8

Dual Varieties of Homogeneous Spaces

Here we study dual varieties and discriminants of several special homogeneous spaces. We start in 8.1 by showing how to use standard results of representation theory such as the Borel–Weyl–Bott theorem, the BGG homomorphism, identities with Schur functors, and formulas of the Schubert calculus to find the codegree of Grassmannians or full and partial flag varieties. We give a list of formulas for the degree of hyperdeterminants and sketch the proof of a Theorem of Zak about varieties of codegree 3. In 8.2 we generalize the Theorem of Matsumura and Monsky about automorphisms of smooth hypersurfaces to automorphisms of smooth very ample divisors on flag varieties. In 8.3 we study commutative algebras without identities from the "discriminantal" point of view. As a corollary we prove that the algebra of diagonal matrices does not have quasiderivations. In 8.4 we study anticommutative algebras (nets of skewsymmetric forms). We show that they have beatiful geometric properties related to cubic surfaces, Del Pezzo surfaces, representation theory of S_5, etc. In 8.5 we show that the discriminant in a simple Lie algebra defined by analogy with the disciminant of a linear operator is equal to the discriminant of the minimal orbit, the so-called adjoint variety. Finally, in 8.6 we study related questions about schemes of zeros of irreducible homogeneous vector bundles. In particular, we address a question of classifying irreducible homogeneous vector bundles with a trivial line subbundle, find the maximal dimension of an isotropic subspace of a generic symmetric or skewsymmetric form, and study properties of the related Moore–Penrose involution.

8.1 Calculations of $\deg X^*$

8.1.1 Borel–Weyl–Bott Theorem

If P is an algebraic group then the exact sequence

$$\{e\} \to R_u(P) \to P \to P/R_u(P) \to \{e\}$$

always splits, i.e. there exists a reductive subgroup $L \subset P$ that maps isomorphically on $P/R_u(P)$. In other words, P is a semi-direct product $P = L \leftthreetimes R_u(P)$, called the *Levi decomposition*. L is called the *Levi subgroup*. A subgroup is a Levi subgroup if and only if it is a maximal reductive subgroup of P. All Levi subgroups are conjugated to each other.

If G is a reductive group, $P \subset G$ is a parabolic subgroup and $L \subset P$ is a Levi subgroup, then we abuse language and call L a Levi subgroup of G.

We choose the maximal torus $T \subset L$, and then T is also a maximal torus of G. Let $B \supset T$ be a Borel subgroup such that $B \cap P = B \cap L$. Then $B \cap L$ is a Borel subgroup in L. We choose the corresponding systems $\Delta_G \supset \Delta_L$ of roots, $\Delta_G^+ \supset \Delta_L^+$ of positive roots, and $\Pi_G \supset \Pi_L$ of simple roots in G and L. The semigroup $\mathcal{P}_L^+$ of dominant weights of L contains the semigroup $\mathcal{P}_G^+$ of dominant weights of G. For example, if $P = B$ is a Borel subgroup then $L = T$ and $\mathcal{P}_L^+ = \mathcal{X}(T)$.

We take any $\lambda \in \mathcal{P}_L^+$, and let U_λ be the corresponding irreducible L-module. We may consider U_λ as P-module with the trivial action of P_u. Now we define the fiber bundle $G \times_P U_\lambda$ as the quotient of $G \times U_\lambda$ by the action of P given by:

$$p \cdot (g, z) = (gp^{-1}, p \cdot z).$$

Projection onto the first factor induces the map $G \times_P U_\lambda \to G/P$, which is an equivariant vector bundle $\mathcal{E}_\lambda$ on G/P. ("Equivariant" means that G acts on $\mathcal{E}_\lambda$ compatibly with the action of G on the base G/P.)

The Borel–Weyl–Bott theorem shows how to calculate cohomology of $\mathcal{E}_\lambda$. Let V_λ be the irreducible G-module with the highest weight λ for any $\lambda \in \mathcal{P}_G^+$. Let ρ be the half-sum of positive roots of G. Then $(\rho, \alpha_i) = 1$ for any simple root $\alpha_i \in \Pi_G$. The element $z \in \mathfrak{t}_{\mathbb{R}}^\vee$ is called *singular* if there exists a root $\alpha \in \Delta_G$ such that $(z, \alpha) = 0$, i.e. if z belongs to the wall of a Weyl chamber. If z is not singular then there exists a unique element $w \in W_G$ of the Weyl group of G such that $w \cdot z$ belongs to the fixed Weyl chamber C^+.

Theorem 8.1 ([Bo])

- *If $\lambda \in \mathcal{P}_G^+$ then $H^0(G/P, \mathcal{E}_\lambda) = V_\lambda^\vee$ and $H^i(G/P, \mathcal{E}_\lambda) = 0$ for $i > 0$.*
- *If $\lambda + \rho$ is singular then $H^i(G/P, \mathcal{E}_\lambda) = 0$ for any i.*
- *If $\lambda + \rho$ is not singular, then, for the unique element $w \in W_G$,*

$$\lambda' = w(\lambda + \rho) - \rho \in \mathcal{P}_G^+.$$

Then $H^{l(w)}(G/P, \mathcal{E}_\lambda) = V_{\lambda'}^\vee$ and $H^i(G/P, \mathcal{E}_\lambda) = 0$ for $i \neq l(w)$.

Corollary 8.2

$$\chi(\mathcal{E}_\lambda) = \sum (-1)^i \dim H^i(G/P, \mathcal{E}_\lambda) = \prod_{\alpha \in \Delta_G^+} (\lambda + \rho, \alpha)/(\rho, \alpha).$$

Proof. Indeed, if $\lambda \in \mathcal{P}_G^+$, then this is a well-known Weyl formula for the dimension of V_λ; see [OV]. If λ is singular then this formula is obvious. Suppose that λ is not singular, and let $\lambda' = w(\lambda+\rho)-\rho \in \mathcal{P}_G^+$ for the unique element $w \in W_G$. Then

$$\chi(\mathcal{E}_\lambda) = (-1)^{l(w)} \dim V_{\lambda'} = (-1)^{l(w)} \prod_{\alpha\in\Delta_G^+} (\lambda'+\rho,\alpha)/(\rho,\alpha) =$$

$$(-1)^{l(w)} \prod_{\alpha\in\Delta_G^+} (w(\lambda+\rho),\alpha)/(\rho,\alpha) = (-1)^{l(w)} \prod_{\alpha\in\Delta_G^+} (\lambda+\rho, w^{-1}\alpha)/(\rho,\alpha) =$$

$$\prod_{\alpha\in\Delta_G^+} (\lambda+\rho,\alpha)/(\rho,\alpha).$$

The corollary is proved. $\square$

8.1.2 Representation Theory of $\mathbf{GL}_n$

We choose the standard maximal torus T of diagonal matrices and the Borel subgroup of upper-triangular matrices. The group of characters $\mathcal{X}(T)$ is freely generated by characters $\varepsilon_1,\ldots,\varepsilon_n$, where

$$\varepsilon_k\big(\mathrm{diag}\{x_1,\ldots,x_n\}\big) = x_k.$$

The roots are $\alpha_{ij} = \varepsilon_i - \varepsilon_j$. The root subspace of α_{ij} is $\mathbb{C}E_{ij}$. The simple roots are $\varepsilon_1-\varepsilon_2,\ldots,\varepsilon_{n-1}-\varepsilon_n$. A weight $\lambda = \sum \lambda_i\varepsilon_i$ is dominant if and only if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. For example, the highest weight of $S^d\mathbb{C}^n$ is $d\varepsilon_1$ and the corresponding highest weight vector is e_1^d. Similarly, the highest weight of $\Lambda^d\mathbb{C}^n$ is $\varepsilon_1+\ldots+\varepsilon_d$ and the corresponding highest weight vector is $e_1\wedge\ldots\wedge e_d$.

One-dimensional representations of GL_n correspond to characters of GL_n, i.e. to functions $\det^p$, $p \in \mathbb{Z}$. The corresponding highest weights are $p\sum\varepsilon_i$. Therefore any irreducible representation, after tensoring by a suitable character, has the highest weight λ with the property $\lambda_n \geq 0$. We identify these dominant weights with the set $\mathcal{P}_n$ of all partitions

$$\lambda_1 \geq \ldots \geq \lambda_n \geq 0.$$

Let $\mathcal{P}_n^m \subset \mathcal{P}_n$ be the subset of partitions such that $\lambda_1 \leq m$. For any partition $\lambda \in \mathcal{P}_n$ we set

$$|\lambda| = \lambda_1 + \ldots + \lambda_n.$$

For any partition $\lambda \in \mathcal{P}_n^m$ we denote by $\lambda^t \in \mathcal{P}_m^n$ the transposed partition given by

$$\lambda_i^t = \max\{j \mid \lambda_j \geq i\}.$$

For any partition $\lambda \in \mathcal{P}_n$ (or just the dominant weight λ), we denote by $S_\lambda(\mathbb{C}^n)$ the irreducible representation of GL_n with the highest weight λ.

We will need the *Cauchy formula*

$$\Lambda^d(\mathbb{C}^n \otimes \mathbb{C}^m) = \bigoplus_{\lambda \in \mathcal{P}_n^m,\ |\lambda|=d} S_\lambda(\mathbb{C}^n) \otimes S_{\lambda^t}(\mathbb{C}^m),$$

the decomposition of $\mathrm{GL}_n \times \mathrm{GL}_m$-modules.

A parabolic subgroup $P \subset \mathrm{GL}_n$ consists of block triangular matrices

$$\begin{pmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{k1} & \dots & \dots & A_{kk} \end{pmatrix}$$

and its Levi subgroup L consists of block diagonal matrices with $A_{ij} = 0$ for $i > j$. Here $A_{ij} \in \mathrm{Mat}_{n_i,n_j}$ and $\sum n_i = n$. Dominant weights λ for L are given by sequences $\lambda_1, \dots, \lambda_n$ such that

$$\lambda_1 \geq \dots \geq \lambda_{n_1}, \quad \dots, \quad \lambda_{n-n_k+1} \geq \dots \geq \lambda_n.$$

By Corollary 8.2, the Euler characteristic $\chi(\mathcal{E}_\lambda)$ is given by the formula

$$\chi(\mathcal{E}_\lambda) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

8.1.3 Dual Variety of the Grassmannian

Let $X = \mathrm{Gr}(n_1, \mathbb{C}^{n_1+n_2})$ be the Grassmannian of n_1-dimensional subspaces in $\mathbb{C}^{n_1+n_2}$ in the Plücker embedding $X \subset \mathbb{P}(\mathbb{C}^N)$, where $\mathbb{C}^N = \Lambda^{n_1}\mathbb{C}^{n_1+n_2}$. We are going to calculate degree of the dual variety X^*.

Theorem 8.3 ([Las]) *Let $X = \mathrm{Gr}(n_1, \mathbb{C}^{n_1+n_2}) \subset \mathbb{P}(\Lambda^{n_1}\mathbb{C}^{n_1+n_2})$ be the Grassmannian of n_1-dimensional subspaces in $\mathbb{C}^{n_1+n_2}$ in the Plücker embedding. Then*

$$\deg \Delta_X = \sum_{\lambda \in \mathcal{P}_{n_1}^{n_2}, S} \frac{(-1)^S (S+1) \nabla(-\lambda_m^t, \dots, -\lambda_1^t, \lambda_1 - p + S, \dots, \lambda_n - p + S)}{\nabla(1, \dots, n_1 + n_2)},$$

where $S = n_1 n_2 - |\lambda|$ or $S = n_1 n_2 - |\lambda| - 1$, $p \in \mathbb{Z}$ is arbitrary, and

$$\nabla(l_1, \dots, l_{n_1+n_2}) = \prod_{1 \leq i < j \leq n_1+n_2} (l_i - l_j + j - i).$$

Proof. Let $\mathcal{L} = \mathcal{O}_X(1)$, and we have $\dim X = n_1 n_2$. By Theorem 6.13,

$$\deg \Delta_X = \chi\big(\Lambda^{n_1 n_2}([T_X^\vee] - 2[\mathcal{L}^\vee] + [\mathcal{O}_X])[\mathcal{L}]^p\big)$$

for any $p \in \mathbb{Z}$. We have

$$\Lambda^{n_1 n_2}([T_X^\vee] - 2[\mathcal{L}^\vee] + [\mathcal{O}_X]) = \sum_{i=0}^{n_1 n_2} \Lambda^i(T_X^\vee)\Lambda^{n_1 n_2 - i}([\mathcal{O}_X] - 2[\mathcal{L}^\vee]),$$

$$\Lambda^k([\mathcal{O}_X] - 2[\mathcal{L}^\vee]) = (-1)^k\big((k+1)[\mathcal{L}^\vee]^k - k[\mathcal{L}^\vee]^{k-1}\big).$$

Let $\mathcal{S}$ be the tautological vector bundle on X, and let $V/\mathcal{S}$ be the quotient tautological bundle. Then $T_X^\vee = \mathcal{S} \otimes (V/\mathcal{S})^\vee$ and by the Cauchy formula,

$$\Lambda^i(\mathcal{S} \otimes (V/\mathcal{S})^\vee) = \sum_{\lambda \in \mathcal{P}_{n_1}^{n_2},\ |\lambda| = i} S_\lambda(\mathcal{S}) \otimes S_{\lambda^t}((V/\mathcal{S})^\vee).$$

Therefore we have

$$\deg \Delta_X = \sum_{\lambda \in \mathcal{P}_{n_1}^{n_2}, S} (-1)^S (S+1)\chi\Big(S_\lambda(\mathcal{S}) \otimes S_{\lambda^t}((V/\mathcal{S})^\vee)[\mathcal{L}]^{p-S}\Big),$$

where $S = n_1 n_2 - |\lambda|$ or $S = n_1 n_2 - |\lambda| - 1$ and $p \in \mathbb{Z}$ is arbitrary.

