5

Integration of Transcendental Functions

Having developped the required machinery in the previous chapters, we can now describe the integration algorithm. In this chapter, we define formally the integration problem in an algebraic setting, prove the main theorem of symbolic integration (Liouville's Theorem), and describe the main part of the integration algorithm.

From now on, and without further mention, all the fields in this book are of characteristic 0. We also use the convention throughout that $\deg(0) = -\infty$.

5.1 Elementary and Liouvillian Extensions

We give in this section precise definitions of elementary functions, and of the problem of integrating functions in finite terms. Throughout this section, let k be a differential field and K a differential extension of k.

Definition 5.1.1. *$t \in K$ is a* primitive over k *if $Dt \in k$. $t \in K^*$ is an* hyperexponential over k *if $Dt/t \in k$. $t \in K$ is* Liouvillian over k *if t is either algebraic, or a primitive or an hyperexponential over k. K is a* Liouvillian extension of k *if there are $t_1, \dots, t_n$ in K such that $K = k(t_1, \dots, t_n)$ and t_i is Liouvillian over $k(t_1, \dots, t_{i-1})$ for i in $\{1, \dots, n\}$.*

We write $t = \int a$ when t is a primitive over k such that $Dt = a$, and $t = e^{\int a}$ when t is an hyperexponential over k such that $Dt/t = a$. Given that t is Liouvillian over k, we need to know whether t is algebraic or transcendental over k. We show that there are simple necessary and sufficient conditions that guarantee that a primitive or hyperexponential is in fact a monomial over k.

Lemma 5.1.1. *If t is a primitive over k and Dt is not the derivative of an element of k, then Dt is not the derivative of an element of any algebraic extension of k.*

Proof. Let t be a primitive over k, $a = Dt$, and suppose that a is not the derivative of an element of k. Let E be any algebraic extension of k, and suppose that $D\alpha = a$ for some $\alpha \in E$. Let Tr be the trace map from $k(\alpha)$ to k, $n = [k(\alpha) : k]$, and $b = Tr(\alpha)/n \in k$. By Theorem 3.2.4,

$$Db = \frac{1}{n} D(Tr(\alpha)) = \frac{1}{n} Tr(D\alpha) = \frac{1}{n} Tr(a) = a$$

in contradiction with $Du \neq a$ for any $u \in k$. □

Theorem 5.1.1. *If t is a primitive over k and Dt is not the derivative of an element of k, then t is a monomial over k,* $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$*, and $\mathcal{S} = k$ (i.e. $\mathcal{S}^{\mathrm{irr}} = \mathcal{S}_1^{\mathrm{irr}} = \emptyset$). Conversely, if t is transcendental and primitive over k, and* $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$*, then Dt is not the derivative of an element of k.*

Proof. Let t be a primitive over k, $a = Dt$, $\overline{k}$ be the algebraic closure of k, and suppose that a is not the derivative of an element of k. Then, $D\alpha \neq a$ for any $\alpha \in \overline{k}$ by Lemma 5.1.1, so t must be transcendental over k, hence it is a monomial over k. Suppose that $p \in \mathcal{S} \setminus k$. Let then $\beta \in \overline{k}$ be a root of p. Then, $D\beta = Dt = a$ by Theorem 3.4.3, in contradiction with $D\alpha \neq a$ for any $\alpha \in \overline{k}$, so $p \in k$. Conversely, $k \subseteq \mathcal{S}$ by definition. Let $c \in \mathrm{Const}(k(t))$. By Lemma 3.4.5, both the numerator and denominator of c must be special, hence in k, so $c \in k$, which implies that $\mathrm{Const}(k(t)) \subseteq \mathrm{Const}(k)$. The reverse inclusion is given by Lemma 3.3.1, so $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$.

Conversely, let t be a transcendental primitive over k and suppose that $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$. If there exists $b \in k$ such that $Dt = Db$, then $c = t - b \in \mathrm{Const}(k(t))$, so $c \in k$ in contradiction with t transcendental over k. Hence Dt is not the derivative of an element in k. □

Theorem 5.1.2. *If t is an hyperexponential over k and Dt/t is not a logarithmic derivative of a k-radical, then t is a monomial over k,* $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$*, and $\mathcal{S}^{\mathrm{irr}} = \mathcal{S}_1^{\mathrm{irr}} = \{t\}$. Conversely, if t is transcendental and hyperexponential over k, and* $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$*, then Dt/t is not a logarithmic derivative of a k-radical.*

Proof. Let t be an hyperexponential over k, $a = Dt/t$, $\overline{k}$ be the algebraic closure of k, and suppose that a is not a logarithmic derivative of a k-radical. We have $Dt/t = a$ and a is not a logarithmic derivative of a $\overline{k}$-radical by Lemma 3.4.8, so t must be transcendental over k, hence it is a monomial over k since $Dt = at$.

Let $p = bt^m$ for $b \in k$ and $m \geq 0$. Then, $Dp = (Db + mab)t^m$, so $p \mid Dp$, which means that $p \in \mathcal{S}$. Let now $p \in \mathcal{S}^{\mathrm{irr}}$ and suppose that p has a nonzero root $\beta \in \overline{k}^*$. Then, $D\beta/\beta = Dt/t = a$ by Theorem 3.4.3, in contradiction with $D\alpha/\alpha \neq a$ for any $\alpha \in \overline{k}^*$. Hence the only root of p in $\overline{k}$ is 0, so $p = t$.

We have $\mathcal{S}_1^{\mathrm{irr}} \subseteq \mathcal{S}^{\mathrm{irr}}$ by definition. Conversely, let $p \in \mathcal{S}^{\mathrm{irr}}$. Then $p = t$, so the only root of p in $\overline{k}$ is $\beta = 0$. We have $p_\beta = p_0 = Dt/t = a$, which is not a logarithmic derivative of a k-radical, so $p \in \mathcal{S}_1^{\mathrm{irr}}$, which implies that $\mathcal{S}_1^{\mathrm{irr}} = \mathcal{S}^{\mathrm{irr}}$.

Let $c \in \mathrm{Const}(k(t))$. By Lemma 3.4.5, both the numerator and denominator of c must be special, hence $c = bt^q$ for $b \in k$ and $q \in \mathbb{Z}$. Suppose that $b \neq 0$ and $q \neq 0$. Then, $0 = Dc = (Db + qab)t^q$, so $Db/b = qa$, which implies that a is a logarithmic derivative of a k-radical, in contradiction with our hypothesis. Hence, $b = 0$ or $q = 0$, so $c \in k$, which implies that $\mathrm{Const}(k(t)) \subseteq \mathrm{Const}(k)$. The reverse inclusion is given by Lemma 3.3.1, so $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$.

Conversely, let t be a transcendental hyperexponential over k and suppose that $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$. If there exist $b \in k^*$ and an integer $n \neq 0$ such that $nDt/t = Db/b$, then $c = t^n/b \in \mathrm{Const}(k(t))$, so $c \in k$ in contradiction with t transcendental over k. Hence Dt/t is not a logarithmic derivative of a k-radical. $\square$

In practice, we only consider primitives and hyperexponentials that satisfy the hypotheses of Theorems 5.1.1 or 5.1.2. As we have seen, such primitives and hyperexponentials are monomials that satisfy the extra condition $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$. Those monomials are traditionally called *Liouvillian monomials* in the literature.

Definition 5.1.2. *$t \in K$ is a* Liouvillian monomial over k *if t is transcendental and Liouvillian over k and* $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$.

One should be careful that our definition of monomial in Chap. 3 does not require $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$, so it is possible for a monomial in the sense of Chap. 3 to be Liouvillian over k and yet *not* a Liouvillian monomial in the sense of Definition 5.1.2 (for example $\log(2)$ over $\mathbb{Q}$). Theorems 5.1.1 and 5.1.2 can be seen as necessary and sufficient conditions for a primitive or hyperexponential to be a Liouvillian monomial. Furthermore, those theorems describe all the special polynomials in such extensions, and they are all of the first kind. We also have:

$$k\langle t\rangle = \begin{cases} k[t], & \text{if } Dt \in k, \\ k[t, t^{-1}], & \text{if } Dt/t \in k. \end{cases} \tag{5.1}$$

The fact that k and $k(t)$ have the same field of constants allows us to refine the relationship between the degree of a polynomial and its derivative in a Liouvillian monomial extension, and to strenghten Theorem 4.4.4.

Lemma 5.1.2. *Let t be a Liouvillian monomial over k, $f \in k(t)$ be such that $Df \neq 0$, and write $f = p/q$ where $p, q \in k[t]$ and q is monic. If $\nu_\infty(f) = 0$, then $\nu_\infty(Df) \geq 0$. Otherwise, $\nu_\infty(f) \neq 0$ and*

$$\nu_\infty(Df) = \begin{cases} \nu_\infty(f), & \text{if } Dt/t \in k \text{ or } D(\mathrm{lc}(p)) \neq 0\,, \\ \nu_\infty(f) + 1, & \text{if } Dt \in k \text{ and } D(\mathrm{lc}(p)) = 0\,. \end{cases}$$

Proof. If $\nu_\infty(f) = 0$, then $\nu_\infty(Df) \geq 0$ by Theorem 4.4.4, so suppose from now on that $\nu_\infty(f) \neq 0$. Then, $n - m \neq 0$ where $n = \deg(p)$ and $m = \deg(q)$. We have

$$Df = \frac{qDp - pDq}{q^2}$$

hence $\nu_\infty(Df) = 2m - \deg(qDp - pDq)$, so we need to compute $\deg(qDp - pDq)$. Write $p = bt^n + r$ and $q = t^m + s$ where $b \in k^*$ and $r, s \in k[t]$ satisfy $\deg(r) < n$ and $\deg(s) < m$. We treat the primitive and hyperexponential cases separately.

Primitive case: Suppose that $Dt = a \in k$. Then,

$$Dp = (Db)t^n + nabt^{n-1} + Dr \tag{5.2}$$

and

$$Dq = mat^{m-1} + Ds$$

so $\deg(Dq) < m$ since $\deg(Ds) < m$ by Lemma 3.4.2.

Suppose first that $Db \neq 0$. Then, $\deg(Dp) = n$ since $\deg(Dr) < n$ by Lemma 3.4.2, so $\deg(qDp) = m + n$ and $\deg(pDq) < m + n$, which implies that $\deg(qDp - pDq) = m + n$, hence that

$$\nu_\infty(Df) = 2m - (m + n) = m - n = \nu_\infty(f)\,.$$

Suppose now that $Db = 0$, and write $r = ct^{n-1} + u$ and $s = dt^{m-1} + v$, where $c, d \in k$ and $u, v \in k[t]$ satisfy $\deg(u) < n - 1$ and $\deg(v) < m - 1$. We have

$$\begin{aligned} qDp - pDq &= (Dc + nab)t^{n+m-1} + (n-1)act^{n+m-2} + t^m Du \\ &\quad +(dt^{m-1} + v)Dp - b(Dd + ma)t^{n+m-1} \\ &\quad -(m-1)abdt^{n+m-2} - bt^n Dv - (ct^{n-1} + u)Dq \\ &= (Dc - bDd + (n-m)ab)\, t^{n+m-1} \\ &\quad + ((n-1)c - (m-1)bd)\, at^{n+m-2} \\ &\quad +(dt^{m-1} + v)Dp + t^m Du - bt^n Dv - (ct^{n-1} + u)Dq\,. \end{aligned}$$

Since $n - m \neq 0$ and $b \neq 0$, $c - bd + (n-m)bt \notin k$, so $D\,(c - bd + (n-m)bt) \neq 0$ since $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$. But

$$D\,(c - bd + (n-m)bt) = Dc - bDd + (n-m)ab$$

since $b \in \mathrm{Const}(k)$, hence $Dc - bDd + (n-m)ab \neq 0$. In addition, (5.2) and $Db = 0$ imply that $\deg(Dp) < n$, and Lemma 3.4.2 imply that $\deg(Du) < n-1$ and $\deg(Dv) < m-1$. Hence, $(dt^{m-1}+v)Dp$, $t^m Du$, $bt^n Dv$ and $(ct^{n-1}+u)Dq$ all have degrees strictly smaller than $n+m-1$, which implies that $\deg(qDp - pDq) = n + m - 1$, hence that $\nu_\infty(Df) = 2m - (n + m - 1) = m - n + 1 = \nu_\infty(f) + 1$.

Hyperexponential case: Suppose that $Dt/t = a \in k$. Then,

$$\begin{aligned} qDp - pDq &= (Db + nab)t^{n+m} + t^m Dr + sDp - bmat^{n+m} - bt^n Ds - rDq \\ &= (Db + (n-m)ab)\, t^{n+m} + (sDp - rDq + t^m Dr - bt^n Ds)\,. \end{aligned}$$

Since $n - m \neq 0$ and $b \neq 0$, $bt^{n-m} \notin k$, so $D\left(bt^{n-m}\right) \neq 0$ since $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$. But $D\left(bt^{n-m}\right) = (Db + (n-m)ab)\,t^{n-m}$, so $Db + (n-m)ab \neq 0$. In addition, $\deg(Dp) \leq n$, $\deg(Dq) \leq m$, $\deg(Dr) < n$ and $\deg(Ds) < m$ by Lemma 3.4.2, so sDp, rDq, $t^m Dr$ and $bt^n Ds$ all have degrees strictly smaller than $n + m$, which implies that $\deg(qDp - pDq) = n + m$, hence that $\nu_\infty(Df) = 2m - (n+m) = m - n = \nu_\infty(f)$. □

Note that when applied to polynomials $p \in k[t]$ when t is a Liouvillian monomial over k, Lemma 5.1.2 implies that

$$\deg(Dp) = \begin{cases} \deg(p), & \text{if } Dt/t \in k \text{ or } D(\mathrm{lc}(p)) \neq 0\,, \\ \deg(p) - 1, & \text{if } Dt \in k \text{ and } D(\mathrm{lc}(p)) = 0 \end{cases}$$

whenever $Dp \neq 0$, and we often use it in this context in the sequel.

We now introduce the particular Liouvillian extensions that define the integration in finite terms problem, namely the elementary extensions.

Definition 5.1.3. *$t \in K$ is a* logarithm over k *if $Dt = Db/b$ for some $b \in k^*$. $t \in K^*$ is an* exponential over k *if $Dt/t = Db$ for some $b \in k$. $t \in K$ is* elementary over k *if t is either algebraic, or a logarithm or an exponential over k. $t \in K$ is an* elementary monomial over k *if t is transcendental and elementary over k, and* $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$*.*

We write $t = \log(b)$ when t is a logarithm over k such that $Dt = Db/b$, and $t = e^b$ when t is an exponential over k such that $Dt/t = b$. Since logarithms are primitives and exponentials are hyperexponentials, elementary monomials are Liouvillian monomials and all the results of this section apply to them.

Definition 5.1.4. *K is an* elementary extension of k *if there are $t_1, \ldots, t_n$ in K such that $K = k(t_1, \ldots, t_n)$ and t_i is elementary over $k(t_1, \ldots, t_{i-1})$ for i in $\{1, \ldots, n\}$. We say that $f \in k$* has an elementary integral over k *if there exists an elementary extension E of k and $g \in E$ such that $Dg = f$. An* elementary function *is any element of any elementary extension of* $(\mathbb{C}(x), d/dx)$*.*

We can now define precisely the *problem of integration in closed form*: given a differential field k and an integrand $f \in k$, to decide in a finite number of steps whether f has an elementary integral over k, and to compute one if it has any. Note that there is a difference between having an elementary integral over k and having an elementary antiderivative: consider $k = \mathbb{C}(x, t_1, t_2)$ where x, t_1, t_2 are indeterminates over $\mathbb{C}$, with the derivation D given by $Dx = 1$, $Dt_1 = t_1$ and $Dt_2 = t_1/x$ (*i.e.* $t_1 = e^x$ and $t_2 = \mathrm{Ei}(x)$). Then,

$$\int \frac{e^x \mathrm{Ei}(x)}{x}\, dx = \frac{\mathrm{Ei}(x)^2}{2} \in k$$

so $e^x \mathrm{Ei}(x)/x$ has an elementary integral over k even though its integral is not an elementary function. The two notions coincide only when k itself is a field of elementary functions.

Remark that the elementary functions of Definition 5.1.4 include all the usual elementary functions of analysis, since the trigonometric functions and their inverses can be rewritten in terms of complex exponential and logarithms by the usual formulas derived from Euler's formula $e^{f\sqrt{-1}} = \cos(f) + \sin(f)\sqrt{-1}$. Those transformations have the computational inconvenience that they introduce $\sqrt{-1}$, and it turns out that they can be avoided when integrating real trigonometric functions (Sections 5.8 and 5.10).

5.2 Outline and Scope of the Integration Algorithm

We outline in this section the integration algorithm so that the structure of the remaining sections and chapters will be easier to follow. Given an integrand $f(x)dx$, we first need to construct a differential field containing f, and the integration algorithm we describe requires that f be contained in a differential field of the form $K = C(t_1, t_2, \dots, t_n)$ where $C = \mathrm{Const}(K)$, $Dt_1 = 1$ (*i.e.* $t_1 = x$ is the integration variable), and each t_i is a monomial over $C(t_1, \dots, t_{i-1})$. If the formula for $f(x)$ contains only Liouvillian operations, this requirement can be checked by integrating recursively the argument of each primitive or hyperexponential before adjoining it[1], and verifying using Theorem 5.1.1 or Theorem 5.1.2 that it is a Liouvillian monomial. Another alternative, which is in general more efficient, is to apply the algorithms that are derived from the various structure theorems, whenever they are applicable (Chap. 9).

Example 5.2.1. Consider

$$\int \log(x)\log(x+1)\log(2x^2+2x)dx\,.$$

We construct the differential field $K = \mathbb{Q}(x, t_1, t_2, t_3)$ with

$$Dx = 1, \quad Dt_1 = \frac{1}{x}, \quad Dt_2 = \frac{1}{x+1} \quad \text{and} \quad Dt_3 = \frac{2x+1}{x^2+x}\,.$$

As we construct K, we integrate at each step and make the following verifications:

- $\int dx \notin \mathbb{Q}$, so x is a Liouvillian monomial over $\mathbb{Q}$;
- $\int dx/x \notin \mathbb{Q}(x)$, so t_1 is a Liouvillian monomial over $\mathbb{Q}(x)$;
- $\int dx/(x+1) \notin \mathbb{Q}(x, t_1)$ so t_2 is a Liouvillian monomial over $\mathbb{Q}(x, t_1)$;
- $$\int \frac{2x+1}{x^2+x}\,dx = t_1 + t_2 \in \mathbb{Q}(x, t_1, t_2)$$
  so t_3 is not a Liouvillian monomial over $\mathbb{Q}(x, t_1, t_2)$, and K is isomorphic as a differential field to $\mathbb{Q}(c)(x, t_1, t_2)$ where $c = t_3 - t_1 - t_2 \in \mathrm{Const}(K)$.

[1] A simpler version of the integration algorithm can be used for those verifications, see Sect. 5.12

- Alternatively, applying the Risch structure Theorem (Corollary 9.3.1), we find that the linear equation (9.8) for $a = 2x^2 + 2x$ becomes

$$\frac{r_1}{x} + \frac{r+2}{x+1} = \frac{2x+1}{x^2+x}$$

which has the rational solution $r_1 = r_2 = 1$. This implies that Dt_3 is the derivative of an element of K and that $c = t_3 - t_1 - t_2 \in \mathrm{Const}(K)$.

