading strategy $\phi \in \Phi^f$ is an *arbitrage opportunity* if $\mathbb{P}\{V_0^f(\phi) = 0\} = 1$, and the terminal wealth of ϕ satisfies

$$V_{T^*}^f(\phi) \geq 0 \quad \text{and} \quad \mathbb{P}\{V_{T^*}^f(\phi) > 0\} > 0.$$

We say that a futures market $\mathcal{M}^f = (f, B, \Phi^f)$ is *arbitrage-free* if there are no arbitrage opportunities in the class Φ^f of all futures trading strategies. The notions of a contingent claim, replication and completeness, as well as of a wealth process of an attainable contingent claim, remain the same, with obvious terminological modifications. For instance, we say that a claim X which settles at time T is *attainable* in \mathcal{M}^f if there exists a self-financing futures trading strategy ϕ such that $V_T^f(\phi) = X$. The following result can be proved along the same lines as Proposition 2.6.1.

Proposition 2.7.1 *Suppose that the market \mathcal{M}^f is arbitrage-free. Then any attainable contingent claim X is uniquely replicated in \mathcal{M}^f .*

The next definition, which introduces the arbitrage price in a futures market, is merely a reformulation of Definition 2.6.8.

Definition 2.7.3 Suppose that the futures market \mathcal{M}^f is arbitrage-free. Then the wealth process of an attainable contingent claim X which settles at time T is called the *arbitrage price process* of X in the market model \mathcal{M}^f . We denote it by $\pi_t^f(X)$, $t \leq T$.

2.7.2 Martingale Measures for a Futures Market

The next step is to examine the arbitrage-free property of a futures market model. Recall that a probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_{T^*})$, equivalent to \mathbb{P} , is called a martingale measure for f if the process f follows a $\tilde{\mathbb{P}}$ -martingale with respect to the filtration

\mathbb{F} . Note that we take here simply the futures prices, as opposed to the case of a spot market in which we dealt with relative prices. We denote by $\mathcal{P}(f)$ the class of all martingale measures for f , in the sense of the following definition (cf. Definition 2.6.10).

Definition 2.7.4 A probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_{T^*})$ equivalent to \mathbb{P} is called a *martingale measure for $\mathcal{M}^f = (f, B, \Phi^f)$* if the relative wealth process $\tilde{V}^f(\phi) = V_t^f(\phi)B_t^{-1}$ of any self-financing futures trading strategy ϕ is a $\tilde{\mathbb{P}}$ -martingale with respect to the filtration \mathbb{F} . The class of all martingale measures for \mathcal{M}^f is denoted by $\mathcal{P}(\mathcal{M}^f)$.

In the next result, it is essential to assume that the discrete-time process B is *predictable* with respect to the filtration \mathbb{F} , meaning that for every $t = 0, 1, \dots, T^* - 1$, the random variable B_{t+1} is measurable with respect to the σ -field \mathcal{F}_t . Intuitively, predictability of B means that the future value of B_{t+1} is known already at time t .

Remarks. Such a specific property of a *savings account* B may arise naturally in a discrete-time model with an uncertain rate of interest. Indeed, it is common to assume that at any date t , the rate of interest r_t that prevails over the next time period $[t, t+1]$ is known already at the beginning of this period. For instance, if the σ -field \mathcal{F}_0 is trivial, the rate r_0 is a real number, so that the value B_1 of a savings account at time 1 is also deterministic. Then, at time 1, at any state the rate r_1 is known, so that B_2 is a \mathcal{F}_1 -measurable random variable and so forth. In this way, we construct a savings account process B which is predictable with respect to the filtration \mathbb{F} (for brevity, we shall simply say that B is predictable).

Lemma 2.7.2 A futures trading strategy ϕ is self-financing if and only if the relative wealth process $\tilde{V}^f(\phi)$ admits the following representation, for every $t = 0, 1, \dots, T^*$,

$$\tilde{V}_t^f(\phi) = \tilde{V}_0^f(\phi) + \sum_{u=0}^{t-1} B_{u+1}^{-1} \phi_u^f \cdot \Delta_u f,$$

where $\Delta_u f = f_{u+1} - f_u$. Consequently, for any martingale measure $\tilde{\mathbb{P}} \in \mathcal{P}(f)$, the relative wealth process $\tilde{V}^f(\phi)$ of any self-financing futures trading strategy ϕ is a martingale under $\tilde{\mathbb{P}}$.

Proof. Let us denote $V^f = V^f(\phi)$ and $\tilde{V}^f = \tilde{V}^f(\phi)$. For the first statement, it is sufficient to check that, for every $t = 0, 1, \dots, T^* - 1$,

$$\tilde{V}_{t+1}^f - \tilde{V}_t^f = B_{t+1}^{-1} \phi_t^f \cdot (f_{t+1} - f_t). \quad (2.75)$$

To this end, it suffices to observe that

$$\tilde{V}_{t+1}^f - \tilde{V}_t^f = V_{t+1}^f B_{t+1}^{-1} - V_t^f B_t^{-1} = \phi_{t+1}^k - \phi_t^k,$$

since $V_t^f = \phi_t^k B_t$ for every t . It is thus clear that (2.75) is equivalent to (2.73), as expected.

For the second assertion, we need to show that for any self-financing futures trading strategy ϕ we have, for every $t = 0, 1, \dots, T^* - 1$,

$$\mathbb{E}_{\tilde{\mathbb{P}}}(\tilde{V}_{t+1}^f - \tilde{V}_t^f | \mathcal{F}_t) = 0.$$

But

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{P}}}(\tilde{V}_{t+1}^f - \tilde{V}_t^f | \mathcal{F}_t) &= \mathbb{E}_{\tilde{\mathbb{P}}}(B_{t+1}^{-1} \phi_t^f \cdot (f_{t+1} - f_t) | \mathcal{F}_t) \\ &= B_{t+1}^{-1} \phi_t^f \cdot \mathbb{E}_{\tilde{\mathbb{P}}}(f_{t+1} - f_t | \mathcal{F}_t) = 0, \end{aligned}$$

as the random variable $B_{t+1}^{-1} \phi_t^f$ is \mathcal{F}_t -measurable (this holds since B is \mathbb{F} -predictable and ϕ^f is \mathbb{F} -adapted), and the price process f is a martingale under the probability measure $\tilde{\mathbb{P}}$. \square

The following corollary can be easily established.

Corollary 2.7.1 *A probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_{T^*})$ is a martingale measure for the futures market model \mathcal{M}^f if and only if $\tilde{\mathbb{P}}$ is a martingale measure for the futures price process f ; that is, the equality $\mathcal{P}(f) = \mathcal{P}(\mathcal{M}^f)$ holds.*

2.7.3 Risk-neutral Valuation Formula

The next result shows that the existence of a martingale measure is a sufficient condition for the absence of arbitrage in the futures market model \mathcal{M}^f . In addition, the risk-neutral valuation formula is valid. Note that a claim X is interpreted as an ordinary “spot” contingent claim which settles at time T . In other words, X is a \mathcal{F}_T -measurable random payoff denominated in units of spot security B . Of course, X may explicitly depend on the futures prices. For instance, in the case of a path-independent European claim that settles at time T we have $X = h(f_T^1, f_T^2, \dots, f_T^{k-1}, S_T^k)$ for some function $h : \mathbb{R}^k \rightarrow \mathbb{R}$.

Proposition 2.7.2 *Assume that the class $\mathcal{P}(\mathcal{M}^f)$ of futures martingale measures is non-empty. Then the futures market model \mathcal{M}^f is arbitrage-free. Moreover, the arbitrage price in \mathcal{M}^f of any attainable contingent claim X which settles at time T is given by the risk-neutral valuation formula*

$$\pi_t^f(X) = B_t \mathbb{E}_{\tilde{\mathbb{P}}}(X B_T^{-1} | \mathcal{F}_t), \quad \forall t \leq T, \quad (2.76)$$

where $\tilde{\mathbb{P}}$ is any martingale measure from the class $\mathcal{P}(\mathcal{M}^f)$.

Proof. It is easy to check the absence of arbitrage opportunities in \mathcal{M}^f . Also, equality (2.76) is a straightforward consequence of Lemma 2.7.2. Indeed, it follows immediately from the martingale property of the discounted wealth of a strategy that replicates X . \square

The last result of this section corresponds to Theorems 2.6.1 and 2.6.2.

Theorem 2.7.1 *The following statements are true.*

- (i) *A finite futures market \mathcal{M}^f is arbitrage-free if and only if the class $\mathcal{P}(\mathcal{M}^f)$ of martingale measures is non-empty.*
- (ii) *An arbitrage-free futures market \mathcal{M}^f is complete if and only if the uniqueness of a martingale measure $\tilde{\mathbb{P}}$ for \mathcal{M}^f holds.*

Proof. Both statements can be proved by means of the same arguments as those used in the case of a spot market. \square

2.7.4 Futures Prices Versus Forward Prices

In the preceding section, the evolution of a futures price was not derived but postulated. Here, we start with an arbitrage-free model of a spot market and we examine the forward price of a spot asset. Subsequently, we investigate the relation between forward and futures prices of such an asset. As before, we write S^i , $i \leq k$, to denote spot prices of primary assets. Also, we assume the strictly positive process $B = S^k$ plays the role of a numeraire asset – intuitively, a savings account.

It is convenient to introduce an auxiliary family of derivative spot securities, referred to as (default-free) *zero-coupon bonds* (or *discount bonds*). For any $t \leq T \leq T^*$, we denote by $B(t, T)$ the time t value of a security which pays to its holder one unit of cash at time T (no intermediate cash flows are paid before this date). We refer to $B(t, T)$ as the spot price at time t of a zero-coupon bond of maturity T , or briefly, the price of a T -maturity bond. Assume that the spot market model is arbitrage-free so that the class $\mathcal{P}(\mathcal{M})$ of spot martingale measures is non-empty. Taking for granted the attainability of the European claim $X = 1$ which settles at time T (and which represents the bond's payoff), we obtain

$$B(t, T) = B_t \mathbb{E}_{\mathbb{P}^*}(B_T^{-1} | \mathcal{F}_t), \quad \forall t \leq T.$$

Let us fix i and let us denote $S^i = S$ (as usual, we shall refer to S as the stock price process). We know already that a forward contract (with the settlement date T) is represented by the contingent claim $X = S_T - K$ that settles at time T . Recall that by definition, the forward price $F_S(t, T)$ at time $t \leq T$ is defined as that level of the delivery price K , which makes the forward contract worthless at time t equal to 0. Since the forward price $F_S(t, T)$ is determined at time t , it is necessarily a \mathcal{F}_t -measurable random variable. It appears that the forward price of a (non-dividend-paying) asset can be expressed in terms of its spot price at time t and the price at time t of a zero-coupon bond of maturity T .

Proposition 2.7.3 *The forward price at time $t \leq T$ of a stock S for the settlement date T equals*

$$F_S(t, T) = \frac{S_t}{B(t, T)}, \quad \forall t \leq T. \quad (2.77)$$

Proof. In view of (2.63), we obtain

$$\pi_t(X) = B_t \mathbb{E}_{\mathbb{P}^*}(B_T^{-1}(S_T - F_S(t, T)) | \mathcal{F}_t) = 0.$$

On the other hand, since by assumption $F_S(t, T)$ is \mathcal{F}_t -measurable, we get

$$\pi_t(X) = B_t \mathbb{E}_{\mathbb{P}^*}(S_T^* | \mathcal{F}_t) - B_t F_S(t, T) \mathbb{E}_{\mathbb{P}^*}(B_T^{-1} | \mathcal{F}_t) = S_t - B(t, T) F_S(t, T),$$

since $\mathbb{E}_{\mathbb{P}^*}(S_T^* | \mathcal{F}_t) = S_t^*$. \square

Though we have formally derived (2.77) using risk-neutral valuation approach, it is clear that equality (2.77) can also be easily established using standard no-arbitrage arguments. In such an approach, it is enough to assume that the underlying asset and the T -maturity bond are among traded securities (the existence of a savings account is not required). On the other hand, if the reference asset S pays dividends (coupons, etc.) during the lifetime of a forward contract, formula (2.77) should be appropriately modified. For instance, in the case of a single (non-random) dividend D to be received by the owner of S at time U , where $t < U < T$, equality (2.77) becomes

$$F_S(t, T) = \frac{S_t - DB(t, U)}{B(t, T)}, \quad \forall t \leq T. \quad (2.78)$$

Let us denote by $f_S(t, T)$ the futures price of the stock S – that is, the price at which a futures contract written on S with the settlement date T is entered into at time t (in particular, $f_S(T, T) = S_T$). We change slightly our setting, namely, instead on focusing on a savings account, we assume that we are given the price process of the T -maturity bond. We make a rather strong assumption that $B(t, T)$ follows a predictable process. In other words, for any $t \leq T - 1$ the random variable $B(t + 1, T)$ is \mathcal{F}_t -measurable. Intuitively, this means that on each date we know the bond price which will prevail on the next date (though hardly a realistic assumption, it is nevertheless trivially satisfied in any security market model which assumes a deterministic⁴ savings account).