By Corollary 8.2,

$$\chi\Big(S_\lambda(\mathcal{S}) \otimes S_{\lambda^t}((V/\mathcal{S})^\vee)[\mathcal{L}]^{p-S}\Big) =$$

$$\frac{\nabla(-\lambda_m^t, -\lambda_{m-1}^t, \ldots, -\lambda_1^t, \lambda_1 - p + S, \ldots, \lambda_n - p + S)}{\nabla(1, \ldots, n_1 + n_2)}.$$

The theorem is proved. □

If $m = 2$ then this degree was already calculated in Example 2.13. For small values of m and n the degree can be found from the following table.

(m,n)	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$	$(3,7)$	$(3,8)$	$(4,4)$	$(4,5)$
d	4	7	16	120	640	3608	126	2943

Table 8.1.

8.1.4 Codegree of G/B

Let G be a simple algebraic group of rank r with a Borel subgroup B. The minimal equivariant projective embedding of G/B corresponds to the line bundle $\mathcal{L}_\rho$, where

$$\rho = \omega_1 + \ldots + \omega_r$$

is a half sum of positive roots. We shall follow [CW] here and calculate the degree of the corresponding discriminant.

Let $\{\beta_1, \ldots, \beta_N\} = \Delta_+$ be the collection of all positive roots. We consider the matrix

$$M = \left(\frac{\langle \beta_j \,|\, \beta_i \rangle}{\langle \rho \,|\, \beta_i \rangle} \right)_{i,j=1,\dots,N}.$$

Let $P_s(M)$ be the sum of all permanents of $s \times s$ submatrices of M. Recall that the permanent of a $n \times n$ matrix (a_{ij}) is equal to $\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$.

Theorem 8.4 ([CW])

$$\deg \Delta_{(G/B, \mathcal{L}_\rho)} = \sum_{s=0}^{N} (s+1)! P_{N-s}(M).$$

Proof. Recall that by Theorem 6.2 we have

$$\deg \Delta_{(G/B, \mathcal{L}_\rho)} = \sum_{i=0}^{N} (i+1) \int_{G/B} c_{n-i}(T^\vee_{G/B}) \cdot c_1(\mathcal{L}_\rho)^i.$$

Therefore we need to calculate these integrals. The correspondence $\lambda \mapsto \mathcal{L}_\lambda$ identifies $\mathcal{P} \otimes \mathbb{Q}$ with $\mathrm{Pic}(G/B) \otimes \mathbb{Q}$. Therefore we have a homomorphism of commutative algebras

$$c:\ S^\bullet(\mathcal{P} \otimes \mathbb{Q}) \to A^\bullet(G/B),$$

where $S^\bullet(\mathcal{P} \otimes \mathbb{Q})$ is the symmetric algebra of $\mathcal{P} \otimes \mathbb{Q}$ and $A^\bullet(G/B)$ is the rational Chow ring. It is well-known (see e.g. [BGG]) that this homomorphism is surjective and its kernel is generated as an ideal by W-invariants of positive degree $(S^\bullet(\mathcal{P} \otimes \mathbb{Q}))^W_+$, where W is the Weyl group. For example,

$$c_1(\mathcal{L}_\rho) = c(\rho).$$

Using an appropriate filtration of the cotangent bundle $T^\vee_{G/B}$ and 'the splitting principle' it is easy to see that

$$c_{n-i}(T^\vee_{G/B}) = c(X_i), \quad X_i = \sum_{\substack{\Gamma \subset \Delta_+ \\ |\Gamma| = n-i}} \prod_{\alpha \in \Gamma} \alpha.$$

Therefore it suffices to compute $c(\rho^i X_i)$.

For any root α we define a linear function

$$D_\alpha(\lambda) = \langle \lambda \,|\, \alpha \rangle.$$

It extends by a Leibniz rule to the differential operator on $S^\bullet(\mathcal{P} \otimes \mathbb{Q})$. Consider the differential operator D given by

$$D = \frac{\prod_{\alpha > 0} D_\alpha}{\prod_{\alpha > 0} \langle \rho \,|\, \alpha \rangle}.$$

D decreases the degree by N, and so in particular we get a linear form

$$D:\ S^N(\mathcal{P}\otimes\mathbb{Q})\to\mathbb{Q}.$$

One can show (see [CW]) that for any $x\in S^N(\mathcal{P}\otimes\mathbb{Q})$ we have

$$D(x)=\int_{G/B}c(x).$$

Now the claim of the theorem follows by an easy calculation. $\square$

8.1.5 A Closed Formula

In most circumstances, known formulas for the degree of the discriminant depend on a certain set of discrete parameters. There are two possibilities for these parameters. First, we may fix a projective variety and vary its polarizations, see Theorem 6.8. The second possibility is to change a variety and to fix a polarization (in a certain sense). For example, consider the irreducible representation of SL_{n_0} with the highest weight λ. Then this weight can be considered as a highest weight of SL_n for any $n\geq n_0$ with respect to the natural embedding $\mathrm{SL}_n\subset\mathrm{SL}_{n+1}\subset\ldots$. As a result, we obtain a tower of flag varieties with "the same" polarization. The degree of the corresponding discriminants will be a function in n. This function can be very complicated, see e.g. Theorem 8.3.

However, sometimes this function has a closed expression – the Boole formula (6.4) is a formula of this kind. Another simple example is the formula (2.13) for the degree of the dual variety to $\mathrm{Gr}(2,n)$ in the Plücker embedding. The following theorem is a mixture of these two cases.

Theorem 8.5 ([T1]) *Let V be an irreducible SL_n-module with the highest weight $(a-1)\varphi_1+\varphi_2$, where $a\geq 2$. Then the variety $X^*\subset\mathbb{P}(V^\vee)$ projectively dual to the projectivization X of the orbit of the highest vector is a hypersurface of degree*

$$\frac{(n^2-n)a^{n+1}-(n^2+n)a^{n-1}-2n(-1)^n}{(a+1)^2}.$$

Proof. We have to calculate the degree of Δ_X. We use Kleiman's formula (6.2) for the degree of the dual variety. In our case $X=G/P$, where $G=\mathrm{SL}_n$ and $P\subset G$ is the parabolic subgroup of matrices

$$\begin{pmatrix} * & 0 & 0 & \dots & 0\\ * & * & 0 & \dots & 0\\ * & * & * & \dots & *\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ * & * & * & \dots & * \end{pmatrix}. \tag{8.1}$$

Assume that $T \subset G$ is the diagonal torus, B is the Borel subgroup of lower-triangular matrices, $x_1, \ldots, x_n$ are the weights of the tautological representation, $X(T)$ is the lattice of characters of T, S is the symmetric algebra of $X(T)$ (over $\mathbb{Q}$), $W \simeq S_n$ is the Weyl group of G, and $W_P \simeq S_{n-2}$ is the Weyl group of P. It is well known (see [BGG]) that the map $c:\ X(T) \to \mathrm{Pic}(G/B)$ that assigns to λ the first Chern class of the invertible sheaf $\mathcal{L}_\lambda$ can be extended to a surjective homomorphism $c:\ S \to A^\bullet(G/B)$ in the (rational) Chow ring, and its kernel coincides with $S_+^W S$. The projection $\alpha:\ G/B \to G/P$ induces an embedding $\alpha^*:\ A^\bullet(G/P) \to A^\bullet(G/B)$. The image coincides with the subalgebra of W_P-invariants. Hence $A^\bullet(G/P) = S^{W_P}/S_+^W S^{W_P}$. We denote the homomorphism $S^{W_P} \to A^\bullet(G/P)$ by the same letter c.

To apply Kleiman's formula we need $c_1(\mathcal{L})$ (which is equal to $c(ax_1+x_2)$) and the total Chern class of $T_Z^\vee$, which is equal to

$$c(T_Z^\vee) = c\left((1-x_1+x_2)\prod_{i=3,\ldots,n}(1-x_1+x_i)\prod_{i=3,\ldots,n}(1-x_2+x_i)\right).$$

(This can be shown using a filtration of $T_Z^\vee$ and the splitting principle.)

Let $\alpha_1, \ldots, \alpha_{n-2}$ be the elementary symmetric polynomials in $x_3, \ldots, x_n$. Then $S^{W_P} = \mathbb{Q}[x_1, x_2, \alpha_1, \ldots, \alpha_{n-2}]$, and the ideal $S_+^W S^{W_P}$ is generated by

$$\begin{gathered} x_1+x_2+\alpha_1, \quad x_1x_2+x_1\alpha_1+x_2\alpha_1+\alpha_2, \\ x_1x_2\alpha_1+x_1\alpha_2+x_2\alpha_2+\alpha_3,\ \ldots,\ x_1x_2\alpha_{n-4}+x_1\alpha_{n-3}+x_2\alpha_{n-3}+\alpha_{n-2}, \\ x_1x_2\alpha_{n-3}+x_1\alpha_{n-2}+x_2\alpha_{n-2}, \quad x_1x_2\alpha_{n-2}. \end{gathered}$$

Hence $\alpha_i = (-1)^i(x_1^i + x_1^{i-1}x_2 + \ldots + x_2^i) \mod S_+^W S^{W_P}$, and $A^\bullet(G/P)$ is isomorphic to the quotient ring $\mathbb{Q}[x_1,x_2]/\langle f_1, f_2\rangle$, where $f_1 = x_1^{n-1}+x_1^{n-2}x_2+\ldots+x_2^{n-1}$ and $f_2 = x_1^n$. Note that f_1, f_2 is the Gröbner basis of the ideal $\langle f_1, f_2\rangle$ with respect to the ordering $x_2 > x_1$ (see [Berg]). Hence the set of X^iY^j is a basis of the quotient algebra, where $X = x_1 \mod \langle f_1, f_2\rangle$ and $Y = x_2 \mod \langle f_1, f_2\rangle$, $i = 1, \ldots, n-1$, $j = 1, \ldots, n-2$.

To calculate the degree of the discriminant by Kleiman's formula, we have to calculate $\int_Z c(X^{n-1}Y^{n-2})$. Let $\tilde{w}_0$ be the longest element in W, and let w_0 be the shortest element in $\tilde{w}_0 W_P$ with the reduced factorization

$$\begin{aligned} w_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ n & n-1 & 1 & 2 & \ldots & n-2 \end{pmatrix} = \\ (n-1,n)(n-2,n-1)\ldots(12)\cdot(n-1,n)(n-2,n-1)\ldots(23). \end{aligned}$$

Let

$$A_{w_0} = A_{(n-1,n)}A_{(n-2,n-1)}\ldots A_{(12)}A_{(n-1,n)}A_{(n-2,n-1)}\ldots A_{(23)}$$

be the corresponding endomorphism of S of degree $-(2n-3)$, where

$$A_{(ij)} = \frac{\mathrm{Id} - s_{(ij)}}{x_i - x_j},$$

and $s_{(ij)}$ is the reflection that transposes x_i and x_j. Then

$$\int_Z c(X^{n-1}Y^{n-2}) = A_{w_0}(x_1^{n-1}x_2^{n-2})$$

(see [BGG]). It is obvious that

$$\begin{aligned} A_{w_0}(x_1^{n-1}x_2^{n-2}) = A_{\rho_1}A_{\rho_2}(x_1^{n-1}x_2^{n-2}) = \\ A_{\rho_1}(x_1^{n-1}A_{\rho_2}(x_2^{n-2})) = A_{\rho_1}(x_1)A_{\rho_2}(x_2), \end{aligned}$$

where $\rho_k = (n{-}1, n)(n{-}2, n{-}1)\dots(k, k{+}1)$. These factors are both equal to 1, since they are equal to $\int_{\mathbb{P}^{n-1}} c_1(\mathcal{O}(1))^{n-1}$ and $\int_{\mathbb{P}^{n-2}} c_1(\mathcal{O}(1))^{n-2}$, respectively. We finally see that $\int_Z c(X^{n-1}Y^{n-2}) = 1$.

It remains to calculate the polynomial

$$\sum_{i=0}^{2n-3}(i+1)c_{2n-3-i}(aX+Y)^i, \tag{8.2}$$

in the ring $\mathbb{Q}[X,Y]$ (with the basis X^iY^j, $i = 1, \dots, n{-}1$, $j = 1, \dots, n{-}2$, and relations $X^{n-1} + X^{n-2}Y + \dots + Y^{n-1} = 0$, $Y^n = 0$ and $X^n = 0$, which follow from the preceding relations), where c_k is the kth homogeneous component of the polynomial

$$(1 - X + Y)\prod_{i=3}^{n}(1 - X + x_i)\prod_{i=3}^{n}(1 - Y + x_i),$$

in which the ith symmetric function of $x_3, \dots, x_n$ must be replaced by $(-1)^i(X^i + X^{i-1}Y + \dots + Y^i)$. By the previous discussion, the result of this calculation will be $\deg(\Delta_X)X^{n-1}Y^{n-2}$.