Example 5.2.2. Consider

$$\int \left(e^{2x} + e^{x+\log(x)/2}\right) dx\,.$$

We construct the differential field $K = \mathbb{Q}(x, t_1, t_2, t_3)$ with

$$Dx = 1, \quad Dt_1 = 2t_1, \quad Dt_2 = \frac{1}{x} \quad \text{and} \quad Dt_3 = \left(1 + \frac{1}{2x}\right) t_3\,.$$

As we construct K we integrate at each step and make the following verifications:

- $\int dx \notin \mathbb{Q}$, so x is a Liouvillian monomial over $\mathbb{Q}$;
- $\int 2dx \neq \log(v)/n$ for any $v \in \mathbb{Q}(x)$ and $n \in \mathbb{Z}$, so 2 is not the logarithmic derivative of a $\mathbb{Q}(x)$-radical, which implies that t_1 is a Liouvillian monomial over $\mathbb{Q}(x)$;
- $\int dx/x \notin \mathbb{Q}(x, t_1)$, so t_2 is a Liouvillian monomial over $\mathbb{Q}(x, t_1)$;
- $$\int \left(1 + \frac{1}{2x}\right) dx = \frac{1}{2}\log(xt_1)$$

  so $1 + 1/(2x)$ is the logarithmic derivative of a $\mathbb{Q}(x, t_1, t_2)$-radical, so t_3 is not a Liouvillian monomial over $\mathbb{Q}(x, t_1, t_2)$, and K is isomorphic as a differential field to $\mathbb{Q}\left(x, t_1, t_2, \sqrt{xt_1}\right)$.
- Alternatively, applying the Risch structure Theorem (Corollary 9.3.1), we find that the linear equation (9.9) for $b = x + t_2/2$ becomes

$$\frac{r_2}{x} + 2r_1 = 1 + \frac{1}{2x}$$

which has the rational solution $r_1 = r_2 = 1/2$. This implies that Dt_3/t_3 is the logarithmic derivative of a K-radical, and that $c = t_3^2/(xt_1) \in \mathrm{Const}(K)$.

Note that the requirement that each t_i be a monomial eliminates expressions containing algebraic functions from the algorithm presented here. Although the problem of integrating elementary functions containing algebraic functions is also decidable, the algorithms used in the algebraic function case are beyond the scope of this book [8, 9, 11, 14, 29, 73, 74, 76, 91].

Once we have a tower of monomials $K = C(t_1, \ldots, t_n)$, the algorithms of this chapter reduce the problem of integrating an element of K to various integration-related problems involving elements of $C(t_1, \ldots, t_{n-1})$, thereby eliminating the monomial t_n. Since the reduced problems involve integrands in a tower of smaller transcendence degree over C, we can use the algorithm recursively on them, and termination is ensured. In order to avoid writing the full tower of extensions throughout this book, we write $K = k(t)$ where $k = C(t_1, \ldots, t_{n-1})$ and $t = t_n$ is a monomial over k, and the task of the algorithms of this chapter is to reduce integrating a given element of $k(t)$ to integration-related problems over k. If t is elementary over k, then having an elementary integral over $k(t)$ is equivalent to having an elementary integral over k, so the algorithms we present in this book provide a complete decision procedure for the problem of deciding whether an element of a purely transcendental elementary extension of $(C(x), d/dx)$ has an elementary integral over $C(x)$. For more general functions, when t is not elementary over k, it can be proven that if t is either an hyperexponential monomial or nonlinear monomial over k with $\mathcal{S}_1^{\mathrm{irr}} = \mathcal{S}^{\mathrm{irr}}$, then having an elementary integral over $k(t)$ is equivalent to having an elementary integral over k (Exercise 5.5), so the algorithm is complete for integrands built from transcendental logarithms, arc-tangents, hyperexponentials and tangents. The only obstruction to a complete algorithm for Liouvillian integrands is the case where t is a nonelementary primitive over k: even though we can reduce the problem to an integrand in k, the problem becomes however to determine whether $f \in k$ has an elementary integral over $k(t)$, and although there are algorithms for special types of primitive monomials [6, 21, 22, 52, 53, 94], this problem has not been solved for general monomials (Exercise 5.5f)). As will be seen from numerous examples in this book, the algorithm can still be used successfully on many integrands involving nonelementary monomials. It cannot however always provide a proof on nonexistence of an elementary integral over $k(t)$ when t is a nonelementary primitive over k. The reduction from $k(t)$ to t is also incomplete for general nonlinear monomials, but is complete for tangents and hyperbolic tangents.

The general line of the integration algorithm is to perform successive reductions, which all transform the integrand to a "simpler" one, until the remaining integrand is in k (Fig. 5.1):

- The *Hermite reduction* (Sect. 5.3), which can be applied to arbitrary monomials, transforms a general integrand to the sum of a simple and a reduced integrand;
- The *polynomial reduction* (section 5.4), which can be applied to nonlinear monomials, reduces the degree of the polynomial part of an integrand;
- The *residue criterion* (Sect. 5.6), which can be applied to arbitrary monomials, either proves that an integrand does not have an elementary integral over $k(t)$, or transforms it to a reduced integrand (*i.e.* an integrand in $k\langle t\rangle$);

- Reduced integrands are integrated by specific algorithms for each case of Liouvillian or hypertangent monomial (Sect. 5.8, 5.9 and 5.10). Those algorithms either prove that there is no elementary integral over $k(t)$, or reduce the problem to various integration-related problems over k. Algorithms for solving those related problems are described in Chap. 6, 7 and 8.

Except for the last part, the various reductions are applicable to arbitrary monomial extensions.

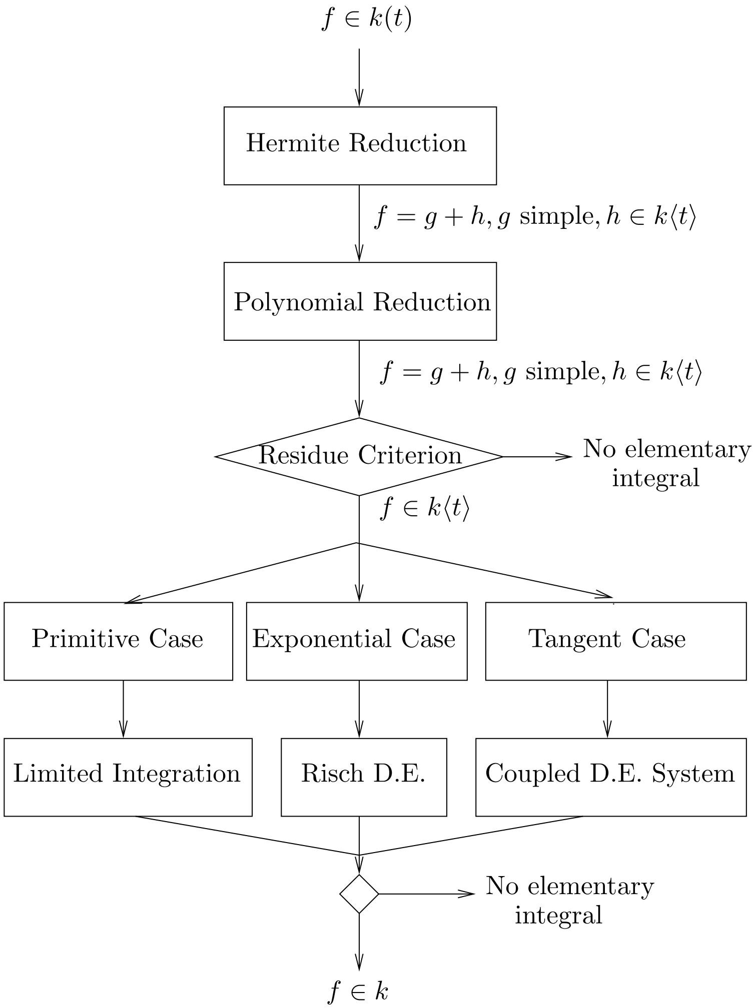

Fig. 5.1. General outline of the integration algorithm

5.3 The Hermite Reduction

We have seen in Sect. 2.2 that the Hermite reduction rewrites any rational function as the sum of a derivative and a rational function with a squarefree denominator. In this section, we show that the Hermite reduction can be applied to the normal part of any element of a monomial extension. Let (k, D) be a differential field and t a monomial over k for the next two sections.

Definition 5.3.1. *For $f \in k(t)$, we define the* polar multiplicity of f *to be*

$$\mu(f) = - \min_{p \in k[t] \setminus k} (\nu_p(f)).$$

Note that $\mu(0) = -\infty$ and that $\mu(f) \geq 0$ for any $f \neq 0$, since in that case there is always some polynomial $p \in k[t]$ for which $\nu_p(f) = 0$. Also, the minimum in the above definition can be taken over all the irreducible or squarefree factors of the denominator of f. It is easy to see that for $f \neq 0$, $\mu(f)$ is exactly the highest power appearing in any squarefree factorization of the denominator of f (Exercise 5.1).

Theorem 5.3.1. *Let $f \in k(t)$. Using only the extended Euclidean algorithm in $k[t]$, one can find $g, h, r \in k(t)$ such that h is simple, r is reduced, and $f = Dg + h + r$. Furthermore, the denominators of g, h and r divide the denominator of f, and either $g = 0$ or $\mu(g) < \mu(f)$.*

Proof. Let $f = f_p + f_s + f_n$ be the canonical representation of f, and write $f_n = a/d$ with $a, d \in k[t]$ and $\gcd(a, d) = 1$. We proceed by induction on $m = \mu(f_n)$. Let $d = d_1 d_2^2 \cdots d_m^m$ be a squarefree factorization of d. If $m \leq 1$, then either $f_n = 0$ or d is normal. In both cases, f_n is simple, so $g = 0$, $h = f_n$ and $r = f_p + f_s \in k\langle t \rangle$ satisfy the theorem.

Otherwise, $m > 1$, so assume that the theorem holds for any nonzero $g = g_p + g_n + g_s$ with $\mu(g_n) < m$, and let $v = d_m$ and $u = d/v^m$. Since every squarefree factor of d is normal by the definition of the canonical representation, v is normal, so $\gcd(Dv, v) = 1$. In addition, $\gcd(u, v) = 1$ by the definition of a squarefree factorization, so $\gcd(uDv, v) = 1$. Hence, we can use the extended Euclidean algorithm to find $b, c \in k[t]$ such that

$$\frac{a}{1-m} = b\,u\,Dv + c\,v.$$

Multiplying both sides by $(1-m)/(uv^m)$ gives

$$f_n = \frac{a}{uv^m} = \frac{(1-m)bDv}{v^m} + \frac{(1-m)c}{uv^{m-1}}$$

so, adding and subtracting Db/v^{m-1} to the right hand side, we get

$$f_n = \left(\frac{Db}{v^{m-1}} - \frac{(m-1)bDv}{v^m} \right) + \frac{(1-m)c - uDb}{uv^{m-1}} = Dg_0 + w$$

where $g_0 = b/v^{m-1}$ and $w = ((1-m)c - uDb)/(uv^{m-1})$. Since the denominator of w divides uv^{m-1}, w has no special part, so let $w = w_p + w_n$ be the canonical representation of w. Since $\mu(w) \le m-1$, we have $\mu(w_n) \le m-1$, so by induction we can find g_1, h_1 and r_1 in $k(t)$ such that $w_n = Dg_1 + h + r_1$, h is simple, r_1 is reduced, the denominators of g_1, h and r_1 divide uv^{m-1}, and $\mu(g_1) < \mu(w)$ if $g_1 \neq 0$. Let then $g = g_0 + g_1$ and $r = f_p + w_p + f_s + r_1$, and write e for the denominator of f. Note that $d \mid e$ by the definition of the canonical representation. The denominator of g_1 divides uv^{m-1} and $g_0 = b/v^{m-1}$, so the denominator of g divides d hence e. The denominator of h divides uv^{m-1}, so it divides d hence e. The denominator of w divides d and the denominator of r_1 divides uv^{m-1}, so the denominator of r divides e. In addition, f_p, w_p, f_s and r_1 are in $k\langle t\rangle$, which is a subring of $k(t)$ by Corollary 4.4.1, so $r \in k\langle t\rangle$. Finally, we have

$$\begin{aligned} f = f_p + f_s + f_n &= f_p + f_s + Dg_0 + w \\ &= f_p + f_s + Dg_0 + w_p + Dg_1 + h + r_1 = Dg + h + r \end{aligned}$$

which proves the theorem. □

Although we have used the quadratic version of the Hermite reduction in the above proof, the other versions are also valid in monomial extensions (Exercise 5.2). Instead of splitting a rational function into a derivative and a simple rational function, the Hermite reduction splits any element of $k(t)$ into a derivative, a simple and a reduced element. Thus, it reduces any integration problem to integrands that are the sum of a simple and a reduced element.

HermiteReduce(f, D) (* Hermite Reduction – quadratic version *)

(* Given a derivation D on $k(t)$ and $f \in k(t)$, return $g, h, r \in k(t)$ such that $f = Dg + h + r$, h is simple and r is reduced. *)

$(f_p, f_s, f_n) \leftarrow$ **CanonicalRepresentation**(f, D)
$(a, d) \leftarrow (\text{numerator}(f_n), \text{denominator}(f_n))$ (* d is monic *)
$(d_1, \ldots, d_m) \leftarrow$ **SquareFree**(d)
$g \leftarrow 0$
for $i \leftarrow 2$ **to** m such that $\deg(d_i) > 0$ **do**
  $v \leftarrow d_i$
  $u \leftarrow d/v^i$
  for $j \leftarrow i-1$ **to** 1 **step** -1 **do**
    $(b, c) \leftarrow$ **ExtendedEuclidean**$(u\,Dv, v, -a/j)$
    $g \leftarrow g + b/v^j$
    $a \leftarrow -jc - u\,Db$
  $d \leftarrow uv$
$(q, r) \leftarrow$ **PolyDivide**(a, uv)
return$(g, r/(uv), q + f_p + f_s)$

Example 5.3.1. Let $k = \mathbb{Q}(x)$ with $D = d/dx$, and let t be a monomial over k satisfying $Dt = 1 + t^2$, *i.e.* $t = \tan(x)$, and consider

$$f = \frac{x - \tan(x)}{\tan(x)^2} = \frac{x - t}{t^2} \in k(t).$$

Since f has no polynomial part and t is normal in $k[t]$, the canonical representation of f is $(f_p, f_s, f_n) = (0, 0, f)$ so we get $a = x - t$ and $d = t^2 = d_2^2$ where $d_2 = t$. We then have:

i	v	u	j	b	c	a
2	t	1	1	$-x$	$xt + 1$	$-xt$

and $a/uv = -xt/t = -x$, so the Hermite reduction returns $(-x/t, 0, -x)$, which means that

$$\int \frac{x - \tan(x)}{\tan(x)^2}\, dx = -\frac{x}{\tan(x)} - \int x\, dx$$

and the remaining integrand is in $k\langle t \rangle$.

The Hermite reduction can also be iterated, yielding a decomposition of f into a sum of higher-order derivatives of reduced and simple elements of $k(t)$ (Exercise 5.3).

5.4 The Polynomial Reduction

In the case of nonlinear monomials, another reduction allows us to rewrite any polynomial in $k[t]$ as the sum of a derivative and a polynomial of degree less than $\delta(t)$.

Theorem 5.4.1. *If t is a nonlinear monomial, then for any $p \in k[t]$, we can find $q, r \in k[t]$ such that $p = Dq + r$ and $\deg(r) < \delta(t)$.*

Proof. We proceed by induction on $n = \deg(p)$. If $n < \delta(t)$, then $q = 0$ and $r = p$ satisfy the theorem. Otherwise $n \geq \delta(t)$ so assume that the theorem holds for any $a \in k[t]$ with $\deg(a) < n$. Let

$$c = \frac{\mathrm{lc}(p)}{(n - \delta(t) + 1)\lambda(t)} \in k,$$

$q_0 = ct^{n-\delta(t)+1}$, and $r_0 = p - Dq_0$. Since t is nonlinear and $\deg(q_0) > 0$, Lemma 3.4.2 implies that $\deg(Dq_0) = \deg(q_0) + \delta(t) - 1 = n$, and that the leading coefficient of Dq_0 is $(n - \delta(t) + 1)\, c\, \lambda(t) = \mathrm{lc}(p)$. Hence, $\deg(r_0) < n$, so by induction we can find $q_1, r \in k[t]$ such that $r_0 = Dq_1 + r$ and $\deg(r) < \delta(t)$. Therefore,

$$p = Dq_0 + r_0 = Dq_0 + Dq_1 + r = Dq + r$$

where $q = q_0 + q_1 \in k[t]$. $\square$

PolynomialReduce(p, D) (* Polynomial Reduction *)

(* Given a derivation D on $k(t)$ and $p \in k[t]$ where t is a nonlinear monomial over k, return $q, r \in k[t]$ such that $p = Dq + r$, and $\deg(r) < \delta(t)$. *)

if $\deg(p) < \delta(t)$ **then return**$(0, p)$
$m \leftarrow \deg(p) - \delta(t) + 1$
$q_0 \leftarrow (\mathrm{lc}(p)/(m\lambda(t)))\, t^m$
$(q, r) \leftarrow$ **PolynomialReduce**$(p - Dq_0, D)$
return$(q_0 + q, r)$

Example 5.4.1. Let $k = \mathbb{Q}(x)$ with $D = d/dx$, and let t be a monomial over k satisfying $Dt = 1 + t^2$, *i.e.* $t = \tan(x)$, and consider

$$p = 1 + x\tan(x) + \tan(x)^2 = 1 + xt + t^2 \in k[t].$$

We have $\delta(t) = 2$, $\lambda(t) = 1$, and applying **PolynomialReduce**, we get $m = \deg(p) - 1 = 1$, $q_0 = t$, $Dq_0 = 1 + t^2$, so $p - Dq_0 = xt$, which has degree 1. Thus,

$$\int (1 + x\tan(x) + \tan(x)^2)\, dx = \tan(x) + \int x\tan(x)dx$$

and it will be proven later that the remaining integral is not an elementary function.

If $\mathcal{S} \neq k$, *i.e.* $\mathcal{S}^{\mathrm{irr}} \neq \emptyset$, then any nontrivial element of $\mathcal{S}$ can be used to eliminate the term of degree $\delta(t) - 1$ from a polynomial.

Theorem 5.4.2. *Suppose that t is a nonlinear monomial. Let $p \in k[t]$ with $\deg(p) < \delta(t)$, $a \in k$ be the coefficient of $t^{\delta(t)-1}$ in p, and $c = a/\lambda(t)$. Then,*

$$\deg\left(p - \frac{c}{\deg(q)}\frac{Dq}{q}\right) < \delta(t) - 1$$

for any $q \in \mathcal{S} \setminus k$.

Proof. Let $q \in \mathcal{S} \setminus k$, then $Dq/q \in k[t]$ and by Lemma 3.4.2, $\deg(Dq/q) = \deg(Dq) - \deg(q) = \delta(t) - 1$, and the leading coefficient of Dq is $\deg(q)\mathrm{lc}(q)\lambda(t)$. Hence,

$$\mathrm{lc}\left(\frac{c}{\deg(q)}\frac{Dq}{q}\right) = \frac{c}{\deg(q)}\frac{\deg(q)\mathrm{lc}(q)\lambda(t)}{\mathrm{lc}(q)} = c\lambda(t) = a$$

which implies that the degree of $p - c/\deg(q)\, Dq/q$ is at most $\delta(t) - 2$. □

5.5 Liouville's Theorem

Given a differential field K and an integrand $f \in K$, if an elementary integral is found, it can be easily proven correct by differentiation. Furthermore, there are usually several ways to find elementary integrals when they exist. Proving that f has no elementary integral is however quite a different problem, since we need results that connect the existence of an elementary integral to a special form of the integrand. The first such result is Laplace's principle [55], which states roughly that we can simplify the integration problem by allowing only new logarithms to appear linearly in the integral, all the other functions must be in the integrand already[2]. Liouville was the first to state and prove a precise theorem from this observation, first in the case of algebraic integrands [57, 58], then for more general integrands [59]. See Chap. IX of [61] for the fascinating history of Liouville's Theorem in the 19$^{\text{th}}$ century. This theorem has become the main tool used in proving that no elementary integral exists for a given function. Furthermore, since it provides an explicit class of elementary extensions to search for an integral, it forms the basis of the integration algorithm. While Liouville used analytic arguments, it is now possible to prove it algebraically in the context of differential fields. Algebraic techiques were first used by Ostrowski [69], who presented a modern proof of Liouville's Theorem, together with an algorithm that reduces integrating in $k(t)$ to integrating in k when t is a primitive monomial over k. The first complete algebraic proof of Liouville's Theorem was then published by Rosenlicht [79] and the first proof of the strong version of Liouville's Theorem by Risch, who published it together with a complete integration algorithm for purely transcendental elementary functions [75]. We follow both of them here, first presenting essentially Rosenlicht's proof of the weak Liouville Theorem, and then progressively removing the restrictions on the constant fields, obtaining Risch's proof of the strong Liouville Theorem. We remark that Liouville's Theorem has been extended in various directions [17, 71, 81, 86], but those extensions go beyond the scope of this book. Integration algorithms that yield nonelementary integrals [21, 22, 52, 53] are based on such extensions [86].