Proposition 2.7.4 *Let the bond price $B(t, T)$ follow a predictable process. The combined spot-futures market is arbitrage-free if and only if the futures and forward prices agree; that is, $f_S(t, T) = F_S(t, T)$ for every $t \leq T$.*

Proof. We first aim show that the asserted equality is necessary for the absence of arbitrage in the combined spot-futures market. As mentioned above, the T -maturity bond is considered as the basic spot asset; in particular, all proceeds from futures contracts are immediately reinvested in this bond. Let us consider a self-financing futures strategy ψ for which $V_0^f(\psi) = 0$. It is not difficult to check that the terminal wealth of a strategy ψ satisfies

$$V_T^f(\psi) = \sum_{t=0}^{T-1} \psi_t B^{-1}(t+1, T) (f_{t+1} - f_t).$$

For instance, the gains/losses $\psi_0(f_1 - f_0)$ incurred at time 1 are used to purchase $B^{-1}(1, T)$ units of the bond that matures at T . This investment results in $\psi_0(f_1 - f_0)B^{-1}(1, T)$ units of cash when the bond expires at time T .

⁴ It was observed by Lutz Schlögl that if all bond prices with different maturities are predictable then the bond prices (and thus also the savings account) are deterministic.

Let us now consider a specific futures trading strategy; namely, we take $\psi_t = -B(t+1, T)$ for every $t \leq T-1$ (note that the bond coordinate of this strategy is uniquely determined by the self-financing condition). It is easy to see that in this case we get simply $V_T^f(\psi) = f_0 - S_T$. In addition, we shall employ the following spot trading strategy ϕ : buy-and-hold the stock S , using the proceeds from the sale of T -maturity bonds. It is clear that in order to purchase one share of stock at the price S_0 , one needs to sell $S_0 B^{-1}(0, T)$ units of the bond at the price $B(0, T)$. Therefore, the initial wealth of ϕ is zero, and the terminal wealth equals

$$V_T(\phi) = S_T - S_0 B^{-1}(0, T).$$

Combining these two strategies, we obtain the spot-futures strategy with zero initial wealth, and terminal wealth that equals

$$V_T(\phi) + V_T^f(\psi) = f_0 - S_0 B^{-1}(0, T).$$

Since arbitrage opportunities in the combined spot-futures market were ruled out, we conclude that $f_0 = S_0 B^{-1}(0, T)$; that is, the futures price at time 0 coincides with the forward price $F_S(0, T)$. Similar reasoning leads to the general equality. The proof of the converse implication is left to the reader as an exercise. \square

2.8 American Contingent Claims

We will now address the issue of valuation and hedging of American contingent claims within the framework of a finite market in which k primary assets are traded at spot prices S^1, S^2, \dots, S^k . As usual, we denote $B = S^k$ and we postulate that $B > 0$. Let us first recall the concept of a stopping time.

Definition 2.8.1 A *stopping time with respect to a filtration* \mathbb{F} (an \mathbb{F} -*stopping time* or, simply, a *stopping time*) is an arbitrary function $\tau : \Omega \rightarrow \{0, 1, \dots, T^*\}$ such that for any $t = 0, 1, \dots, T^*$ the random event $\{\tau = t\}$ belongs to the σ -field \mathcal{F}_t . We denote by $\mathcal{T}_{[0, T^*]}$ the class of all *stopping times* with values in $\{0, 1, \dots, T^*\}$ defined on the filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$.

Any stopping time τ is manifestly a random variable taking values in $\{0, 1, \dots, T^*\}$, but the converse is not true, in general. Intuitively, the property $\{\tau = t\} \in \mathcal{F}_t$ means that the decision whether to stop a given process at time t or to continue (in our case, whether to exercise an American claim at time t or to hold it alive at least till the next date, $t+1$) is made exclusively on the basis of information available at time t . It is worth mentioning that the property that $\{\tau = t\} \in \mathcal{F}_t$ for any $t = 0, 1, \dots, T^*$ is equivalent to the condition that $\{\tau \leq t\} \in \mathcal{F}_t$ for any $t = 0, 1, \dots, T^*$. Therefore, for any stopping time τ , the event $\{\tau > t\} = \{\tau \geq t+1\}$ belongs to \mathcal{F}_t for any $t = 0, 1, \dots, T^*$.

The following lemma is easy to established and thus its proof is left to the reader as an exercise.

Lemma 2.8.1 *Let τ and σ be two stopping times. Then the random variables $\tau \wedge \sigma = \min(\tau, \sigma)$ and $\tau \vee \sigma = \max(\tau, \sigma)$ are stopping times as well.*

For convenience, we define the following subclass of stopping times.

Definition 2.8.2 We write $\mathcal{T}_{[t,T]}$ to denote the subclass of those stopping times τ with respect to \mathbb{F} that satisfy the inequalities $t \leq \tau \leq T$, that is, take values in the set $\{t, t+1, \dots, T\}$.

Let us recall the definition of an American contingent claim (see Definition 2.4.1).

Definition 2.8.3 An *American contingent claim* $X^a = (X, \mathcal{T}_{[0,T]})$ expiring at time T consists of an \mathbb{F} -adapted payoff process $X = (X_t)_{t=0}^T$ and of the set $\mathcal{T}_{[0,T]}$ of admissible exercise times for the holder of an American claim.

For any stopping time $\tau \in \mathcal{T}_{[0,T]}$, the random variable X_τ , defined as

$$X_\tau = \sum_{t=0}^T X_t \mathbb{1}_{\{\tau=t\}},$$

represents the payoff received at time τ by the holder of an American claim X^a , if she decides to exercise the claim at time τ .

Note that we will adopt a convention that stipulates that an American claim is exercised at its expiry date T if it was not exercised prior to that date. We will frequently assume that a given American claim is *alive* at time t , meaning that, by assumption, it was not exercised at times $0, 1, \dots, t-1$ (notwithstanding whether it should have been exercised at some date $u \leq t-1$ according to some rationality principle) and we will examine the valuation of the claim at this date, as well as its hedging from time t onwards.

Let us thus suppose that an American claim is alive at time t . Then, the set of exercise times available to the holder of an American claim from time t onwards is represented by the class $\mathcal{T}_{[t,T]}$ of stopping times τ taking values in the set $\{t, t+1, \dots, T\}$. In other words, the set of admissible exercise times from time t onwards equals $\mathcal{T}_{[t,T]}$.

Assume that the finite market model $\mathcal{M} = (S, \Phi)$ is arbitrage-free, but not necessarily complete. Then the concept of an arbitrage price in \mathcal{M} of an American claim is introduced through the following definition.

Definition 2.8.4 By an *arbitrage price* in \mathcal{M} of an American claim X^a we mean any price process $\pi_t(X^a)$, $t \leq T$, such that the extended market model – that is, the market model with trading in assets S^1, S^2, \dots, S^k and an American claim – remains arbitrage-free.

Note that an American claim is traded only up to its exercise time, which is chosen at discretion of its holder, so it is essentially any stopping time from the class $\mathcal{T}_{[0,T]}$. If market completeness is not postulated, we cannot expect an arbitrage

price of an American claim to be unique. Indeed, in that case, the uniqueness of an arbitrage price is not ensured even for European claims. Therefore, to simplify the analysis we will work hereafter under the standing assumption that the market model is not only arbitrage-free, but also complete, in the sense discussed in Sect. 2.6.1 (see, in particular, Definition 2.6.5).

In the case of an arbitrage-free and complete finite model, we have the following result, which can be seen as an extension of Corollary 1.7.1, in which we examined the valuation of American claims for a single-period model (see also Proposition 2.4.1 for the CRR model case).

Proposition 2.8.1 *Assume that the finite market model $\mathcal{M} = (S, \Phi)$ is arbitrage-free and complete. Let \mathbb{P}^* be the unique martingale measure for the process $S^* = S/B$. Then the arbitrage price $\pi(X^a)$ in \mathcal{M} of an American claim X^a , with the payoff process X and expiry date T is unique and it is given by the formula, for every $t = 0, 1, \dots, T$,*

$$\pi_t(X^a) = \max_{\tau \in \mathcal{T}_{[t, T]}} B_t \mathbb{E}_{\mathbb{P}^*}(B_\tau^{-1} X_\tau | \mathcal{F}_t). \quad (2.79)$$

In particular, the arbitrage prices of American call and put options, written on the asset S^l for some $l \neq k$, are given by the following expressions, for every $t = 0, 1, \dots, T$,

$$C_t^a = \max_{\tau \in \mathcal{T}_{[t, T]}} B_t \mathbb{E}_{\mathbb{P}^*}(B_\tau^{-1} (S_\tau^l - K)^+ | \mathcal{F}_t)$$

and

$$P_t^a = \max_{\tau \in \mathcal{T}_{[t, T]}} B_t \mathbb{E}_{\mathbb{P}^*}(B_\tau^{-1} (K - S_\tau^l)^+ | \mathcal{F}_t).$$

The proof of Proposition 2.8.1 is postponed to Sect. 2.8.2 (see, in particular, Corollary 2.8.3), since it requires several preliminary results regarding the so-called *optimal stopping problems*. The valuation and hedging of an American claim is clearly a more involved issue than just solving some optimal stopping problem, so let us first make some comments.

On the one hand, we note that formula (2.79) is intuitively plausible from the perspective of the holder. Indeed, the holder of an American claim may choose freely her exercise time τ , and thus it is natural to expect that she will try select a stopping time τ that would maximize the price of a European claim with the terminal payoff X_τ to be received at a random maturity τ . We will thus refer to any stopping time that realizes the maximum in (2.79) for a given t as a *rational exercise time* as of time t . Note that we do not claim that this time is unique, even when the date t is fixed.

On the other hand, it is not evident whether the price level given by (2.79) will be also acceptable by the seller of an American claim. Let us make clear that the seller's strategy should not rely on the assumption that the claim will necessarily be exercised by its holder at a rational exercise time. Therefore, starting from the initial wealth $\pi_0(X^a)$, the seller should be able to construct a trading strategy, such that the wealth process will cover his potential liabilities due to her short position in an American claim at any time t , and not only at a rational exercise time.

2.8.1 Optimal Stopping Problems

The goal of this section is to establish the most fundamental results concerning the optimal stopping problem in an abstract setup with a finite horizon. Let thus $Z = (Z_t)_{t=0}^n$ be a sequence of random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathbb{F} = (\mathcal{F}_t)_{t=0}^n$ be some filtration such that the *payoff process* Z is \mathbb{F} -adapted. We assume, as usual, that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and we denote by $\mathcal{T}_{[t,n]}$ the class of all \mathbb{F} -stopping times taking values in the set $\{t, t+1, \dots, n\}$.

The assumption that the underlying probability space has a finite number of elementary events is not crucial here, and thus it can be relaxed. Indeed, it suffices to make the technical assumption that $\mathbb{E}_{\mathbb{P}}|Z_t| < \infty$ for every $t = 1, 2, \dots, n$, so that for any stopping time $\tau \in \mathcal{T}_{[t,n]}$, we have that

$$\mathbb{E}_{\mathbb{P}}|Z_{\tau}| < n \max_{t=0,1,\dots,n} \mathbb{E}_{\mathbb{P}}|Z_t| < \infty.$$

Of course, this condition is satisfied by any process Z when Ω is finite, since in that case the random variables $|Z_0|, |Z_1|, \dots, |Z_n|$ are all bounded by a common constant.

We will examine the following *optimal stopping problems*.

Problem (A) For $t = 0$, we wish to maximize the expected value $\mathbb{E}_{\mathbb{P}}(Z_{\tau})$ over the class $\mathcal{T}_{[0,n]}$ of stopping times τ . In other words, we search for an *optimal stopping time* $\tau_0^* \in \mathcal{T}_{[0,n]}$ such that

$$\mathbb{E}_{\mathbb{P}}(Z_{\tau_0^*}) \geq \mathbb{E}_{\mathbb{P}}(Z_{\tau}), \quad \forall \tau \in \mathcal{T}_{[0,n]},$$

and for the corresponding expectation $\mathbb{E}_{\mathbb{P}}(Z_{\tau_0^*})$.

Problem (B) For any $t = 0, 1, \dots, n$, we wish to maximize the conditional expectation $\mathbb{E}_{\mathbb{P}}(Z_{\tau} | \mathcal{F}_t)$ over the class $\mathcal{T}_{[t,n]}$ of stopping times. We thus search for a collection of *optimal stopping times* $\tau_t^* \in \mathcal{T}_{[t,n]}$ that satisfy, for each $t = 0, 1, \dots, n$,

$$\mathbb{E}_{\mathbb{P}}(Z_{\tau_t^*} | \mathcal{F}_t) \geq \mathbb{E}_{\mathbb{P}}(Z_{\tau} | \mathcal{F}_t), \quad \forall \tau \in \mathcal{T}_{[t,n]},$$

and for the corresponding conditional expectations $\bar{Y}_t = \mathbb{E}_{\mathbb{P}}(Z_{\tau_t^*} | \mathcal{F}_t)$.

Remarks. It is clear that the latter problem is a direct extension of the former, and thus for $t = 0$ it suffices to examine problem (A). Note also that the solution to problem (B) for $t = n$ is manifestly $\tau_n^* = n$, since the stopping time $\tau = n$ is the unique element of the class $\mathcal{T}_{[n,n]}$. Therefore, we have that $\bar{Y}_n = \mathbb{E}_{\mathbb{P}}(Z_n | \mathcal{F}_n) = Z_n$.