Note that the polynomial (8.2) is equal to

$$F'(T)|_{T=2X+Y} = (F_1F_2F_3F_4)'(T)|_{T=2X+Y},$$

where

$$\begin{aligned} F_1 = T, \quad F_2 = T - X + Y,& \\ F_3 = \prod_{i=3}^{n}(T - X + x_i) = \sum_{i=0}^{n-2}(T-X)^{n-2-i}(-1)^i(X^i + \dots + Y^i),& \\ F_4 = \prod_{i=3}^{n}(T - Y + x_i) = \sum_{i=0}^{n-2}(T-Y)^{n-2-i}(-1)^i(X^i + \dots + Y^i).& \end{aligned}$$

We have, further,

$$F_3(T) = \left(\sum_{i=0}^{n-2}(T-X)^{n-2-i}(-1)^i\frac{X^{i+1} - Y^{i+1}}{X - Y}\right) =$$

$$\frac{X(T-X)^{n-2}}{X-Y}\left(\sum_{i=0}^{n-2}(T-X)^{-i}(-1)^i X^i\right)-$$

$$\frac{Y(T-X)^{n-2}}{X-Y}\left(\sum_{i=0}^{n-2}(T-X)^{-i}(-1)^i Y^i\right)=$$

$$\frac{X\left((T-X)^{n-1}-(-X)^{n-1}\right)}{T(X-Y)}-\frac{Y\left((T-X)^{n-1}-(-Y)^{n-1}\right)}{(T-X+Y)(X-Y)}.$$

We obtain, likewise, that

$$F_4(T)=\left(\sum_{i=0}^{n-2}(T-Y)^{n-2-i}(-1)^i\frac{X^{i+1}-Y^{i+1}}{X-Y}\right)=$$

$$\frac{X(T-Y)^{n-2}}{X-Y}\left(\sum_{i=0}^{n-2}(T-Y)^{-i}(-1)^i X^i\right)-$$

$$\frac{Y(T-Y)^{n-2}}{X-Y}\left(\sum_{i=0}^{n-2}(T-Y)^{-i}(-1)^i Y^i\right)=$$

$$\frac{X\left((T-Y)^{n-1}-(-X)^{n-1}\right)}{(T+X-Y)(X-Y)}-\frac{Y\left((T-Y)^{n-1}-(-Y)^{n-1}\right)}{T(X-Y)}.$$

We deduce from the latter formula that

$$F'(T)=\frac{n(X-Y)(T-X)^{n-1}+(-X)^n-(-Y)^n}{X-Y}\times$$

$$\frac{(X-Y)(T-Y)^n+T(-X)^n-(T+X-Y)(-Y)^n}{T(T+X-Y)(X-Y)}+$$

$$\frac{(X-Y)(T-X)^n+(T-X+Y)(-X)^n-T(-Y)^n}{T(X-Y)}\times$$

$$\frac{\Big(n(T+X-Y)-(T-Y)\Big)(T-Y)^{n-1}+(-X)^n}{(T+X-Y)^2}-$$

$$\frac{(X-Y)(T-X)^n+(T-X+Y)(-X)^n-T(-Y)^n}{T(X-Y)}\times$$

$$\frac{(X-Y)(T-Y)^n+T(-X)^n-(T+X-Y)(-Y)^n}{T(T+X-Y)(X-Y)}.$$

Using the elementary formulas

$$\frac{\alpha^n(\gamma-\beta)+\beta^n(\alpha-\gamma)+\gamma^n(\beta-\alpha)}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}=\sum_{i+j+k=n-2}\alpha^i\beta^j\gamma^k,$$

$$\frac{\alpha^n(\gamma-\beta)+\beta^n(\alpha-\gamma)+\gamma^n(\beta-\alpha)}{(\alpha-\beta)(\gamma-\beta)} = \sum_{i=0}^{n-2} \beta^i(\alpha^{n-1-i}-\gamma^{n-1-i}),$$

$$\frac{(n(\alpha-\beta)-\alpha)\alpha^{n-1}+\beta^n}{(\alpha-\beta)^2} = \sum_{i=0}^{n-2}(n-1-i)\alpha^{n-2-i}\beta^i,$$

we obtain

$$\begin{aligned} F'(T) = \Big(n(T-X)^{n-1} + (-1)^n \sum_{i=0}^{n-1} X^{n-1-i}Y^i\Big)\times \\ \Big(\sum_{i+j+k=n-2}(T-Y)^i(-X)^j(-Y)^k\Big)+ \\ \Big(\sum_{i=0}^{n-2}(-X)^i((T-X)^{n-1-i}-(-Y)^{n-1-i})\Big)\times \\ \Big(\sum_{i=0}^{n-2}(n-1-i)(T-Y)^{n-2-i}(-X)^i\Big)- \\ \Big(\sum_{i=0}^{n-2}(-X)^i((T-X)^{n-1-i}-(-Y)^{n-1-i})\Big)\times \\ \Big(\sum_{i+j+k=n-2}(T-Y)^i(-X)^j(-Y)^k\Big). \end{aligned}$$

Putting $T = aX + bY$ in the latter formula, we obtain

$$\begin{aligned} \Big(\sum_{i=0}^{n-1}\big(1+(-1)^n n(a-1)^{n-1-i}b^i\binom{n-1}{i}\big)X^{n-1-i}Y^i\Big)\times \\ \Big(\sum_{i=0}^{n-2}\begin{bmatrix} n-2 \\ i\end{bmatrix}_{a,b-1} X^iY^{n-2-i}\Big)- \\ \Big(\sum_{i=0}^{n-1}\big(\begin{bmatrix} n-1 \\ i\end{bmatrix}_{a-1,b} - \begin{bmatrix} n-2 \\ i-1\end{bmatrix}_{a-1,b} - 1\big)X^{n-1-i}Y^i\Big)\times \\ \Big(\sum_{i=0}^{n-2}\big(\begin{Bmatrix} n-2 \\ i\end{Bmatrix}_{a,b-1} - \begin{Bmatrix} n-3 \\ i\end{Bmatrix}_{a,b-1}\big)X^{n-2-i}Y^i\Big)+ \\ \Big(\sum_{i=0}^{n-1}\big(\begin{bmatrix} n-1 \\ i\end{bmatrix}_{a-1,b} - \begin{bmatrix} n-2 \\ i-1\end{bmatrix}_{a-1,b} - 1\big)X^iY^{n-1-i}\Big)\times \\ \Big(\sum_{i=0}^{n-2}\begin{bmatrix} n-2 \\ i\end{bmatrix}_{a,b-1} X^iY^{n-2-i}\Big), \end{aligned}$$

where

$$\begin{bmatrix} n \\ k\end{bmatrix}_{xy} = \sum_{p=0}^{k}\sum_{q=0}^{n-k}(-1)^{p+q}x^py^q\binom{p+q}{p},$$

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{xy} = \sum_{p=0}^{k} \sum_{q=0}^{n-k} (p+q+1)(-1)^{p+q} x^p y^q \binom{p+q}{p}.$$

It remains to calculate the degree of the discriminant, that is, the difference between the coefficients of $X^{n-1}Y^{n-2}$ and $X^{n-2}Y^{n-1}$ in the expression for $F'(aX+bY)$. After some transformations we obtain

$$\sum_{i=0}^{n-1} \left((-1)^n n(a-1)^{n-1-i} b^i \binom{n-1}{i} + W_i \right) \times \left(\begin{bmatrix} n-2 \\ i \end{bmatrix}_{a,b-1} - \begin{bmatrix} n-2 \\ i-1 \end{bmatrix}_{a,b-1} \right) -$$
$$\sum_{i=0}^{n-1} (W_i - 1) \times \left(\left\{ \begin{matrix} n-2 \\ i \end{matrix} \right\}_{a,b-1} - \left\{ \begin{matrix} n-3 \\ i \end{matrix} \right\}_{a,b-1} - \left\{ \begin{matrix} n-2 \\ i-1 \end{matrix} \right\}_{a,b-1} + \left\{ \begin{matrix} n-3 \\ i-1 \end{matrix} \right\}_{a,b-1} \right),$$

where

$$W_i = \begin{bmatrix} n-1 \\ n-1-i \end{bmatrix}_{a-1,b} - \begin{bmatrix} n-2 \\ n-1-i \end{bmatrix}_{a-1,b}.$$

This is the degree of the discriminant of the irreducible SL_n-module with the highest weight $(a-b)\varphi_1 + b\varphi_2$.

Substituting $b = 1$ in the last formula, we obtain

$$\frac{(n^2-n)a^{n+1} - (n^2+n)a^{n-1} - 2n(-1)^n}{(a+1)^2}.$$

□

8.1.6 Degree of Hyperdeterminants

Consider the flag variety $\mathbb{P}^{k_1} \times \ldots \times \mathbb{P}^{k_r}$ of the group $\mathrm{SL}_{k_1+1} \times \ldots \times \mathrm{SL}_{k_r+1}$. The projectively dual variety of its 'minimal' equivariant projective embedding is related to a nice theory of hyperdeterminants initiated by Cayley and Schläfli [Ca1, Ca2, Schl].

Let $r \geq 2$ be an integer, and let $A = (a_{i_1 \ldots i_r})$, $0 \leq i_j \leq k_j$ be an r-dimensional complex matrix of *format* $(k_1+1) \times \ldots \times (k_r+1)$. The hyperdeterminant of A is defined as follows; cf. Example 3.8. Consider the product $X = \mathbb{P}^{k_1} \times \ldots \times \mathbb{P}^{k_r}$ of several projective spaces embedded in $\mathbb{P}^{(k_1+1)\times\ldots\times(k_r+1)-1}$ via the Segre embedding. Let X^* be the projectively dual variety. If X^* is a hypersurface then it is defined by a corresponding discriminant Δ_X, which in this case is called the *hyperdeterminant* Det. Clearly $\mathrm{Det}(A)$ is a polynomial function in matrix entries of A invariant under the action of $\mathrm{SL}_{k_1+1} \times \ldots \times \mathrm{SL}_{k_r+1}$. If X^* is not a hypersurface then we set $\mathrm{Det} = 1$.

Example 3.8 shows that X^* is a hypersurface (and, hence, defines a hyperdeterminant) if and only if $2k_j \le k_1 + \ldots + k_r$ for $j = 1, \ldots, r$. If for one of the j we have an equality $2k_j = k_1 + \ldots + k_r$ then the format is called *boundary*. Let $N(k_1, \ldots, k_r)$ be the degree of the hyperdeterminant of format $(k_1 + 1) \times \ldots \times (k_r + 1)$. The proof of the following theorem can be found in [GKZ2] or [GKZ3].

Theorem 8.6

(i) *The generating function for* $N(k_1, \ldots, k_r)$ *is given by*

$$\sum_{k_1,\ldots,k_r \ge 0} N(k_1, \ldots, k_r) z_1^{k_1} \ldots z_r^{k_r} = \frac{1}{\left(1 - \sum_{i=2}^{r} (i-1) e_i(z_1, \ldots, z_r)\right)^2}$$

where $e_i(z_1, \ldots, z_r)$ *is the* i*-th elementary symmetric polynomial.*

(ii) *The degree* $N(k_1, \ldots, k_r)$ *of the boundary format is given by* (*assuming that* $k_1 = k_2 + \ldots + k_r$)

$$N(k_2 + \ldots + k_r, k_2, \ldots, k_r) = \frac{(k_2 + \ldots + k_r + 1)!}{k_2! \ldots k_r!}.$$

(iii) *The degree of the hyperdeterminant of the cubic format is given by*

$$N(k, k, k) = \sum_{0 \le j \le k/2} \frac{(j + k + 1)!}{(j!)^3 (k - 2j)!} \cdot 2^{k-2j}.$$

(iv) *The exponential generating function for the degree* N_r *of the hyperdeterminant of format* $2 \times 2 \times \ldots \times 2$ (r *times*) *is given by*

$$\sum_{r \ge 0} N_r \frac{z^r}{r!} = \frac{e^{-2z}}{(1-z)^2}.$$

8.1.7 Varieties of Small Codegree

Let X be a projective variety. We define the *codegree* $\operatorname{codeg} X$ by

$$\operatorname{codeg} X = \deg X^*.$$

If X^* is a hypersurface, then $\operatorname{codeg} X$ is also called the *class* of X. This is a classical invariant playing an important role in enumerative geometry. If X^* is not a hypersurface and X' is a generic hyperplane section of X, then by Theorem 5.3 it is clear that $\operatorname{codeg} X' = \operatorname{codeg} X$.

With regard to codegree, the most simple nonsingular projective varieties are those whose codegree is small. The problem of classification of varieties of small codegree should be compared with that of classification of varieties of small degree. Much is known about this last problem. The case of varieties of degree two is classical (quadrics). The complete description of varieties of degree three was given by A. Weil in the anonymous publication [XXX]. Swinnerton–Dyer [Sw] classified all varieties of degree four. In the smooth case the classification up to degree 8 has been completed in several papers by Okonek [O1, O2, O3, O4] and Ionescu [I, I1, I2]. Fania and Livorni [FaL1, FaL2] classified varieties of degree 9, 10.

The same question can be asked for the codegree, but this problem is quite different. For example, in the case of varieties of small degree one can proceed by induction using the fact that the degree is stable with respect to passing to hyperplane sections, whereas there is no such inductive procedure for codegree.