Theorem 5.5.1 (Liouville's Theorem). *Let K be a differential field and $f \in K$. If there exist an elementary extension E of K with* $\mathrm{Const}(E) = \mathrm{Const}(K)$ *and $g \in E$ such that $Dg = f$, then there are $v \in K$, $u_1, \dots, u_n \in K^*$ and $c_1, \dots, c_n \in \mathrm{Const}(K)$ such that*

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}\,. \tag{5.3}$$

[2] "…la différentiation laissant subsister les quantités exponentielles et radicales, et ne faisant disparaitre les quantités logarithmiques qu'autant qu'elles ont multipliées par des constantes, on doit en conclure que l'intégrale d'une fonction différentielle ne peut contenir d'autres quantités exponentielles et radicales que celles qui sont contenues dans cette fonction…"

Proof. Write $C = \mathrm{Const}(K)$ and let E be an elementary extension of K with $\mathrm{Const}(E) = C$ and $g \in E$ be such that $Dg = f$. Then, there are $t_1, \dots, t_m \in E$ such that $E = K(t_1, \dots, t_m)$ and each t_i is elementary over $K(t_1, \dots, t_{i-1})$. We proceed by induction on m. For $m = 0$, we have $E = K$, so letting $v = g \in K$, we get $f = Dv$, which is of the form (5.3) with $n = 0$. Suppose now that $m > 0$ and that the theorem holds for any elementary extension generated by $m - 1$ elements. Let $t = t_1$ and $F = K(t)$. Since $K \subseteq F \subseteq E$, then $C \subseteq \mathrm{Const}(F) \subseteq \mathrm{Const}(E) = C$, so $\mathrm{Const}(F) = C$. In addition, $f \in F$, and $E = F(t_2, \dots, t_m)$ is an elementary extension of F generated by $m - 1$ elements, so by induction there are $v \in F$, $u_1, \dots, u_n \in F^*$ and $c_1, \dots, c_n \in C$ such that

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i} \,. \tag{5.4}$$

Case 1: t transcendental over K. Then, since $\mathrm{Const}(F) = C$, t is Liouvillian monomial over K by Theorems 5.1.1 and 5.1.2. Let $p \in K[t]$ be normal and irreducible. We have $\nu_p(Du_i/u_i) \ge -1$ by Corollary 4.4.2, hence $\nu_p(\sum_{i=1}^n c_i Du_i/u_i) \ge -1$ by Theorem 4.1.1. Suppose that $\nu_p(v) < 0$. Then, $\nu_p(Dv) = \nu_p(v) - 1 < -1$ by Theorem 4.4.2, so $\nu_p(f) = \min(\nu_p(Dv), -1) < -1$ by Theorem 4.1.1, in contradiction with $f \in K$. Hence $\nu_p(v) \ge 0$, so, since this holds for any normal irreducible p, $v \in K\langle t\rangle$. Hence, $Dv \in K\langle t\rangle$ by Corollary 4.4.1. Write now $u_i = w_i \prod_{j=1}^{n_i} p_{ij}^{e_{ij}}$ where $w_i \in K^*$, each $p_{ij} \in K[t]$ is monic irreducible, and the e_{ij}'s are integers. Then, using the logarithmic derivative identity and grouping together all the terms involving the same p_{ij}, we get

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Dw_i}{w_i} + \sum_{j=1}^{N} d_j \frac{Dq_j}{q_j} \tag{5.5}$$

where the q_j's are in $K[t]$, monic, irreducible and coprime. Write

$$g = \sum_{i=1}^{n} c_i \frac{Dw_i}{w_i} \in K, \quad h = \sum_{j=1}^{N} d_j \frac{Dq_j}{q_j} \,,$$

and suppose that one of the q_j's, say q_k, is normal. We have $\nu_{q_k}(q_k) = 1$ and $\nu_{q_k}(q_j) = 0$ for $j \ne k$, so $\nu_{q_k}(d_k Dq_k/q_k) = -1$ and $\nu_{q_k}(d_j Dq_j/q_j) = 0$ by Corollary 4.4.2. This implies that $\nu_{q_k}(\sum_{j \ne k} d_j Dq_j/q_j) \ge 0$, hence that $\nu_{q_k}(h) = -1$. But q_k is normal and $Dv \in K\langle t\rangle$, hence $\nu_{q_k}(Dv) \ge 0$, so $\nu_{q_k}(f) = -1$, in contradiction with $f \in K$. Hence all the q_j's in equation (5.5) are special.

Case 1a: t is a logarithm over K. Then, $Dt = Da/a$ for some $a \in K^*$, and every irreducible $p \in K[t]$ is normal by Theorem 5.1.1, so $N = 0$ in equation (5.5) and $v, Dv \in K[t]$. From (5.5) we get $Dv = f - g \in K$. By Lemma 5.1.2, this implies that or $v = ct + b$ where $b, c \in K$ and $Dc = 0$ (otherwise $\deg(Dv) \ge 1$). Hence,

$$f = Db + c\frac{Da}{a} + \sum_{i=1}^{n} c_i \frac{Dw_i}{w_i}$$

which is of the form (5.3).

Case 1b: t is an exponential over K. Then, $Dt/t = Da$ for some $a \in K$, and the only special monic irreducible $p \in K[t]$ is $p = t$ by Theorem 5.1.2, so $N = 1$ in equation (5.5) and $q_1 = t$ (with d_1 possibly 0). Hence, $d_1 Dq_1/q_1 = d_1 Dt/t = d_1 Da$, so $f = Dw + g$ where $w = v + d_1 a \in K\langle t\rangle$. Suppose that $\nu_t(w) < 0$, then $\nu_t(Dw) = \nu_t(w) < 0$ by Theorem 4.4.2 since $t \in \mathcal{S}^{\mathrm{irr}}$, so $\nu_t(f) < 0$ in contradiction with $f \in K$. Hence, $\nu_t(w) \ge 0$ so $w \in K[t]$. By Lemma 5.1.2, $\nu_\infty(Dw) = \nu_\infty(w)$, so $\deg(Dw) = \deg(w)$, which implies that $\deg(w) = 0$ since $f = Dw + g \in K$. Hence $w \in K$ and

$$f = Dw + \sum_{i=1}^{n} c_i \frac{Dw_i}{w_i}$$

which is of the form (5.3).

Case 2: t algebraic over K. Let $Tr : F \to K$ and $N : F \to K$ be the trace and norm maps from F to K and $d = [F : K]$. Applying Tr to both sides of equation (5.4) we get:

$$Tr(f) = Tr(Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}) = Tr(Dv) + \sum_{i=1}^{n} c_i\, Tr(\frac{Du_i}{u_i})$$

since Tr is K-linear and the c_i's are in K. We have $Tr(f) = df$ since $f \in K$, and

$$Tr(Dv) = D(Tr(v)) \quad \text{and} \quad Tr\left(\frac{Du_i}{u_i}\right) = \frac{DN(u_i)}{N(u_i)}$$

by Theorem 3.2.4, so

$$f = Dw + \sum_{i=1}^{n} \frac{c_i}{d}\frac{Dw_i}{w_i}$$

which is of the form (5.3) with $w = Tr(v)/d \in K$ and $w_i = N(u_i) \in K^*$. □

Of course, in practice we may have to adjoin new constants in order to compute integrals, as we have seen in Chap. 2. We first show that new transcendental constants are not necessary in order to express an elementary integral.

Theorem 5.5.2. *Let K be a differential field with algebraically closed constant field and $f \in K$. If there exist an elementary extension E of K and $g \in E$ such that $Dg = f$, then there are $v \in K$, $u_1, \dots, u_n \in K^*$ and $c_1, \dots, c_n \in \mathrm{Const}(K)$ such that*

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}\,.$$

Proof. Suppose that there exist an elementary extension E of K and $g \in E$ such that $Dg = f$. Write $\mathrm{Const}(K) = C$, $\mathrm{Const}(E) = C(a_1, \dots, a_m)$ for some constants $a_1, \dots, a_m$ in E, and let $F = K(a_1, \dots, a_m)$. Since $C(a_1, \dots, a_m) \subseteq F \subseteq E$, $C(a_1, \dots, a_m) \subseteq \mathrm{Const}(F) \subseteq \mathrm{Const}(E)$, so F and E have the same constant subfield. In addition, $f \in F$ and E is elementary over F, so by Theorem 5.5.1, there are $v \in F$, $u_1, \dots, u_m \in F^*$ and $c_1, \dots, c_n \in \mathrm{Const}(F)$ such that

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i} \,. \tag{5.6}$$

Let $X_1, \dots, X_m$ be independent indeterminates over K. Since the elements of F are rational functions in $a_1, \dots, a_n$, we can write

$$v = \frac{p(a_1, \dots, a_m)}{q(a_1, \dots, a_m)}, \quad c_i = \frac{r_i(a_1, \dots, a_m)}{s_i(a_1, \dots, a_m)} \quad \text{and } u_i = \frac{p_i(a_1, \dots, a_m)}{q_i(a_1, \dots, a_m)} \tag{5.7}$$

where p, q, p_i, q_i are in $K[X_1, \dots, X_m]$, and r_i, s_i are in $C[X_1, \dots, X_m]$. In addition, $g(a_1, \dots, a_m) \neq 0$, where

$$g = q \left(\prod_{i=1}^{n} s_i\right) \left(\prod_{i=1}^{n} p_i\right) \left(\prod_{i=1}^{n} q_i\right) \quad \in K[X_1, \dots, X_m] \,.$$

Replacing v, $c_1, \dots, c_m$ and $u_1, \dots, u_m$ by the fractions (5.7) in (5.6), and clearing denominators, we obtain a polynomial $h \in K[X_1, \dots, X_m]$ such that $h(a_1, \dots, a_m) = 0$. By Lemma 3.3.6 applied to g and $S = \{h\}$, there are $b_1, \dots, b_m \in C$ such that $g(b_1, \dots, b_m) \neq 0$ and $h(b_1, \dots, b_m) = 0$. But this implies that

$$f = Dw + \sum_{i=1}^{n} d_i \frac{Dw_i}{w_i}$$

where

$$w = \frac{p(b_1, \dots, b_m)}{q(b_1, \dots, b_m)}, \quad d_i = \frac{r_i(b_1, \dots, b_m)}{s_i(b_1, \dots, b_m)} \quad \text{and } w_i = \frac{p_i(b_1, \dots, b_m)}{q_i(b_1, \dots, b_m)} \,.$$

Since $p, q, p_i, q_i \in K[X_1, \dots, X_m]$ and $r_i, s_i \in C[X_1, \dots, X_m]$, we get $w \in K$, $w_1, \dots, w_n \in K^*$ and $d_1, \dots, d_n \in C$, which proves the theorem. □

We can finally remove all the constant restrictions in Liouville's Theorem, showing that for arbitrary constant subfields, v in (5.3) can be taken in K, and the u_i's can be taken in $K(c_1, \dots, c_n)$.

Theorem 5.5.3 (Liouville's Theorem – Strong version). *Let K be a differential field, $C = \mathrm{Const}(K)$, and $f \in K$. If there exist an elementary extension E of K and $g \in E$ such that $Dg = f$, then there are $v \in K$, $c_1, \dots, c_n \in \overline{C}$, and $u_1, \dots, u_n \in K(c_1, \dots, c_n)^*$ such that*

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i} \,.$$

Proof. Suppose that there exist an elementary extension E of K and $g \in E$ such that $Dg = f$. Since $\overline{C}K$ is algebraic over K, $\mathrm{Const}(\overline{C}K) = \overline{C} \cap \overline{C}K = \overline{C}$ by Corollary 3.3.1. Hence, $\overline{C}K$ has an algebraically closed constant subfield, $f \in \overline{C}K$, $g \in \overline{C}E$, which is an elementary extension of $\overline{C}K$, so by Theorem 5.5.2, there are $v \in \overline{C}K$, $u_1, \dots, u_n \in (\overline{C}K)^*$ and $c_1, \dots, c_n \in \overline{C}$ such that

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i} .$$

$F = K(v, u_1, \dots, u_n, c_1, \dots, c_n)$ is finite algebraic over K, so let $Tr_K^F : F \to K$ be the trace from F to K, $\overline{K}$ be the algebraic closure of K and $\sigma_1, \dots, \sigma_m$ be the distinct embeddings of F in $\overline{K}$ over K. Each σ_j can be extended to a field automorphism of $\overline{K}$ over K, and since Tr_K^F and each σ_j commute with D by Theorem 3.2.4, we have

$$mf = \sum_{j=1}^{m} f^{\sigma_j} = Tr_K^F(Dv) + \sum_{j=1}^{m}\sum_{i=1}^{n} c_i^{\sigma_j} \frac{Du_i^{\sigma_j}}{u_i^{\sigma_j}}$$

so

$$f = Dw + \sum_{j=1}^{m}\sum_{i=1}^{n} d_{ij} \frac{Dw_{ij}}{w_{ij}}$$

$$\text{with} \quad w = \frac{1}{m} Tr_K^F(v) \in K, \quad d_{ij} = \frac{1}{m} c_i^{\sigma_j} \in \overline{K} \quad \text{and} \quad w_{ij} = u_i^{\sigma_j} \in \overline{K}^* .$$

In addition, $\mathrm{Const}(\overline{K}) = \overline{C} \cap \overline{K} = \overline{C}$ by Corollary 3.3.1, and $Dd_{ij} = D(c_i^{\sigma_j}/m) = (Dc_i)^{\sigma_j}/m = 0$, so $d_{ij} \in \overline{C}$ for each i and j. Let now $L = K(d_{11}, \dots, d_{mn})$ and $M = L(w_{11}, \dots, w_{mn})$. Since L is algebraic over K, $\overline{K}$ is the algebraic closure of L. Since M is finite algebraic over L, let $Tr_L^M : M \to L$ and $N : M \to L$ be the trace and norm maps from M to L. Since $d_{ij} \in L$ and Tr_L^M is L-linear, we have

$$Tr_L^M \left(d_{ij} \frac{Dw_{ij}}{w_{ij}} \right) = d_{ij}\, Tr_L^M \left(\frac{Dw_{ij}}{w_{ij}} \right) = d_{ij}\, \frac{DN(w_{ij})}{N(w_{ij}}$$

by Theorem 3.2.4, so

$$\begin{aligned} kf = Tr_L^M(f) &= Tr_L^M(Dw) + Tr_L^M(\sum_{j=1}^{m}\sum_{i=1}^{n} d_{ij} \frac{Dw_{ij}}{w_{ij}}) \\ &= kDw + \sum_{j=1}^{m}\sum_{i=1}^{n} d_{ij} \frac{DN(w_{ij})}{N(w_{ij})} \end{aligned}$$

hence

$$f = Dw + \sum_{j=1}^{m}\sum_{i=1}^{n} \frac{d_{ij}}{k} \frac{Dz_{ij}}{z_{ij}}$$

which is of the form (5.3) with $w \in K$, $d_{ij} \in \overline{C}$ and $z_{ij} = N(w_{ij})$ in $K(d_{11}, \dots, d_{mn})^*$. $\square$

5.6 The Residue Criterion

Now that Liouville's Theorem gives us a way of proving that a function has no elementary integral over a given field, we can complete the integration algorithm. For the rest of this chapter, let (k, D) be a differential field and t a monomial over k. From the Hermite reduction, we can assume without loss of generality that the integrand is given as the sum of a simple and a reduced element of $k(t)$.

We have seen in Sect. 2.4 that the Rothstein–Trager algorithm expresses the integral of a simple rational function with no polynomial part as a sum of logarithms. In this section, we show that this algorithm can be generalized to any monomial extension, where it will either prove that a function has no elementary integral, or reduce the problem to integrating elements of $k\langle t\rangle$. Rothstein had already generalized this algorithm to elementary transcendental extensions in his dissertation [83].

Lemma 5.6.1. *Let $f \in k(t)$ be simple. If there are $h \in k\langle t\rangle$, an algebraic extension E of* Const(k)*, $v \in k(t)$, $c_1, \dots, c_n \in E$, and $u_1, \dots, u_n \in Ek(t)$ such that*

$$f + h = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}$$

then

$$\mathrm{residue}_p(f) = \sum_{i=1}^{n} c_i \nu_p(u_i)$$

for any normal irreducible $p \in Ek[t]$.

Proof. Let $f \in k(t)$ be simple, and suppose that there are $h \in k\langle t\rangle$, an algebraic extension E of Const(k), $v \in k(t)$, $c_1, \dots, c_n \in E$, and $u_1, \dots, u_n \in Ek(t)$ such that

$$f + h = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}\,.$$

Note that $f + h$ is simple since $h \in k\langle t\rangle$. Let $p \in Ek[t]$ be normal and irreducible. Then, for each i, $\nu_p(Du_i/u_i) \ge -1$ and $\mathrm{residue}_p(Du_i/u_i) = \nu_p(u_i)$ by Corollary 4.4.2. Suppose that $\nu_p(v) < 0$. Then $\nu_p(Dv) = \nu_p(v) - 1 < -1$ by Theorem 4.4.2, which implies that $\nu_p(f + h) < -1$ in contradiction with $f + h$ being simple. Hence $\nu_p(v) \ge 0$, so $\nu_p(Dv) \ge 0$, which implies that $\mathrm{residue}_p(Dv) = 0$. Furthermore, $\nu_p(h) \ge 0$, so $\mathrm{residue}_p(h) = 0$. Since $\mathrm{residue}_p$ is Ek-linear, we get

$$\begin{aligned}\mathrm{residue}_p(f) &= \mathrm{residue}_p(f) + \mathrm{residue}_p(h) = \mathrm{residue}_p(f+h)\\ &= \mathrm{residue}_p(Dv) + \sum_{i=1}^{n} c_i\, \mathrm{residue}_p\left(\frac{Du_i}{u_i}\right) = \sum_{i=1}^{n} c_i\, \nu_p(u_i)\,.\end{aligned}$$

□

Lemma 5.6.2. *Suppose that* $\mathrm{Const}(k)$ *is algebraically closed and let* $f \in k(t)$ *be simple. If there exists* $h \in k\langle t \rangle$ *such that* $f + h$ *has an elementary integral over* $k(t)$*, then* $\mathrm{residue}_p(f) \in \mathrm{Const}(k)$ *for any normal irreducible* $p \in k[t]$*.*

Proof. Let $C = \mathrm{Const}(k)$, and suppose C is algebraically closed and that $f+h$ has an elementary integral over $k(t)$ where $f \in k(t)$ is simple and $h \in k\langle t \rangle$. By Theorem 5.5.1, there are $v, u_1, \dots, u_n \in k$ and $c_1, \dots, c_n \in C$ such that

$$f + h = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}\,.$$

Let $p \in k[t]$ be normal and irreducible. By Lemma 5.6.1 we have

$$\mathrm{residue}_p(f) = \sum_{i=1}^{n} c_i\, \nu_p(u_i) \in C\,.$$

□

Example 5.6.1. Let $k = \mathbb{Q}$, t be a monomial over k with $Dt = 1$ (*i.e.* $D = d/dt$), and

$$f = \frac{2t-2}{t^2+1} \in k(t)\,.$$

Then, f has an elementary integral over $k(t)$:

$$\int \frac{2t-2}{t^2+1}\,dt = (1+\sqrt{-1})\log(1+t\sqrt{-1}) + (1-\sqrt{-1})\log(1-t\sqrt{-1})\,.$$

On the other hand, t^2+1 is irreducible over $\mathbb{Q}$, but

$$\mathrm{residue}_{t^2+1}(f) = \pi_{t^2+1}\left(\frac{2t-2}{2t}\right) = t+1$$

which is not a constant. This shows that the hypothesis that the constant field of k be algebraically closed is required in Lemma 5.6.2. If we replace $\mathbb{Q}$ by $\mathbb{C}$, then $t^2+1 = (t-\sqrt{-1})(t+\sqrt{-1})$,

$$\mathrm{residue}_{t-\sqrt{-1}}(f) = \pi_{t-\sqrt{-1}}\left(\frac{2t-2}{t+\sqrt{-1}}\right) = 1+\sqrt{-1}$$

and

$$\mathrm{residue}_{t+\sqrt{-1}}(f) = \pi_{t+\sqrt{-1}}\left(\frac{2t-2}{t-\sqrt{-1}}\right) = 1-\sqrt{-1}$$

which are constants. This shows that the hypothesis that p be irreducible is also required in Lemmas 5.6.1 and 5.6.2.