Definition 2.8.5 Suppose that for every $t = 0, 1, \dots, n$, the problem (B) admits a solution τ_t^* . Then the \mathbb{F} -adapted process \bar{Y} , given by the formula $\bar{Y}_t = \mathbb{E}_{\mathbb{P}}(Z_{\tau_t^*} | \mathcal{F}_t)$ for $t = 0, 1, \dots, n$, is called the *value process* for the optimal stopping problem (B).

As we shall see in what follows, an optimal stopping time τ_t^* is not unique, in general. It is clear, however, that the value process is uniquely defined.

The main tool in solving the optimal stopping problem (B) is the concept of the *Snell envelope* of the payoff process Z .

Definition 2.8.6 A process $Y = (Y_t)_{t=0}^n$ is the *Snell envelope* of an \mathbb{F} -adapted process $Z = (Z_t)_{t=0}^n$ whenever the following conditions are satisfied:

- (i) Y is a supermartingale with respect to the filtration \mathbb{F} ,
- (ii) Y dominates Z , that is, the inequality $Y_t \geq Z_t$ is valid for any $t = 0, 1, \dots, n$,
- (iii) for any other supermartingale \tilde{Y} dominating Z , we have that $\tilde{Y}_t \geq Y_t$ for every $t = 0, 1, \dots, n$.

Conditions (i)–(iii) of Definition 2.8.6 can be summarized as follows: the Snell envelope of Z is the minimal supermartingale dominating Z . In particular, property (iii) implies that the Snell envelope, if it exists, is unique.

To address the issue of existence of the Snell envelope of Z , we define the process $Y = (Y_t)_{t=0}^n$ by setting, for every $t = 1, 2, \dots, n$,

$$Y_n = Z_n, \quad Y_{t-1} = \max \{Z_{t-1}, \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1})\}. \quad (2.80)$$

Lemma 2.8.2 *The process Y given by formula (2.80) is the Snell envelope of the payoff process Z .*

Proof. First, we note that Y is an \mathbb{F} -adapted process and that (2.80) implies immediately that $Y_{t-1} \geq \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1})$ for $t = 1, 2, \dots, n$. This shows that Y is an \mathbb{F} -supermartingale. Formula (2.80) implies also that Y dominates Z . It remains to check that condition (iii) in Definition 2.8.6 is satisfied as well. Let thus \tilde{Y} be any supermartingale that dominates Z . Of course, condition (iii) holds for $t = n$, since obviously $\tilde{Y}_n \geq Z_n = Y_n$. We now proceed by induction. Assume that $\tilde{Y}_t \geq Y_t$ for some $t = 1, 2, \dots, n$. Since \tilde{Y} is a supermartingale dominating Z , we have that $\tilde{Y}_{t-1} \geq \mathbb{E}_{\mathbb{P}}(\tilde{Y}_t | \mathcal{F}_{t-1})$ and $\tilde{Y}_{t-1} \geq Z_{t-1}$. Therefore,

$$\tilde{Y}_t \geq \max \{Z_{t-1}, \mathbb{E}_{\mathbb{P}}(\tilde{Y}_t | \mathcal{F}_{t-1})\} \geq \max \{Z_{t-1}, \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1})\} = Y_{t-1},$$

since the assumed inequality $\tilde{Y}_t \geq Y_t$ and monotonicity of the conditional expectation imply that

$$\mathbb{E}_{\mathbb{P}}(\tilde{Y}_t | \mathcal{F}_{t-1}) \geq \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1}).$$

We thus conclude that Y given by (2.80) is the minimal supermartingale dominating Z . \square

We claim that the optimal stopping time is the first moment when the Snell envelope Y hits the payoff process Z and that Y is in fact the value process of an optimal stopping problem. Formally, for any fixed $t = 0, 1, \dots, n$, we define the stopping time τ_t^* belonging to the class $\mathcal{T}_{[t,n]}$ by setting

$$\tau_t^* = \min \{u \in \{t, t+1, \dots, n\} | Y_u = Z_u\}. \quad (2.81)$$

It is worth noting that the process Y satisfies, for any $t = 0, 1, \dots, n-1$,

$$\mathbb{1}_{\{\tau_t^* > t\}} Y_t = \mathbb{1}_{\{\tau_t^* > t\}} \mathbb{E}_{\mathbb{P}}(Y_{t+1} | \mathcal{F}_t)$$

since clearly $\{\tau_t^* > t\} = \{Y_t > Z_t\}$ and the inequality $Y_t > Z_t$ implies in turn that $Y_t = \mathbb{E}_{\mathbb{P}}(Y_{t+1} | \mathcal{F}_t)$.

Proposition 2.8.2 *For any $t = 0, 1, \dots, n$, the stopping time τ_t^* is a solution to problem (B), that is, τ_t^* is an optimal stopping time as of time t . Moreover, for any $t = 0, 1, \dots, n$,*

$$Y_t = \mathbb{E}_{\mathbb{P}}(Z_{\tau_t^*} | \mathcal{F}_t) = \bar{Y}_t. \quad (2.82)$$

In other words, the Snell envelope Y of Z is equal to the value process \bar{Y} for the optimal stopping problem (B).

First proof of Proposition 2.8.2. We first present a demonstration based on the induction with respect to t . Of course, the statement is trivially satisfied for $t = n$, since we know that $Y_n = Z_n$ and that the stopping time $\tau_n^* = n$ is optimal.

Let us now assume that the assertions are valid for some t . Our goal is to show that they also hold for time $t - 1$.

We first check that (2.82) holds for $t - 1$ if it is satisfied for t . We start by observing that the stopping time τ_{t-1}^* can be represented as follows

$$\tau_{t-1}^* = (t - 1)\mathbb{1}_{\{\tau_{t-1}^* = t-1\}} + \tau_t^*\mathbb{1}_{\{\tau_{t-1}^* \geq t\}}.$$

Indeed,

$$\begin{aligned} \tau_{t-1}^* &= \min \{u \in \{t - 1, t, \dots, n\} \mid Y_u = Z_u\} \\ &= (t - 1)\mathbb{1}_{\{Y_{t-1} = Z_{t-1}\}} + \min \{u \geq t \mid Y_u = Z_u\}\mathbb{1}_{\{Y_{t-1} > Z_{t-1}\}} \\ &= (t - 1)\mathbb{1}_{\{\tau_{t-1}^* = t-1\}} + \tau_t^*\mathbb{1}_{\{\tau_{t-1}^* \geq t\}}. \end{aligned}$$

It is important to observe that the events

$$\{\tau_{t-1}^* = t - 1\} = \{Z_{t-1} \geq \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1})\}$$

and

$$\{\tau_{t-1}^* > t - 1\} = \{\tau_{t-1}^* \geq t\} = \{Z_{t-1} < \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1})\}$$

belong to the σ -field \mathcal{F}_{t-1} . Consequently, we have that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(Z_{\tau_{t-1}^*} | \mathcal{F}_{t-1}) &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau_{t-1}^* = t-1\}} Z_{t-1} + \mathbb{1}_{\{\tau_{t-1}^* \geq t\}} Z_{\tau_t^*} | \mathcal{F}_{t-1}) \\ &= \mathbb{1}_{\{\tau_{t-1}^* = t-1\}} Z_{t-1} + \mathbb{1}_{\{\tau_{t-1}^* \geq t\}} \mathbb{E}_{\mathbb{P}}(Z_{\tau_t^*} | \mathcal{F}_{t-1}) \\ &= \mathbb{1}_{\{\tau_{t-1}^* = t-1\}} Z_{t-1} + \mathbb{1}_{\{\tau_{t-1}^* \geq t\}} \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(Z_{\tau_t^*} | \mathcal{F}_t) | \mathcal{F}_{t-1}) \\ &= \mathbb{1}_{\{Z_{t-1} \geq \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1})\}} Z_{t-1} + \mathbb{1}_{\{Z_{t-1} < \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1})\}} \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1}) \\ &= \max(Z_{t-1}, \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1})) = Y_{t-1}, \end{aligned}$$

where we have used, in particular, the assumption that $\mathbb{E}_{\mathbb{P}}(Z_{\tau_t^*} | \mathcal{F}_t) = Y_t$ and formula (2.80).

We will now show that the optimality of τ_t^* implies that τ_{t-1}^* is optimal as well. For any stopping time $\tau \in \mathcal{T}_{[t-1, n]}$, we have that

$$\tau = (t - 1)\mathbb{1}_{\{\tau = t-1\}} + \tau\mathbb{1}_{\{\tau \geq t\}},$$

where the events $\{\tau = t - 1\}$ and $\{\tau \geq t\} = \{\tau > t - 1\}$ manifestly belong to the σ -field \mathcal{F}_{t-1} . Let us set $\tilde{\tau} = \tau \vee t = \max(\tau, t)$. It is easy to check that $\tilde{\tau}$ is a stopping time belonging to the class $\mathcal{T}_{[t,n]}$. We now obtain

$$\begin{aligned}
 \mathbb{E}_{\mathbb{P}}(Z_{\tau_{t-1}^*} | \mathcal{F}_{t-1}) &= \max(Z_{t-1}, \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1})) \\
 &\geq \mathbb{1}_{\{\tau=t-1\}} Z_{t-1} + \mathbb{1}_{\{\tau \geq t\}} \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1}) \\
 &= \mathbb{1}_{\{\tau=t-1\}} Z_{t-1} + \mathbb{1}_{\{\tau \geq t\}} \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(Z_{\tau^*} | \mathcal{F}_t) | \mathcal{F}_{t-1}) \\
 &\geq \mathbb{1}_{\{\tau=t-1\}} Z_{t-1} + \mathbb{1}_{\{\tau \geq t\}} \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(Z_{\tilde{\tau}} | \mathcal{F}_t) | \mathcal{F}_{t-1}) \\
 &= \mathbb{1}_{\{\tau=t-1\}} Z_{t-1} + \mathbb{1}_{\{\tau \geq t\}} \mathbb{E}_{\mathbb{P}}(Z_{\tilde{\tau}} | \mathcal{F}_{t-1}) \\
 &= \mathbb{1}_{\{\tau=t-1\}} Z_{t-1} + \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \geq t\}} Z_{\tau} | \mathcal{F}_{t-1}) \\
 &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau=t-1\}} Z_{t-1} + \mathbb{1}_{\{\tau \geq t\}} Z_{\tau} | \mathcal{F}_{t-1}) = \mathbb{E}_{\mathbb{P}}(Z_{\tau} | \mathcal{F}_{t-1}),
 \end{aligned}$$

where the second inequality holds since $\tilde{\tau}$ belongs to $\mathcal{T}_{[t,n]}$ and thus, by assumption, we have that

$$\mathbb{E}_{\mathbb{P}}(Z_{\tau^*} | \mathcal{F}_t) \geq \mathbb{E}_{\mathbb{P}}(Z_{\tilde{\tau}} | \mathcal{F}_t).$$

We conclude that the stopping time τ_{t-1}^* is optimal for problem (B) as of time $t - 1$, as was required to demonstrate. \square

An alternative derivation of Proposition 2.8.2 will hinge on the following lemma, which is of independent interest.

Lemma 2.8.3 *The stopped process $Y^{\tau_0^*}$ is an \mathbb{F} -martingale.*

Proof. Let us denote $M = Y^{\tau_0^*}$. Then we have, for any $t = 0, 1, \dots, n - 1$,

$$\begin{aligned}
 \mathbb{E}_{\mathbb{P}}(M_{t+1} | \mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau_0^* > t\}} M_{t+1} + \mathbb{1}_{\{\tau_0^* \leq t\}} M_{t+1} | \mathcal{F}_t) \\
 &= \mathbb{1}_{\{\tau_0^* > t\}} \mathbb{E}_{\mathbb{P}}(M_{t+1} | \mathcal{F}_t) + \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau_0^* \leq t\}} M_t | \mathcal{F}_t) \\
 &= \mathbb{1}_{\{\tau_0^* > t\}} M_t + \mathbb{1}_{\{\tau_0^* \leq t\}} M_t = M_t
 \end{aligned}$$

since manifestly $M_{t+1} = Y_{\tau_0^*} = M_t$ on the event $\{\tau_0^* \leq t\} \in \mathcal{F}_t$, and

$$M_t = Y_t = \mathbb{E}_{\mathbb{P}}(Y_{t+1} | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(Y_{t+1}^{\tau_0^*} | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(M_{t+1} | \mathcal{F}_t)$$

on the event $\{\tau_0^* > t\} = \{\tau_0^* \geq t + 1\} \in \mathcal{F}_t$. The second equality above follows from the inclusion $\{\tau_0^* > t\} \subset \{Y_t > Z_t\}$ and the observation that the inequality $Y_t > Z_t$ in (2.80) implies that $Y_t = \mathbb{E}_{\mathbb{P}}(Y_{t+1} | \mathcal{F}_t)$. \square

We are ready to present an alternative proof of Proposition 2.8.2.