Theorem 8.7 ([Z6]) *There exist exactly* 10 *non-degenerate nonsingular complex projective varieties of codegree three, namely*

- *the self-dual Segre threefold* $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$*,*
- *its hyperplane section* $F_1 \subset \mathbb{P}^4$ *obtained by blowing up a point in* $\mathbb{P}^2$ *by means of the map defined by the linear system of conics passing through this point,*
- *the four Severi varieties, i.e. the Veronese surface* $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$*, the Segre variety* $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$*, the Grassmann variety* $Gr_2(\mathbb{C}^6) \subset \mathbb{P}^{14}$ *of lines in* $\mathbb{P}^5$ *and the* 16*-dimensional variety* $E \subset \mathbb{P}^{26}$ *corresponding to the orbit of highest weight vector for the standard representation of the algebraic group of type* E_6*,*
- *the four varieties obtained by projecting the Severi varieties from generic points of their ambient linear spaces.*

Sketch of the proof. If X^* is not a hypersurface, then we can either apply Weil's classification theorem to X^* or consider the intersection of X with a general hyperplane, thus reducing the problem to the case when X^* is a (singular) cubic in $(\mathbb{P}^N)^\vee$. Suppose that X^* is a hypersurface, and let $\Sigma = X^*_{sing}$. Then each point of Σ has multiplicity two and, since $\deg X^* = 3$, we conclude that $\mathrm{Sec}(\Sigma) \subseteq X^*$. Let pr_1 and pr_2 denote the projections of the conormal variety $I_X \subset \mathbb{P}^N \times (\mathbb{P}^N)^\vee$ on X and X^*. Let x be a generic point of X, let $\mathcal{P}_x = \mathrm{pr}_2(\mathrm{pr}_1^{-1}(x))$, and let

$$\Sigma_x = \{H \in \mathcal{P}_x \,|\, x \text{ is not a non-degenerate quadratic singularity of } H \cap X\}\,.$$

It is easy to see that $\Sigma_x \subset \Sigma \cap \mathcal{P}_x$ is either a hyperplane or a quadric in $\mathcal{P}_x$. In the first case one can show that Σ is a linear subspace in $(\mathbb{P}^N)^\vee$ from which it is possible to deduce that $X = F_1$. In the second case $\mathrm{Sec}(\Sigma) = X^*$ and one can show that Σ is nonsingular and, either $\dim \Sigma = n$ and X is a Severi variety, or $\dim \Sigma = n-1$ and X is a nonsingular projection of a Severi variety from a point. $\square$

The classification of smooth varieties of codegree 4 is still unknown. All known examples arise from Pyasetskii pairing (2.2). The conjectural list consists of

- The Segre embedding of $\mathbb{P}^1 \times Q$, where Q is a quadric hypersurface.
- The twisted cubic curve $v_3(\mathbb{P}^1)$.
- The Plücker embedding of the Grassmannian $\mathrm{Gr}_3(\mathbb{C}^6)$.
- The isotropic Grassmannian $\mathrm{Gr}_3^0(\mathbb{C}^6)$ of isotropic 3 dimensional subspaces in the symplectic space $\mathbb{C}^3$ in its minimal equivariant embedding.
- The spinor variety $\mathbb{S}_6$.
- 27-dimensional variety $E \subset \mathbb{P}^{55}$ corresponding to the orbit of highest weight vector for the minimal representation of the algebraic group of type E_7.
- The Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$ (self-dual variety with defect 2), its non-singular hyperplane section and its non-singular section by two hyperplanes (which is either $\mathbb{P}^1 \times Q^1$ or the blow-up of a projective quadratic cone in its vertex).

8.2 Matsumura–Monsky Theorem

Let $D \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. It was first proved in [MM] that the group of projective automorphisms preserving D is finite if $d > 2$. In fact, it was also proved that the group of biregular automorphisms of D is finite if $d > 2$ (except the cases $d = 3$, $n = 2$ and $d = 4$, $n = 3$). Though this generalization looks much stronger, actually it is an easy consequence of the "projective" version and the Bart Theorem [Ba].

More generally, let G/P be a flag variety of a simple Lie group and $D \subset G/P$ be a smooth ample divisor. Let $L_\lambda = \mathcal{O}(D)$ be the corresponding ample line bundle, where $\lambda \in \mathcal{P}^+$ is a dominant weight. Then one might expect that the normalizer $N_G(D)$ of D in G is finite if λ is big enough. Indeed, if $N_G(D)$ (or actually any linear algebraic group of transformations of D) contains a one-parameter subgroup of automorphisms of D then D is covered by rational curves. However, if λ is big enough then the canonical class K_D is nef by the adjunction formula and, therefore, D can not be covered by rational curves. Unfortunately, this transparent approach does not give strong estimates on λ. Much better estimates can be obtained generalizing the original proof of the Matsumura-Monsky theorem.

The problem can be reformulated as follows. Suppose that V_λ is an irreducible G-module with the highest weight λ. Let $\mathcal{D} \subset V_\lambda$ be the discriminant variety (the dual variety to the orbit of the highest weight vector in $V_\lambda^\vee$). We will show that if λ is big enough then any point $x \in V_\lambda \setminus \mathcal{D}$ has a finite stabilizer G_x and the orbit of x is closed, $Gx = \overline{Gx}$ (therefore x is a stable point of V_λ in the sense of Geometric Invariant Theory, see [MFK]). This result can be compared with the results of [AVE], where all irreducible modules of simple algebraic groups with infinite stabilizers of generic points were found (this classification was extended later in [El1] and [El2] to handle irreducible representations of semisimple groups and any representations of simple groups).

If $\mathcal{D}$ is not a hypersurface then an easy inspection using Theorem 7.56 shows that the stabilizer of any point is infinite. So from now on we shall assume that $\mathcal{D}$ is a hypersurface defined by vanishing of the discriminant Δ.

A dominant weight λ is called *self-dual* if V_λ is isomorphic to $V_\lambda^\vee$ as a G-module. Let $\mathcal{P}_S^+ \subset \mathcal{P}^+$ be the subcone of self-dual dominant weights. Let γ be the highest root.

Theorem 8.8 *Let V_λ be an irreducible representation of a simple algebraic group G with the highest weight λ such that $\mathcal{D}$ is a hypersurface. Suppose that $(\lambda-\gamma,\mu) > 0$ for any $\mu \in \mathcal{P}_S^+$. Let $x \in V_\lambda \backslash \mathcal{D}$. Then G_x is finite and $Gx = \overline{Gx}$. Moreover, $G_{[x]}$ is also finite, where $[x]$ is the line spanned by x.*

Proof. If G_x is finite and $G_{[x]}$ is infinite then $[x]\backslash\{0\} \subset Gx$. Therefore $0 \in \overline{Gx}$ and $\Delta(x) = \Delta(0) = 0$, and hence, $x \in \mathcal{D}$. The same argument shows that if $x \in V_\lambda \setminus \mathcal{D}$ then $\overline{Gx} \subset V_\lambda \setminus \mathcal{D}$. If Gx is not closed then G_y is infinite for any point $y \in \overline{Gx} \setminus Gx$. Therefore, in order to prove the theorem, it suffices to prove that G_x is finite for any $x \in V_\lambda \setminus \mathcal{D}$.

Suppose that G_x is infinite. Then G_y is infinite and reductive (see e.g. [PV]) for any point y from the closed orbit in $\overline{Gx}$. Therefore, it suffices to prove that if $S \subset G$ is a one-dimensional torus and $Sx = x$ then $x \in \mathcal{D}$.

Without loss of generality we may assume that $S \subset T$, where T is the fixed maximal torus, and $\mathfrak{t} = \operatorname{Lie} T$ is the Cartan subalgebra. Let $x = \sum_{\pi \in \mathcal{P}} x_\pi$ be the weight decomposition of x. Let $\operatorname{Supp}(x) = \{\pi \in \mathcal{P} \,|\, x_\pi \neq 0\}$ be the *support* of x. For any $\mu \in \mathcal{P}$ let H_μ denote the hyperplane of weights perpendicular to μ. Then there exists $\mu \in \mathcal{P}$ such that $\operatorname{Supp}(x) \subset H_\mu$. Using the action of the Weyl group we may assume that $\mu \in \mathcal{P}^+$.

For any $\lambda \in \mathcal{P}^+$ let $\lambda^\vee$ be the highest weight of the dual module $V_\lambda^\vee$. Then $\lambda^\vee = -w_0(\lambda)$, where w_0 is the longest element of the Weyl group.

Suppose that $(\mu, \lambda^\vee - \gamma) > 0$. Then for any positive root α we have $(\mu, \lambda^\vee - \alpha) > 0$. Therefore x is perpendicular to $[\mathfrak{g}, v_{\lambda^\vee}]$, where $v_{\lambda^\vee}$ is the highest weight vector of $V_\lambda^\vee$. It follows that $x \in \mathcal{D}$.

Suppose that $(\mu^\vee, \lambda^\vee - \gamma) > 0$. Then for any positive root α we have $(\mu^\vee, \lambda^\vee - \alpha) > 0$. Therefore $w_0(x)$ is perpendicular to $[\mathfrak{g}, v_{\lambda^\vee}]$. It follows that $w_0(x) \in \mathcal{D}$ and hence $x \in \mathcal{D}$.

Suppose now that $(\mu, \lambda^\vee - \gamma) \leq 0$ and $(\mu^\vee, \lambda^\vee - \gamma) \leq 0$. Then

$$(\mu + \mu^\vee, \lambda^\vee - \gamma) \leq 0.$$

But $\mu + \mu^\vee$ is a self-dual weight, and hence this contradicts assumptions of the theorem. $\square$

Example 8.9 If $G = \mathrm{SL}_n$ and $\lambda = \sum n_i \omega_i$, where $\omega_1, \ldots, \omega_n$ are the fundamental weights, then the conditions of the theorem are satisfied if and only if $\sum n_i > 2$; for example if $\lambda = n\omega_1$ and $n > 2$. In particular, we recover the original Matsumura–Monsky Theorem.

8.3 Discriminants of Commutative Algebras

8.3.1 Commutative Algebras Without Identities

Let $V = \mathbb{C}^n$. Consider the vector space $\mathcal{A} = S^2V^\vee \otimes V$ parametrizing bilinear commutative multiplications in V. In the sequel we identify points of $\mathcal{A}$ with the corresponding commutative algebras.

Definition 8.10 Let $A \in \mathcal{A}$. A non-zero element $v \in A$ is called a *quadratic nilpotent* if $v^2 = 0$. Let $\mathcal{D}_1 \subset \mathcal{A}$ be the subset of all algebras that contain quadratic nilpotents. A one-dimensional subalgebra $U \subset A$ is called *singular* if there exists linear independent vectors $u \in U$ and $v \in A$ such that

$$u^2 = \alpha u, \quad uv = \frac{\alpha}{2}v, \quad \text{where } \alpha \in \mathbb{C}.$$

Let $\mathcal{D}_2 \subset \mathcal{A}$ be the subset of all algebras that contain singular subalgebras.

Then the following theorem holds.

Theorem 8.11 ([T6])

(i) *$\mathcal{D}_1$ and $\mathcal{D}_2$ are irreducible hypersurfaces.*
(ii) *Let $A \in \mathcal{A}$. Then A contains a one-dimensional subalgebra.*
(iii) *Let $A \in \mathcal{A} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$. Then A contains exactly $2^n - 1$ one-dimensional subalgebras; all these subalgebras are spanned by idempotents.*

Proof. Clearly $\mathcal{A}$ can be identified with a set of at most n-dimensional linear systems of quadrics in V. Namely, any linear function $f \in V^\vee$ defines a homogeneous quadratic function

$$v \to f(v^2).$$

Then $\mathcal{D}_1$ corresponds to the set of linear systems of quadrics with zero resultant. This proves that $\mathcal{D}_1$ is an irreducible hypersurface.

There exists, however, another useful identification. Any algebra A determines the n-dimensional linear system of *affine* quadrics in V. Namely, any linear form $f \in V^\vee$ defines an affine quadratic form

$$v \to f(v - v^2).$$

We can embed V into a projective space $\mathbb{P}$ as an affine chart. Then this linear system is naturally identified with a n-dimensional linear system of quadrics in $\mathbb{P}$. The base points of this linear system that do not lie on the infinite hyperplane coincide with idempotents of A. Infinite base points are the projectivizations of lines spanned by quadratic nilpotents. It is easy to see that quadrics from our linear system intersect transversally at 0. Moreover, quadrics intersect non-transversally at some point $v \neq 0$ if and only if the subalgebra spanned by v (if v is finite) or the subalgebra with the projectivization v (if v is infinite) is singular. Therefore (iii) follows from the Bezout theorem.

$\mathcal{D}_2$ is irreducible since $\mathcal{D}_2 = \mathrm{GL}_n \cdot \mathcal{D}_2'$, where $D_2' \subset \mathcal{A}$ is the linear subset of all algebras that have a fixed singular subalgebra and a fixed line spanned by the vector v from the definition of a singular subalgebra. It is also quite easy to check that D_2 is actually a closed hypersurface.