Theorem 5.6.1. *Let $f \in k(t)$ be simple, and write $f = p + a/d$ where $p, a, d \in k[t]$, $d \neq 0$, $\deg(a) < \deg(d)$, and $\gcd(a, d) = 1$. Let z be an indeterminate over k,*

$$r = \text{resultant}_t(a - zDd, d) \in k[z]\,,$$

$r = r_s r_n$ be a splitting factorization of r w.r.t. the coefficient lifting κ_D of D to $k[z]$, and

$$g = \sum_{r_s(\alpha)=0} \alpha \, \frac{Dg_\alpha}{g_\alpha} \tag{5.8}$$

where $g_\alpha = \gcd(a - \alpha Dd, d) \in k(\alpha)[t]$ and the sum is taken over all the distinct roots of r_s. Then,

(i) $g \in k(t)$, the denominator of g divides d, and $f - g$ is simple.
(ii) If there exists $h \in k\langle t\rangle$ such that $f + h$ has an elementary integral over $k(t)$, then $r_n \in k$ and $f - g \in k[t]$.
(iii) If there are $h \in k\langle t\rangle$, an algebraic extension E of $\mathrm{Const}(k)$, $v \in k(t)$, $c_1, \ldots, c_n \in E$, and $u_1, \ldots, u_n \in Ek(t)$ such that

$$f + h = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}$$

then r_s factors linearly over E.

Proof. (i) Let $r_s = c r_1^{e_1} \cdots r_n^{e_n}$ be the irreducible factorization of r_s in $k[z]$. Then, g can be rewritten as

$$g = \sum_{i=1}^{n} \sum_{r_i(\alpha)=0} \alpha \, \frac{Dg_\alpha}{g_\alpha}\,.$$

For each i, let k_i be $k(t)$ extended by all the roots of r_i, and α_i be a given root of r_i. Since k_i is a finitely generated algebraic extension of $k(t)$, the field automorphisms of k_i over $k(t)$ commute with D by Theorem 3.2.4, so we get

$$g = \sum_{i=1}^{n} Tr_i \left(\alpha_i \, \frac{Dg_{\alpha_i}}{g_{\alpha_i}} \right)$$

by Theorem 3.2.4 where Tr_i is the trace map from $k(t)(\alpha_i)$ to $k(t)$. Hence, $g \in k(t)$. Furthermore, since $g_\alpha \mid d$ for each root α of r_s, $\mathrm{lcm}_{r_s(\alpha)=0}(g_\alpha) \mid d$, so the denominator of g also divides d. Hence the denominator of $f - g$ divides d, which implies that $f - g$ is simple since d is normal.
(ii) Suppose that $f + h$ has an elementary integral over $k(t)$ for some $h \in k\langle t\rangle$, and let $\overline{k}$ be the algebraic closure of k. By Corollary 3.4.1, t is a monomial over $\overline{k}$, and simple (resp. reduced) elements of $k(t)$ remain simple (resp. reduced) when viewed as elements of $\overline{k}(t)$. Furthermore $f + h$ has an elementary integral over $\overline{k}(t)$, so we work with $\overline{k}(t)$ in the rest of this proof. Let $\alpha \in \overline{k}$ be any

root of r. If $\alpha = 0$, then $D\alpha = 0$. Otherwise $\alpha \neq 0$ and $\alpha = \mathrm{residue}_q(f)$ for some normal irreducible $q \in \overline{k}[x]$ by Theorem 4.4.3, hence $D\alpha = 0$ by Lemma 5.6.2. Thus $r_s(\alpha) = 0$ in both cases by Theorem 3.5.2, so $r_n(\alpha) \neq 0$ since $\gcd(r_n, r_s) = 1$. Since this holds for all the roots of r, we have $r_n \in k$.

For any $\alpha \in \overline{k}$, write $g_\alpha = \gcd(d, a - \alpha Dd)$. Note that all the irreducible factors of g_α must be normal, since $g_\alpha \mid d$, which is normal. Let $\alpha, \beta \in \overline{k}$, and $q \in \overline{k}[t]$ be a normal irreducible common factor of g_α and g_β. Then $\alpha = \mathrm{residue}_q(a/d) = \beta$ by Lemma 4.4.3, so $\gcd(g_\alpha, g_\beta) = 1$ when $\alpha \neq \beta$. Let now $q \in \overline{k}[t]$ be irreducible and normal, and $\beta = \mathrm{residue}_q(f)$. If $\beta = 0$, then q does not divide d, so q does not divide any g_α, which implies that $\nu_q(g) \geq 0$, hence that $\mathrm{residue}_q(g) = 0 = \mathrm{residue}_q(f - g)$. If $\beta \neq 0$, then $r(\beta) = 0$ by Theorem 4.4.3, and $q \mid g_\beta$ by Lemma 4.4.3, so $r_s(\beta) = 0$ since $r_n \in k$. Since d is squarefree, g_β is squarefree, so $\nu_q(g_\beta) = 1$. By Theorem 4.4.1, $\mathrm{residue}_q$ is $\overline{k}$-linear, so we get

$$\mathrm{residue}_q(f - g) = \beta - \sum_{r_s(\alpha)=0} \alpha \, \mathrm{residue}_q\left(\frac{Dg_\alpha}{g_\alpha}\right) = \beta - \sum_{r_s(\alpha)=0} \alpha \, \nu_q(g_\alpha)$$

by Corollary 4.4.2. Since $\nu_q(g_\alpha) = 0$ for $\alpha \neq \beta$, this gives $\mathrm{residue}_q(s) = \beta - \beta = 0$. Since this holds for any normal irreducible $q \in \overline{k}[t]$ and $f - g$ is simple, we have $f - g \in \overline{k}[t]$, hence $f - g \in k[t]$.
(iii) Suppose that there are $h \in k\langle t \rangle$, an algebraic extension E of $\mathrm{Const}(k)$, $v \in k(t)$, $c_1, \dots, c_n \in E$, and $u_1, \dots, u_n \in Ek(t)$ such that

$$f + h = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}\,. \tag{5.9}$$

Let $\overline{k}$ be the algebraic closure of k. As explained in part (ii), we can replace $k(t)$ by $\overline{k}(t)$ and view (5.9) as an equality in $\overline{k}(t)$. Let $\alpha \in \overline{k}$ be any root of r_s. By Theorem 4.4.3, $\alpha = \mathrm{residue}_p(f)$ for some normal irreducible $p \in \overline{k}[t]$, so by Lemma 5.6.1

$$\alpha = \mathrm{residue}_p(f) = \sum_{i=1}^{n} c_i \nu_p(u_i) \in E\,.$$

Hence, E contains all the roots of r_s in $\overline{k}$, so r_s factors linearly over E. □

Note that since the roots of r_s are all constants by Theorem 3.5.2, g as given by (5.8) always has an elementary integral, namely

$$\int g = \sum_{r_s(\alpha)=0} \alpha \log(\gcd(d, a - \alpha Dd))$$

which is the Rothstein–Trager formula in the case of rational functions. Part (iii) of Theorem 5.6.1 applied to the rational function case proves part (iii) of Theorem 2.4.1, thereby completing the proof of that theorem. As in the rational function case, a prime factorization $r_s = u\, s_1^{e_1} \cdots s_m^{e_m}$ is required, as well as a gcd computation in $k(\alpha_i)[t]$ for each i, where α_i is a root of s_i. There is no need however to compute the splitting field of r_s. Furthermore, the monic part of r_s always has constant coefficients.

ResidueReduce(f, D) (* Rothstein–Trager resultant reduction *)

(* Given a derivation D on $k(t)$ and $f \in k(t)$ simple, return g elementary over $k(t)$ and a Boolean $b \in \{0,1\}$ such that $f - Dg \in k[t]$ if $b = 1$, or $f + h$ and $f + h - Dg$ do not have an elementary integral over $k(t)$ for any $h \in k\langle t \rangle$ if $b = 0$. *)

$d \leftarrow$ denominator(f)
$(p, a) \leftarrow$ **PolyDivide**(numerator$(f), d)$ $(*\ f = p + a/d\ *)$
$z \leftarrow$ a new indeterminate over $k(t)$
$r \leftarrow \text{resultant}_t(d, a - zDd)$
$(r_n, r_s) \leftarrow$ **SplitFactor**(r, κ_D)
$u\, s_1^{e_1} \cdots s_m^{e_m} \leftarrow$ **factor**(r_s) (* factorization into irreducibles *)
for $i \leftarrow 1$ **to** m **do**
  $\alpha \leftarrow \alpha \mid s_i(\alpha) = 0$
  $g_i \leftarrow \gcd(d, a - \alpha Dd)$ (* algebraic gcd computation *)
if $r_n \in k$ **then** $b \leftarrow 1$ **else** $b \leftarrow 0$
return$(\sum_{i=1}^m \sum_{\alpha \mid s_i(\alpha)=0} \alpha \log(g_i), b)$

Example 5.6.2. Consider

$$\int \frac{2\log(x)^2 - \log(x) - x^2}{\log(x)^3 - x^2 \log(x)}\, dx\,.$$

Let $k = \mathbb{Q}(x)$ with $D = d/dx$, and let t be a monomial over k satisfying $Dt = 1/x$, *i.e.* $t = \log(x)$. Our integrand is then

$$f = \frac{2t^2 - t - x^2}{t^3 - x^2 t} \in k(t)$$

which is simple since $t^3 - x^2 t$ is squarefree. We get

$$d = t^3 - x^2 t, \quad p = 0, \quad a = 2t^2 - t - x^2$$

and

$$\begin{aligned} r &= \text{resultant}_t\left((t^3 - x^2 t, \frac{2x - 3z}{x} t^2 + (2xz - 1)t + x(z - x)\right) \\ &= 4x^3(1 - x^2)\left(z^3 - xz^2 - \frac{1}{4} z + \frac{x}{4}\right) \end{aligned}$$

which is squarefree. Then,

$$\kappa_D r = -x^2(4(5x^2+3)z^3 + 8x(3x^2-2)z^2 + (5x^2-3)z - 2x(3x^2-2))$$

so the splitting factorization of r w.r.t. κ_D is

$$r_s = \gcd(r, \kappa_D r) = x^2\left(z^2 - \frac{1}{4}\right)$$

and

$$r_n = \frac{r}{r_s} = -4x(x^2-1)(z-x) \notin k\,.$$

Hence, f does not have an elementary integral. Proceeding further we get

$$g_1 = \gcd\left(t^3 + x^2 t, \frac{2x-3\alpha}{x}t^2 + (2x\alpha - 1)t + x(\alpha - x)\right) = t + 2\alpha x$$

where $\alpha^2 - 1/4 = 0$, so

$$g = \sum_{\alpha|\alpha^2-1/4=0} \alpha \log(t + 2\alpha x) = \frac{1}{2}\log(t+x) - \frac{1}{2}\log(t-x)\,.$$

Computing $f - Dg$ we find

$$\begin{aligned}\int \frac{2\log(x)^2 - \log(x) - x^2}{\log(x)^3 - x^2\log(x)}\,dx &= \frac{1}{2}\log\left(\frac{\log(x)+x}{\log(x)-x}\right) + \int \frac{dx}{\log(x)} \\ &= \frac{1}{2}\log\left(\frac{\log(x)+x}{\log(x)-x}\right) + \mathrm{Li}(x)\end{aligned}$$

where $\mathrm{Li}(x)$ is the logarithmic integral, which has been proven to be nonelementary since $r_n \notin k$.

With the notation as in Theorem 5.6.1, we have $\gcd(r_s, r_n) = 1$, so any root α of r_s with multiplicity n is also a root of r with multiplicity n. Since $\gcd(a,d) = \gcd(d, Dd) = 1$ and $\deg(a) < \deg(d)$, we can apply Theorem 2.5.1 with $A = a$, $B = Dd$ and $C = d$, and we get that for any root α of r of multiplicity $i > 0$,

$$\gcd(d, a - \alpha Dd) = \mathrm{pp}_t(R_m)(\alpha, t)$$

where $\deg_t(R_m) = i$ and R_m is in the subresultant PRS of d and $a - zDd$ if $\deg(Dd) \le \deg(d)$, or of $a - zDd$ and d if $\deg(Dd) > \deg(d)$. Thus, the Lazard–Rioboo–Trager algorithm is applicable in arbitrary monomial extensions, and it is not necessary to compute the prime factorization of r_s, or the g_α's appearing in (5.8), we can use the various remainders appearing in the subresultant PRS instead. As in the case of rational functions, we use a squarefree factorization of $r_s = \prod_{i=1}^n q_i^i$ to split the sum appearing in (5.8) into several summands, each indexed by the roots of q_i. We can also avoid computing $\mathrm{pp}_t(R_m)$, ensuring instead that its leading coefficient is coprime with the corresponding q_i. And since multiplying any g_α in (5.8) by an arbitrary nonzero element of $k(\alpha)$ does not change the conclusion of Theorem 5.6.1, we can make $\mathrm{pp}_t(R_m)(\alpha, t)$ monic in order to simplify the answer. This last step requires inverting an element of $k[\alpha]$ and is optional. As in the rational function case, it turns out that the leading coefficients of the $\mathrm{pp}_t(R_m)(\alpha, t)$'s are always invertible in $k[\alpha]$ (Exercise 2.7).

ResidueReduce(f, D)
(* Lazard–Rioboo–Rothstein–Trager resultant reduction *)

(* Given a derivation D on $k(t)$ and $f \in k(t)$ simple, return g elementary over $k(t)$ and a Boolean $b \in \{0,1\}$ such that $f - Dg \in k[t]$ if $b = 1$, or $f + h$ and $f + h - Dg$ do not have an elementary integral over $k(t)$ for any $h \in k\langle t\rangle$ if $b = 0$. *)

$d \leftarrow$ denominator(f)
$(p, a) \leftarrow$ **PolyDivide**(numerator$(f), d)$ $\qquad$ $(*\ f = p + a/d\ *)$
$z \leftarrow$ a new indeterminate over $k(t)$
if $\deg(Dd) \le \deg(d)$
then $(r, (R_0, R_1, \dots, R_q, 0)) \leftarrow$ **SubResultant**$_x(d, a - zDd)$
else $(r, (R_0, R_1, \dots, R_q, 0)) \leftarrow$ **SubResultant**$_x(a - zDd, d)$
$((n_1, \dots, n_n), (s_1, \dots, s_n)) \leftarrow$ **SplitSquarefreeFactor**(r, κ_D)
for $i \leftarrow 1$ **to** n such that $\deg(s_i) > 0$ **do**
  if $i = \deg(d)$ **then** $S_i \leftarrow d$
  else
    $S_i \leftarrow R_m$ where $\deg_t(R_m) = i, \quad 1 \le m < q$
    $(A_1, \dots, A_s) \leftarrow$ **SquareFree**$(\mathrm{lc}_t(S_i))$
    for $j \leftarrow 1$ **to** s **do** $S_i \leftarrow S_i / \gcd_z(A_j, s_i)^j$ $\qquad$ (* exact quotient *)
if $\prod_{i=1}^n n_i \in k$ **then** $b \leftarrow 1$ **else** $b \leftarrow 0$
return$(\sum_{i=1}^m \sum_{\alpha | s_i(\alpha)=0} \alpha \log(S_i(\alpha, t)), b)$

Example 5.6.3. Consider the same integrand as in example 5.6.2. We have $\deg(Dd) < \deg(d)$ and the subresultant PRS of d and $a - zDd$ is

i	R_i
0	$t^3 - x^2 t$
1	$(2 - 3z/x)t^2 + (2xz - 1)t + x(z - x)$
2	$(4x^2 - 6)z^2 + 3xz - 2x^2 + 1)t + x(z - x)(2xz - 1)$
3	$4x^3(1 - x^2)\left(z^3 - xz^2 - \frac{1}{4}z + \frac{1}{4}x\right)$

The Rothstein–Trager resultant is $r = R_3$, and its split–squarefree factorization w.r.t. κ_D is

$$s_1 = \gcd(r, \kappa_D r) = x^2 \left(z^2 - \frac{1}{4}\right), \qquad n_1 = \frac{r}{s_1} = -4x(x^2-1)(z-x) \notin k\,.$$

Hence, f does not have an elementary integral. Proceeding further we find that s_1 is squarefree, and the remainder of degree 1 in t in the PRS is

$$R_2 = ((4x^2-6)z^2 + 3xz - 2x^2 + 1)t + x(z-x)(2xz-1)\,.$$

Since

$$\gcd(\mathrm{lc}_t(R_2), s_1) = \gcd\left((4x^2-6)z^2 + 3xz - 2x^2 + 1, x^2\left(z^2 - \frac{1}{4}\right)\right) = 1\,,$$

$S_1 = R_2$. Evaluating for z at a root α of $z^2 - 1/4 = 0$ we get

$$S_1(\alpha, t) = -\frac{1}{2}((2x^2 - 6\alpha x + 1)t + 4\alpha x^3 - 3x^2 + 2\alpha x)$$

so

$$\begin{aligned} g &= \sum_{\alpha \mid \alpha^2 - 1/4 = 0} \alpha \log\left(-\frac{1}{2}((2x^2 - 6\alpha x + 1)t + x(4\alpha x^2 - 3x + 2\alpha))\right) \\ &= \frac{1}{2}\log\left(-\frac{(2x^2-3x+1)(t+x)}{2}\right) - \frac{1}{2}\log\left(-\frac{(2x^2+3x+1)(t-x)}{2}\right)\,. \end{aligned}$$

Computing $f - Dg$ we find

$$\begin{aligned} \int \frac{2\log(x)^2 - \log(x) - x^2}{\log(x)^3 - x^2\log(x)}\,dx &= \frac{1}{2}\log\left(\frac{(2x^2-3x+1)(\log(x)+x)}{(2x^2+3x+1)(\log(x)-x)}\right) \\ &\quad + \int\left(\frac{1}{\log(x)} - \frac{6x^2-3}{4x^4-5x^2+1}\right)dx \end{aligned}$$

where the remaining integral has been proven to be nonelementary. In fact, it is the integral of a rational function plus the logarithmic integral $\mathrm{Li}(x)$.
If we had decided to make $S_1(\alpha, t)$ monic, we would have obtained

$$S_1(\alpha, x) = -\frac{1}{2}(2x^2 - 6\alpha x + 1)(t + 2\alpha x)$$

so the integral is then the same as in example 5.6.2.

5.7 Integration of Reduced Functions

From the results of the previous sections, we are left with the problem of integrating reduced elements of a monomial extension. We use a specialized version of Liouville's Theorem for such elements.

Theorem 5.7.1. *Let k be a differential field, t be a monomial over k, $C = \mathrm{Const}(k(t))$, and $f \in k\langle t\rangle$. If there exist an elementary extension E of $k(t)$ and $g \in E$ such that $Dg = f$, then there are $v \in k\langle t\rangle$, $c_1, \dots, c_n \in \overline{C}$, and $u_1, \dots, u_n \in \mathcal{S}_{k(c_1,\dots,c_n)[t]:k(c_1,\dots,c_n)}$ such that*

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}\,.$$

Proof. Suppose that there exist an elementary extension E of $k(t)$ and $g \in E$ such that $Dg = f$. Then, by Theorem 5.5.3, there are $v \in k(t)$, $c_1, \dots, c_n \in \overline{C}$, and $u_1, \dots, u_n \in k(c_1, \dots, c_n)(t)$ such that $f = Dv + \sum_{i=1}^{n} c_i D(u_i)/u_i$. Write $g = \sum_{i=1}^{n} c_i D(u_i)/u_i$. Since $g = f - Dv$, it follows that $g \in k(t)$. Let $p \in k[t]$ be normal and irreducible, and $q \in k(c_1, \dots, c_n)[t]$ be any irreducible factor of p over $k(c_1, \dots, c_n)$. Then, $\nu_p(f) \geq 0$ by Corollary 4.4.1, and $\nu_q(c_i Du_i/u_i) \geq -1$ for each i by Corollary 4.4.2, so $\nu_q(g) \geq -1$. Since this holds for any irreducible factor q of p and $g \in k(t)$, Theorem 4.1.2 implies that $\nu_p(g) \geq -1$. Suppose that $\nu_p(v) < 0$. Then, $\nu_p(Dv) = \nu_p(v) - 1 < -1$ by Theorem 4.4.2, which implies that $\nu_p(Dv + g) < -1$, hence that $\nu_p(f) < -1$, in contradiction with f reduced. Hence, $\nu_p(v) \geq 0$ for all normal irreducible $p \in k[t]$, which means that $v \in k\langle t\rangle$ and $Dv \in k\langle t\rangle$ by Corollary 4.4.1.