Second proof of Proposition 2.8.2. For simplicity, let us focus on the case $t = 0$. From Lemma 2.8.3, the stopped process $Y^{\tau_0^*}$ is a martingale, whereas for any stopping time $\tau \in \mathcal{T}_{[0,T]}$ the stopped process Y^{τ} is a supermartingale (this follows from the fact that Y is a supermartingale). Since $Y \geq Z$, we thus obtain, for any stopping time $\tau \in \mathcal{T}_{[0,T]}$,

$$\mathbb{E}_{\mathbb{P}}(Z_{\tau}) \leq \mathbb{E}_{\mathbb{P}}(Y_{\tau}) = \mathbb{E}_{\mathbb{P}}(Y_{\tau}^{\tau}) \leq \mathbb{E}_{\mathbb{P}}(Y_0) = \mathbb{E}_{\mathbb{P}}(Y_{\tau_0^*}^{\tau_0^*}) = \mathbb{E}_{\mathbb{P}}(Z_{\tau_0^*}),$$

where the last equality follows from the observation that $Y_{\tau_0^*} = Z_{\tau_0^*}$. This proves that τ_0^* is a solution to the optimal stopping problem (A). As we shall see in what follows (cf. Corollary 2.8.1(ii)), the properties that the stopped process $Y^{\tilde{\tau}}$ is a martingale and $Y_{\tilde{\tau}} = Z_{\tilde{\tau}}$ characterize in fact any optimal stopping time $\tilde{\tau}$ for problem (A).

The case of any date t can be dealt with in an analogous way. It suffices to extend slightly Lemma 2.8.3 by checking that, for any fixed t , the process $(Y_u^{\tau_t^*})_{u=t}^n$ is an \mathbb{F} -martingale. The details are left to the reader. \square

We have shown in Proposition 2.8.2 that, for any fixed t , the stopping time τ_t^* is a solution to problem (B). This does not mean, of course, that the uniqueness of an optimal stopping problem holds, in general. We will now address this issue and we will show that τ_t^* is the *minimal solution* to problem (B), in the sense that for any fixed t and any other optimal stopping time $\tilde{\tau}_t$ for problem (B) we have that $\tilde{\tau}_t \geq \tau_t^*$. Let us observe that a stopping time $\tilde{\tau}_t$ is optimal for problem (B) at time t if $\tilde{\tau}_t$ belongs to $\mathcal{T}_{[t,n]}$ and

$$\mathbb{E}_{\mathbb{P}}(Z_{\tilde{\tau}_t} | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(Z_{\tau_t^*} | \mathcal{F}_t).$$

We will first find the maximal solution to the optimal stopping problem. To this end, recall that the value process Y is a supermartingale and denote by $Y = M - A$ its Doob-Meyer decomposition. This means, in particular, that M is a martingale, whereas A is an \mathbb{F} -predictable process, so that A_{t+1} is \mathcal{F}_t -measurable for any $t = 0, 1, \dots, n-1$. As usual, we postulate that $M_0 = Y_0$ and $A_0 = 0$ to ensure the uniqueness of the Doob-Meyer decomposition of Y .

Since M is a martingale, it follows that, for every $t = 0, 1, \dots, n-1$,

$$A_{t+1} - A_t = Y_t - \mathbb{E}_{\mathbb{P}}(Y_{t+1} | \mathcal{F}_t) \geq 0. \quad (2.83)$$

We first consider problem (A). Let the stopping time $\hat{\tau}_0$ be given by the formula (recall that $A_0 = 0$)

$$\hat{\tau}_0 = \min \{t \in \{0, 1, \dots, n-1\} \mid A_{t+1} > 0\},$$

where, by convention, we set $\inf \emptyset = n$, so that $\hat{\tau}_0 = n$ if the set in the right-hand side of the last formula is empty.

Lemma 2.8.4 *The stopping time $\hat{\tau}_0$ is a solution to the optimal stopping problem (A).*

Proof. In view of (2.83), the inequality $A_{t+1} - A_t > 0$ holds if and only if $Y_t > \mathbb{E}_{\mathbb{P}}(Y_{t+1} | \mathcal{F}_t)$, which in turn implies that $Y_t = Z_t$. Hence the equality $Y_{\hat{\tau}_0} = Z_{\hat{\tau}_0}$. It is clear that the stopped process $Y^{\hat{\tau}_0}$ satisfies the equality $Y^{\hat{\tau}_0} = M^{\hat{\tau}_0}$, and thus the supermartingale Y stopped at $\hat{\tau}_0$ is a martingale. This implies that

$$\mathbb{E}_{\mathbb{P}}(Z_{\hat{\tau}_0}) = \mathbb{E}_{\mathbb{P}}(Y_{\hat{\tau}_0}) = \mathbb{E}_{\mathbb{P}}(Y_{\hat{\tau}_0}^{\hat{\tau}_0}) = Y_0,$$

and thus $\hat{\tau}_0$ is an optimal stopping time for problem (A). Note also that $\mathbb{E}_{\mathbb{P}}(M_{\hat{\tau}_0}) = Y_0$. \square

Let us observe that $\tau_0^* \leq \hat{\tau}_0$. Indeed, for every $t = 0, 1, \dots, n-1$, on the event $\{\tau_0^* > t\}$ we have that $Y_t > Z_t$ and thus $Y_t = \mathbb{E}_{\mathbb{P}}(Y_{t+1} | \mathcal{F}_t)$, which manifestly implies that $A_{t+1} = 0$, and thus the inequality $\hat{\tau}_0 > t$ holds on this event. Since $\tau_0^* \leq \hat{\tau}_0$, it is clear that the supermartingale Y stopped at τ_0^* is a martingale.

Lemma 2.8.5 (i) *The stopping time τ_0^* is the minimal solution to problem (A), specifically, if τ is any stopping time such that $\mathbb{P}\{\tau < \tau_0^*\} > 0$ then $Y_0 > \mathbb{E}_{\mathbb{P}}(Z_\tau)$.*
(ii) *The stopping time $\hat{\tau}_0$ is the maximal solution to problem (A), meaning that if τ is any stopping time such that $\mathbb{P}\{\tau > \hat{\tau}_0\} > 0$ then $Y_0 > \mathbb{E}_{\mathbb{P}}(Z_\tau)$.*

Proof. For part (i), we note that on the event $\{\tau < \tau_0^*\}$ we have $Y_\tau > Z_\tau$ (by the definition of τ_0^*). It is thus easy to check that the equalities

$$Y_0 = \mathbb{E}_{\mathbb{P}}(Z_{\tau_0^*}) = \mathbb{E}_{\mathbb{P}}(Z_\tau)$$

would imply that for the stopping time $\tilde{\tau} = \tau \vee \tau_0^*$ we would have $\mathbb{E}_{\mathbb{P}}(Z_{\tilde{\tau}}) > \mathbb{E}_{\mathbb{P}}(Z_{\tau_0^*})$. This would contradict the optimality of τ_0^* .

To prove part (ii), we note that since M is a martingale, for stopping times τ and $\hat{\tau}_0$, we obtain

$$\mathbb{E}_{\mathbb{P}}(M_\tau) = \mathbb{E}_{\mathbb{P}}(M_{\hat{\tau}_0}) = Y_0.$$

However, since $\mathbb{P}\{\tau > \hat{\tau}_0\} > 0$, we obtain

$$\mathbb{E}_{\mathbb{P}}(Z_\tau) \leq \mathbb{E}_{\mathbb{P}}(Y_\tau) < \mathbb{E}_{\mathbb{P}}(M_\tau) = Y_0,$$

since on the event $\{\tau > \hat{\tau}_0\}$ we have $A > 0$, and thus $Y = M - A < M$. Hence τ is not optimal. \square

Let $\mathcal{T}_{[0,n]}^*$ stand for the class of all *optimal stopping times* for problem (A), so that $\tilde{\tau} \in \mathcal{T}_{[0,n]}^*$ whenever

$$\mathbb{E}_{\mathbb{P}}(Z_{\tilde{\tau}}) = \max_{\tau \in \mathcal{T}_{[0,n]}} \mathbb{E}_{\mathbb{P}}(Z_\tau).$$

Note that on the stochastic interval $[\tau_0^*, \hat{\tau}_0]$ we merely have that $Y_t \geq Z_t$. Therefore, it is not true, in general, that any stopping time τ such that $\tau_0^* \leq \tau \leq \hat{\tau}_0$ belongs to the class $\mathcal{T}_{[0,n]}^*$. The following result provides some useful characterizations of the class $\mathcal{T}_{[0,n]}^*$.

Corollary 2.8.1 (i) *A stopping time $\tau \in \mathcal{T}_{[0,n]}$ belongs to $\mathcal{T}_{[0,n]}^*$ whenever $\tau_0^* \leq \tau \leq \hat{\tau}_0$ and $Y_\tau = Z_\tau$.*

(ii) *A stopping time $\tau \in \mathcal{T}_{[0,n]}$ belongs to $\mathcal{T}_{[0,n]}^*$ whenever the stopped process Y^τ is a martingale and $Y_\tau = Z_\tau$.*

Proof. For part (i), it suffices to note that the stopped process Y^τ is a martingale and thus

$$Y_0 = \mathbb{E}_{\mathbb{P}}(Y_\tau) = \mathbb{E}_{\mathbb{P}}(Z_\tau),$$

where the last equality holds if and only if $Y_\tau = Z_\tau$ (recall that $Y \geq Z$).

For part (ii), it suffices to observe that the stopped process Y^τ is always a supermartingale, so that $Y_0 \geq \mathbb{E}_{\mathbb{P}}(Y_\tau)$. If a stopping time τ is optimal then $Y_0 = \mathbb{E}_{\mathbb{P}}(Z_\tau) \leq \mathbb{E}_{\mathbb{P}}(Y_\tau)$ and thus $Y_0 = \mathbb{E}_{\mathbb{P}}(Y_\tau)$. This implies that Y^τ is a martingale. The converse implication is also clear. \square

It is sometimes convenient to focus on the minimal solution only and to formally define the optimal stopping time as the *first* moment when the decision to stop is optimal. This convention would ensure the uniqueness of a solution to optimal stopping problem (A), but it would hide an important observation that it is sometimes possible to hold an American claim after time τ_0^* , and still exercise it later at some optimal time.

The analysis above can be extended without difficulty to the case of any $t = 1, 2, \dots, n$. For any fixed t , we define the stopping time $\hat{\tau}_t \in \mathcal{T}_{[t,n]}$ by setting

$$\hat{\tau}_t = \min \{u \in \{t, t+1, \dots, n-1\} \mid A_{u+1} - A_u > 0\}.$$

For any fixed $t = 0, 1, \dots, n$, the stopping time $\hat{\tau}_t$ is an optimal stopping time for problem (B) and it satisfies the inequality $\hat{\tau}_t \geq \tau_t^*$. For any fixed $t = 0, 1, \dots, n$, we write $\mathcal{T}_{[t,n]}^*$ to denote the class of all optimal stopping times for problem (B). Let us formulate a counterpart of Corollary 2.8.1.

Corollary 2.8.2 *For a fixed $t = 0, 1, \dots, n$, a stopping time τ belongs to $\mathcal{T}_{[t,n]}^*$ whenever $\tau_t^* \leq \tau \leq \hat{\tau}_t$ and $Y_\tau = Z_\tau$.*

For any stopping time $\tau \in \mathcal{T}_{[t,n]}$ that does not belong to $\mathcal{T}_{[t,n]}^*$ the optimality of τ_t^* and $\hat{\tau}_t$ implies immediately that

$$\mathbb{P} \{ \mathbb{E}_{\mathbb{P}}(Z_{\tau_t^*} \mid \mathcal{F}_t) > \mathbb{E}_{\mathbb{P}}(Z_\tau \mid \mathcal{F}_t) \} > 0$$

and

$$\mathbb{P} \{ \mathbb{E}_{\mathbb{P}}(Z_{\hat{\tau}_t} \mid \mathcal{F}_t) > \mathbb{E}_{\mathbb{P}}(Z_\tau \mid \mathcal{F}_t) \} > 0.$$

For a fixed t , let us consider problem (B) under various assumptions about the payoff process Z . The following consequences of results established in this subsection are worth stating.

(i) First, a rather trivial example of non-uniqueness of an optimal stopping time is furnished by a payoff process Z that follows an \mathbb{F} -martingale under \mathbb{P} . In that case, we manifestly have that $\mathcal{T}_{[t,n]}^* = \mathcal{T}_{[t,n]}$, that is, any stopping time is an optimal solution to problem (B). In particular, we have that $\tau_t^* = t$ and $\hat{\tau}_t = n$ for every $t = 0, 1, \dots, n$.

(ii) Second, if a payoff process Z is a strict submartingale, that is,

$$Z_{t-1} < \mathbb{E}_{\mathbb{P}}(Z_t \mid \mathcal{F}_{t-1}), \quad \forall t = 0, 1, \dots, n,$$

then $\tau_t^* = \hat{\tau}_t = n$, so that $\mathcal{T}_{[t,n]}^* = \{n\}$. If Z is a strict supermartingale, so that

$$Z_{t-1} > \mathbb{E}_{\mathbb{P}}(Z_t \mid \mathcal{F}_{t-1}), \quad \forall t = 0, 1, \dots, n,$$

we have that $\tau_t^* = \hat{\tau}_t = t$ and thus $\mathcal{T}_{[t,n]}^* = \{t\}$.

(iii) Finally, if payoff process Z is a (not necessarily strict) submartingale then the stopping time $\hat{\tau}_t = n$ is optimal. If Z is a (not necessarily strict) supermartingale then the stopping time $\tau_t^* = t$ is optimal.