Since quadrics of our linear system intersect transversally at 0, it follows that there exist other base points, i.e. there exists at least one 1-dimensional subalgebra. □

Algebras that do not belong to discriminant varieties $\mathcal{D}_1$ and $\mathcal{D}_2$ are called *regular*. Both hypersurfaces $\mathcal{D}_1$ and $\mathcal{D}_2$ can be interpreted as standard discriminants. First, we can enlarge the symmetry group and consider $S^2(\mathbb{C}^n)^\vee \otimes \mathbb{C}^n$ as an $\mathrm{SL}_n \times \mathrm{SL}_n$-module. Then this module is irreducible and its discriminant variety (the dual variety of the projectivization of the highest weight vector orbit) coincides with $\mathcal{D}_1$. Now, consider $\mathcal{A} = S^2V^\vee \otimes V$ as an $\mathrm{SL}(V)$-module. Then this module is reducible, $\mathcal{A} = \mathcal{A}_0 + \tilde{\mathcal{A}}$, where $\mathcal{A}_0$ is a set of algebras with zero trace and $\tilde{\mathcal{A}}$ is isomorphic to $V^\vee$ as an $\mathrm{SL}(V)$-module. Consider the discriminant of $\mathcal{A}_0$ as a function on $\mathcal{A}$ (forgetting other coordinates). Then the corresponding hypersurface is exactly $\mathcal{D}_2$. If we consider the set of linear operators $\mathrm{Hom}(V, V) = V^\vee \otimes V$ instead of $\mathcal{A}$, then these constructions give rise to the determinant and the discriminant of a linear operator (see Example 2.11, Theorem 8.25).

8.3.2 Quasiderivations

Let $\mathfrak{g}$ be a Lie algebra with representation

$$\rho : \mathfrak{g} \to \mathrm{End}(V).$$

Consider any $v \in V$. A subalgebra

$$\mathfrak{g}_v = \{g \in \mathfrak{g} \mid \rho(g)v = 0\}$$

is called the *annihilator* of v. The subset

$$Q\mathfrak{g}_v = \{g \in \mathfrak{g} \mid \rho(g)^2 v = 0\}$$

is called the *quasi-annihilator* of v. Clearly, $\mathfrak{g}_v \subset Q\mathfrak{g}_v$. Of course, the quasi-annihilator is not a linear subspace in general, However, we have the following version of a Jordan decomposition:

Lemma 8.12 *Suppose that ρ is the differential of the representation of an algebraic group. Let $g \in Q\mathfrak{g}_v$. Consider the Jordan decomposition in $\mathfrak{g}$, $g = g_s + g_n$, where g_s is semisimple and g_n is nilpotent, and $[g_s, g_n] = 0$. Then $g_s \in \mathfrak{g}_v$ and $g_n \in Q\mathfrak{g}_v$.*

The proof is easy.

Now we apply this construction for $\mathfrak{g} = \mathfrak{gl}_n$ and ρ being the natural representation in the vector space $V^\vee \otimes V^\vee \otimes V$ that parametrizes algebras (bilinear multiplications in V). Let A be any algebra. Then $\mathfrak{g}_A$ is identified with the Lie algebra of derivations $\operatorname{Der}(A)$. Operators $D \in Q\mathfrak{g}_A$ are called *quasiderivations*. Of course, it is possible to write down explicit equations that determine $Q\operatorname{Der}(A) = Q\mathfrak{g}_A$ in $\operatorname{End}(A)$, but this formula is quite useless (see [Vi1]). We shall use a particular case of it that is quite easy to verify:

Lemma 8.13 *Let A be an algebra, $D \in \operatorname{End}(A)$, where $D^2 = 0$. Then $D \in Q\operatorname{Der}(A)$ if and only if*

$$D(x)D(y) = D(D(x)y) + D(xD(y)) \tag{8.3}$$

for any $x, y \in A$.

Example 8.14 It was conjectured in [Vi1] that all quasiderivations of the algebra of matrices Mat_n have the form $D(x) = ax+xb$, where $(a+b)^2 = [a, b]$. Let us give a counterexample. Consider the linear operator $D(x) = exe$, where $e^2 = 0$, $e \neq 0$. Then it is easy to check using (8.3) that D is a quasiderivation. However, D of course can not be written in the form $ax + xb$.

Quasiderivations can be used to define naive deformations. Namely, suppose that A is any algebra with the multiplication $u \cdot v$ and D is its quasiderivation. Consider the algebra A_D with the same underlying vector space but with the new multiplication

$$u \star v = u \cdot D(v) + D(u) \cdot v - D(u \cdot v).$$

Then, if A satisfies a polynomial identity, A_D satisfies this identity as well. More generally, let $\rho : \mathfrak{g} \to \operatorname{End}(V)$ be the differential of a representation of an algebraic group G, and let $v \in V$ and $D \in Q\mathfrak{g}_v$. Suppose that $H \subset V$ is a closed conical G-equivariant hypersurface and $v \in H$. Then $\rho(D)v$ also belongs to H. Indeed, since H is equivariant, $\exp(\lambda\rho(D))v$ belongs to H for any $\lambda \in \mathbb{C}$. Since D is a quasiderivation, $\exp(\lambda\rho(D))v = v + \lambda\rho(D)v$. Since H is conical, $v/\lambda + \rho(D)v$ belongs to H. Since H is closed, $\rho(D)v$ also belongs to H.

Now suppose that A is a regular commutative algebra. Then we claim that $Q\operatorname{Der}(A) = 0$.

Theorem 8.15 ([T6]) *Let $A \in \mathcal{A}$ and $A \notin \mathcal{D}_1 \cup \mathcal{D}_2$. Then $Q\operatorname{Der}(A) = 0$.*

Proof. By Lemma 8.12 it suffices to check that A has no semisimple derivations and no nilpotent quasiderivations. Suppose that $\operatorname{Der}(A)$ contains a nonzero semisimple element. Then the group $\operatorname{Aut}(A)$ contains a one-dimensional algebraic torus T. Let $\mathfrak{t}$ be its Lie algebra, $H \in \mathfrak{t}$, and $H \neq 0$. We way assume that the spectrum of H in A is integer-valued. Let $A_n \subset A$ be a weight space of weight n. Then $A = \oplus A_n$ is a $\mathbb{Z}$-grading. Let $v \in A$ be a homogeneous element of a maximal positive (or minimal negative) degree. Then $v^2 = 0$, and hence $A \in \mathcal{D}_1$.

Suppose now that E is a non-zero nilpotent quasiderivation. We can embed E in an $\mathfrak{sl}_2$-triple $\langle F, H, E\rangle \subset \mathrm{End}(A)$. Let

$$A = \oplus m_d R_d$$

be an $\mathfrak{sl}_2$-module decomposition, where R_d is an irreducible $(d+1)$-dimensional module. Consider also the weight decomposition

$$A = \oplus A^n \quad \text{and} \quad \mathcal{A} = \oplus \mathcal{A}^n$$

with respect to H. Let

$$J^n = A^n + A^{n+1} + A^{n+2} + \dots .$$

To avoid the abuse of notations, denote by $\alpha \in \mathcal{A}$ the point corresponding to the algebra A. Let Supp α be the support of α (i.e. all $n \in \mathbb{Z}$ such that $\alpha_n \neq 0$, where $\alpha = \sum \alpha_n$, $\alpha_n \in \mathcal{A}^n$). Since $E^2\alpha = 0$, the vector $E\alpha$ is a linear combination of highest weight vectors, and therefore Supp $\alpha \subset \{-1, 0, 1, 2, \dots\}$. This is equivalent to $J^n J^m \subset J^{n+m-1}$. In particular, if $v \in A$ is a weight vector of the weight n then $v^2 = 0$ since $n > 1$. Since A is regular, it follows that

$$A = m_0 R_0 \oplus m_1 R_1 = A^{-1} \oplus A^0 \oplus A^1.$$

Since $A^1 = J^1$, A^1 is a subalgebra in A. By Theorem 8.11 (ii), A^1 has a one-dimensional subalgebra U spanned by an idempotent u (since A has no quadratic nilpotents). Let $v \in A^{-1}$ be the unique vector such that $Ev = u$. Since $E^2 = 0$, we can apply formula (8.3) with $x = y = v$. We get $E(v)^2 = 2E(E(v)v)$, and hence $u = 2E(uv)$. Therefore $uv - \frac{1}{2}v \in J^0$. Notice that the operator of left multiplication by u preserves J^0. Since this operator has an eigenvector in A/J^0 with eigenvector $1/2$, it has such an eigenvector in A. Therefore U is a singular subalgebra in A. □

The following Corollary was proved in [An] using other methods.

Corollary 8.16 ([An]) *Let A be an n-dimensional semisimple commutative algebra, i.e. a direct sum of n copies of $\mathbb{C}$, i.e. the algebra of diagonal $n \times n$ matrices. Then A has no nonzero quasiderivations.*

Proof. It is sufficient to check that $A \notin \mathcal{D}_1 \cup \mathcal{D}_2$. Clearly, A has no nilpotents. Suppose that $U \subset A$ is a one-dimensional subalgebra spanned by an idempotent $e \in U$. Since the spectrum of the operator of left multiplication by e consists of 0 and 1, it follows that U is not singular. □

8.4 Discriminants of Anticommutative Algebras

8.4.1 Generic Anticommutative Algebras

If the set of certain objects is parametrized by an algebraic variety X then we can speak about generic objects. Namely, we say that a generic object satisfies some property if there exists a dense Zariski-open subset $X_0 \subset X$ such that objects parametrized by points of X_0 share this property. Sometimes it is possible to find a discriminant-type closed subvariety $Y \subset X$ and to study properties of 'regular' objects parametrized by points from $X \setminus Y$. For example, instead of studying generic hypersurfaces it is beneficial to study smooth hypersurfaces. In this section we implement this program for quite non-geometric objects, namely anticommutative algebras.

Let $V = \mathbb{C}^n$. We fix an integer k, $1 < k < n-1$. Let $\mathcal{A}_{n,k} = \Lambda^k V^\vee \otimes V$ be the vector space of k-linear anticommutative maps from V to V. We identify points of $\mathcal{A}_{n,k}$ with the corresponding algebras, that is, we assume that $A \in \mathcal{A}_{n,k}$ is the space V equipped with the structure of a k-argument anticommutative algebra. Subalgebras in generic algebras with $k = 2$ were studied in [T2]. The following theorem is a generalization of these results.

Theorem 8.17 ([T1]) *Let $A \in \mathcal{A}_{n,k}$ be a generic algebra. Then*

(i) *Every m-dimensional subspace is a subalgebra if $m < k$.*
(ii) *A contains no m-dimensional subalgebras with $k+1 < m < n$.*
(iii) *The set of k-dimensional subalgebras is a smooth irreducible $(k-1)(n-k)$-dimensional subvariety in the Grassmannian $\mathrm{Gr}(k, A)$.*
(iv) *There are finitely many $(k+1)$-dimensional subalgebras, and their number is*

$$\sum_{\substack{n-k-1 \geq \mu_1 \geq \ldots \geq \mu_{k+1} \geq 0 \\ n-k-1 \geq \lambda_1 \geq \ldots \geq \lambda_{k+1} \geq 0 \\ \mu_1 \leq \lambda_1, \ldots, \mu_{k+1} \leq \lambda_{k+1}}} (-1)^{|\mu|} \frac{(\lambda_1 + k)!(\lambda_2 + k - 1)! \ldots \lambda_{k+1}!}{(\mu_1 + k)!(\mu_2 + k - 1)! \ldots \mu_{k+1}!} (|\lambda| - |\mu|)! \times \left| \frac{1}{(i - j + \lambda_j - \mu_i)!} \right|^2_{i,j=1,\ldots,k+1},$$

where $|\lambda| = \lambda_1 + \ldots + \lambda_{k+1}$, $|\mu| = \mu_1 + \ldots + \mu_{k+1}$, $1/N! = 0$ if $N < 0$.
(v) *A contains a $(k+1)$-dimensional subalgebra.*
(vi) *If $k = n-2$, then the number of $(k+1)$-dimensional subalgebras is equal to*

$$\frac{2^n - (-1)^n}{3.}$$

Proof. The GL_n-module $\mathcal{A}_{n,k}$ is a sum of two irreducible submodules:

$$\mathcal{A}_{n,k} = \mathcal{A}^0_{n,k} \oplus \tilde{\mathcal{A}}_{n,k}. \tag{8.4}$$

Here $\tilde{\mathcal{A}}_{n,k}$ is isomorphic to $\Lambda^{k-1}V^\vee$: we assign to every $(k-1)$-form ω the algebra with multiplication

$$[v_1, \ldots, v_k] = \sum_{i=1}^{k} (-1)^{i-1} \omega(v_1, \ldots, \hat{v}_i, \ldots, v_k) v_i.$$

Note that every subspace of this algebra is a subalgebra. Hence the lattice of subalgebras of $A \in \mathcal{A}_{n,k}$ coincides with the lattice of subalgebras of A^0, where $A \mapsto A^0$ is the GL_n-equivariant projector on the first summand in (8.4). Algebras in $\mathcal{A}^0_{n,k}$ will be called *zero trace algebras*, since $A \in \mathcal{A}^0_{n,k}$ if and only if the $(k-1)$-form $\mathrm{Tr}[v_1, \ldots, v_{k-1}, \cdot]$ is equal to zero. Hence the theorem will be proved once we have proved it for generic algebras in $\mathcal{A}^0_{n,k}$.