Write now $u_i = w_i \prod_{j=1}^{n_i} p_{ij}^{e_{ij}}$ where $w_i \in k(c_1, \dots, c_n)$, each p_{ij} is a monic irreducible element of $k(c_1, \dots, c_n)[t]$, and the e_{ij}'s are integers. Then, using the logarithmic derivative identity and grouping together all the terms involving the same p_{ij}, we get

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Dw_i}{w_i} + \sum_{j=1}^{N} d_j \frac{Dq_j}{q_j} \tag{5.10}$$

where the q_j's are in $k(c_1, \dots, c_n)[t]$, monic, irreducible and coprime. Each w_i is special since it is in $k(c_1, \dots, c_n)$. Suppose that q_s is normal for some s. Then, Lemma 5.6.1 applied to (5.10) implies that

$$\mathrm{residue}_{q_s}(f) = \sum_{i=1}^{n} c_i \nu_{q_s}(w_i) + \sum_{j=1}^{n} d_j \nu_{q_s}(q_j)\,.$$

But $\mathrm{residue}_{q_s}(f) = 0$ since $f \in k\langle t\rangle$, and $\nu_{q_s}(w_i) = 0$ since $w_i \in k(c_1, \dots, c_n)$, and $\nu_{q_s}(q_j) = 0$ for $j \neq s$ since the q_j's are coprime. Hence, $0 = d_s \nu_{q_s}(q_s) = d_s$, so $d_s = 0$ whenever q_s is normal. Keeping only the nonzero summands in (5.10), we get that each q_j is special, which proves the theorem. $\square$

In the case of nonlinear monomials, we have seen that we can always rewrite a polynomial $p \in k[t]$ as the sum of a derivative and a polynomial of degree less than $\delta(t)$. We then have an analogue of the residue criterion that either proves that such a reduced function does not have an elementary integral, or eliminates the term of degree $\delta(t) - 1$ from its polynomial part.

Theorem 5.7.2. *Suppose that t is a nonlinear monomial. Let $f \in k\langle t \rangle$ and write $f = p + a/d$ where $p, a, d \in k[t]$, $d \neq 0$, $\deg(p) < \delta(t)$ and $\deg(a) < \deg(d)$. Let $b \in k$ be the coefficient of $t^{\delta(t)-1}$ in p, and $c = b/\lambda(t)$. If f has an elementary integral over $k(t)$ then $Dc = 0$.*

Proof. Let $C = \mathrm{Const}(k)$. Replacing C by its algebraic closure, we can assume without loss of generality that C is algebraically closed. Suppose that f has an elementary integral over $k(t)$. Then, by Theorem 5.7.1, there are $v \in k\langle t \rangle$, $c_1, \dots, c_n \in C$, and $u_1, \dots, u_n \in \mathcal{S}$ such that

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}\,. \tag{5.11}$$

By Theorem 4.4.4, $\nu_\infty(Du_i/u_i) \geq -m$ and $\pi_\infty(t^{-m}Du_i/u_i) = -\nu_\infty(u_i)\lambda(t)$ for each i where $m = \delta(t) - 1$. Furthermore, $\nu_\infty(a/d) > 0$ since $\deg(a) < \deg(d)$, so $\nu_\infty(f) \geq -\deg(p) \geq -m$ and $\pi_\infty(a/d) = \pi_\infty(t^{-m}a/d) = 0$, which implies that $\pi_\infty(t^{-m}f) = b$. Suppose that $\nu_\infty(v) < 0$, then $\nu_\infty(Dv) < -m$ by Theorem 4.4.4, so $\nu_\infty(Dv + \sum_{i=1}^n c_i Du_i/u_i) < -m$, in contradiction with $\nu_\infty(f) \geq -m$. Hence $\nu_\infty(v) \geq 0$. If $\nu_\infty(v) > 0$, then $\nu_\infty(Dv) > -m$ by Theorem 4.4.4. Otherwise, $\nu_\infty(v) = 0$ and $\nu_\infty(Dv) > -m$ also by Theorem 4.4.4. Hence $\nu_\infty(t^{-m}Dv) > 0$ in any case, so $\pi_\infty(t^{-m}Dv) = 0$. Multiplying both sides of (5.11) by t^{-m} and applying π_∞, we get

$$b = \pi_\infty(t^{-m}f) = \sum_{i=1}^{n} c_i\, \pi_\infty\left(t^{-m}\frac{Du_i}{u_i}\right) = -\sum_{i=1}^{n} c_i\, \nu_\infty(u_i)\, \lambda(t)$$

hence $c = b/\lambda(t) = -\sum_{i=1}^n c_i\, \nu_\infty(u_i)$, so $Dc = 0$. □

If c is a constant, then Theorem 5.4.2 implies that

$$f - D\left(\frac{c}{\deg(q)}\log(q)\right)$$

has degree at most $\delta(t) - 2$ for any $q \in \mathcal{S} \setminus k$, so in the case of nonlinear monomials, we are left with reduced integrands with polynomial parts of degree at most $\delta(t) - 2$, provided that we know at least one nontrivial special polynomial. If we know that there are no nontrivial special polynomials, then integrating reduced elements of such nonlinear extensions is in fact easier, and an algorithm for that purpose will be presented in Sect. 5.11.

We have now all the necessary tools to complete the integration algorithm. In the following sections, we give algorithms that, given an integrand f in $k(t)$ for a monomial t, either prove that f has no elementary integral over $k(t)$, or compute an elementary extension E of $k(t)$ and an element $g \in E$ such that $f - Dg \in k$. This process eliminates t from the integrand, thus reducing the problem to integrating an element of k, which can be done recursively, *i.e.* the algorithms of this chapter can be applied to elements of k until we are left with constants to integrate. Note that when t itself is not elementary over k, then the problems of deciding whether an element of k has an elementary integral over k or over $k(t)$ are fundamentally different, so our algorithms will produce proofs of nonintegrability only if the integrand is itself an elementary function. They can be applied however to much larger classes of functions.

It turns out that it will also be necessary to assume that some related problems are solvable for elements of k. Those problems depend on the kind of monomial we are dealing with, so we need to handle the various cases separately at this point. Algorithms for all those related problems will be presented in later chapters.

5.8 The Primitive Case

In the case of primitive monomials over a differential field k, the related problem we need to solve over k is the *limited integration problem*: recall that the problem of integration in closed form is, given $f \in k$ to determine whether there exist an elementary extension E of k and $g \in E$ such that $\mathrm{Const}(E)$ is algebraic over $\mathrm{Const}(k)$ and $Dg = f$. Let $w_1, \dots, w_n \in k$ be fixed. The *problem of limited integration with respect to* $w_1, \dots, w_n$ is: given $f \in k$, determine whether there are $g \in k$ and $c_1, \dots, c_n \in \mathrm{Const}(k)$ such that $Dg = f - c_1 w_1 - \dots - c_n w_n$, and to compute g and the c_i's if they exist. It is very similar to the problem of integration in closed form, except that the specific differential extension $k(\int w_1, \dots, \int w_n)$ is provided for the integral. We present in this section an algorithm that, with appropriate assumptions on k, integrates elements of $k(t)$ when t is a primitive monomial over k. We first describe an algorithm for integrating elements of $k[t]$.

Theorem 5.8.1. *Let k be a differential field and t a primitive over k. If the problem of limited integration w.r.t. Dt is decidable for elements of k, and Dt is not the derivative of an element of k, then for any $p \in k[t]$ we can either prove that p has no elementary integral over $k(t)$, or compute $q \in k[t]$ such that $p - Dq \in k$.*

Proof. We proceed by induction on $m = \deg(p)$. If $m = 0$, then $p \in k$ and $q = 0$ satisfies the theorem, so suppose that $m > 0$ and that the theorem holds for any polynomial of degree less than m. Since Dt is not the derivative of an element of k, t is a monomial over k, $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$, and $\mathcal{S} = k$ by Theorem 5.1.1. Thus, Theorem 5.7.1 says that if p has an elementary integral over $k(t)$, then there are $v \in k[t]$, $c_1, \dots, c_n \in \overline{C}$ and $u_1, \dots, u_n \in k(c_1, \dots, c_n)$ such that

$$p = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i} \tag{5.12}$$

where $C = \mathrm{Const}(k)$. $K = k(c_1, \dots, c_n)$ is an algebraic extension of k, so t is transcendental over K. Furthermore, Dt is not the derivative of an element of

K by Lemma 5.1.1, so t is a monomial over K and $\mathrm{Const}(K(t)) = \mathrm{Const}(K)$. Equating degrees in (5.12) we get $\deg(Dv) = \deg(p) = m > 0$, so $\deg(v) \le m+1$ by Lemma 5.1.2, so write $p = at^m + s$ and $v = ct^{m+1} + bt^m + w$ where $a, b, c \in k$, $s, w \in k[t]$, $\deg(s) < m$ and $\deg(w) < m$. Equating the coefficients of t^{m+1} and t^m in (5.12) we get $Dc = 0$ and

$$a = Db + (m+1)\, c\, Dt\,. \tag{5.13}$$

Since we can solve the problem of limited integration w.r.t. Dt for elements of k and $a \in k$, we can either prove that (5.13) has no solution $b \in k, c \in \mathrm{Const}(k)$, or find such a solution. If it has no solution, then (5.12) has no solution so p has no elementary integral over $k(t)$. If we have a solution b, c, letting $q_0 = ct^{m+1} + bt^m$, we get

$$p - Dq_0 = (at^m + s) - ((m+1)cDt + Db)\, t^m - (mbDt)t^{m-1} = s - (mbDt)t^{m-1}$$

hence $\deg(p - Dq_0) < m$. By induction we can either prove that $p - Dq_0$ has no elementary integral over $k(t)$, in which case p has no elementary integral over $k(t)$, or we get $q_1 \in k[t]$ such that $p - Dq_0 - Dq_1 \in k$, which implies that $p - Dq \in k$ where $q = q_0 + q_1$. $\square$

IntegratePrimitivePolynomial(p, D)
(* Integration of polynomials in a primitive extension *)

(* Given a is a primitive monomial t over k, and $p \in k[t]$, return $q \in k[t]$ and a Boolean $\beta \in \{0, 1\}$ such that $p - Dq \in k$ if $\beta = 1$, or $p - Dq$ does not have an elementary integral over $k(t)$ if $\beta = 0$. *)

if $p \in k$ **then return**$(0, 1)$
$a \leftarrow \mathrm{lc}(p)$
(* LimitedIntegrate will be given in Chap. 7 *)
$(b, c) \leftarrow$ **LimitedIntegrate**(a, Dt, D) $(*\ a = Db + cDt\ *)$
if $(b, c) =$ "no solution" **then return**$(0, 0)$
$m \leftarrow \deg(p)$
$q_0 \leftarrow ct^{m+1}/(m+1) + bt^m$
$(q, \beta) \leftarrow$ **IntegratePrimitivePolynomial**$(p - Dq_0, D)$
return$(q + q_0, \beta)$

Example 5.8.1. Consider

$$\int \left(\left(\log(x) + \frac{1}{\log(x)} \right) \mathrm{Li}(x) - \frac{x}{\log(x)} \right) dx$$

where $\mathrm{Li}(x) = \int dx / \log(x)$ is the logarithmic integral. Let $k = \mathbb{Q}(x, t_0)$ with $D = d/dx$, where t_0 is a monomial over $\mathbb{Q}(x)$ satisfying $Dt_0 = 1/x$, *i.e.* $t_0 =$

$\log(x)$, and let t be a monomial over k satisfying $Dt = 1/t_0$, *i.e.* $t = \mathrm{Li}(x)$. Our integrand is then

$$p = \left(t_0 + \frac{1}{t_0}\right) t - \frac{x}{t_0} \in k[t]\,.$$

We get

1. $a = \mathrm{lc}(p) = t_0 + 1/t_0$
2. $$\left(t_0 + \frac{1}{t_0}\right) - \frac{1}{t_0} = t_0 = \log(x) = \frac{d}{dx}\left(x\log(x) - x\right) = D(xt_0 - x)$$
   so $(b, c) = \mathbf{LimitedIntegrate}(t_0 + 1/t_0, 1/t_0, D) = (xt_0 - x, 1)$
3. $q_0 = ct^2/2 + bt = t^2/2 + (xt_0 - x)t$
4. $p - Dq_0 = -x \in k$ so the call **IntegratePrimitivePolynomial**$(-x, D)$ returns $(q, \beta) = (0, 1)$.

Hence,

$$\begin{aligned}\int \left(\left(\log(x) + \frac{1}{\log(x)}\right)\mathrm{Li}(x) - \frac{x}{\log(x)}\right) dx & \\ &= \frac{\mathrm{Li}(x)^2}{2} + (x\log(x) - x)\mathrm{Li}(x) - \int x dx \\ &= \frac{\mathrm{Li}(x)^2}{2} + (x\log(x) - x)\mathrm{Li}(x) - \frac{x^2}{2}\,.\end{aligned}$$

Putting all the pieces together, we get an algorithm for integrating elements of $k(t)$.

Theorem 5.8.2. *Let k be a differential field and t a primitive over k. If the problem of limited integration w.r.t. Dt is decidable for elements of k, and Dt is not the derivative of an element of k, then for any $f \in k(t)$ we can either prove that f has no elementary integral over $k(t)$, or compute an elementary extension E of $k(t)$ and $g \in E$ such that $f - Dg \in k$.*

Proof. Suppose that Dt is not the derivative of an element of k, then t is a monomial over k and $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$ by Theorem 5.1.1. Let $f \in k(t)$. By Theorem 5.3.1, we can compute $g_1, h, r \in k(t)$ such that $f = Dg_1 + h + r$, h is simple and r is reduced. From h, which is simple, we compute $g_2 \in k(t)$ given by (5.8) in Theorem 5.6.1. Note that $g_0 = g_1 + \int g_2$ lies in some elementary extension of $k(t)$. Let $p = h - g_2$ and $q = p + r$, then $f = Dg_0 + q$ so f has an elementary integral over $k(t)$ if and only if q has one. If $p \notin k[t]$, then $p + r$ does not have an elementary integral over $k(t)$ by Theorem 5.6.1, so f does not have an elementary integral over $k(t)$. Suppose now that $p \in k[t]$. We have $k\langle t\rangle = k[t]$ by (5.1), so $r \in k[t]$, hence $q \in k[t]$. By Theorem 5.8.1 we can either prove that q has no elementary integral over $k(t)$, in which case f has no elementary integral over $k(t)$, or compute $s \in k[t]$ such that $q - Ds \in k$, in which case $f - Dg \in k$ where $g = g_0 + s$. $\square$

IntegratePrimitive(f, D) (* Integration of primitive functions *)

(* Given a is a primitive monomial t over k, and $f \in k(t)$, return g elementary over $k(t)$ and $\beta \in \{0,1\}$ such that $f - Dg \in k$ if $\beta = 1$, or $f - Dg$ does not have an elementary integral over $k(t)$ if $\beta = 0$. *)

$(g_1, h, r) \longleftarrow$ **HermiteReduce**(f, D)
$(g_2, \beta) \longleftarrow$ **ResidueReduce**(h, D)
if $\beta = 0$ **then return**$(g_1 + g_2, 0)$
$(q, \beta) \longleftarrow$ **IntegratePrimitivePolynomial**$(h - Dg_2 + r, D)$
return$(g_1 + g_2 + q, \beta)$

5.9 The Hyperexponential Case

In the case of hyperexponential monomials over a differential field k, the related problem we need to solve over k is the *Risch differential equation problem*: given $f, g \in k$, determine whether there exists $y \in k$ such that

$$Dy + fy = g \tag{5.14}$$

and to compute y if it exists. It may happen in general that (5.14) has more than one solution in k, so we first need to examine when this can happen.

Lemma 5.9.1. *Let (K, D) be a differential field. If there are $\alpha, y, z \in K$ such that $y \neq z$ and $Dy + \alpha y = Dz + \alpha z$, then $\alpha = Du/u$ for some $u \in K^*$.*

Proof. Let $u = 1/(y - z) \in K^*$. Then,

$$Du - \alpha u = -\frac{Dy - Dz}{(y-z)^2} - \frac{\alpha}{y - z} = \frac{(Dz + \alpha z) - (Dy + \alpha y)}{(y-z)^2} = 0$$

so $\alpha = Du/u$. □

We present in this section an algorithm that, with appropriate assumptions on k, integrates elements of $k(t)$ when t is a hyperexponential monomial over k. We first describe an algorithm for integrating elements of $k\langle t\rangle$.

Theorem 5.9.1. *Let k be a differential field and t an hyperexponential over k. If we can solve Risch differential equations over k, and Dt/t is not a logarithmic derivative of a k-radical, then for any $p \in k\langle t\rangle$ we can either prove that p has no elementary integral over $k(t)$, or compute $q \in k\langle t\rangle$ such that $p - Dq \in k$.*

Proof. Since Dt/t is not a logarithmic derivative of a k-radical, t is a monomial over k, $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$, and $\mathcal{S}^{\mathrm{irr}} = \{t\}$ by Theorem 5.1.2. Thus $k\langle t\rangle = k[t, t^{-1}]$ by (5.1), and Theorem 5.7.1 says that if p has an elementary integral over $k(t)$, then there are $v \in k\langle t\rangle$, $c_1, \ldots, c_n \in \overline{C}$, $b_1, \ldots, b_n \in k(c_1, \ldots, c_n)$, and $m_1, \ldots, m_n \in \mathbb{Z}$ such that

$$p = Dv + \sum_{i=1}^{n} c_i \frac{Db_i t^{m_i}}{b_i t^{m_i}} = Dv + \frac{Dt}{t} \sum_{i=1}^{n} m_i c_i + \sum_{i=1}^{n} c_i \frac{Db_i}{b_i} \tag{5.15}$$

where $C = \text{Const}(k)$. $K = k(c_1, \ldots, c_n)$ is an algebraic extension of k, so t is transcendental over K. Furthermore, Dt/t is not a logarithmic derivative of a K-radical by Lemma 3.4.8, so t is a monomial over K and $\text{Const}(K(t)) = \text{Const}(K)$. Since $p, v \in k[t, t^{-1}]$, write $p = \sum_{i=m}^{M} a_i t^i$ and $v = \sum_{i=r}^{R} v_i t^i$ where $a_i, v_i \in k$, $m, M, r, R \in \mathbb{Z}$, $m \le M$ and $r \le R$. Let $p_1 = \sum_{i=m}^{0} a_i t^i$. If $M = 0$, then $p - Dq_0 = p_1$ where $q_0 = 0 \in k\langle t \rangle$. If $M > 0$, then $\nu_\infty(p) = -M < 0$, which implies that $\nu_\infty(Dv) = -M < 0$, so $\nu_\infty(v) = -M$ by Lemma 5.1.2, hence $R = M$. Equating the coefficients of $t, \ldots, t^M$ in (5.15) we get

$$a_i = Dv_i + i\frac{Dt}{t} v_i \qquad \text{for } 1 \le i \le M\,. \tag{5.16}$$

Since we can solve Risch differential equations over k and $a_i, Dt/t \in k$, we can either prove that (5.16) has no solution $v_i \in k$, or find such a solution[3] If it has no solution for some i, then (5.15) has no solution so p has no elementary integral over $k(t)$. If we have solutions v_i for $1 \le i \le M$, letting $q_0 = v_1 t + \ldots v_M t^M$, we get

$$p - Dq_0 = \sum_{i=1}^{M} a_i t^i + \sum_{i=m}^{0} a_i t^i - \sum_{i=1}^{M} \left(Dv_i + i\frac{Dt}{t} v_i \right) t^i = \sum_{i=m}^{0} a_i t^i = p_1\,.$$

If $m = 0$, then $p_1 \in k$ so $q = q_0$ satisfies the theorem. If $m < 0$, then $\nu_t(p_1) = -m < 0$, which implies that $\nu_t(Dv) = -m < 0$, so $\nu_t(v) = -m$ by Theorem 4.4.2 (since $t \in \mathcal{S}^{\text{irr}}$), hence $r = m$. Equating the coefficients of $t^{-1}, \ldots, t^{-m}$ in (5.15) we get

$$a_i = Dv_i + i\frac{Dt}{t} v_i \qquad \text{for } m \le i \le -1\,. \tag{5.17}$$

Since we can solve Risch differential equations over k and $a_i, Dt/t \in k$, we can either prove that (5.17) has no solution $v_i \in k$, or find such a solution. If it has no solution for some i, then (5.15) has no solution, so p_1 and p have no elementary integrals over $k(t)$. If we have solutions v_i for $m \le i \le -1$, letting $q_1 = v_{-1} t^{-1} + \ldots v_{-m} t^{-m}$ and $q = q_0 + q_1 \in k\langle t \rangle$, we get

$$p - Dq = p_1 - Dq_1 = \sum_{i=m}^{-1} a_i t^i + a_0 - \sum_{i=m}^{-1} \left(Dv_i + i\frac{Dt}{t} v_i \right) t^i = a_0 \in k\,.$$

□

[3] Although this fact is not needed by the algorithm, we remark that Lemma 5.9.1 implies that (5.16) has at most one solution in k.