2.8.2 Valuation and Hedging of American Claims

We assume that the market model $\mathcal{M} = (S, \Phi)$ is arbitrage-free and complete. Let \mathbb{P}^* be the unique martingale measure for the process $S^* = S/B$. We will first consider the concept of hedging of an American claim from the seller's perspective.

Definition 2.8.7 We say that a self-financing trading strategy $\phi \in \Phi$ is the (seller's) *super-hedging strategy* for an American claim X^a expiring at T whenever the inequality $V_t(\phi) \geq X_t$ holds for every $t = 0, 1, \dots, T$. In other words, $\phi \in \Phi$ *super-hedges* X^a for the seller if the wealth process $V(\phi)$ dominates the payoff process X of an American claim X^a .

Of course, there is no reason to expect that a super-hedging strategy for X^a is unique. Hence it is natural to define the seller's price of X^a through minimization of the initial cost of super-hedging.

Definition 2.8.8 The *seller's price* of an American claim X^a is the minimal cost of a super-hedging strategy for X^a . Let $\Phi(X^a)$ denote the class of all super-hedging strategies for X^a . Similarly, let $\Phi_t(X^a)$ be the class of all super-hedging strategies for X^a from time t onwards. Then the seller's price of X^a equals, for $t = 0$,

$$\pi_0^s(X^a) = \inf \{ V_0(\phi) \mid \phi \in \Phi(X^a) \}$$

and, for every $t = 1, 2, \dots, T$,

$$\pi_t^s(X^a) = \inf \{ V_t(\phi) \mid \phi \in \Phi_t(X^a) \}.$$

Let us define the process U by setting $U_T = X_T$ and, for every $t = 1, 2, \dots, T$,

$$U_{t-1} = \max \left\{ X_{t-1}, B_{t-1} \mathbb{E}_{\mathbb{P}^*}(U_t B_t^{-1} \mid \mathcal{F}_{t-1}) \right\}.$$

Also, let $U^* = B^{-1}U$ and $X^* = B^{-1}X$ stand for the relative processes. Then, clearly, $U_T^* = X_T^*$ and, for every $t = 1, 2, \dots, T$,

$$U_{t-1}^* = \max \left\{ X_{t-1}^*, \mathbb{E}_{\mathbb{P}^*}(U_t^* \mid \mathcal{F}_{t-1}) \right\}. \quad (2.84)$$

It follows immediately from Lemma 2.8.2 that the process U^* is the Snell envelope of the discounted payoff process X^* . Furthermore, from Proposition 2.8.2, we obtain

$$U_t^* = \max_{\tau \in \mathcal{T}_{[t, T]}} \mathbb{E}_{\mathbb{P}^*}(X_\tau^* \mid \mathcal{F}_t)$$

or, equivalently,

$$U_t = \max_{\tau \in \mathcal{T}_{[t, T]}} B_t \mathbb{E}_{\mathbb{P}^*}(B_\tau^{-1} X_\tau \mid \mathcal{F}_t). \quad (2.85)$$

Proposition 2.8.3 For any fixed $t = 0, 1, \dots, T$, there exists a trading strategy $\phi^* \in \Phi_t(X^a)$ such that $V_t(\phi^*) = U_t$. Hence $\pi_t^s(X^a) \leq U_t$ for every $t = 0, 1, \dots, T$. Moreover, for any super-hedging strategy $\phi \in \Phi_t(X^a)$ for X^a we have that $V_t(\phi) = U_t$ and thus $\pi_t^s(X^a) \geq U_t$ for every $t = 0, 1, \dots, T$. Consequently, the seller's price $\pi^s(X^a)$ coincides with the process U .

Proof. Let us first examine the case $t = 0$. Let $U^* = M - A$ be the Doob-Meyer decomposition of the (bounded) supermartingale U^* . Here, M is a martingale with $M_0 = U_0^*$ and A an \mathbb{F} -predictable, increasing process with $A_0 = 0$. In particular, A_t is \mathcal{F}_{t-1} -measurable (recall that, by convention, $\mathcal{F}_{-1} = \mathcal{F}_0$) and, for every $t = 1, 2, \dots, T$,

$$A_t - A_{t-1} = U_{t-1}^* - \mathbb{E}_{\mathbb{P}^*}(U_t^* | \mathcal{F}_{t-1}).$$

Let us set $Z = M_T B_T$ and let us consider a replicating strategy $\phi^* \in \Phi$ for Z . From Lemma 2.6.2, the relative wealth process $V^*(\phi^*)$ is a martingale. Hence

$$V_t^*(\phi^*) = \mathbb{E}_{\mathbb{P}^*}(V_T^*(\phi^*) | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}(V_T(\phi^*) B_T^{-1} | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}(M_T | \mathcal{F}_t) = M_t.$$

Consequently, for every $t = 0, 1, \dots, T$,

$$V_t^*(\phi^*) = M_t = U_t^* + A_t^* \geq U_t^* \geq X_t^*.$$

We conclude that the wealth process of ϕ^* satisfies, for every $t = 0, 1, \dots, T$,

$$V_t(\phi^*) \geq U_t \geq X_t$$

so that it super-hedges an American claim X^a . Also, $V_0^*(\phi^*) = M_0 = U_0^* + A_0 = U_0^*$ and thus $V_0(\phi^*) = U_0$. This shows that $U_0 \geq \pi_0^s(X^a)$.

To establish the desired inequality $U_t \geq \pi_t^s(X^a)$ for $t = 1, 2, \dots, T$, it is enough to consider the Doob-Meyer decomposition of the supermartingale U^* restricted to the time set $\{t, t+1, \dots, T\}$ and to introduce the corresponding trading strategy ϕ^* (note that the strategy ϕ^* will depend here on the initial date t).

Finally, since U^* is the smallest supermartingale dominating X^* and the relative wealth process of any super-hedging strategy ϕ for X^a is a martingale dominating X^* , we see that the inequality $V^*(\phi) \geq U^*$ for any super-hedging strategy ϕ for X^a . Since U^* is given by the recurrence relationship (2.84), this argument can be applied to any start date t of a super-hedging strategy for X^a . We thus conclude that $\pi_t^s(X^a) \geq U_t$ for every $t = 0, 1, \dots, T$. \square

The next goal is to examine an American claim from the perspective of the buyer. We shall argue that, for any fixed $t = 0, 1, \dots, T$, any stopping time that solves the optimization problem

$$\max_{\tau \in \mathcal{T}_{[t, T]}} B_t \mathbb{E}_{\mathbb{P}^*}(B_\tau^{-1} X_\tau | \mathcal{F}_t)$$

is the *rational exercise time* for the buyer of an American claim, in the sense that if the buyer purchases the claim at time t for $\pi_0^s(X^a)$ and fails to exercise it at some stopping time from the class $\mathcal{T}_{[t, T]}^*$ of optimal stopping times, then an arbitrage opportunity for the seller of the claim will arise.

Let the stopping time τ_t^* be given by

$$\tau_t^* = \min \{u \in \{t, t+1, \dots, T\} | U_u = X_u\}$$

and let the stopping time $\hat{\tau}_t$ satisfy

$$\hat{\tau}_t = \min \{u \in \{t, t+1, \dots, T-1\} \mid A_{u+1} - A_u > 0\}$$

where, by convention, $\min \emptyset = T$.

We know already that τ_t^* and $\hat{\tau}_t$ are optimal stopping times for the problem: for a fixed $t = 0, 1, \dots, n$, maximize the conditional expectation

$$B_t \mathbb{E} \mathbb{P}^*(B_\tau^{-1} X_\tau \mid \mathcal{F}_t).$$

Moreover, a stopping time τ such that $\tau_t^* \leq \tau \leq \hat{\tau}_t$ is optimal if and only if $U_\tau = X_\tau$. Another characterization of an optimal stopping time reads: a stopping time τ is optimal if and only if the process U^* stopped at τ is a martingale and $U_\tau = X_\tau$.

Hereafter, we will refer to any optimal stopping time as a *rational exercise time* for the buyer of an American claim and we will focus on the case $t = 0$.

Proposition 2.8.4 *Assume that an American claim X^a was sold at time 0 at the initial price $\pi_0^s(X^a)$. Then there exists an arbitrage opportunity for the seller of the claim if and only if the holder does not exercise the claim at some rational exercise time.*

Proof. Suppose first that the claim is exercised by its holder at some rational exercise time τ . Then the wealth process $V(\phi^*) = U$ of the optimal superhedging strategy ϕ^* for the seller matches exactly the payoff process X at time τ .

If, on the contrary, the claim is exercised some stopping time τ , which is not a rational exercise time then either of the following three cases is valid:

- (a) $\tau_0^* \leq \tau \leq \hat{\tau}_0$ and $\mathbb{P}\{U_\tau > X_\tau\} > 0$ (equivalently, $\mathbb{P}^*\{U_\tau > X_\tau\} > 0$),
- (b) $\mathbb{P}\{\tau < \tau_0^*\} > 0$ and, by the definition of τ_0^* , we have $U_\tau > X_\tau$ on the event $\{\tau < \tau_0^*\}$,
- (c) $\mathbb{P}\{\tau > \hat{\tau}_0\} > 0$ and, by the definition of $\hat{\tau}_0$, we have, on the event $\{\tau > \hat{\tau}_0\}$,

$$V_\tau^*(\phi^*) = M_\tau > U_\tau^* \geq X_\tau^*,$$

so that $U_\tau > X_\tau$ on this event.

Recall that $Y \geq X$. Hence in either case the strategy ϕ^* is an arbitrage opportunity for the seller.

□

Let us finally observe that no arbitrage opportunity exists for the buyer who purchases at time 0 an American claim at the seller's price $\pi_0^s(X^a)$. This is a consequence of the following lemma.

Lemma 2.8.6 *Suppose that a trading strategy $\phi \in \Phi$ is such that $V_0(\phi) = -\pi_0^s(X^a)$ and for some stopping time $\tau \in \mathcal{T}_{[0, T]}$ we have $V_\tau(\phi) + X_\tau \geq 0$. Then $V_\tau(\phi) + X_\tau = 0$.*

Proof. Assume, on the contrary, that the inequality $\mathbb{P}\{V_\tau(\phi) + X_\tau > 0\} > 0$ is valid or, equivalently, $\mathbb{P}^*\{V_\tau^*(\phi) + X_\tau^* > 0\} > 0$. Then the martingale property of the relative wealth $V^*(\phi)$ under \mathbb{P}^* yields

$$\mathbb{E}_{\mathbb{P}^*}(V_\tau^*(\phi) + X_\tau^*) = -\pi_0^s(X^a)B_0^{-1} + \mathbb{E}_{\mathbb{P}^*}(X_\tau^*) > 0.$$

Consequently,

$$B_0 \mathbb{E}_{\mathbb{P}^*}(B_\tau^{-1} X_\tau) > \pi_0^s(X^a) = U_0.$$

This clearly contradicts (2.85). \square

However, if either the seller is able to sell an American claim at the price higher than $\pi_0^s(X^a)$, or the buyer is able to buy at the price lower than $\pi_0^s(X^a)$, then an arbitrage opportunity arises. Hence the following result, which establishes also Proposition 2.8.1.

Corollary 2.8.3 *The unique arbitrage price of an American claim X^a is given by the following expression, for every $t = 0, 1, \dots, T$,*

$$\pi_t(X^a) = \max_{\tau \in \mathcal{T}_{[t, T]}} B_t \mathbb{E}_{\mathbb{P}^*}(B_\tau^{-1} X_\tau | \mathcal{F}_t) = B_t \mathbb{E}_{\mathbb{P}^*}(B_{\tilde{\tau}_t}^{-1} X_{\tilde{\tau}_t} | \mathcal{F}_t),$$

where $\tilde{\tau}_t$ is any stopping time from $\mathcal{T}_{[t, T]}^*$. Equivalently, the relative price $\pi^*(X^a) = B^{-1}\pi(X^a)$ satisfies $\pi_T^*(X^a) = X_T^*$ and, for every $t = 1, 2, \dots, T$,

$$\pi_{t-1}^*(X^a) = \max \{X_{t-1}^*, \mathbb{E}_{\mathbb{P}^*}(\pi_t^*(X^a) | \mathcal{F}_{t-1})\}. \quad (2.86)$$

The last result can also be supported by introducing the concept of super-hedging strategy for the buyer (this proves useful when dealing with American claims in an incomplete market model).

Definition 2.8.9 We say that a self-financing trading strategy $\phi \in \Phi$ is a *super-hedging strategy for the buyer* of an American claim X^a whenever there exists a stopping time $\tau \in \mathcal{T}_{[0, T]}$ such that the inequality $V_\tau(\phi) \geq -X_\tau$ holds.

The definition above can be easily extended to any date $t = 0, 1, \dots, T$.

Definition 2.8.10 For a fixed $t = 0, 1, \dots, T$, let $V_t(\phi^*)$ be the minimal cost of a super-hedging strategy for the buyer from time t onwards. The *buyer's price* at time t of an American claim X^a is defined as $\pi_t^b(X^a) = -V_t(\phi^*)$.

The proof of the following result is left to the reader.