We choose a basis $\{e_1, \ldots, e_n\}$ in V, identify GL_n with the group of matrices, consider the standard diagonal maximal torus T, and take for B and B_- the subgroups of upper- and lower-triangular matrices. We fix an $m \geq k$. Consider the parabolic subgroup of matrices

$$P = \begin{pmatrix} A & 0 \\ * & B \end{pmatrix},$$

where B is an $m \times m$ matrix and A is an $(n-m) \times (n-m)$ matrix. Then G/P coincides with $\mathrm{Gr}(m, V)$. Consider the vector bundle

$$\mathcal{L} = \Lambda^k \mathcal{S}^\vee \otimes V/\mathcal{S}$$

on G/P, where $\mathcal{S}$ is the tautological bundle and $V/\mathcal{S}$ is the factor-tautological bundle. Then the assumptions of Theorem 8.27 are fulfilled since $\mathcal{L} = \mathcal{L}_\lambda$, where λ is the highest weight of $\mathcal{A}^0_{n,k}$. Therefore

$$\mathcal{A}^0_{n,k} = H^0(\mathrm{Gr}(m, V), \Lambda^k \mathcal{S}^\vee \otimes V/\mathcal{S}).$$

Let $A \in \mathcal{A}^0_{n,k}$, and let s_A be the corresponding global section. Then the scheme of zeros $(Z_{s_A})_{red}$ coincides with the variety of m-dimensional subalgebras of A.

Let us return to the theorem. Statement (i) is obvious. (ii) follows from Theorem 8.27 (i). Theorem 8.27 (ii) implies that if every k-argument anti-commutative algebra A contains a k-dimensional subalgebra, then the variety of k-dimensional subalgebras of a generic algebra is a smooth unmixed $(k-1)(n-k)$-dimensional subvariety in $\mathrm{Gr}(k, A)$. We claim that any $(k-1)$-dimensional subspace U can be included in a k-dimensional subalgebra. The multiplication in the algebra defines a linear map from V/U to V/U. Let $v + U$ be a non-zero eigenvector. It is obvious that $\mathbb{C}v \oplus U$ is a k-dimensional subalgebra.

We have only to prove that the variety of k-dimensional subalgebras is irreducible. Assume that a section s of the bundle $\mathcal{L} = \Lambda^k \mathcal{S}^\vee \otimes V/\mathcal{S}$ over $\mathrm{Gr}(k, V)$ corresponding to A has transversal intersection with the zero section. Then the Koszul complex

$$0 \to \Lambda^{n-k}\mathcal{L}^\vee \xrightarrow{s} \ldots \xrightarrow{s} \Lambda^2\mathcal{L}^\vee \xrightarrow{s} \mathcal{L}^\vee \xrightarrow{s} \mathcal{O} \to \mathcal{O}_{Z(s)} \to 0$$

is exact by Theorem 5.17.

Note that $\Lambda^p\mathcal{L}^\vee$ is isomorphic to the bundle $S^p\Lambda^k\mathcal{S} \otimes \Lambda^p(V/\mathcal{S})^\vee$. This is a homogeneous bundle over G/P of the form $\mathcal{L}_\mu$, where

$$\mu = -\varepsilon_1 - \ldots - \varepsilon_p + \varepsilon_{n-k+1} + \ldots + \varepsilon_n$$

and ε_i are the weights of the diagonal torus in the tautological representation. Note that the weight $\mu + \rho$ (where ρ is the half-sum of the positive roots) is singular (belongs to the wall of the Weyl chamber) for any p, $1 \le p \le n-k$. By the Borel–Weyl–Bott Theorem 8.1, $H^\bullet(\mathrm{Gr}(k,V), \Lambda^p \mathcal{L}^\vee) = 0$ for $1 \le p \le n-k$. Hence $H^0(Z(s), \mathcal{O}_{Z(s)}) = H^0(\mathrm{Gr}(k,V), \mathcal{O}) = \mathbb{C}$, as was to be shown.

We postpone the proof of (v) till the end of this section and consider (iv) and (vi), that is, we shall calculate the highest Chern class of the bundle $\Lambda^k \mathcal{S}^\vee \otimes V/\mathcal{S}$ over $\mathrm{Gr}(k+1, V)$.

We use the standard notation, facts, and formulae from the Schubert calculus (cf. [Fu1]. The letters λ and μ always denote Young diagrams in the rectangle with $k+1$ rows and $n-k-1$ columns. It is well known that such diagrams parametrize the basis of the Chow ring of $\mathrm{Gr}(k+1, V)$. The cycle corresponding to λ is denoted by σ_λ, where $\sigma_\lambda \in A^{|\lambda|}(\mathrm{Gr}(k+1,V))$. (We grade the Chow ring by the codimension, $|\lambda| = \lambda_1 + \ldots + \lambda_{k+1}$, where λ_i is the length of the ith row of λ.) We need the total Chern class of the bundles $\mathcal{S}$ and $V/\mathcal{S}$:

$$c(\mathcal{S}) = 1 - \sigma_1 + \sigma_{1,1} - \ldots + (-1)^{k+1}\sigma_{1,\ldots,1},$$
$$c(V/\mathcal{S}) = 1 + \sigma_1 + \sigma_2 + \ldots + \sigma_{n-k-1}.$$

We begin by calculating the total Chern class of $\mathcal{S} \otimes V/\mathcal{S}$. The standard formula for the total Chern class of the tensor products of two bundles implies that

$$c(\mathcal{S} \otimes V/\mathcal{S}) = \sum_{\mu \subset \lambda} d_{\lambda\mu} \Delta_{\tilde{\mu}}(c(\mathcal{S})) \Delta_{\lambda'}(c(V/\mathcal{S})), \tag{8.5}$$

where

$$d_{\lambda\mu} = \left| \binom{\lambda_i + k + 1 - i}{\mu_j + k + 1 - j} \right|_{1 \le i,j \le k+1}, \quad \Delta_\lambda(c) = |c_{\lambda_i + j - i}|,$$

$\lambda' = (n-k-1-\lambda_{k+1}, n-k-1-\lambda_k, \ldots, n-k-1-\lambda_1)$, and $\tilde{\mu}$ is the diagram obtained from μ by transposition. We shall use the fact that $\Delta_{\tilde{\mu}}(c(\mathcal{S})) = \Delta_\mu(s(\mathcal{S}))$, where $s(E)$ is the Segre class of E. Another fact is that $\Delta_\lambda(c(V/\mathcal{S})) = \sigma_\lambda$ and $\Delta_\lambda(s(\mathcal{S})) = (-1)^{|\lambda|}\sigma_\lambda$ (since $s(\mathcal{S}) = 1 - \sigma_1 + \sigma_2 + \ldots + (-1)^{n-k-1}\sigma_{n-k-1}$). Therefore (8.5) can be written as

$$c(\mathcal{S} \otimes V/\mathcal{S}) = \sum_{\mu \subset \lambda} d_{\lambda\mu} (-1)^{|\mu|} \sigma_\mu \sigma_{\lambda'}. \tag{8.6}$$

To calculate the highest Chern class of $\mathcal{L} = \Lambda^k \mathcal{S}^\vee \otimes V/\mathcal{S}$, we use the formula $\mathcal{L} = \Lambda^{k+1}\mathcal{S}^\vee \otimes (\mathcal{S} \otimes V/\mathcal{S})$. The total Chern class of the first factor is equal to $c(\Lambda^{k+1}\mathcal{S}^\vee) = 1 + \sigma_1$. The total Chern class of the second is given by (8.6). Hence the highest Chern class of $\mathcal{L}$ is

$$c_{top}(\mathcal{L}) = \sum_{\mu \subset \lambda} d_{\lambda\mu}(-1)^{|\mu|}\sigma_\mu \sigma_{\lambda'} \sigma_1^{|\lambda|-|\mu|}. \tag{8.7}$$

The last result of the Schubert calculus that we need is the exact formula for the degree of a product of two cycles. In our case this can be written as

$$\sigma_\mu \sigma_{\lambda'} \sigma_1^{|\lambda|-|\mu|} = \deg(\sigma_\mu \sigma_{\lambda'}) = (|\lambda| - |\mu|)! \left| \frac{1}{(i - j + \lambda_j - \mu_i)!} \right|_{i,j=1,\dots,n}, \tag{8.8}$$

where $1/N! = 0$ if $N < 0$. The formula in part (iv) of the theorem can be obtained from (8.7) and (8.8) by a slight modification of the determinant in the formula for $d_{\lambda\mu}$.

It remains to verify the formula in part (vi). Let $k = n - 2$. Then the formula in (iv) can be written as

$$\sum_{0 \le i \le j \le k+1} (-1)^i \frac{(k+1)!k!\dots(k-j+2)!(k-j)!\dots 0!}{(k+1)!k!\dots(k-i+2)!(k-i)!\dots 0!}(j-i)! \det{}^2 A, \tag{8.9}$$

where

$$A = \begin{pmatrix} X & 0 & 0 \\ * & Y & 0 \\ * & * & Z \end{pmatrix},$$

X is an $i \times i$ matrix, Y is a $(j-i) \times (j-i)$ matrix, and the format of Z is a $(k+1-j) \times (k+1-j)$ matrix. X and Z are lower-triangular matrices with 1s on the diagonal and Y is given by

$$Y = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1/2! & 1 & 1 & 0 & \cdots & 0 \\ 1/3! & 1/2! & 1 & 1 & \cdots & 0 \\ 1/4! & 1/3! & 1/2! & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/(j-i)! & 1/(j-i-1)! & 1/(j-i-2)! & 1/(j-3)! & \cdots & 1 \end{pmatrix}.$$

It is easy to verify that $\det Y = 1/(j-i)!$, which enables us to rewrite (8.9) as

$$\sum_{i \le j} (-1)^i \frac{(1+(k-i))!\dots(1+(k-j+1))!}{(k-i)!\dots(k-j+1)!}/(j-i)! =$$
$$\sum_{i \le j} (-1)^i \binom{k+1-i}{j-i} = \sum_i (-1)^i 2^{k+1-i} = \frac{2^{k+2} - (-1)^{k+2}}{3} = \frac{2^n - (-1)^n}{3},$$

as was to be shown.

It remains to prove part (v) of the theorem. We have to prove that every $A \in \mathcal{A}_{n,k}$ has a $(k+1)$-dimensional subalgebra. We fix a basis $\{e_1, \dots, e_n\}$ in V and consider the subspace $U = \langle e_{n-k}, \dots, e_n \rangle$. Let $M \subset \mathcal{A}_{n,k}$ be the subspace that consists of the algebras for which U is a $(k+1)$-dimensional subalgebra. We claim that $\mathcal{A}_{n,k} = \mathrm{GL}_n \cdot M$. It is sufficient to prove that the differential of the canonical morphism $\phi : \mathrm{GL}_n \times M \to \mathcal{A}_{n,k}$ is surjective at a point (e, A). Consider the algebra $A \in M$ in which $[e_{n-k}, \ldots, \hat{e}_i, \ldots, e_n] = e_i$ for all $n-k \leq i \leq n$ and the other products are zero. We claim that $d\phi$ is surjective at (e, A). Consider the map $\pi : \mathcal{A}_{n,k} \to \mathcal{A}_{n,k}/M$. It is sufficient to verify that $\pi \circ d\phi_{(e,A)}(\mathfrak{gl}_n, 0) = \mathcal{A}_{n,k}/M \simeq \Lambda^k U^\vee \otimes V/U$. But this is obvious, since multiplication in $d\phi_{(e,A)}(E_{ji}, 0)$ for $n-k \leq i \leq n$, $1 \leq j \leq n-k-1$ (E_{ji} is the matrix identity) is given by $[e_{n-k}, \ldots, \hat{e}_i, \ldots, e_n] = e_j$ with the other products equal to zero. This completes the proof of the theorem. □

8.4.2 Regular Algebras

It is an essential drawback of Theorem 8.17 that we cannot use it to study the structure of subalgebras of any particular algebra. To correct this, we introduce an explicit class of 'regular' algebras instead of the implicit class of generic algebras. The natural way to remove degeneracies is to consider discriminants.

Let $(\mathcal{A}^0_{n,k})^*$ be the GL_n-module dual to $\mathcal{A}^0_{n,k}$, and let $S_D \subset (\mathcal{A}^0_{n,k})^*$ be the closure of the orbit of the highest vector. Let $PS_D \subset \mathbb{P}(\mathcal{A}^0_{n,k})^*$ be its projectivization, let $P\mathcal{D} \subset \mathbb{P}\mathcal{A}^0_{n,k}$ be the subvariety projectively dual to the subvariety PS_D, and let $\mathcal{D} \subset \mathcal{A}^0_{n,k}$ be the cone over it. We call $\mathcal{D}$ the *D-discriminant subvariety.* Algebras $A \in \mathcal{D}$ are called *D-singular*, and algebras $A \notin \mathcal{D}$ are called *D-regular.*

Theorem 8.18 ([T1])

(i) *$\mathcal{D}$ is a hypersurface.*
(ii) *If A is D-regular then k-dimensional subalgebras of A form a smooth irreducible $(k-1)(n-k)$-dimensional variety.*
(iii) *Let $k = n-2$. Then the degree of $\mathcal{D}$ is equal to*

$$\frac{(3n^2 - 5n)2^n - 4n(-1)^n}{18.} \tag{8.10}$$

It follows that D-singularity of A is equivalent to vanishing of the SL_n-invariant polynomial D called the *D-discriminant.*

Proof. The fact that $\mathcal{D}$ is a hypersurface follows immediately from the results of Section 7.4. (ii) can be deduced from the corresponding assertion of Theorem 8.17 by an easy calculation with differentials. Namely, let A be a D-regular algebra. We use the arguments in the proof of Theorem 8.27 (ii) and Theorem 8.17 (iii). According to these calculations, it is sufficient to verify that, if U is a k-dimensional subalgebra of A, then the map

$$\psi : \mathfrak{gl}_n \to \Lambda^k U^\vee \otimes A/U,$$

$$\psi(g)(v_1 \wedge \ldots \wedge v_k) = g[v_1, \ldots, v_k] - [gv_1, \ldots, v_k] - \ldots - [v_1, v_2, \ldots, gv_k] + U$$

is surjective. Assume the opposite. Then there is a hyperplane $H \supset U$ such that the image of ψ lies in $\Lambda^k U^\vee \otimes H/U$. Consider a non-zero algebra $\tilde{A}$ in $S_D \subset (\mathcal{A}^0_{n,k})^*$ such that $[U^\perp, V^\vee, \ldots, V^\vee] = 0$, $[V^\vee, \ldots, V^\vee] \subset H^\perp$, where $U^\perp$ and $H^\perp$ are the annihilators of U and H in $(\mathcal{A}^0_{n,k})^*$, and square brackets denote multiplication in the algebra. $\tilde{A}$ is defined by these conditions uniquely up to a scalar. Then $\tilde{A}$ annihilates $[\mathfrak{gl}_n, A]$, which is equivalent to the fact that A annihilates $[\mathfrak{gl}_n, \tilde{A}]$, that is, the tangent space to S_D at $\tilde{A}$. This means that A lies in $\mathcal{D}$, i.e. A is a D-singular algebra.