IntegrateHyperexponentialPolynomial(p, D)
(* Integration of hyperexponential polynomials *)

(* Given an hyperexponential monomial t over k and $p \in k[t, t^{-1}]$ return $q \in k[t, t^{-1}]$ and a Boolean $\beta \in \{0, 1\}$ such that $p - Dq \in k$ if $\beta = 1$, or $p - Dq$ does not have an elementary integral over $k(t)$ if $\beta = 0$. *)

```
q ← 0, β ← 1
for i ← ν_t(p) to −ν_∞(p) such that i ≠ 0 do
    a ← coefficient(p, t^i)
    (* RischDE will be given in Chap. 6 *)
    v ← RischDE(iDt/t, a)                    (* a = Dv + ivDt/t *)
    if v = "no solution" then β ← 0 else q ← q + vt^i
return(q, β)
```

Example 5.9.1. Consider

$$\int \left(\left(\tan(x)^3 + (x+1)\tan(x)^2 + \tan(x) + x + 2 \right) e^{\tan(x)} + \frac{1}{x^2+1} \right) dx\,.$$

Let $k = \mathbb{Q}(x, t_0)$ with $D = d/dx$, where t_0 is a monomial over $\mathbb{Q}(x)$ satisfying $Dt_0 = 1 + t_0^2$, *i.e.* $t_0 = \tan(x)$, and let t be a monomial over k satisfying $Dt = (1 + t_0^2)t$, *i.e.* $t = e^{\tan(x)}$. Our integrand is then

$$p = \left(t_0^3 + (x+1)t_0^2 + t_0 + x + 2 \right) t + \frac{1}{x^2+1} \in k[t]\,.$$

We get

1. $q = 0$, $\beta = 1$
2. $\nu_t(p) = -\nu_\infty(p) = 1$
3. $i = 1$
4. $a = \mathrm{lc}(p) = t_0^3 + (x+1)t_0^2 + t_0 + x + 2$
5. $D(t_0 + x) + (1 + t_0^2)(t_0 + x) = a$, so $v = \mathbf{RischDE}(1 + t_0^2, a) = t_0 + x$
6. $q = vt = (t_0 + x)t$
7. $p - Dq = 1/(x^2 + 1)$.

Hence,

$$\begin{aligned} \int \left(\left(\tan(x)^3 + (x+1)\tan(x)^2 + \tan(x) + x + 2 \right) e^{\tan(x)} + \frac{1}{x^2+1} \right) dx \\ = (\tan(x) + x)e^{\tan(x)} + \int \frac{dx}{x^2+1} \\ = (\tan(x) + x)e^{\tan(x)} + \arctan(x)\,. \end{aligned}$$

Putting all the pieces together, we get an algorithm for integrating elements of $k(t)$.

Theorem 5.9.2. *Let k be a differential field and t an hyperexponential over k. If we can solve Risch differential equations over k, and Dt/t is not a logarithmic derivative of a k-radical, then for any $f \in k(t)$ we can either prove that f has no elementary integral over $k(t)$, or compute an elementary extension E of $k(t)$ and $g \in E$ such that $f - Dg \in k$.*

Proof. Suppose that Dt/t is not a logarithmic derivative of a k-radical, then t is a monomial over k and $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$ by Theorem 5.1.2. Let $f \in k(t)$. By Theorem 5.3.1, we can compute $g_1, h, r \in k(t)$ such that $f = Dg_1 + h + r$, h is simple and r is reduced. From h, which is simple, we compute $g_2 \in k(t)$ given by (5.8) in Theorem 5.6.1. Note that $g_0 = g_1 + \int g_2$ lies in some elementary extension of $k(t)$. Let $p = h - g_2$ and $q = p + r$, then $f = Dg_0 + q$ so f has an elementary integral over $k(t)$ if and only if q has one. If $p \notin k[t]$, then $p + r$ does not have an elementary integral over $k(t)$ by Theorem 5.6.1, so f does not have an elementary integral over $k(t)$. Suppose now that $p \in k[t]$. We have $k\langle t\rangle = k[t, t^{-1}]$ by (5.1), so $r \in k[t, t^{-1}]$, hence $q \in k[t, t^{-1}]$. By Theorem 5.9.1 we can either prove that q has no elementary integral over $k(t)$, in which case f has no elementary integral over $k(t)$, or compute $s \in k[t, t^{-1}]$ such that $q - Ds \in k$, in which case $f - Dg \in k$ where $g = g_0 + s$. □

IntegrateHyperexponential(f, D)
(* Integration of hyperexponential functions *)

(* Given an hyperexponential monomial t over k and $f \in k(t)$, return g elementary over $k(t)$ and a Boolean $\beta \in \{0, 1\}$ such that $f - Dg \in k$ if $\beta = 1$, or $f - Dg$ does not have an elementary integral over $k(t)$ if $\beta = 0$. *)

$(g_1, h, r) \leftarrow$ **HermiteReduce**(f, D)
$(g_2, \beta) \leftarrow$ **ResidueReduce**(h, D)
if $\beta = 0$ **then return**$(g_1 + g_2, 0)$
$(q, \beta) \leftarrow$ **IntegrateHyperexponentialPolynomial**$(h - Dg_2 + r, D)$
return$(g_1 + g_2 + q, \beta)$

5.10 The Hypertangent Case

Tangents and trigonometric functions can be integrated by transforming them to complex logarithms and exponentials, but the theory of monomial extensions allows us to integrate them directly without introducing the algebraic number $\sqrt{-1}$. We start by defining tangent monomials and computing the special polynomials. Let k be a differential field and K a differential extension of k.

Definition 5.10.1. *Let $t \in K$ be such that $t^2+1 \neq 0$. t is a* hypertangent over *k if $Dt/(t^2+1) \in k$. t is a* tangent over *k if $Dt/(t^2+1) = Db$ for some $b \in k$. t is a* hypertangent (resp. tangent) monomial over *k if t is a hypertangent (resp. tangent) over k, transcendental over k, and* $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$.

We write $t = \tan(\int a)$ when t is a hypertangent over k such that $Dt/(t^2+1) = a$, and $t = \tan(b)$ when t is a tangent over k such that $Dt/(t^2+1) = Db$.

Lemma 5.10.1. *Let (F, D) be a differential field containing $\sqrt{-1}$, $a \in F$ be such that $a^2+1 \neq 0$, and $b = (\sqrt{-1}-a)/(\sqrt{-1}+a)$. Then, $b \neq 0$ and*

$$\frac{Db}{b} = 2\sqrt{-1}\frac{Da}{a^2+1}\,.$$

Proof. $b \neq 0$ since $a^2+1 \neq 0$, and we have

$$\begin{aligned}\frac{Db}{b} &= D\left(\frac{\sqrt{-1}-a}{\sqrt{-1}+a}\right)\frac{\sqrt{-1}+a}{\sqrt{-1}-a}\\ &= -2\sqrt{-1}\frac{Da}{(\sqrt{-1}+a)^2}\frac{\sqrt{-1}+a}{\sqrt{-1}-a} = 2\sqrt{-1}\frac{Da}{1+a^2}\,.\end{aligned}$$

□

Theorem 5.10.1. *If t is an hypertangent over k and $\sqrt{-1}Dt/(t^2+1)$ is not a logarithmic derivative of a $k(\sqrt{-1})$-radical, then t is a monomial over k,* $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$, *and any $p \in \mathcal{S}^{\mathrm{irr}}$ divides t^2+1 in $k[t]$. Furthermore, $\mathcal{S}_1^{\mathrm{irr}} = \mathcal{S}^{\mathrm{irr}}$. Conversely, if t is transcendental and hypertangent over k, and* $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$, *then $\sqrt{-1}Dt/(t^2+1)$ is not a logarithmic derivative of a $k(\sqrt{-1})$-radical.*

Proof. Let t be an hypertangent over k, $a = Dt/(t^2+1)$, and suppose that $a\sqrt{-1}$ is not a logarithmic derivative of a $k(\sqrt{-1})$-radical. Let $\theta = \frac{\sqrt{-1}-t}{\sqrt{-1}+t} \in k(\sqrt{-1})(t)$. By Lemma 5.10.1, we have

$$\frac{D\theta}{\theta} = 2\sqrt{-1}\frac{Dt}{1+t^2} = 2a\sqrt{-1} \in k(\sqrt{-1})$$

so θ is hyperexponential over $k(\sqrt{-1})$. Since $a\sqrt{-1}$ is not a logarithmic derivative of a $k(\sqrt{-1})$-radical, $2a\sqrt{-1}$ is not one either, so by Theorem 5.1.2, θ is a monomial over $k(\sqrt{-1})$, and $\mathrm{Const}(k(\sqrt{-1})(\theta)) = \mathrm{Const}(k(\sqrt{-1}))$. But $t = \sqrt{-1}(\theta-1)/(\theta+1)$, so t is transcendental over $k(\sqrt{-1})$, hence a monomial over k since $Dt = a + at^2$. Furthermore, $k(\sqrt{-1})(\theta) = k(\sqrt{-1})(t)$, so

$$\mathrm{Const}(k(\sqrt{-1})(t)) = \mathrm{Const}(k(\sqrt{-1})(\theta)) = \mathrm{Const}(k(\sqrt{-1})) = \overline{C} \cap k(\sqrt{-1})$$

by Corollary 3.3.1 where $\overline{C}$ is the algebraic closure of $\mathrm{Const}(k)$. This implies that $\mathrm{Const}(k(t)) \subseteq \overline{C} \cap k(\sqrt{-1}) \cap k(t) \subseteq k$ since t is transcendental over k. Hence, $\mathrm{Const}(k(t)) \subseteq \mathrm{Const}(k)$. The reverse inclusion is given by Lemma 3.3.1, so $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$, which implies that $\mathrm{Const}(\overline{k}(t)) = \mathrm{Const}(\overline{k})$ by Lemma 3.3.3.

We have $D(t^2+1) = 2tDt = 2at(t^2+1)$ so $t^2+1 \in \mathcal{S}$, hence any factor of t^2+1 is special by Theorem 3.4.1. Suppose now that $p \in \mathcal{S}$, and let $\beta \in \overline{k}$ be any root of p. $D\beta = a\beta^2 + a$ by Theorem 3.4.3, so

$$\begin{aligned} D\left(\frac{t-\beta}{\beta t+1}\right) &= a\frac{(t^2-\beta^2)(\beta t+1) - (t-\beta)(t\beta^2+t+\beta t^2+\beta)}{(\beta t+1)^2} \\ &= a(t-\beta)\frac{(\beta t^2+t+\beta^2 t+\beta) - (t\beta^2+t+\beta t^2+\beta)}{(\beta t+1)^2} = 0 \end{aligned}$$

which implies that $c = (t-\beta)/(\beta t+1) \in \mathrm{Const}(\overline{k}(t)) \subseteq \overline{k}$. Since t is transcendental over $\overline{k}$, $(c\beta-1)t + (c+\beta) = 0$ implies that $c\beta - 1 = c+\beta = 0$, so $\beta^2+1 = 0$. Since this holds for every root of p, this implies that every irreducible factor of p divides t^2+1 in $k[t]$.

We have $\mathcal{S}_1^{\mathrm{irr}} \subseteq \mathcal{S}^{\mathrm{irr}}$ by definition. Conversely, let $p \in \mathcal{S}^{\mathrm{irr}}$. Then p divides t^2+1, so all the roots of p in $\overline{k}$ satisfy $\beta^2 = -1$. Hence,

$$p_\beta = \frac{Dt - D\beta}{t-\beta} = a\frac{t^2+1}{t-\beta} = a(t+\beta)$$

which implies that $p_\beta(\beta) = 2a\beta = \pm 2\sqrt{-1}a$, which is not a logarithmic derivative of a $k(\sqrt{-1})$-radical, hence not a logarithmic derivative of a $k(\beta)$-radical. Thus, $p \in \mathcal{S}_1^{\mathrm{irr}}$ which implies that $\mathcal{S}_1^{\mathrm{irr}} = \mathcal{S}^{\mathrm{irr}}$.

Conversely, let t be a transcendental hypertangent over k and suppose that $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$. Then, $\mathrm{Const}(\overline{k}(t)) = \mathrm{Const}(\overline{k})$ by Lemma 3.3.3. If there exist $b \in k(\sqrt{-1})^*$ and an integer $n > 0$ such that

$$n\sqrt{-1}\frac{Dt}{t^2+1} = \frac{Db}{b}$$

then, taking

$$\theta = \frac{\sqrt{-1}-t}{\sqrt{-1}+t} \quad \text{and} \quad c = \frac{\theta^n}{b^2} \in k(\sqrt{-1})(t)$$

we get

$$\frac{Dc}{c} = n\frac{D\theta}{\theta} - 2\frac{Db}{b} = 2n\sqrt{-1}\frac{Dt}{t^2+1} - 2\frac{Db}{b} = 0$$

so $c \in \mathrm{Const}(\overline{k}(t)) \subseteq \overline{k}$ in contradiction with t transcendental over k. Hence, $\sqrt{-1}Dt/(t^2+1)$ is not a logarithmic derivative of a $k(\sqrt{-1})$-radical. $\square$

As a consequence, we have

$$k\langle t\rangle = \{f \in k(t) \text{ such that } (t^2+1)^n f \in k[t] \text{ for some integer } n \geq 0\}$$

when t is a hypertangent monomial over k. We now present an algorithm that, with appropriate assumptions on k, integrates elements of $k(t)$ when t

is a hypertangent monomial over k. Note first that if the polynomial $X^2 + 1$ factors over k, then $\sqrt{-1} \in k$, so $k(t) = k(\theta)$ where $\theta = (\sqrt{-1} - t)/(\sqrt{-1} + t)$ is a hyperexponential monomial over k. Hence we can use the algorithm for integrating elements of hyperexponential extensions in this case, so we can assume for the rest of this section that $X^2 + 1$ is irreducible over k, in other words that $\sqrt{-1} \notin k$. Since hypertangents are nonlinear monomials, integrating elements of $k[t]$ is straightforward.

Theorem 5.10.2. *Let k be a differential field not containing $\sqrt{-1}$, and t an hypertangent over k. If $\sqrt{-1}Dt/(t^2+1)$ is not a logarithmic derivative of a $k(\sqrt{-1})$-radical, then for any $p \in k[t]$ we can compute $q \in k[t]$ and $c \in k$ such that*

$$p - Dq - c\,\frac{D(t^2+1)}{t^2+1} \in k\,.$$

Furthermore, if $Dc \neq 0$, then p has no elementary integral over k.

Proof. Let $\alpha = Dt/(t^2+1) \in k$. Since $\alpha\sqrt{-1}$ is not a logarithmic derivative of a $k(\sqrt{-1})$-radical, t is a monomial over k, $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$, and all the special irreducible polynomials divide t^2+1 in $k[t]$ by Theorem 5.10.1. Since $\sqrt{-1} \notin k$, t^2+1 is irreducible over k, so $\mathcal{S}^{\mathrm{irr}} = \{t^2+1\}$. Since $\delta(t) = 2$, Theorem 5.4.1 shows how to compute $q, r \in k[t]$ such that $p - Dq = r$ and $\deg(r) \le 1$. Write $r = at + b$ where $a, b \in k$, and let $c = a/(2\alpha) \in k$. Since $h = t^2 + 1 \in \mathcal{S}$, Theorem 5.4.2 says that $\deg(r - cDh/h) < 1$, hence that

$$p - Dq - c\,\frac{D(t^2+1)}{t^2+1} \in k\,.$$

Suppose now that $Dc \neq 0$, and that r has an elementary integral over $k(t)$. Then, by Theorem 5.7.1, there are $v \in k\langle t\rangle$, $c_1, \ldots, c_n \in \overline{C}$, $b_1, \ldots, b_n \in k(c_1, \ldots, c_n)$, and $m_1, \ldots, m_n \in \mathbb{Z}$ such that

$$at + b = Dv + \sum_{i=1}^{n} c_i \frac{Db_i(t^2+1)^{m_i}}{b_i(t^2+1)^{m_i}} = Dv + 2t\alpha \sum_{i=1}^{n} m_i c_i + \sum_{i=1}^{n} c_i \frac{Db_i}{b_i}\,. \quad (5.18)$$

If $\nu_\infty(v) < 0$, then $\nu_\infty(Dv) = \nu_\infty(v) - 1 < -1$ by Theorem 4.4.4, in contradiction with (5.18), hence $\nu_\infty(v) \ge 0$, which implies that $\nu_\infty(Dv) \ge 0$ by Theorem 4.4.4. Let $c = a/(2\alpha) \in k$. Equating the coefficients of t in (5.18), we get $a = 2\alpha \sum_{i=1}^n m_i c_i$, so

$$c = \frac{a}{2\alpha} = \sum_{i=1}^{n} m_i c_i \in \mathrm{Const}(k)$$

in contradiction with $Dc \neq 0$. Hence (5.18) has no solution if $Dc \neq 0$, which implies that r, and hence p, have no elementary integral over $k(t)$. □

IntegrateHypertangentPolynomial(p, D)
(* Integration of hypertangent polynomials *)

(* Given a differential field k such that $\sqrt{-1} \notin k$, a hypertangent monomial t over k and $p \in k[t]$, return $q \in k[t]$ and $c \in k$ such that $p - Dq - cD(t^2+1)/(t^2+1) \in k$ and $p - Dq$ does not have an elementary integral over $k(t)$ if $Dc \neq 0$. *)

$(q, r) \leftarrow$ **PolynomialReduce**(p, D) (* $\deg(r) \leq 1$ *)
$\alpha \leftarrow Dt/(t^2+1)$
$c \leftarrow$ **coefficient**$(r, t)/(2\alpha)$
return(q, c)

Example 5.10.1. Consider

$$\int \left(\tan(x)^2 + x\tan(x) + 1\right) dx$$

Let $k = \mathbb{Q}(x)$ with $D = d/dx$, and and let t be a monomial over k satisfying $Dt = 1 + t^2$, *i.e.* $t = \tan(x)$. Our integrand is then

$$p = t^2 + xt + 1 \in k[t]\,.$$

We get

1. $(q, r) =$ **PolynomialReduce**$(t^2 + xt + 1) = (t, xt)$
2. $\alpha = Dt/(t^2+1) = 1$
3. $c = x/2$.

Since $Dc = 1/2 \neq 0$, we conclude that

$$\int \left(\tan(x)^2 + x\tan(x) + 1\right) dx = \tan(x) + \int x\tan(x) dx$$

and the latter integral is not an elementary function.

For reduced elements in an hypertangent extension, the related problem we need to solve over k is the *coupled differential system problem*: given $f_1, f_2, g_1, g_2 \in k$, determine whether there are $y_1, y_2 \in k$ such that

$$\begin{pmatrix} Dy_1 \\ Dy_2 \end{pmatrix} + \begin{pmatrix} f_1 & -f_2 \\ f_2 & f_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

and to compute y_1 and y_2 if they exist.