Corollary 2.8.4 *For an American claim X^a the equality $\pi_t^s(X^a) = \pi_t^b(X^a)$ holds for every $t = 0, 1, \dots, T$. Hence $\pi(X^a) = \pi^s(X^a) = \pi^b(X^a)$.*

It is worth recalling that we work in this section under the standing assumption that the underlying finite market model is arbitrage-free and complete. In the case of a market incompleteness, the seller's price is typically strictly greater than the buyer's price and thus the bid-ask spread arises.

2.8.3 American Call and Put

Let us fix some $l < k$, and let us consider an *American call option* with the payoff $X_t = (S_t^l - K)^+$ for every $t = 0, 1, \dots, T$. Assume that the process $B = S^k$ is increasing, so that $B_{t+1} \geq B_t$ for $t = 0, 1, \dots, T-1$. When B represents the savings account, this corresponds to the assumption that interest rates are non-negative.

Using the conditional Jensen's inequality (see Lemma A.1.3), one can show that the discounted payoff

$$X_t^* = B_t^{-1} X_t = B_t^{-1} (S_t^l - K)^+ = (S_t^{l*} - B_t^{-1} K)^+$$

is a submartingale under \mathbb{P}^* and thus $\hat{\tau}_0 = T$ is a rational exercise time (not unique, in general) for an American call. Hence an American call option is equivalent to a European call with expiry date T , in the sense that $\pi_t(X^a) = C_t$, where C is the price process of a European call with terminal payoff $C_T = (S_T^l - K)^+$.

This property is not shared by an *American put option* with the payoff $X_t = (K - S_t^l)^+$ for every $t = 0, 1, \dots, T$, unless the process $B = S^k$ is decreasing, that is, unless the inequality $B_{t+1} \leq B_t$ holds for every $t = 0, 1, \dots, T-1$. In that case, the discounted payoff

$$X_t^* = B_t^{-1} X_t = B_t^{-1} (K - S_t^l)^+ = (B_t^{-1} K - S_t^{l*})^+$$

is a submartingale under \mathbb{P}^* and thus American and European put options are equivalent, in the sense described above.

Note that if the process B is intended to model the savings account, then the assumption that B is decreasing is equivalent to the property that (possibly random) interest rates are non-positive.

2.9 Game Contingent Claims

The concept of a game contingent claim was introduced by Kiefer (2000), as a natural extension of the notion of an American contingent claim to a situation where both parties to a contract may exercise it prior to its maturity date. The payoff of a game contingent claim depends not only on the moment when it is exercised, but also on which party takes the decision to exercise. In order to make a clear distinction between the exercise policies of the two parties, we will refer to the decision of the seller (also termed the issuer) as the *cancellation policy*, whereas the decision of the buyer (also referred to as the holder) is called the *exercise policy*.

Definition 2.9.1 A *game contingent claim* $X^g = (L, H, T^e, T^c)$ expiring at time T consists of \mathbb{F} -adapted payoff processes L and H , a set T^e admissible exercise times, and a set T^c of admissible cancellation times.

Remarks. A game contingent claim is also known in the literature as a *game option* or an *Israeli option*. In practical applications, the terminology for exercise times is

frequently modified to reflect the feature of a particular class of derivatives that fall within the general scope of game contingent claims. For instance, in the case of a *convertible bond*, the cancelation and exercise times are referred to as the *call time* and the *put/conversion time*, respectively (see Example 2.9.1).

It is assumed throughout that $L \leq H$, meaning that the \mathcal{F}_t -measurable random variables L_t and H_t satisfy the inequality $L_t \leq H_t$ for every $t = 0, 1, \dots, T$.

Unless explicitly otherwise stated, the sets of admissible exercise and cancelation times are restricted to the class $\mathcal{T}_{[0,T]}$ of all stopping times of the filtration \mathbb{F}^S with values in $\{0, 1, \dots, T\}$ – that is, it is postulated that $\mathcal{T}^e = \mathcal{T}^c = \mathcal{T}_{[0,T]}$.

We interpret L_t as the payoff received by the holder upon exercising a game contingent claim at time t . The random variable H_t represents the payoff received by the holder if a claim is canceled (i.e., exercised by its issuer) at time t . More formally, if a game contingent claim is exercised by either of the two parties at some date $t \leq T$, that is, on the event $\{\tau = t\} \cup \{\sigma = t\}$, the random payoff equals

$$X_t = \mathbb{1}_{\{\tau=t\leq\sigma\}} L_t + \mathbb{1}_{\{\sigma=t<\tau\}} H_t, \quad (2.87)$$

where τ and σ are the exercise and cancelation times, respectively. Note that both the holder and the issuer may choose freely their exercise and cancelation times. If they decide to exercise their right at the same moment, we adopt the convention that a game contingent claim is exercised, rather than canceled, so that the payoff is given by the process L . This feature is already reflected in the payoff formula (2.87).

Remarks. A comparison of Definitions 2.8.3 and 2.9.1 shows that if the class of all admissible cancelation times is assumed to be $\mathcal{T}^c = \{T\}$ then a game contingent claim X^g becomes an American claim X^a with the payoff process $X = L$. Therefore, Definition 2.9.1 of a game contingent claim covers as a special case the notion of an American contingent claim. If we postulate, in addition, that $\mathcal{T}^e = \{T\}$ then a game contingent claim reduces to a European claim L_T maturing at time T .

From the previous section, we know that valuation and hedging of American claims is related to optimal stopping problems. Game contingent claims are associated with the so-called *Dynkin games*, which in turn can be seen as natural extensions of optimal stopping problems. Therefore, before analyzing in some detail the game contingent claims, we first present a brief survey of basic results concerning Dynkin games.

2.9.1 Dynkin Games

In this section, we assume that $t = 0, 1, \dots, n$ and we investigate the *Dynkin game* (also known as the *optimal stopping game*) associated with the payoff

$$Z(\sigma, \tau) = \mathbb{1}_{\{\tau \leq \sigma\}} L_\tau + \mathbb{1}_{\{\sigma < \tau\}} H_\sigma,$$

where $L \leq H$ are \mathbb{F} -adapted stochastic processes defined on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration \mathbb{F} . Stochastic games, as described by the

foregoing definition, were first studied in the classic papers by Dynkin (1969) and Kiefer (1971).

Definition 2.9.2 For any fixed date $t = 0, 1, \dots, n$, by the *Dynkin game* started at time t and associated with the payoff $Z(\sigma, \tau)$, we mean a stochastic game in which the *min-player*, who controls a stopping time $\sigma \in \mathcal{T}_{[t,n]}$, wishes to minimize the conditional expectation

$$\mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau) \mid \mathcal{F}_t), \quad (2.88)$$

while the *max-player*, who controls a stopping time $\tau \in \mathcal{T}_{[t,n]}$, wishes to maximize the conditional expectation (2.88).

Let us fix some date t . Since the stopping times σ and τ are assumed to belong to the class $\mathcal{T}_{[t,n]}$, formula (2.87) yields

$$\mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau) \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\sum_{u=t}^n (\mathbb{1}_{\{\tau=u \leq \sigma\}} L_u + \mathbb{1}_{\{\sigma=u < \tau\}} H_u) \mid \mathcal{F}_t\right). \quad (2.89)$$

We are interested in finding the so-called *value process* of a Dynkin game and the corresponding *optimal stopping times*. We start by formulating the definition of the *upper* and *lower* value processes.

Definition 2.9.3 The \mathbb{F} -adapted process \bar{Y}^u given by the formula

$$\bar{Y}_t^u = \min_{\sigma \in \mathcal{T}_{[t,n]}} \max_{\tau \in \mathcal{T}_{[t,n]}} \mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau) \mid \mathcal{F}_t)$$

is called the *upper value process*. The *lower value process* \bar{Y}^l is an \mathbb{F} -adapted process given by the formula

$$\bar{Y}_t^l = \max_{\tau \in \mathcal{T}_{[t,n]}} \min_{\sigma \in \mathcal{T}_{[t,n]}} \mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau) \mid \mathcal{F}_t).$$

Lemma 2.9.1 Let $L^0(\Omega, \mathcal{F}, \mathbb{P})$ stand for the class of all random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let A and B be two finite sets and let $g : A \times B \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$ be an arbitrary map. Then

$$\min_{a \in A} \max_{b \in B} g(a, b) \geq \max_{b \in B} \min_{a \in A} g(a, b).$$

Proof. It is clear that, for any $a_0 \in A$ and $b_0 \in B$,

$$G(a_0) := \max_{b \in B} g(a_0, b) \geq g(a_0, b_0) \geq \min_{a \in A} g(a, b_0) =: H(b_0),$$

and thus $\min_{a_0 \in A} G(a_0) \geq H(b_0)$ for every $b_0 \in B$. Consequently,

$$\min_{a \in A} \max_{b \in B} g(a, b) = \min_{a_0 \in A} G(a_0) \geq \max_{b_0 \in B} H(b_0) = \max_{b \in B} \min_{a \in A} g(a, b),$$

and thus we have established the required result. \square

It follows immediately from Definition 2.9.3 and Lemma 2.9.1 that the upper value process \bar{Y}^u always dominates the lower value process \bar{Y}^l , in the sense that the inequality $\bar{Y}_t^u \geq \bar{Y}_t^l$ is valid for every $t = 0, 1, \dots, n$.

Definition 2.9.4 If the equality $\bar{Y}^u = \bar{Y}^l$ is satisfied, we say that the *Stackelberg equilibrium* holds for a Dynkin game. Then the process $\bar{Y} = \bar{Y}^u = \bar{Y}^l$ is called the *value process*.

The next definition is based on the concept of a *saddle point* of a Dynkin game.

Definition 2.9.5 We say that the *Nash equilibrium* holds for a Dynkin game if for any t there exist stopping times $\sigma_t^*, \tau_t^* \in \mathcal{T}_{[t,n]}$, such that the inequalities

$$\mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tau) | \mathcal{F}_t) \leq \mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tau_t^*) | \mathcal{F}_t) \leq \mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau_t^*) | \mathcal{F}_t) \quad (2.90)$$

are satisfied for arbitrary stopping times $\tau, \sigma \in \mathcal{T}_{[t,n]}$, that is, the pair (σ_t^*, τ_t^*) is a *saddle point* of a Dynkin game.

The next result shows that the Nash equilibrium for a Dynkin game implies the Stackelberg equilibrium.

Lemma 2.9.2 Assume that the Nash equilibrium holds. Then the Stackelberg equilibrium holds and

$$\bar{Y}_t = \mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tau_t^*) | \mathcal{F}_t),$$

so that σ_t^* and τ_t^* are optimal stopping times as of time t .

Proof. From (2.90), we obtain

$$\max_{\tau \in \mathcal{T}_{[t,n]}} \mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tau) | \mathcal{F}_t) \leq \mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tau_t^*) | \mathcal{F}_t) \leq \min_{\sigma \in \mathcal{T}_{[t,n]}} \mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau_t^*) | \mathcal{F}_t).$$

Consequently,

$$\begin{aligned} \bar{Y}_t^u &= \min_{\sigma \in \mathcal{T}_{[t,n]}} \max_{\tau \in \mathcal{T}_{[t,n]}} \mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau) | \mathcal{F}_t) \leq \mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tau_t^*) | \mathcal{F}_t) \\ &\leq \max_{\tau \in \mathcal{T}_{[t,n]}} \min_{\sigma \in \mathcal{T}_{[t,n]}} \mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau) | \mathcal{F}_t) = \bar{Y}_t^l. \end{aligned}$$

Since the inequality $\bar{Y}_t^u \geq \bar{Y}_t^l$ is known to be always satisfied, we conclude that the value process \bar{Y} is well defined and satisfies $\bar{Y}_t = \mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tau_t^*) | \mathcal{F}_t)$ for any $t = 0, 1, \dots, n$. \square

The following definition introduces a plausible candidate for the value process of a Dynkin game.

Definition 2.9.6 The process Y is defined by setting $Y_n = L_n$ and, for any $t = 0, 1, \dots, n-1$,

$$Y_t = \min \left\{ H_t, \max \left\{ L_t, \mathbb{E}_{\mathbb{P}}(Y_{t+1} | \mathcal{F}_t) \right\} \right\}. \quad (2.91)$$

Remarks. The assumption that $L \leq H$ ensures that, for any $t = 0, 1, \dots, n-1$,

$$\begin{aligned} Y_t &= \min \left\{ H_t, \max \left\{ L_t, \mathbb{E}_{\mathbb{P}}(Y_{t+1} \mid \mathcal{F}_t) \right\} \right\} \\ &= \max \left\{ L_t, \min \left\{ H_t, \mathbb{E}_{\mathbb{P}}(Y_{t+1} \mid \mathcal{F}_t) \right\} \right\}. \end{aligned}$$

It is also clear from (2.91) that $L_t \leq Y_t \leq H_t$ for $t = 0, 1, \dots, n$. In particular, $Y_t = L_t = H_t$ if the equality $L_t = H_t$ holds.