Finally, (iii) follows from Theorem 8.5. □

We will also define the E-discriminant and E-regularity but only for $(n-2)$-argument n-dimensional anticommutative algebras. Let $\mathcal{A} = \mathcal{A}^0_{n,n-2}$. Consider the projection $\pi : \mathrm{Gr}(n-1, V) \times P\mathcal{A} \to P\mathcal{A}$ on the second summand and the incidence subvariety $Z \subset \mathrm{Gr}(n-1, V) \times P\mathcal{A}$ that consists of pairs $S \subset PA$, where S is a subalgebra in A. Let $\tilde{\pi} = \pi|_Z$. By Theorem 8.17, we have $\tilde{\pi}(Z) = \mathcal{A}$. Let $\tilde{\mathcal{E}} \subset Z$ be the set of critical points of $\tilde{\pi}$, let $P\mathcal{E} = \tilde{\pi}(\tilde{\mathcal{E}})$ be the set of critical values of $\tilde{\pi}$, and let $\mathcal{E} \subset \mathcal{A}$ be the cone over $P\mathcal{E}$. Then $\mathcal{E}$ is called the *E-discriminant subvariety*. The algebras $A \in \mathcal{E}$ are said to be *E-singular*. The algebras $A \notin \mathcal{E}$ are said to be *E-regular*.

Theorem 8.19 ([T1])

(i) *$\mathcal{E}$ is an irreducible hypersurface.*
(ii) *Let A be an E-regular algebra. Then A has precisely*

$$\frac{2^n - (-1)^n}{3}$$

$(n-1)$-dimensional subalgebras.
(iii) *The map $\tilde{\pi} : \tilde{\mathcal{E}} \to P\mathcal{E}$ is birational.*

Hence the E-singularity of A is determined by the vanishing of the SL_n-invariant polynomial that defines $\mathcal{E}$. This polynomial is called the *E-discriminant*. (iii) can be formulated as follows: a generic E-singular algebra has precisely one "critical" $(n-1)$-dimensional subalgebra.

Proof. The proof of (ii) is similar to the proof of Theorem 8.17 (iv). We claim that (i) follows from (iii). We choose a basis $\{f_1, \ldots, f_n\}$ in $V^\vee$ dual to the basis $\{e_1, \ldots, e_n\}$ in V. Let $U \in \mathrm{Gr}(n-1, V)$ be the hyperplane $f_1 = 0$. It is clear that $\tilde{\mathcal{E}} = \mathrm{GL}_n \cdot M_0$, where $M_0 = \tilde{\mathcal{E}} \cap (U, P\mathcal{A})$. Moreover, $M = Z \cap (U, P\mathcal{A})$ is the linear subspace of the algebras for which U is an $(n-1)$-dimensional subalgebra. Let P be the parabolic subgroup of matrices $\begin{pmatrix} A & 0 \\ * & B \end{pmatrix}$, and let $\mathfrak{u}$ be the Lie algebra of matrices $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$, where B is an $(n-1) \times (n-1)$ matrix and X is an $1 \times (n-1)$ matrix. Every algebra $A \in M$ defines a linear map $\mathfrak{u} \to \Lambda^{n-1}U^\vee \otimes V/U$. Since $A \in M_0$ if and only if this map is degenerate, we have $\operatorname{codim}_M M_0 = 1$. It is obvious that M_0 is irreducible, since M_0 is the spreading of the subspace

$$M_0^1 = \{A \in M_0 \mid \text{the algebra } E_{12}A \text{ has a subalgebra } U\}$$

by the group P. Hence $\tilde{\mathcal{E}}$ is an irreducible divisor in Z, and (i) follows from (iii).

We prove (iii). We say that an $(n-1)$-dimensional subalgebra U' of $A \in \mathcal{E}$ is *critical* if (U', A) lies in $\tilde{\mathcal{E}}$. In this case there is an $(n-2)$-dimensional subspace $W' \subset U'$ such that, if $V = U' \oplus \mathbb{C}e$ and $v \in \mathfrak{gl}_n$ is a non-zero linear operator such that $v(V) \subset \mathbb{C}e$ and $v(W') = 0$, then U' is a subalgebra of vA. To prove (iii) it suffices to prove that in generic algebras in M_0^1 the subalgebra U is the unique critical subalgebra. Let $N \subset M_0^1$ be the subvariety of all algebras that have another critical subalgebra.

Note that M_0^1 is normalized by the parabolic subgroup

$$Q = \begin{pmatrix} * & 0 & 0 & \dots & 0 \\ * & * & 0 & \dots & 0 \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & * \end{pmatrix}.$$

Then N is the spreading of the subvarieties N_2 and N_3 by the group Q, where $A \in N_i$ if and only if the hyperplane $f_i = 0$ is a critical subalgebra. In turn, N_2 is the spreading of the vector spaces N_2^1 and N_2^3, and N_3 is the spreading of the subspaces N_3^1, N_3^2, and N_3^4, where $N_i^j \subset N_i$ is the subspace of all algebras such that the $(n-2)$-dimensional subspace W' (mentioned above) is given by $f_i = f_j = 0$. Let $Q_i^j \subset Q$ be the subgroup that normalizes the flag $f_i \subset \langle f_i, f_j \rangle$. It is easy to verify that

$$\operatorname{codim}_Q Q_2^1 = 1,\ \operatorname{codim}_Q Q_2^3 = n-1,\ \operatorname{codim}_Q Q_3^1 = n-1,$$

$$\operatorname{codim}_Q Q_3^2 = n,\ \operatorname{codim}_Q Q_3^4 = 2n-3.$$

On the other hand,

$$\operatorname{codim}_{M_0^1} N_2^1 = n,$$

$$\operatorname{codim}_{M_0^1} N_2^3 = \operatorname{codim}_{M_0^1} N_3^1 = \operatorname{codim}_{M_0^1} N_3^2 = \operatorname{codim}_{M_0^1} N_3^4 = 2n-2.$$

Hence $\operatorname{codim}_{M_0^1} QN_i^j \geq 1$ in all cases, which completes the proof. □

8.4.3 Regular 4-dimensional Anticommutative Algebras

An $(n-2)$-argument n-dimensional anticommutative algebra is called *regular* if it is both D-regular and E-regular. In this subsection we consider 2-argument 4-dimensional algebras. The corresponding generic algebras were studied in [T2].

Theorem 8.20 ([T1]) *Let A be a 4-dimensional regular anticommutative algebra. Then*

(i) *A has precisely five 3-dimensional subalgebras. The set of these subalgebras is a generic configuration of five hyperplanes. In particular, A has a pentahedral normal form, that is, it can be reduced by a transformation that belongs to GL_4 to an algebra such that the set of its five subalgebras is a Sylvester pentahedron $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_1+x_2+x_3+x_4 = 0$.*
(ii) *A has neither one- nor two-dimensional ideals.*
(iii) *The set of two-dimensional subalgebras of A is a Del Pezzo surface of degree 5 (a blowing up of $\mathbb{P}^2$ at four generic points).*
(iv) *A has precisely 10 fans, that is, flags $V_1 \subset V_3$ of 1-dimensional and 3-dimensional subspaces such that every intermediate subspace U, $V_1 \subset U \subset V_3$, is a two-dimensional subalgebra.*

Proof. We begin with (i). We have to prove that, if A is a 4-dimensional regular anticommutative algebra with zero trace, then the set of its three-dimensional subalgebras $S_1, S_2, \ldots, S_5$ is a generic configuration of hyperplanes, that is, the intersection of any three of them is one-dimensional and the intersection of any four of them is zero-dimensional. Indeed, assume, for example, that $U = S_1 \cap S_2 \cap S_3$ is two-dimensional. Let $v \in U$ and $v \neq 0$. Then $[v, \cdot]$ induces a linear operator on A/U since U is a subalgebra. S_1/U, S_2/U, and S_3/U are one-dimensional eigenspaces. Since $\dim A/U = 2$, the operator is a dilation. Since this is true for any $v \in U$, any three-dimensional subspace that contains U is a three-dimensional subalgebra, which contradicts the fact that there are precisely five such subalgebras.

Now assume that $U = S_1 \cap S_2 \cap S_3 \cap S_4$ is one-dimensional, and let $v \in U$, $v \neq 0$. Then the operator $[v, \cdot]$ induces an operator on A/U. This operator has four two-dimensional eigenspaces $S_1/U, \ldots, S_4/U$ of which any three have zero intersection. Hence this operator is a dilation. Let $W \supset U$ be an arbitrary two-dimensional subspace, and let $w \in W$ be a vector that is not proportional to v. Then the operator $[w, \cdot]$ induces a linear operator on A/W. Let z be a non-zero eigenvector. Then $\langle v, w, z\rangle$ is a three-dimensional subalgebra. Therefore every vector can be included in a three-dimensional subalgebra, which contradicts the fact that there are only five such subalgebras.

This argument also shows that A has no one-dimensional ideals. Since every three-dimensional subspace that contains a two-dimensional ideal is a subalgebra, there are no two-dimensional ideals, which completes the proof of (ii).

To prove (iii), we consider the subvariety X of two-dimensional subalgebras in A. Then $X \subset \mathrm{Gr}(2,4)$. Consider the Plücker embedding $\mathrm{Gr}(2,4) \subset \mathbb{P}^5 = P(\Lambda^2\mathbb{C}^4)$. First we claim that the embedding $X \subset \mathbb{P}^5$ is non-degenerate, that is, the image is contained in no hyperplane. Let S_1, S_2, S_3, S_4 be four three-dimensional subalgebras. Since it is a generic configuration, we can choose a basis $\{e_1, e_2, e_3, e_4\}$ in A such that $S_i = \langle e_1, \ldots, \hat{e}_i, \ldots, e_4\rangle$. Since the intersection of three-dimensional subalgebras is a two-dimensional subalgebra, A has six subalgebras $\langle e_i, e_j\rangle$, $i \neq j$. The set of corresponding bivectors $e_i \wedge e_j$ is a basis in $\Lambda^2\mathbb{C}^4$. Therefore they can lie in no hyperplane. Simple calculation with Koszul complexes (cf. Theorem 5.17) shows that $H^0(X, \mathcal{O}_X(1))^\vee = \Lambda^2\mathbb{C}^4$.

To prove that X is a Del Pezzo surface of degree five, we have only to verify that $\mathcal{O}_X(1)$ coincides with the anticanonical sheaf (see [Ma]). Since $Y = \mathrm{Gr}(2,4)$ is a quadric in $\mathbb{P}^5$, we have

$$\omega_Y = \mathcal{O}_Y(2-5-1) = \mathcal{O}_Y(-4).$$

The set X is a non-singular subvariety of codimension 2 in Y. Therefore $\omega_X = \omega_Y \otimes \Lambda^2\mathcal{N}_{X/Y}$, where $\mathcal{N}_{/Y}$ is the normal sheaf. Further, X is the scheme of zeros of a regular section of the vector bundle $\mathcal{L} = \Lambda^2\mathcal{S}^\vee \otimes V/\mathcal{S}$, and hence $\mathcal{N}_{X/Y} = \mathcal{L}|_Y$ and $\Lambda^2\mathcal{N}_{X/Y} = \mathcal{O}_X(3)$, since $c_1(\mathcal{L}) = 3H$. We obtain that $\omega_X = \mathcal{O}_X(-4) \otimes \mathcal{O}_X(3) = \mathcal{O}_X(-1)$, as was to be shown.

It remains to prove (iv). Since X is a del Pezzo surface of degree five, it contains ten straight lines. Since the embedding $X \subset P(\Lambda^2\mathbb{C}^4)$ is anticanonical, these straight lines are ordinary straight lines in $P(\Lambda^2\mathbb{C}^4)$ that lie in $\mathrm{Gr}(2,4)$. It remains to establish a bijection between these straight lines and fans. If $b \in \Lambda^2\mathbb{C}^4$, then b belongs to the cone over $\mathrm{Gr}(2,4)$ if and only if $b \wedge b = 0$. If b_1 and b_2 belong to this cone, then the straight line that joins them belongs to this cone if and only if $b_1 \wedge b_2 = 0$, which coincides with the fan condition. $\square$

8.4.4 Dodecahedral Section

Let us start with some definitions. Let X be an irreducible G-variety (a variety with an action of algebraic group G), $S \subset X$ be an irreducible subvariety. Then S is called a *section* of X if $\overline{G \cdot S} = X$. The section S is called a *relative section* if the following condition holds: there exists a dense Zariski-open subset $U \subset S$ such that if $x \in U$ and $gx \in S$ then $g \in H$, where $H = N_G(S) = \{g \in G \,|\, gS \subset S\}$ is the normalizer of S in G (see [PV]). In this case for any invariant function $f \in \mathbb{C}(X)^G$ the restriction $f|_S$ is well-defined and the map

$$\mathbb{C}(X)^G \to \mathbb{C}(S)^H, \quad f \mapsto f|_S,$$

is an isomorphism. Any relative section defines a G-equivariant rational map $\psi : X \to G/H$: if $g^{-1}x \in S$ then $x \mapsto gH$. Conversely, any G-equivariant rational map $\psi : X \to G/H$ with irreducible fibers defines the relative section $\overline{\psi^{-1}(eH)}$.