Theorem 5.10.3. *Let k be a differential field not containing $\sqrt{-1}$, and t an hypertangent over k. If we can solve coupled differential systems over k, and $\sqrt{-1}Dt/(t^2+1)$ is not a logarithmic derivative of a $k(\sqrt{-1})$-radical, then for any $p \in k\langle t\rangle$ we can either prove that p has no elementary integral over $k(t)$, or compute $q \in k\langle t\rangle$ such that $p - Dq \in k[t]$.*

Proof. Let $\alpha = Dt/(t^2+1) \in k$. Since $\alpha\sqrt{-1}$ is not a logarithmic derivative of a $k(\sqrt{-1})$-radical, t is a monomial over k, $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$, and all the special irreducible polynomials divide t^2+1 in $k[t]$ by Theorem 5.10.1. Since $\sqrt{-1} \notin k$, t^2+1 is irreducible over k, so $\mathcal{S}^{\mathrm{irr}} = \mathcal{S}_1^{\mathrm{irr}} = \{t^2+1\}$. Thus, Theorem 5.7.1 says that if p has an elementary integral over $k(t)$, then there are $v \in k\langle t\rangle$, $c_1,\dots,c_n \in \overline{C}$, $b_1,\dots,b_n \in k(c_1,\dots,c_n)$, and $m_1,\dots,m_n \in \mathbb{Z}$ such that

$$\begin{aligned} p &= Dv + \sum_{i=1}^n c_i \frac{Db_i(t^2+1)^{m_i}}{b_i(t^2+1)^{m_i}} \\ &= Dv + 2t\alpha\sum_{i=1}^n m_i c_i + \sum_{i=1}^n c_i \frac{Db_i}{b_i} = Dv + w \end{aligned} \tag{5.19}$$

where $C = \mathrm{Const}(k)$, and

$$w = 2t\alpha\sum_{i=1}^n m_i c_i + \sum_{i=1}^n c_i \frac{Db_i}{b_i} \in k(c_1,\dots,c_n)[t]\,.$$

$K = k(c_1,\dots,c_n)$ is an algebraic extension of k, so t is transcendental over K. Furthermore, $\alpha\sqrt{-1}$ is not a logarithmic derivative of a $K(\sqrt{-1})$-radical by Lemma 3.4.8, so t is a monomial over K and $\mathrm{Const}(K(t)) = \mathrm{Const}(K)$. We proceed by induction on $-\nu_{t^2+1}(p)$. If $\nu_{t^2+1}(p) \ge 0$, then $p - Dq \in k[t]$ where $q = 0 \in k\langle t\rangle$, so suppose that $m = -\nu_{t^2+1}(p) > 0$ and that the theorem holds for all $h \in k\langle t\rangle$ with $-\nu_{t^2+1}(h) < m$. Since $p \in k\langle t\rangle$ and $m = -\nu_{t^2+1}(p) > 0$, we have $p = r/(t^2+1)^m$ where $r \in k[t]$ and $\gcd(r, t^2+1) = 1$. Since $\nu_{t^2+1}(p) = -m < 0$, (5.19) implies that $\nu_{t^2+1}(Dv) = -m < 0$, hence that $\nu_{t^2+1}(v) = -m$ by Theorem 4.4.2, since $t^2+1 \in \mathcal{S}_1$. Thus, $v = s/(t^2+1)^m$ where $s \in k[t]$ and $\gcd(s, t^2+1) = 1$. Dividing r and s by t^2+1, we get $r = r_0(t^2+1) + at + b$ and $s = s_0(t^2+1) + ct + d$, where $r_0, s_0 \in k[t]$, $a,b,c,d \in k$, $at+b \ne 0$, and $ct+d \ne 0$. From (5.19), we get

$$\begin{aligned} \frac{at+b}{(t^2+1)^m} + \frac{r_0}{(t^2+1)^{m-1}} &= D\left(\frac{ct+d}{(t^2+1)^m} + \frac{s_0}{(t^2+1)^{m-1}}\right) + w \\ &= \frac{tDc + c\alpha(t^2+1) + Dd}{(t^2+1)^m} - \frac{2m\alpha t(t^2+1)(ct+d)}{(t^2+1)^{m+1}} + Dw_0 + w \\ &= \frac{tDc + Dd}{(t^2+1)^m} - 2m\alpha\frac{ct^2+dt}{(t^2+1)^m} + c\alpha\frac{1}{(t^2+1)^{m-1}} + Dw_0 + w \\ &= \frac{tDc + Dd}{(t^2+1)^m} - 2m\alpha\frac{dt-c}{(t^2+1)^m} + c\alpha\frac{1-2m}{(t^2+1)^{m-1}} + Dw_0 + w \end{aligned}$$

where $w_0 = s_0/(t^2+1)^{m-1}$. Since $\nu_{t^2+1}(w_0) > -m$, $\nu_{t^2+1}(Dw_0) > -m$ by Theorem 4.4.2, so, equating the coefficients of $(t^2+1)^{-m}$ we get

$$at + b = (Dc - 2m\alpha d)t + Dd + 2m\alpha c$$

which implies that

$$\begin{pmatrix} Dc \\ Dd \end{pmatrix} + \begin{pmatrix} 0 & -2m\alpha \\ 2m\alpha & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{5.20}$$

Since we can solve coupled differential systems over k and $a, b, \alpha \in k$, we can either prove that (5.20) has no solution $c, d \in k$, or find such a solution. If it has no solution in k, then (5.19) has no solution, so p has no elementary integral over $k(t)$. If we have a solution $c, d \in k$, letting $q_0 = (ct + d)/(t^2 + 1)^m \in k\langle t \rangle$, we get

$$p - Dq_0 = \frac{u}{(t^2 + 1)^{m-1}}$$

for some $u \in k[t]$, so $\nu_{t^2+1}(p - Dq_0) > -m$. By induction we can either prove that $p - Dq_0$ has no elementary integral over $k(t)$, in which case p has no elementary integral over $k(t)$, or we get $q_1 \in k\langle t \rangle$ such that $p - Dq_0 - Dq_1 \in k[t]$, which implies that $p - Dq \in k[t]$ where $q = q_0 + q_1$. □

IntegrateHypertangentReduced(p, D)
(* Integration of hypertangent reduced elements *)

(* Given a differential field k such that $\sqrt{-1} \notin k$, a hypertangent monomial t over k and $p \in k\langle t \rangle$, return $q \in k\langle t \rangle$ and a Boolean $\beta \in \{0, 1\}$ such that $p - Dq \in k[t]$ if $\beta = 1$, or $p - Dq$ does not have an elementary integral over $k(t)$ if $\beta = 0$. *)

$m \leftarrow -\nu_{t^2+1}(p)$
if $m \le 0$ **then return**$(0, 1)$
$h \leftarrow (t^2 + 1)^m p$ (* $h \in k[t]$ *)
$(q, r) \leftarrow$ **PolyDivide**$(h, t^2 + 1)$ (* $h = (t^2 + 1)q + r, \deg(r) \le 1$ *)
$a \leftarrow$ **coefficient**(r, t), $b \leftarrow r - at$ (* $r = at + b$ *)
(* CoupledDESystem will be given in Chap. 8 *)
$(c, d) \leftarrow$ **CoupledDESystem**$(0, 2mDt/(t^2 + 1), a, b)$
(* $Dc - 2mDt/(t^2 + 1)d = a, Dd + 2mDt/(t^2 + 1)c = b$ *)
if $(c, d) =$ "no solution" **then return**$(0, 0)$
$q_0 \leftarrow (ct + d)/(t^2 + 1)^m$
$(q, \beta) \leftarrow$ **IntegrateHypertangentReduced**$(p - Dq_0, D)$
return$(q + q_0, \beta)$

Example 5.10.2. Consider

$$\int \frac{\sin(x)}{x}\, dx\,.$$

Let $k = \mathbb{Q}(x)$ with $D = d/dx$, and and let t be a monomial over k satisfying $Dt = (1 + t^2)/2$, *i.e.* $t = \tan(x/2)$. Using the classical half-angle formula, our integrand is then

$$p = \frac{\sin(x)}{x} = \frac{2\tan(x/2)}{x(\tan(x/2)^2+1)} = \frac{2t/x}{t^2+1} \in k\langle t\rangle\,.$$

We get $Dt/(t^2+1) = 1/2$ and

1. $m = -\nu_{t^2+1}(p) = 1$
2. $h = p(t^2+1) = 2t/x$
3. $(q,r) =$ **PolyDivide**$(2t/x, t^2+1) = (0, 2t/x)$, so $(a,b) = (2/x, 0)$
4. Since
$$\begin{pmatrix} Dc \\ Dd \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 2/x \\ 0 \end{pmatrix}$$
has no solution in $\mathbb{Q}(x)$, **CoupledDESystem**$(0,1,2/x,0)$ returns "no solution".

Hence,

$$\int \frac{\sin(x)}{x}\,dx$$

is not an elementary function.

Example 5.10.3. Consider

$$\int \frac{\tan(x)^5 + \tan(x)^3 + x^2\tan(x) + 1}{\left(\tan(x)^2+1\right)^3}\,dx\,.$$

Let $k = \mathbb{Q}(x)$ with $D = d/dx$, and and let t be a monomial over k satisfying $Dt = 1 + t^2$, *i.e.* $t = \tan(x)$. Our integrand is then

$$p = \frac{t^5 + t^3 + x^2 t + 1}{\left(t^2+1\right)^3} \in k\langle t\rangle\,.$$

We get $Dt/(t^2+1) = 1$ and

1. $m = -\nu_{t^2+1}(p) = 3$
2. $h = p(t^2+1)^3 = t^5 + t^3 + x^2 t + 1$
3. $(q,r) =$ **PolyDivide**$(h, t^2+1) = (t^3, x^2 t + 1)$, so $(a,b) = (x^2, 1)$
4. Since
$$\begin{pmatrix} Dc \\ Dd \end{pmatrix} + \begin{pmatrix} 0 & -6 \\ 6 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} x^2 \\ 1 \end{pmatrix}$$
has the solution $c = x/18 + 1/6$ and $d = 1/108 - x^2/6$ in $\mathbb{Q}(x)$, $(c,d) =$ **CoupledDESystem**$(0,6,x^2,1) = (x/18 + 1/6, 1/108 - x^2/6)$.
5.
$$q_0 = \frac{ct+d}{\left(t^2+1\right)^3} = \frac{(1+x/3)\,t - \left(x^2 - 1/18\right)}{6\left(t^2+1\right)^3}\,,$$
$$p - Dq_0 = \frac{t^3 + 5x/18 + 15/18}{\left(t^2+1\right)^2}$$

6. Recursively calling $(q, \beta) =$ **IntegrateHypertangentReduced**$(p - Dq_0)$, we get $\beta = 1$ and
$$q = \frac{5\,(1 + x/3)\,t + 77/12}{24\,(t^2+1)^2} + \frac{5\,(1 + x/3)\,t - 43/6}{16\,(t^2+1)}$$
7.
$$p - D(q + q_0) = \frac{5}{16}\left(1 + \frac{x}{3}\right)$$

Hence,
$$\int \frac{\tan(x)^5 + \tan(x)^3 + x^2\tan(x) + 1}{(\tan(x)^2+1)^3}\,dx =$$
$$\frac{(1 + x/3)\tan(x) - \left(x^2 - 1/18\right)}{6\,(\tan(x)^2+1)^3} + \frac{5\,(1 + x/3)\tan(x) + 77/12}{24\,(\tan(x)^2+1)^2}$$
$$+\frac{5\,(1 + x/3)\tan(x) - 43/6}{16\,(\tan(x)^2+1)} + \frac{5}{16}\int\left(1 + \frac{x}{3}\right)\,dx$$

and the remaining integral is of course $x + x^2/6$.

Putting all the pieces together, we get an algorithm for integrating elements of $k(t)$.

Theorem 5.10.4. *Let k be a differential field not containing $\sqrt{-1}$, and t an hypertangent over k. If we can solve coupled differential systems over k, and $\sqrt{-1}Dt/(t^2+1)$ is not a logarithmic derivative of a $k(\sqrt{-1})$-radical, then for any $f \in k(t)$ we can either prove that f has no elementary integral over $k(t)$, or compute an elementary extension E of $k(t)$ and $g \in E$ such that $f - Dg \in k$.*

Proof. Suppose that $\sqrt{-1}Dt/(t^2+1)$ is not a logarithmic derivative of a $k(\sqrt{-1})$-radical, then t is a monomial over k and $\mathrm{Const}(k(t)) = \mathrm{Const}(k)$ by Theorem 5.10.1. Let $f \in k(t)$. By Theorem 5.3.1, we can compute $g_1, h, r \in k(t)$ such that $f = Dg_1 + h + r$, h is simple and r is reduced. From h, which is simple, we compute $g_2 \in k(t)$ given by (5.8) in Theorem 5.6.1. Note that $g_0 = g_1 + \int g_2$ lies in some elementary extension of $k(t)$. Let $p = h - g_2$ and $q = p + r$, then $f = Dg_0 + q$ so f has an elementary integral over $k(t)$ if and only if q has one. If $p \notin k[t]$, then $p + r$ does not have an elementary integral over $k(t)$ by Theorem 5.6.1, so f does not have an elementary integral over $k(t)$. Suppose now that $p \in k[t]$. Then $p \in k\langle t\rangle$ so $q \in k\langle t\rangle$. By Theorem 5.10.3 we can either prove that q has no elementary integral over $k(t)$, in which case f has no elementary integral over $k(t)$, or compute $s \in k\langle t\rangle$ such that $u = q - Ds \in k[t]$, in which case by Theorem 5.10.2, we compute $v \in k[t]$ and $c \in \mathrm{Const}(k)$ such that $u - Dv - cD(t^2+1)/(t^2+1) \in k$. If $Dc \neq 0$, then u, and hence f, have no elementary integral over $k(t)$, otherwise $Dc = 0$ so $f - Dg \in k$ where $g = g_0 + s + v + c\int D(t^2+1)/(t^2+1)$ lies in some elementary extension of $k(t)$. $\square$

IntegrateHypertangent(f, D) (* Integration of hypertangent functions *)

(* Given a differential field k such that $\sqrt{-1} \notin k$, a hypertangent monomial t over k and $f \in k(t)$, return g elementary over $k(t)$ and a Boolean $\beta \in \{0,1\}$ such that $f - Dg \in k$ if $\beta = 1$, or $f - Dg$ does not have an elementary integral over $k(t)$ if $\beta = 0$. *)

$(g_1, h, r) \leftarrow$ **HermiteReduce**(f, D)
$(g_2, \beta) \leftarrow$ **ResidueReduce**(h, D)
if $\beta = 0$ **then return**$(g_1 + g_2, 0)$
$p \leftarrow h - Dg_2 + r$
$(q_1, \beta) \leftarrow$ **IntegrateHypertangentReduced**(p, D)
if $\beta = 0$ **then return**$(g_1 + g_2 + q_1, 0)$
$(q_2, c) \leftarrow$ **IntegrateHypertangentPolynomial**$(p - Dq_1, D)$
if $Dc = 0$ **then return**$(g_1 + g_2 + q_1 + q_2 + c\log(t^2+1), 1)$
else return$(g_1 + g_2 + q_1 + q_2, 0)$

5.11 The Nonlinear Case with no Specials

In the case of nonlinear monomials over a differential field k, we have seen that we can reduce the problem to integrating reduced elements of the form $p + a/d$ where $p, a \in k[t]$, $d \in \mathcal{S} \setminus \{0\}$, $\deg(p) < \delta(t)$ and $\deg(a) < \deg(d)$. Furthermore, Theorem 5.7.2 provides a criterion for nonintegrability, and if an element of $\mathcal{S} \setminus k$ is known, allows us to reduce the problem to $\deg(p) < \delta(t) - 1$. We address in this section the case $\mathcal{S} = k$, *i.e.* $\mathcal{S}^{\mathrm{irr}} = \emptyset$, which corresponds to interesting classes of functions as will be illustrated in the examples. Note that if $\mathcal{S}^{\mathrm{irr}} = \emptyset$, then $k\langle t\rangle = k[t]$, so as a result of the polynomial reduction (Sect. 5.4), we consider integrands of the form $p \in k[t]$ with $\deg(p) < \delta(t)$. It turns out that if such elements are integrable, then they must be in k.

Corollary 5.11.1. *Suppose that t is a nonlinear monomial and that $\mathcal{S}^{\mathrm{irr}} = \emptyset$. Let $p \in k[t]$ be such that $\deg(p) < \delta(t)$. If p has an elementary integral over $k(t)$, then $p \in k$.*

Proof. Let $C = \mathrm{Const}(k(t))$, $p \in k[t]$ be such that $\deg(p) < \delta(t)$, and suppose that p has an elementary integral over $k(t)$. By Theorem 5.7.1 there are $v \in k[t]$, $c_1, \dots, c_n \in \overline{C}$ and $u_1, \dots, u_n \in \mathcal{S}_{k(c_1,\dots,c_n)[t]:k(c_1,\dots,c_n)}$ such that $p = Dv + g$ where $g = \sum_{i=1}^n c_i D(u_i)/u_i$. Note that $g = p - Dv \in k[t]$. Since $\mathcal{S}^{\mathrm{irr}}_{k[t]:k} = \emptyset$, it follows that $\mathcal{S}^{\mathrm{irr}}_{k(c_1,\dots,c_n)[t]:k(c_1,\dots,c_n)} = \emptyset$ (Exercise 3.5), hence that $\mathcal{S}_{k(c_1,\dots,c_n)[t]:k(c_1,\dots,c_n)} = k(c_1,\dots,c_n)$. This implies that $g \in k(c_1,\dots,c_n)$. Since $g \in k[t]$, we get that $g \in k$. Suppose that $\deg(v) \ge 1$, then,

$$\deg(p) = \deg(Dv + g) = \deg(Dv) = \deg(v) + \delta(t) - 1 \ge \delta(t)$$

in contradiction with $\deg(p) < \delta(t)$. Hence, $v \in k$, so $p = Dv + g \in k$. □

This provides a complete algorithm for integrating elements of $k(t)$.

Theorem 5.11.1. *Let k be a differential field and t be a nonlinear monomial over k be such that $\mathcal{S}^{\text{irr}} = \emptyset$. Then, for any $f \in k(t)$ we can either prove that f has no elementary integral over $k(t)$, or compute an elementary extension E of $k(t)$ and $g \in E$ such that $f - Dg \in k$.*

Proof. Suppose that t is a nonlinear monomial over k and that $\mathcal{S}^{\text{irr}} = \emptyset$. Then, $\text{Const}(k(t)) = \text{Const}(k)$ by Lemma 3.4.5. Let $f \in k(t)$. By Theorem 5.3.1, we can compute $g_1, h, r \in k(t)$ such that $f = Dg_1 + h + r$, h is simple and r is reduced. From h, which is simple, we compute $g_2 \in k(t)$ given by (5.8) in Theorem 5.6.1. Note that $g_0 = g_1 + \int g_2$ lies in some elementary extension of $k(t)$. Let $p = h - g_2$ and $q = p + r$, then $f = Dg_0 + q$ so f has an elementary integral over $k(t)$ if and only if q has one. If $p \notin k[t]$, then $p + r$ does not have an elementary integral over $k(t)$ by Theorem 5.6.1, so f does not have an elementary integral over $k(t)$. Suppose now that $p \in k[t]$. We have $k\langle t \rangle = k[t]$ by (5.1), so $r \in k[t]$, hence $q \in k[t]$. By Theorem 5.4.1 we compute $q_1, q_2 \in k[t]$ such that $q = Dq_1 + q_2$ and $\deg(q_2) < \delta(t)$. We now have $f - Dg = q_2$ where $g = g_0 + q_1$. If $q_2 \in k$, then the theorem is proven, otherwise $0 < \deg(q_2) < \delta(t)$, so q_2, and therefore f, have no elementary integral over k by Corollary 5.11.1. □

IntegrateNonLinearNoSpecial(f, D)
(* Integration of nonlinear monomials with no specials *)

(* Given a is a nonlinear monomial t over k with $\mathcal{S}^{\text{irr}} = \emptyset$, and $f \in k(t)$, return g elementary over $k(t)$ and a Boolean $\beta \in \{0, 1\}$ such that $f - Dg \in k$ if $\beta = 1$, or $f - Dg$ does not have an elementary integral over $k(t)$ if $\beta = 0$. *)

$(g_1, h, r) \leftarrow$ **HermiteReduce**(f, D)
$(g_2, \beta) \leftarrow$ **ResidueReduce**(h, D)
if $\beta = 0$ **then return**$(g_1 + g_2, 0)$
$(q_1, q_2) \leftarrow$ **PolynomialReduce**$(h - Dg_2 + r, D)$
if $q_2 \in k$ **then** $\beta \leftarrow 1$ **else** $\beta \leftarrow 0$
return$(g_1 + g_2 + q_1, \beta)$

Example 5.11.1. Let $\nu \in \mathbb{Z}$ be any integer and consider

$$\int \frac{J_{\nu+1}(x)}{J_\nu(x)}\, dx$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν. From

$$\frac{dJ_\nu(x)}{dx} = -J_{\nu+1}(x) + \frac{\nu}{x} J_\nu(x)$$

we get

$$\int \frac{J_{\nu+1}(x)}{J_\nu(x)}\,dx = \int \nu\,\frac{dx}{x} - \int \frac{dJ_\nu(x)/dx}{J_\nu(x)}\,dx = \nu\log(x) - \int \phi_\nu(x)dx$$

where $\phi_\nu(x)$ is the logarithmic derivative of $J_\nu(x)$. Since $J_\nu(x)$ is a solution of the Bessel equation

$$y''(x) + \frac{1}{x}y'(x) + \left(1 - \frac{\nu^2}{x^2}\right)y(x) = 0 \tag{5.21}$$

it follows that $\phi_\nu(x)$ is a solution of the Riccati equation

$$y'(x) + y(x)^2 + \frac{1}{x}y(x) + \left(1 - \frac{\nu^2}{x^2}\right) = 0\,. \tag{5.22}$$

Let $k = \mathbb{Q}(x)$ with $D = d/dx$, and let t be a monomial over k satisfying $Dt = -t^2 - t/x - (1-\nu^2/x^2)$, *i.e.* $t = \phi_\nu(x)$. It can be proven that $\mathcal{S}^{\mathrm{irr}} = \emptyset$ in this extension[4] so Corollary 5.11.1 implies that t has no elementary integral over k, hence that

$$\int \frac{J_{\nu+1}(x)}{J_\nu(x)}\,dx = \nu\log(x) - \int \phi_\nu(x)dx$$

where the remaining integral is not elementary over $\mathbb{Q}(x, \phi_\nu(x))$.