Lemma 2.9.3 *Let us set, for any fixed $t = 0, 1, \dots, n$,*

$$\sigma_t^* = \min \{u \in \{t, t+1, \dots, n\} \mid Y_u = H_u\} \wedge n \quad (2.92)$$

and

$$\tau_t^* = \min \{u \in \{t, t+1, \dots, n\} \mid Y_u = L_u\}. \quad (2.93)$$

Then the following representations are valid

$$\sigma_{t-1}^* = (t-1)\mathbb{1}_{\{\sigma_{t-1}^* = t-1\}} + \sigma_t^*\mathbb{1}_{\{\sigma_{t-1}^* \geq t\}}$$

and

$$\tau_{t-1}^* = (t-1)\mathbb{1}_{\{\tau_{t-1}^* = t-1\}} + \tau_t^*\mathbb{1}_{\{\tau_{t-1}^* \geq t\}}.$$

Proof. We have

$$\begin{aligned} \sigma_{t-1}^* &= \min \{u \in \{t-1, t, \dots, n\} \mid Y_u = H_u\} \wedge n \\ &= (t-1)\mathbb{1}_{\{Y_{t-1} = H_{t-1}\}} + (\min \{u \geq t \mid Y_u = H_u\} \wedge n)\mathbb{1}_{\{Y_{t-1} < H_{t-1}\}} \\ &= (t-1)\mathbb{1}_{\{\sigma_{t-1}^* = t-1\}} + \sigma_t^*\mathbb{1}_{\{\sigma_{t-1}^* \geq t\}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \tau_{t-1}^* &= \min \{u \in \{t-1, t, \dots, n\} \mid Y_u = L_u\} \\ &= (t-1)\mathbb{1}_{\{Y_{t-1} = L_{t-1}\}} + \min \{u \geq t \mid Y_u = L_u\}\mathbb{1}_{\{Y_{t-1} > L_{t-1}\}} \\ &= (t-1)\mathbb{1}_{\{\tau_{t-1}^* = t-1\}} + \tau_t^*\mathbb{1}_{\{\tau_{t-1}^* \geq t\}}. \end{aligned}$$

This ends the proof. \square

The next result examines some auxiliary properties of the process Y .

Lemma 2.9.4 *For every $t = 0, 1, \dots, n-1$, we have that $Y_t \leq \mathbb{E}_{\mathbb{P}}(Y_{t+1} \mid \mathcal{F}_t)$ on the event $\{\tau_t^* > t\}$ and $Y_t \geq \mathbb{E}_{\mathbb{P}}(Y_{t+1} \mid \mathcal{F}_t)$ on the event $\{\sigma_t^* > t\}$. Therefore, the equality $Y_t = \mathbb{E}_{\mathbb{P}}(Y_{t+1} \mid \mathcal{F}_t)$ holds on the event $\{\sigma_t^* \wedge \tau_t^* > t\} = \{\sigma_t^* > t\} \cap \{\tau_t^* > t\}$.*

Proof. Let us fix t and let us write $a = L_t$, $b = H_t$ and $c = \mathbb{E}_{\mathbb{P}}(Y_{t+1} \mid \mathcal{F}_t)$. We wish to examine the relationship between the value of c and $h(c) = \min(b, \max(a, c))$ for arbitrary a, b such that $a \leq b$. It is clear that $Y_t > L_t$ on the event $\{\tau_t^* > t\}$. This corresponds to the inequality $h(c) > a$. But this in turn implies that $h(c) \leq c$, which means that the inequality $Y_t \leq \mathbb{E}_{\mathbb{P}}(Y_{t+1} \mid \mathcal{F}_t)$ holds. Similarly, the inequality $Y_t < H_t$ is easily seen to hold on the event $\{\sigma_t^* > t\}$. This corresponds to $h(c) < b$, which entails that $h(c) \geq c$. We have thus established that on the event $\{\sigma_t^* > t\}$ we have $Y_t \geq \mathbb{E}_{\mathbb{P}}(Y_{t+1} \mid \mathcal{F}_t)$. \square

By arguing as in the proof of Lemma 2.8.3, one can also establish the following result.

Lemma 2.9.5 *The stopped process $Y^{\sigma_0^* \wedge \tau_0^*}$ is an \mathbb{F} -martingale.*

We are in a position to show that the process Y is equal to the value process \bar{Y} of a Dynkin game.

Theorem 2.9.1 (i) *Let the stopping times σ_t^*, τ_t^* be given by (2.92)–(2.93). Then we have, for arbitrary stopping times $\tau, \sigma \in \mathcal{T}_{[t,n]}$,*

$$\mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tau) | \mathcal{F}_t) \leq Y_t \leq \mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau_t^*) | \mathcal{F}_t) \quad (2.94)$$

and thus also

$$\mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tau) | \mathcal{F}_t) \leq \mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tau_t^*) | \mathcal{F}_t) \leq \mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau_t^*) | \mathcal{F}_t) \quad (2.95)$$

so that the Nash equilibrium holds.

(ii) *The process Y is the value process of a Dynkin game, that is, for every $t = 0, 1, \dots, n$,*

$$\begin{aligned} Y_t &= \min_{\sigma \in \mathcal{T}_{[t,n]}} \max_{\tau \in \mathcal{T}_{[t,n]}} \mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau) | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tau_t^*) | \mathcal{F}_t) \\ &= \max_{\tau \in \mathcal{T}_{[t,n]}} \min_{\sigma \in \mathcal{T}_{[t,n]}} \mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau) | \mathcal{F}_t) = \bar{Y}_t, \end{aligned} \quad (2.96)$$

and thus the stopping times σ_t^* and τ_t^* are optimal as of time t .

Proof. We will first show that the inequality holds for $t = n$. Of course, if $\tau, \sigma \in \mathcal{T}_{[t,n]}$ then $\tau = \sigma = n$. As

$$\sigma_n^* = \min \{u \geq n \mid Y_n = H_n\} \wedge n = n = \min \{u \geq n \mid Y_n = L_n\} = \tau_n^*,$$

it follows from Definition 2.9.6 that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(Z(\sigma_n^*, n) | \mathcal{F}_n) &= \mathbb{E}_{\mathbb{P}}(Z(n, \tau_n^*) | \mathcal{F}_n) = \mathbb{E}_{\mathbb{P}}(Z(n, n) | \mathcal{F}_n) \\ &= \mathbb{E}_{\mathbb{P}}(L_n | \mathcal{F}_n) = L_n = Y_n. \end{aligned}$$

Evidently, for $t = n$, the upper and lower bounds hold as equalities in (2.94).

We now assume that (2.94) holds for some t , so that

$$\mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tilde{\tau}) | \mathcal{F}_t) \leq Y_t \leq \mathbb{E}_{\mathbb{P}}(Z(\tilde{\sigma}, \tau_t^*) | \mathcal{F}_t) \quad (2.97)$$

for arbitrary stopping times $\tilde{\tau}, \tilde{\sigma} \in \mathcal{T}_{[t,n]}$. We wish to prove that (2.94) holds for $t - 1$, that is, for arbitrary stopping times $\tau, \sigma \in \mathcal{T}_{[t-1,n]}$,

$$\mathbb{E}_{\mathbb{P}}(Z(\sigma_{t-1}^*, \tau) | \mathcal{F}_{t-1}) \leq Y_{t-1} \leq \mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau_{t-1}^*) | \mathcal{F}_{t-1}). \quad (2.98)$$

Let us first establish the upper bound in (2.98). For any $\sigma \in \mathcal{T}_{[t-1,n]}$, let us define $\tilde{\sigma} = \sigma \vee t = \max(\sigma, t)$. It is clear that the stopping time $\tilde{\sigma}$ belongs to $\mathcal{T}_{[t,n]}$. We thus have that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(Z(\sigma, \tau_{t-1}^*) | \mathcal{F}_{t-1}) &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\sigma=t-1\}} Z(t-1, \tau_{t-1}^*) | \mathcal{F}_{t-1}) \\
&\quad + \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\sigma \geq t\}} Z(\tilde{\sigma}, \tau_{t-1}^*) | \mathcal{F}_{t-1}) \\
&= \mathbb{1}_{\{\sigma=t-1\}} \mathbb{E}_{\mathbb{P}}(Z(t-1, \tau_{t-1}^*) | \mathcal{F}_{t-1}) + \mathbb{1}_{\{\sigma \geq t\}} \mathbb{E}_{\mathbb{P}}(Z(\tilde{\sigma}, \tau_{t-1}^*) | \mathcal{F}_{t-1}) \\
&= \mathbb{1}_{\{\sigma=t-1\}} \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau_{t-1}^*=t-1\}} L_{t-1} + \mathbb{1}_{\{\tau_{t-1}^* \geq t\}} H_{t-1} | \mathcal{F}_{t-1}) \\
&\quad + \mathbb{1}_{\{\sigma \geq t\}} \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau_{t-1}^*=t-1\}} Z(\tilde{\sigma}, t-1) + \mathbb{1}_{\{\tau_{t-1}^* \geq t\}} Z(\tilde{\sigma}, \tau_t^*) | \mathcal{F}_{t-1}) \\
&= \mathbb{1}_{\{\sigma=t-1=\tau_{t-1}^*\}} L_{t-1} + \mathbb{1}_{\{\sigma=t-1 < \tau_{t-1}^*\}} H_{t-1} \\
&\quad + \mathbb{1}_{\{\tau_{t-1}^*=t-1 < \sigma\}} L_{t-1} + \mathbb{1}_{\{\tau_{t-1}^* \geq t, \sigma \geq t\}} \mathbb{E}_{\mathbb{P}}(Z(\tilde{\sigma}, \tau_t^*) | \mathcal{F}_{t-1}) \\
&= \mathbb{1}_{\{\sigma=t-1=\tau_{t-1}^*\}} L_{t-1} + \mathbb{1}_{\{\sigma=t-1 < \tau_{t-1}^*\}} H_{t-1} + \mathbb{1}_{\{\tau_{t-1}^*=t-1 < \sigma\}} L_{t-1} \\
&\quad + \mathbb{1}_{\{\tau_{t-1}^* \geq t, \sigma \geq t\}} \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(Z(\tilde{\sigma}, \tau_t^*) | \mathcal{F}_t) | \mathcal{F}_{t-1}) \\
&\geq \mathbb{1}_{\{\tau_{t-1}^*=t-1\}} L_t + \mathbb{1}_{\{\sigma=t-1 < \tau_{t-1}^*\}} H_{t-1} + \mathbb{1}_{\{\tau_{t-1}^* \geq t, \sigma \geq t\}} \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1}) \\
&\geq Y_{t-1}
\end{aligned}$$

where we have used the right-hand side of inequality (2.97) to establish the penultimate inequality. Finally, the last inequality follows from Lemma 2.9.4, the equality

$$\mathbb{1}_{\{\tau_{t-1}^*=t-1\}} L_{t-1} = \mathbb{1}_{\{\tau_{t-1}^*=t-1\}} Y_{t-1},$$

and the inequality $H_{t-1} \geq Y_{t-1}$. We have thus established the upper bound of (2.98) and now set about establishing the lower bound in (2.98).

For any $\tau \in \mathcal{T}_{[t-1, n]}$, we set $\tilde{\tau} = \tau \vee t = \max(\tau, t)$ so that $\tilde{\tau}$ is in $\mathcal{T}_{[t, n]}$. Then

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(Z(\sigma_{t-1}^*, \tau) | \mathcal{F}_{t-1}) &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau=t-1\}} Z(\sigma_{t-1}^*, t-1) | \mathcal{F}_{t-1}) \\
&\quad + \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau \geq t\}} Z(\sigma_{t-1}^*, \tilde{\tau}) | \mathcal{F}_{t-1}) \\
&= \mathbb{1}_{\{\tau=t-1\}} \mathbb{E}_{\mathbb{P}}(Z(\sigma_{t-1}^*, t-1) | \mathcal{F}_{t-1}) + \mathbb{1}_{\{\tau \geq t\}} \mathbb{E}_{\mathbb{P}}(Z(\sigma_{t-1}^*, \tilde{\tau}) | \mathcal{F}_{t-1}) \\
&= \mathbb{1}_{\{\tau=t-1\}} \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\sigma_{t-1}^*=t-1\}} H_{t-1} + \mathbb{1}_{\{\sigma_{t-1}^* > t\}} L_{t-1} | \mathcal{F}_{t-1}) \\
&\quad + \mathbb{1}_{\{\tau \geq t\}} \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\sigma_{t-1}^*=t-1\}} Z(t-1, \tilde{\tau}) + \mathbb{1}_{\{\sigma_{t-1}^* \geq t\}} Z(\sigma_t^*, \tilde{\tau}) | \mathcal{F}_{t-1}) \\
&= \mathbb{1}_{\{\tau=t-1=\sigma_{t-1}^*\}} H_{t-1} + \mathbb{1}_{\{\tau=t-1 < \sigma_{t-1}^*\}} L_{t-1} \\
&\quad + \mathbb{1}_{\{\sigma_{t-1}^*=t-1 < \tau\}} H_{t-1} + \mathbb{1}_{\{\tau_{t-1}^* \geq t, \tau \geq t\}} \mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tilde{\tau}) | \mathcal{F}_{t-1}) \\
&= \mathbb{1}_{\{\tau=t-1=\sigma_{t-1}^*\}} H_{t-1} + \mathbb{1}_{\{\tau=t-1 < \sigma_{t-1}^*\}} L_{t-1} + \mathbb{1}_{\{\sigma_{t-1}^*=t-1 < \tau\}} H_{t-1} \\
&\quad + \mathbb{1}_{\{\sigma_{t-1}^* \geq t, \tau \geq t\}} \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tilde{\tau}) | \mathcal{F}_t) | \mathcal{F}_{t-1}) \\
&\leq \mathbb{1}_{\{\sigma_{t-1}^*=t-1\}} H_{t-1} + \mathbb{1}_{\{\tau=t-1 < \sigma_{t-1}^*\}} L_{t-1} + \mathbb{1}_{\{\sigma_{t-1}^* \geq t, \tau \geq t\}} \mathbb{E}_{\mathbb{P}}(Y_t | \mathcal{F}_{t-1}) \\
&\leq \mathbb{1}_{\{\sigma_{t-1}^*=t-1\}} Y_{t-1} + \mathbb{1}_{\{\tau=t-1 < \sigma_{t-1}^*\}} Y_{t-1} + \mathbb{1}_{\{\sigma_{t-1}^* \geq t, \tau \geq t\}} Y_{t-1} \\
&= Y_{t-1}
\end{aligned}$$

where the penultimate inequality follows from the left-hand side inequality in (2.97) and by noting that

$$\{\tau = t-1 = \sigma_{t-1}^*\} \cup \{\sigma_{t-1}^* = t-1 < \tau\} = \{\sigma_{t-1}^* = t-1\}.$$

The last inequality follows from Lemma 2.9.4 and using the fact that

$$\mathbb{1}_{\{\sigma_{t-1}^* = t-1\}} H_{t-1} = \mathbb{1}_{\{\sigma_{t-1}^* = t-1\}} Y_{t-1}, \quad L_{t-1} \leq Y_{t-1}.$$

This proves the lower bound in (2.98) and thus completes the proof of part (i). Part (ii) is an immediate consequence of part (i) and Lemma 2.9.2. \square

Let us assume that $L_0 = H_0$ and let us consider the Dynkin game started at time 0. As was already observed, the equality $L_0 = H_0$ implies that $Y_0 = L_0 = H_0$. Therefore, the stopping times $\sigma_0^* = \tau_0^* = 0$ are optimal and $\bar{Y}_0 = L_0 = H_0$. Note that we only assume here that $L_t \leq H_t$ for $t = 1, 2, \dots, n$.