Example 5.11.2. Let $\nu \in \mathbb{C}$ be any complex number and consider

$$\int \frac{x^2\phi_\nu^5 + x\phi_\nu^4 - \nu^2\phi_\nu^3 - x(x^2+1)\phi_\nu^2 - (x^2-\nu^2)\phi_\nu - x^5/4}{x^2\phi_\nu^4 + x^2(x^2+2)\phi_\nu^2 + x^2 + x^4 + x^6/4}\,dx$$

where $\phi_\nu(x)$ is the logarithmic derivative of $J_\nu(x)$, the Bessel function of the first kind of order ν. Let $k = \mathbb{Q}(x)$ with $D = d/dx$, and let t be a monomial over k satisfying $Dt = -t^2 - t/x - (1 - \nu^2/x^2)$, *i.e.* $t = \phi_\nu(x)$. Our integrand is then

$$f = \frac{x^2t^5 + xt^4 - \nu^2t^3 - x(x^2+1)t^2 - (x^2-\nu^2)t - x^5/4}{x^2t^4 + x^2(x^2+2)t^2 + x^2 + x^4 + x^6/4}$$

and we get

1. Calling $(g_1, h, r) =$ **HermiteReduce**(f, D) we get

$$g_1 = -\frac{1 + x^2/4}{t^2 + 1 + x^2/2},\quad h = -\frac{(\nu^2 + x^4/2)t + x^3 + x}{x^2t^2 + x^2 + x^4/2},\ \text{and } r = t + \frac{1}{x}\,.$$

[4] The fact that (5.21) has no solutions in quadratures for $\nu \in \mathbb{Z}$ (its Galois group is $SL_2(\mathbb{C})$) implies that (5.22) has no algebraic function solution, hence no solution in $\overline{k}$. Theorem 3.4.3 then implies that $\mathcal{S}^{\mathrm{irr}} = \emptyset$.

2. Calling $(g_2, \beta) =$ **ResidueReduce**(h, D) we get $\beta = 1$ and

$$g_2 = -\frac{1}{2}\log\left(t^2 + 1 + \frac{x^2}{2}\right).$$

3. We have $h - Dg_2 + r = 0$, so $(q_1, q_2) = (0, 0)$.

Hence $f = Dg_1 + Dg_2$, which means that

$$\int \frac{x^2\phi_\nu^5 + x\phi_\nu^4 - \nu^2\phi_\nu^3 - x(x^2+1)\phi_\nu^2 - (x^2-\nu^2)\phi_\nu - x^5/4}{x^2\phi_\nu^4 + x^2(x^2+2)\phi_\nu^2 + x^2 + x^4 + x^6/4}\,dx = \\ -\frac{1 + x^2/4}{\phi_\nu(x)^2 + 1 + x^2/2} - \frac{1}{2}\log\left(\phi_\nu(x)^2 + 1 + x^2/2\right).$$

Note that the above integral is valid regardless of whether $\mathcal{S}^{\mathrm{irr}}$ is empty.

The above examples used Bessel functions, but in fact the algorithm of this section can be applied whenever the integrand can be expressed in terms of the logarithmic derivative of a function defined by a second-order linear ordinary differential equation. If the defining equation is known not to have solutions in quadratures (for example for Airy functions), then $\mathcal{S}^{\mathrm{irr}} = \emptyset$, as explained in note 4 of this chapter.

5.12 In–Field Integration

We outline in this section minor variants of the integration algorithm that are used for deciding whether an element of $k(t)$ is either a

- derivative of an element of $k(t)$,
- logarithmic derivative of an element of $k(t)$,
- logarithmic derivative of a $k(t)$-radical.

As we have seen in Sect. 5.2, such procedures are needed when building the tower of fields containing the integrand. Furthermore, they will be needed at various points by the algorithms of the remaining chapters, in particular when bounding orders and degrees.

Note that the structure Theorems of Chap. 9 provide efficient alternatives to the use of modified integration algorithms, and in some cases the only complete algorithms for recognizing logarithmic derivatives.

Recognizing Derivatives

The first problem is, given $f \in k(t)$, to determine whether there exists $u \in k(t)$ such that $Du = f$, and to compute such an u if it exists. We first perform the Hermite reduction on f, obtaining $g \in k(t)$, a simple $h \in k(t)$, and $r \in k\langle t\rangle$ such that $f = Dg + h + r$. At that point, we can prove (see Exercise 4.1) that if $f = Du$ for some $u \in k(t)$, then $h \in k[t]$, so we are left with integrating $h+r$ which is reduced. The algorithms of Sects. 5.7 to 5.11 can then be applied (with a minor modification in the nonlinear case, to prevent introducing a new logarithm), either proving that there is no such u, or reducing the problem to deciding whether an element $a \in k$ has an integral in $k(t)$.

If t is a primitive over k, then it follows from Theorem 4.4.2 and Lemma 5.1.2 that if a has an integral in $k(t)$, then $a = Dv + cDt$ where $v \in k$ and $c \in \mathrm{Const}(k)$, and we are reduced to a limited integration problem in k. Otherwise, $\delta(t) \geq 1$, and it follows from Theorem 4.4.2 and Lemmas 3.4.2 and 5.1.2 that if a has an integral in $k(t)$, then $a = Dv$ where $v \in k$, and we are reduced to a similar problem in k.

When $f = Da/a$ for some $a \in k(t)^*$, then Corollary 9.3.1, 9.3.2 or 9.4.1 provide alternative algorithms: $f = Du$ for $u \in k(t)$ if and only if the linear equation (9.8), (9.12) or (9.21) has a solution in $\mathbb{Q}$. Corollary 9.3.2 also provides an alternative algorithm if $f = Db/(b^2+1)$ for some $b \in k(t)$, *i.e.* $f = \arctan(b)$.

It is obvious that the solution u is not unique, but that if $f = Du = Dv$ for $u, v \in k(t)$, then $u - v \in \mathrm{Const}(k(t))$.

Recognizing Logarithmic Derivatives

The second problem is, given $f \in k(t)$, to determine whether there exists a nonzero $u \in k(t)$ such that $Du/u = f$, and to compute such an u if it exists. We can prove (see Exercise 4.2) that if $f = Du/u$ for some nonzero $u \in k(t)$, then f is simple and that all the roots of the Rothstein–Trager resultant are integers. In that case, the residue reduction produces

$$g = \sum_{r_s(\alpha)=0} \alpha \frac{Dg_\alpha}{g_\alpha} = \frac{D\left(\prod_{r_s(\alpha)=0} g_\alpha^\alpha\right)}{\prod_{r_s(\alpha)=0} g_\alpha^\alpha} = \frac{Dv}{v}$$

where $v \in k(t)$ since the α's are all integers. Furthermore, Theorem 5.6.1 implies that if $f = Du/u$ for $u \in k(t)$, then $f - g \in k[t]$, so we are left with deciding whether an element p of $k[t]$ is the logarithmic derivative of an element of $k(t)$. If $p = Du/u$ for $u \in k(t)$, then it follows from Exercise 4.2 that $\deg(p) < \max(1, \delta(t))$ and from Corollary 4.4.2 that $u = p_1^{e_1} \dots p_n^{e_n}$ where $p_i \in \mathcal{S}$ and $e_i \in \mathbb{Z}$.

If t is a primitive over k, then both p and u must be in k since $\mathcal{S} = k$, so we are reduced to a similar problem in k.

If t is an hyperexponential over k, then $p \in k$ and $u = vt^e$ for $v \in k^*$ and $e \in \mathbb{Z}$, since $\mathcal{S}^{\mathrm{irr}} = \{t\}$. We are thus reduced to deciding whether $p \in k$ can be written as

$$p = \frac{Dv}{v} + e\frac{Dt}{t}$$

for $v \in k^*$ and $e \in \mathbb{Z}$. This is a special case of the parametric logarithmic derivative problem, a variant of the limited integration problem, which is discussed in Chap. 7.

If t is a hypertangent over k and $\sqrt{-1} \notin k$, then $p = a + bt$ for $a, b \in k$, and $u = v(t^2+1)^e$ for $v \in k^*$ and $e \in \mathbb{Z}$, since $\mathcal{S}^{\mathrm{irr}} = \{t^2+1\}$. We are thus reduced to deciding whether $a + bt$ can be written as

$$a + bt = \frac{Dv}{v} + e\frac{D(t^2+1)}{t^2+1} = \frac{Dv}{v} + 2e\frac{Dt}{t^2+1}t$$

which is equivalent to

$$a = \frac{Dv}{v} \quad \text{and} \quad \frac{b}{2}\frac{t^2+1}{Dt} \in \mathbb{Z}\,.$$

The second condition can be immediately verified, while the first is the problem of deciding whether an element of k is the logarithmic derivative of an element of k.

When $f = Db$ for some $b \in k(t)$, then Corollary 9.3.1, 9.3.2 or 9.4.1 provide alternative algorithms: f is the logarithmic derivative of a $k(t)$-radical if and only if the linear equation (9.9), (9.13) or (9.22) has a solution in $\mathbb{Q}$.

The solution u is not unique, but if $f = Du/u = Dv/v$ for $u, v \in k(t) \setminus \{0\}$, then $u/v \in \mathrm{Const}(k(t))$ (this is the case $n = m = 1$ of Lemma 5.12.1 below).

Recognizing Logarithmic Derivatives of $k(t)$-radicals

The third problem is, given $f \in k(t)$, to determine whether there exist a nonzero $n \in \mathbb{Z}$ and a nonzero $u \in k(t)$ such that $Du/u = nf$, and to compute such an n and u if they exist. We can prove (see Exercise 4.2) that if $nf = Du/u$ for some nonzero $n \in \mathbb{Z}$ and $u \in k(t)$, then f is simple and that all the roots of the Rothstein–Trager resultant are rational numbers. In that case, let m be a common denominator for the roots of the Rothstein–Trager resultant. Then, the residue reduction produces

$$g = \sum_{r_s(\alpha)=0} \alpha \frac{Dg_\alpha}{g_\alpha} = \frac{1}{m}\frac{D\left(\prod_{r_s(\alpha)=0} g_\alpha^{m\alpha}\right)}{\prod_{r_s(\alpha)=0} g_\alpha^{m\alpha}} = \frac{1}{m}\frac{Dv}{v}$$

where $v \in k(t)$ since the $m\alpha$ is an integer for each α. Furthermore, Theorem 5.6.1 implies that if $f = Du/(nu)$ for $n \in \mathbb{Z}$ and $u \in k(t)$, then $f - Dg \in k[t]$, so we are left with deciding whether an element p of $k[t]$ is the logarithmic derivative of a $k(t)$-radical. If $p = Du/(nu)$ for $n \in \mathbb{Z}$ and $u \in k(t)$, then it follows from Exercise 4.2 that $\deg(p) < \max(1, \delta(t))$ and from Corollary 4.4.2 that $u = p_1^{e_1} \dots p_s^{e_s}$ where $p_i \in \mathcal{S}$ and $e_i \in \mathbb{Z}$.

If t is a primitive over k, then both p and u must be in k since $\mathcal{S} = k$, so we are reduced to a similar problem in k.

If t is an hyperexponential over k, then $p \in k$ and $u = vt^e$ for $v \in k^*$ and $e \in \mathbb{Z}$, since $\mathcal{S}^{\mathrm{irr}} = \{t\}$. We are thus reduced to deciding whether $p \in k$ can be written as

$$p = \frac{1}{n}\frac{Dv}{v} + \frac{e}{n}\frac{Dt}{t}$$

for $v \in k^*$ and $n, e \in \mathbb{Z}$. This is the parametric logarithmic derivative problem, a variant of the limited integration problem, which is discussed in Chap. 7.

If t is a hypertangent over k and $\sqrt{-1} \notin k$, then $p = a + bt$ for $a, b \in k$, and $u = v(t^2+1)^e$ for $v \in k^*$ and $e \in \mathbb{Z}$, since $\mathcal{S}^{\text{irr}} = \{t^2+1\}$. We are thus reduced to deciding whether $a + bt$ can be written as

$$a + bt = \frac{1}{n}\frac{Dv}{v} + \frac{e}{n}\frac{D(t^2+1)}{t^2+1} = \frac{1}{n}\frac{Dv}{v} + \frac{2e}{n}\frac{Dt}{t^2+1}t$$

which is equivalent to

$$na = \frac{Dv}{v} \quad \text{and} \quad \frac{b}{2}\frac{t^2+1}{Dt} \in \mathbb{Q}\,.$$

The second condition can be immediately verified, while the first is the problem of deciding whether an element of k is the logarithmic derivative of a k-radical.

When $f = Db$ for some $b \in k(t)$, then Corollary 9.3.1, 9.3.2 or 9.4.1 provide alternative algorithms: f is the logarithmic derivative of a $k(t)$-radical if and only if the linear equation (9.9), (9.13) or (9.22) has a solution in $\mathbb{Q}$.

The solution (n, u) is not unique, but any two solutions are related by the following lemma.

Lemma 5.12.1. *Let (K, D) be a differential field and $u, v \in K^*$. If*

$$\frac{1}{n}\frac{Du}{u} = \frac{1}{m}\frac{Dv}{v}$$

for nonzero $n, m \in \mathbb{Z}$, then

$$\frac{u^{\operatorname{lcm}(n,m)/n}}{v^{\operatorname{lcm}(n,m)/m}} \in \operatorname{Const}(K)\,.$$

Proof. Let $c = u^{\operatorname{lcm}(n,m)/n}/v^{\operatorname{lcm}(n,m)/m}$. Then,

$$\frac{Dc}{c} = \frac{\operatorname{lcm}(n,m)}{n}\frac{Du}{u} - \frac{\operatorname{lcm}(n,m)}{m}\frac{Dv}{v} = \operatorname{lcm}(n,m)\left(\frac{1}{n}\frac{Du}{u} - \frac{1}{m}\frac{Dv}{v}\right) = 0$$

so $c \in \operatorname{Const}(K)$. □

Exercises

Exercise 5.1. Let k be a differential field of characteristic 0, t a monomial over k, and $d \in k[t] \setminus \{0\}$. Let $d = d_1 d_2^2 \cdots d_n^n$ be a squarefree factorization of d. Show that $\mu(a/d) \le n$ for any $a \in k[t]$, and that $\mu(a/d) = n$ if and only if $\gcd(a, d) = 1$.

Exercise 5.2. Rewrite the proof of Theorem 5.3.1 using Mack's linear version of the Hermite reduction instead of the quadratic version.

Exercise 5.3. Let k be a differential field of characteristic 0, t a monomial over k, and $f \in k(t)^*$. Show that using only the extended Euclidean algorithm in $k[t]$, one can find $h_0, h_1, \ldots, h_q$ and $r \in k(t)$ such that $q \leq \mu(f)$, each h_i is simple, r is reduced, and $f = r + h_0 + Dh_1 + D^2h_2 + \cdots + D^q h_q$.

Exercise 5.4 (In-field integration). Let k be a differential field of characteristic 0 and t be a monomial over k, Write an algorithm that, given any $f \in k(t)$, returns either $g \in k(t)$ such that $Dg = f$, or "no solution" if f has no antiderivative in $k(t)$ (see Exercise 4.1).

Exercise 5.5 (Generalizations of Liouville's Theorem). Let k be a differential field of characteristic 0, $C = \mathrm{Const}(k)$, $f \in k(t)$, t be a monomial over k, and suppose that there exist an elementary extension E of $k(t)$ and $g \in E$ such that $Dg = f$.

a) Prove that

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i} \tag{5.23}$$

has a solution $v \in k\langle t \rangle$, $c_1, \ldots, c_n \in \overline{C}$, and $u_1, \ldots, u_n \in \mathcal{S}_{\overline{C}k[t]:\overline{C}k} \setminus \{0\}$.

b) Prove that if t is a nonlinear monomial over k, then (5.23) has a solution $v \in k[t]$, $c_1, \ldots, c_n \in \overline{C}$, and $u_1, \ldots, u_n \in \mathcal{S}_{\overline{C}k[t]:\overline{C}k} \setminus \{0\}$.

c) Prove that if $\mathcal{S}_1^{\mathrm{irr}} = \mathcal{S}^{\mathrm{irr}}$, then (5.23) has a solution $v \in k[t]$, $c_1, \ldots, c_n \in \overline{C}$, and $u_1, \ldots, u_n \in \mathcal{S}_{\overline{C}k[t]:\overline{C}k} \setminus \{0\}$.

d) Prove that if t is a nonlinear monomial over k and $\mathcal{S}_1^{\mathrm{irr}} = \mathcal{S}^{\mathrm{irr}}$, then f has an elementary integral over k.

e) Prove that if t is an hyperexponential monomial over k, then f has an elementary integral over k.

f) Prove that if t is a primitive monomial over k, then (5.23) has a solution $v = at + b$, $c_1, \ldots, c_n \in \overline{C}$, and $u_1, \ldots, u_n \in \overline{C}k^*$, where $a \in C$ and $b \in k$.

Exercise 5.6. Decide which of the following integrals are elementary functions, and compute those that are elementary. Since the recursive problems involving the procedures **LimitedIntegrate**, **RischDE** and **CoupledDESystem** are trivial in these exercises, perform the portions allocated to those procedures by elementary methods.

a)

$$\int \tan(ax)^5 dx, \quad a \in \mathbb{C}^* .$$

b)

$$\int x^n e^x dx, \quad n \in \mathbb{Z}, n \neq 0 .$$

c)

$$\int \frac{\log(x+a)}{x+b}\,dx, \quad a, b \in \mathbb{C}, a \neq b\,.$$

d)

$$\int \frac{(x+1)e^{x^2}+1}{\left(e^{x^2}\right)^2-1}\,dx$$

e)

$$\int \left(1+\frac{x^{2-n}}{2-n}+\frac{n-1}{x^n}\right)e^x\,dx, \quad n \in \mathbb{Z}, n \neq 2\,.$$

f)

$$\int \frac{2+\tan(x)^2}{1+\left(\tan(x)+x\right)^2}\,dx$$

g)

$$\int \frac{(3x-2)\log(x)^3+(x-1)\log(x)^2+2x(x-2)\log(x)+x^2}{x\log(x)^6-4x^2\log(x)^5+6x^3\log(x)^4-4x^4\log(x)^3+x^5\log(x)^2}\,dx$$

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