2.9.2 Valuation and Hedging of Game Contingent Claims

In this section, we make the standing assumption that the finite market model is complete or, equivalently, that there exists a unique martingale measure \mathbb{P}^* for relative prices $S^* = (S^{*1}, S^{*2}, \dots, S^{*(k-1)})$. Recall that such a \mathbb{P}^* is also a martingale measure for the discounted wealth process $U^*(\phi)$ of any self-financing trading strategy $\phi \in \Phi$. Our aim is to now show that the value process of the Dynkin game associated with a game contingent claim can be interpreted as its arbitrage price.

To this end, we first define the discounted payoffs $L^* = B^{-1}L$ and $H^* = B^{-1}H$. We will investigate the Dynkin game with the payoff given by the expression

$$Z^*(\sigma, \tau) = \mathbb{1}_{\{\tau \leq \sigma\}} L_\tau^* + \mathbb{1}_{\{\sigma < \tau\}} H_\sigma^*.$$

In view of our financial interpretation, the max-player and the min-player will be referred to as the seller (issuer) and the buyer (holder), respectively.

Let us write $U = BU^*$, where U^* is the value process of the Dynkin game, that is,

$$\begin{aligned} U_t^* &= \min_{\sigma \in \mathcal{I}_{[t, T]}} \max_{\tau \in \mathcal{I}_{[t, T]}} \mathbb{E}_{\mathbb{P}}(Z^*(\sigma, \tau) | \mathcal{F}_t) \\ &= \max_{\tau \in \mathcal{I}_{[t, T]}} \min_{\sigma \in \mathcal{I}_{[t, T]}} \mathbb{E}_{\mathbb{P}}(Z^*(\sigma, \tau) | \mathcal{F}_t). \end{aligned} \quad (2.99)$$

In view of Theorem 2.9.1, this also means that

$$U_t^* = \min \left\{ H_t^*, \max \left\{ L_t^*, \mathbb{E}_{\mathbb{P}}(U_{t+1}^* | \mathcal{F}_t) \right\} \right\} \quad (2.100)$$

or, equivalently,

$$U_t = \min \left\{ H_t, \max \left\{ L_t, B_t \mathbb{E}_{\mathbb{P}}(B_{t+1}^{-1} U_{t+1} | \mathcal{F}_t) \right\} \right\}. \quad (2.101)$$

By a slight abuse of language, we will refer to U (rather than U^*) as the value process of the Dynkin game associated with a game contingent claim $X^g = (L, H, \mathcal{T}^e, \mathcal{T}^c)$. Let us set, for any fixed $t = 0, 1, \dots, T$,

$$\sigma_t^* = \min \left\{ u \in \{t, t+1, \dots, T\} \mid U_u = H_u \right\} \wedge T$$

and

$$\tau_t^* = \min \left\{ u \in \{t, t+1, \dots, T\} \mid U_u = L_u \right\}.$$

We first state the following immediate corollary to results of Sect. 2.8.2.

Proposition 2.9.1 *For any stopping time $\sigma \in \mathcal{T}_{[t,T]}$, there exists a minimal seller's super-hedging strategy for the American contingent claim $X^{a,\sigma}$ with the payoff $X_t^\sigma = Z(\sigma, t)$ for $t = 0, 1, \dots, T$. Moreover, the seller's price of this claim, $\pi^s(X^{a,\sigma})$, is the solution to the following maximization problem*

$$\pi_t^s(X^{a,\sigma}) = \max_{\tau \in \mathcal{T}_{[t,T]}} B_t \mathbb{E}_{\mathbb{P}^*}(Z^*(\sigma, \tau) | \mathcal{F}_t). \quad (2.102)$$

The following simple result, which is easy to establish by induction, will be useful in the proof of Theorem 2.9.2.

Lemma 2.9.6 *Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a finite filtered probability space and let M be an \mathbb{F} -martingale. Then for any $t = 0, 1, \dots, T$ and any stopping time $\tau \in \mathcal{T}_{[t,T]}$*

$$\mathbb{E}_{\mathbb{P}}(M_\tau | \mathcal{F}_t) = M_t.$$

In the next result, we take the perspective of the seller.

Theorem 2.9.2 *The seller's price $\pi_t^s(X^g)$ at time t of a game contingent claim is equal to the seller's price $\pi^s(X^{a,\sigma_t^*})$ of an American claim X^{a,σ_t^*} with the payoff process $X^{\sigma_t^*}$ given by the formula $X_t^{\sigma_t^*} = Z(\sigma_t^*, t)$ for every $t = 0, 1, \dots, T$. Consequently, $\pi_t^s(X^g) = U_t$ for every $t = 0, 1, \dots, T$, and thus $\pi_t^s(X^g)$ is the solution to the problem*

$$\pi_t^s(X^g) = \max_{\tau \in \mathcal{T}_{[t,T]}} B_t \mathbb{E}_{\mathbb{P}^*}(Z^*(\sigma_t^*, \tau) | \mathcal{F}_t) = \max_{\tau \in \mathcal{T}_{[t,T]}} \min_{\sigma \in \mathcal{T}_{[t,T]}} B_t \mathbb{E}_{\mathbb{P}^*}(Z^*(\sigma, \tau) | \mathcal{F}_t).$$

This also means that

$$\pi_t^s(X^g) = B_t \mathbb{E}_{\mathbb{P}^*}(Z^*(\sigma_t^*, \tau_t^*) | \mathcal{F}_t).$$

Proof. Let us fix some date $t = 0, 1, \dots, T$. The seller of a game contingent claim is set to choose her cancellation time $\sigma \in \mathcal{T}_{[t,T]}$ as well as a trading strategy ϕ . We will first show that for any seller's super-hedging strategy (σ, ϕ) for a game contingent claim from time t onwards, the inequality $V_t(\phi) \geq U_t$ is satisfied.

First, as a pair (σ, ϕ) is a super-hedging strategy for the seller, we necessarily have that $V_u(\phi) \geq Z(\sigma, u)$ for every $u = t, t+1, \dots, T$. Furthermore, the discounted wealth of a self-financing strategy is an \mathbb{F} -martingale under \mathbb{P}^* , and thus we obtain, for $\tau = \tau_t^*$,

$$V_t^*(\phi) = \mathbb{E}_{\mathbb{P}^*}(V_{\sigma \wedge \tau_t^*}^*(\phi) | \mathcal{F}_t) \geq \mathbb{E}_{\mathbb{P}^*}(Z^*(\sigma, \tau_t^*) | \mathcal{F}_t) \geq U_t^*,$$

where the equality follows from Lemma 2.9.6, the first inequality follows from the fact that ϕ is a trading strategy that super-hedges the payoff $Z(\sigma, \tau)$, and the last inequality is a consequence of (2.94) applied to Z^* and U^* .

Let us now check that there exists a trading strategy ϕ for a game contingent claim such that the wealth process $V_t(\phi) = U_t$. To this end, it suffices to take $\sigma = \sigma_t^*$ in Proposition 2.9.1. Indeed, the seller's price of the associated American claim X^{a,σ_t^*} with the payoff process $X_t^{\sigma_t^*} = Z(\sigma_t^*, t)$ satisfies

$$\pi_t^s(X^{a, \sigma_t^*}) = \max_{\tau \in \mathcal{T}_{[t, T]}} B_t \mathbb{E}_{\mathbb{P}^*}(Z^*(\sigma_t^*, \tau) | \mathcal{F}_t) = B_t \mathbb{E}_{\mathbb{P}^*}(Z^*(\sigma_t^*, \tau_t^*) | \mathcal{F}_t) = U_t,$$

where the first equality follows from (2.102) and where the second equality is a consequence of part (iii) in Theorem 2.9.1. \square

It remains to examine the buyer's price of a game contingent claim.

Proposition 2.9.2 *The buyer's price $\pi^b(X^g)$ of a game contingent claim X^g is equal to the seller's price $\pi^s(X^g)$, that is, $\pi_t^b(X^g) = U_t$ for every $t = 0, 1, \dots, T$. Hence the arbitrage price $\pi(X^g)$ of a game contingent claim is unique and it is equal to the value process U of the associated Dynkin game.*

Proof. Since the super-hedging problems for the seller and the buyer of a game contingent claim are essentially symmetric, the equality $\pi^b(X^g) = U$ can be established using the same arguments as those employed in the proof of Theorem 2.9.2. We thus obtain the asserted equality $\pi_t(X^g) = U_t$ for every $t = 0, 1, \dots, T$. The details are left to the reader. \square

Similarly as in the case of an American claim, it is possible to argue that the arbitrage price is a fair value of a game contingent claim, in the sense that if the seller and the buyer of a game contingent claim decide to exercise it at their respective rational exercise times, σ_t^* and τ_t^* , then the claim's payoff at the random time $\tau_t^* \wedge \sigma_t^* = \min(\tau_t^*, \sigma_t^*)$ will match perfectly the wealth of the minimal super-hedging strategy for the seller.

Let us note that the latter property hinges on the standing assumption that the underlying finite market model is arbitrage-free and complete. If the market completeness is not postulated, a strictly positive bid-ask spread $\pi_t^s(X^g) - \pi_t^b(X^g)$ is likely to arise when valuing a game contingent claim. For more information on pricing of American and game options in incomplete models, we refer to Kallsen and Kühn (2004) and Dolinsky and Kifer (2007) and the references therein.

Example 2.9.1 As a real life example of a game contingent claim, let us describe briefly a *convertible bond*, that is, a coupon-bearing bond issued by a company in which both the issuer and the holder have an option to exercise the contract before its maturity date T .

First, the issuer has the right to cancel the contract by *calling* the bond; in that case, the payoff at cancellation time is given by the pre-determined *call price* \bar{C} . Second, the holder of the bond is has typically the right to either put the bond at the pre-determined *put price* \bar{P} (which is set to be lower than the call price), or to convert the bond into a pre-determined number $\kappa > 0$ (the so-called *conversion rate*) of shares of the issuer's equity S . We thus deal in fact with a callable and puttable convertible bond.

Suppose momentarily that we deal with a zero-coupon convertible bond with the nominal value N . In that case, the bond can be formally defined as a game contingent claim given by the expression

$$Z(\sigma, \tau) = (\bar{P} \vee \kappa S_\tau) \mathbb{1}_{\{\tau \leq \sigma < T\}} + \bar{C} \mathbb{1}_{\{\sigma < \tau\}} + N \mathbb{1}_{\{\tau = \sigma = T\}},$$

where σ stands for the *call time* and τ represents the *put/conversion time*. Hence the lower payoff satisfies $L_t = \bar{P} \vee \kappa S_t$ for $t < T$ and $L_T = N$, whereas the upper payoff equals $H_t = \bar{C}$ for $t < T$ and $H_T = N$. In fact, to ensure that $L \leq H$, we should consider the adjusted lower payoff process $L \wedge \bar{C}$, rather than L .

Of course, to analyze the actual convertible bond, one has to specify, in addition, the coupon amounts and the coupon payment dates. This leads to a suitable modification of the last formula. Let us conclude by observing that the natural way of dealing with a convertible bond is to combine a model for the equity value with some model of the term structure of interest rates. Issues related to the modeling of the term structure of interest rates are extensively studied in the second part of this book.



<http://www.springer.com/978-3-540-20966-9>

Martingale Methods in Financial Modelling

Musiela, M.; Rutkowski, M.

2005, XX, 720 p., Hardcover

ISBN: 978-3-540-20966-9