We are going to apply Theorem 8.20 and construct a relative section in the SL_4-module $\mathcal{A}_0$ (the module of 4-dimensional anticommutative algebras with zero trace). The action of SL_4 on 'Sylvester pentahedrons' is transitive with finite stabilizer H (which is the central extension of the permutation group S_5). In the sequel the Sylvester pentahedron will always mean the standard configuration formed by the hyperplanes

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0,\ x_1 + x_2 + x_3 + x_4 = 0.$$

Let $S \subset \mathcal{A}_0$ be a linear subspace formed by all algebras such that the hyperplanes of Sylvester pentahedron are their subalgebras. Then Theorem 8.20 implies that S is a 5-dimensional linear relative section of SL_4-module $\mathcal{A}_0$.

It is easy to see that the multiplication in algebras from S is given by formulas

$$[e_i\,, e_j] = a_{ij}e_i + b_{ij}e_j \quad (1 \le i < j \le 4),$$

where a_{ij} and b_{ij} satisfy a certain set of linear conditions. Consider 6 algebras $A_1, \ldots, A_6$ with the following structure constants:

	A_1	A_2	A_3	A_4	A_5	A_6
a_{12}	0	1	−1	1	0	−1
b_{12}	1	0	1	−1	−1	0
a_{13}	1	1	−1	0	−1	0
b_{13}	0	−1	0	1	1	−1
a_{14}	1	0	0	1	−1	−1
b_{14}	−1	1	−1	0	0	1
a_{23}	−1	−1	0	1	0	1
b_{23}	1	0	−1	0	1	−1
a_{24}	0	−1	−1	0	1	1
b_{24}	−1	1	0	1	−1	0
a_{34}	−1	1	1	−1	0	0
b_{34}	0	0	−1	1	−1	1

Table 8.2.

Then it is easy to see that $A_i \in S$ for any i. Moreover, the algebras A_i satisfy the unique linear relation $A_1 + \ldots + A_6 = 0$. It follows that any $A \in S$ can be written uniquely in the form $\alpha_1 A_1 + \ldots + \alpha_6 A_6$, where $\alpha_1 + \ldots + \alpha_6 = 0$. The coordinates α_i are called *dodecahedral coordinates* and S is called the *dodecahedral section* (this name will be clear later).

The stabilizer of the standard Sylvester pentahedron in PGL_4 is isomorphic to $\mathbb{S}_5$ represented by permutations of its hyperplanes. The group S_5 is generated by the transposition (12) and the cycle (12345). The preimages of these elements in GL_4 are given by matrices

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \tau = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The preimage of S_5 in SL_4 is the group H of 480 elements. The representation of H in S induces the projective representation of S_5 in $\mathbb{P}^4$. We have the following

Proposition 8.21 *This projective representation is the projectivization of the 5-dimensional irreducible representation of S_5 (either of two possible).*

Proof. Recall certain 'folklore' facts about the representation theory of S_5. It is well-known that S_5 admits exactly two embeddings in S_6 up to conjugacy. One is standard via permutations of the first five elements of the six-element set permuted by S_6. The other one can be obtained from the first by taking the composition with the unique (up to conjugacy) outer involution of S_6. S_5 has exactly two irreducible 5-dimensional representations, which have the same projectivizations. One of these representations has the following model. One takes the tautological 5-dimensional irreducible representation of S_6 and considers its composition with the non-standard embedding $S_5 \subset S_6$.

Now let us return to our projective representation. The action of σ and τ on algebras A_i is given by the formulas

$$\begin{pmatrix} \sigma A_1 \\ \sigma A_2 \\ \sigma A_3 \\ \sigma A_4 \\ \sigma A_5 \\ \sigma A_6 \end{pmatrix} = \begin{pmatrix} -A_2 \\ -A_1 \\ -A_4 \\ -A_3 \\ -A_6 \\ -A_5 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \tau A_1 \\ \tau A_2 \\ \tau A_3 \\ \tau A_4 \\ \tau A_5 \\ \tau A_6 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_6 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix}. \tag{8.11}$$

Therefore S_5 permutes the lines spanned by the A_i. Moreover, the induced embedding is clearly non-standard: transposition in S_5 maps to the composition of 3 independent transpositions in S_6. It is clear that that the corresponding projective representation of S_5 is isomorphic to the projectivization of the 5-dimensional irreducible representation in the model described above. □

Remark 8.22 The section S is called dodecahedral for the following reason. Though the surjection $H \to S_5$ does not split, the alternating group A_5 can be embedded in H. The induced representation of A_5 in S has the following description. $\mathbb{A}_5$ can be realised as a group of rotations of the dodecahedron. Let $\{\Gamma_1, \ldots, \Gamma_6\}$ be the set of pairs of opposite faces of the dodecahedron. Consider the vector space of functions

$$f : \{\Gamma_1, \ldots, \Gamma_6\} \to \mathbb{C}, \quad \sum_{i=1}^{6} f(\Gamma_i) = 0.$$

Then this vector space is an A_5-module. It is easy to see that this module is isomorphic to S via the identification $A_i \mapsto f_i$, where

$$f_i(\Gamma_i) = 5, \quad f_i(\Gamma_j) = -1, \quad j \neq i.$$

The following proposition follows from the discussion above

Theorem 8.23 *The restriction of invariants induces an isomorphism of invariant fields*

$$\mathbb{C}(\mathcal{A}_0)^{GL_4} \simeq \mathbb{C}(\mathbb{C}^5)^{\mathbb{C}^* \times \mathbb{S}_5},$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^5$ by homotheties and $\mathbb{S}_5$ acts via any of two 5-dimensional irreducible representations.

Remark 8.24 The Sylvester pentahedron also naturally arises in the theory of cubic surfaces. The SL_4-module of cubic forms $S^3(\mathbb{C}^4)^\vee$ admits the relative section (the so-called Sylvester section, or Sylvester normal form). Namely, a generic cubic form in a suitable system of homogeneous coordinates $x_1, \ldots, x_5$, $x_1 + \ldots + x_5 = 0$, can be written as a sum of 5 cubes $x_1^3 + \ldots + x_5^3$. The Sylvester pentahedron can be recovered from a generic cubic form f in a very interesting way: its 10 vertices coincide with 10 singular points of a quartic surface $\det \mathrm{Hes}(f)$. The Sylvester section has the same normalizer H as our dodecahedral section. It can be proved [Bek] that in this case the restriction of invariants induces an isomorphism

$$\mathbb{C}(S^3(\mathbb{C}^4)^\vee)^{GL_4} \simeq \mathbb{C}(\mathbb{C}^5)^{\mathbb{C}^* \times \mathbb{S}_5},$$

where $\mathbb{C}^*$ acts via homotheties and $\mathbb{S}_5$ via permutations of coordinates (i.e. via the *reducible* 5-dimensional representation). Other applications of the Sylvester pentahedron to moduli varieties can be found in [Bar].

These results were used in [T2] in order to prove that the field of invariant functions of the 5-dimensional irreducible representation of S_5 is rational (is isomorphic to the field of invariant functions of a vector space). From this result it is easy to deduce that in fact the field of invariant functions of any representation of S_5 is rational (see [T5]).

8.5 Adjoint Varieties

For the adjoint representation of $\mathrm{SL}_n = \mathrm{SL}(V)$ there is a natural notion of the discriminant defined as follows. For any operator $A \in \mathfrak{sl}(V)$ let

$$P_A = \det(t\,\mathrm{Id} - A)$$

be the characteristic polynomial. This is a polynomial in one variable t and its coefficients are homogeneous forms in the matrix entries of A. The discriminant $D(A) = D(P_A)$ of this polynomial is a homogeneous form on $\mathfrak{sl}(V)$ of degree $n^2 - n$. Clearly $D(A) \neq 0$ if and only if all eigenvalues of A are distinct, in other words, if A is a *regular semisimple operator*.

This discriminant can be defined for other simple Lie algebras as well. Before doing this, notice that $D(A)$ can be also defined as follows. Consider the characteristic polynomial

$$Q_A = \det(t\,\mathrm{Id} - \mathrm{ad}(A)) = \sum_{i=0}^{n^2-1} t^i D_i(A)$$

of the adjoint operator $\mathrm{ad}(A) = [A, \cdot]$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A (counted with multiplicities) then the set of eigenvalues of $\mathrm{ad}(A)$ consists of $n-1$ zeros and $n(n-1)$ differences $\lambda_i - \lambda_j$, $i, j = 1, \dots, n$, $i \neq j$. Therefore $D_0(A) = \dots = D_{n-2}(A) = 0$ and $D_{n-1}(A)$ coincides with $D(A)$ up to a non-zero scalar.

Suppose now that $\mathfrak{g}$ is a simple Lie algebra of rank r. For any $x \in \mathfrak{g}$ let

$$Q_x = \det(t\,\mathrm{Id} - \mathrm{ad}(x)) = \sum_{i=0}^{\dim \mathfrak{g}} t^i D_i(x)$$

be the characteristic polynomial of the adjoint operator $\mathrm{ad}(x) = [x, \cdot]$. Then $D(x) = D_r(x)$ is called the *discriminant* of x. Clearly, D is a homogeneous Ad-invariant polynomial on $\mathfrak{g}$ of degree $n - r$. Since the dimension of the centralizer $\mathfrak{g}_x$ of any element $x \in \mathfrak{g}$ is greater or equal to r, it follows that $D_0 = \dots = D_{r-1} = 0$, and therefore $D(x) = 0$ if and only if $\mathrm{ad}(x)$ has the eigenvalue 0 with multiplicity $> r$. We claim that actually $D(x) \neq 0$ if and only if x is regular semisimple (recall that x is called *regular* if $\dim \mathfrak{g}_x = r$). Indeed, if x is semisimple then $\mathrm{ad}(x)$ is a semisimple operator, and therefore $D(x) = 0$ if and only if the dimension of the centralizer $\mathfrak{g}_x$ is greater than r, i.e. x is not regular. If x is not semisimple then we take the Jordan decomposition $x = x_s + x_n$, where x_s is semisimple, x_n is nilpotent, and $[x_s, x_n] = 0$. Then x_s is automatically not regular, therefore since $Q_x = Q_{x_s}$ we have $D(x) = D(x_s) = 0$ (of course, this last equality also follows from the Ad-invariance of D and the fact that the Ad-orbit of x contains the Ad-orbit of x_s in its closure).

To study $D(x)$ further, we can use the Chevalley restriction theorem (see [PV]) $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{t}]^W$, where $\mathfrak{t} \subset \mathfrak{g}$ is any Cartan subalgebra and W is the Weyl group. Let $\Delta \subset \mathfrak{t}^\vee$ be the root system, $|\Delta| = n - r$. Let $x \in \mathfrak{t}$. Then $D(x) = 0$ if and only if x is not regular if and only if $\alpha(x) = 0$ for some $\alpha \in \Delta$. Since $\deg D = n - r$ and $D|_\mathfrak{t}$ is W-invariant, it easily follows that

$$D|_\mathfrak{t} = \prod_{\alpha \in \Delta} \alpha.$$

The Weyl group acts transitively on the set of roots of the same length. Therefore $D|_\mathfrak{t}$, and hence D, is irreducible if and only if all roots in Δ have the same length, i.e. Δ is of type A, D, or E. If Δ is of type B, C, F, or G, we have $\Delta = \Delta_s \cup \Delta_l$, where Δ_s is the set of short roots and Δ_l is the set of long roots. Then we have $D = D_l D_s$, where D_l and D_s are irreducible polynomials and

$$D_l|_\mathfrak{t} = \prod_{\alpha \in \Delta_l} \alpha, \quad D_s|_\mathfrak{t} = \prod_{\alpha \in \Delta_s} \alpha.$$

In the $A - D - E$ case we also set $D_l = D$ to simplify notations.

We are going to show that D_l is also the discriminant in our usual sense. The adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is irreducible. In $A - D - E$ case let $\mathcal{O} = \mathcal{O}_l$ be the Ad-orbit of any root vector. In $B-C-F-G$ case let $\mathcal{O}_l \subset \mathfrak{g}$ (resp. $\mathcal{O}_s \subset \mathfrak{g}$) be the Ad-orbit of any long root vector (resp. any short root vector). Then $\mathcal{O}_l$ is the orbit of the highest weight vector. Its projectivization is called the adjoint variety. Both orbits $\mathcal{O}_l$ and $\mathcal{O}_s$ are conical. Let $X_l = \mathbb{P}(\mathcal{O}_l)$ and $X_s = \overline{\mathbb{P}(\mathcal{O}_s)}$.

Theorem 8.25 *D_l is the discriminant of X_l. D_s is the discriminant of X_s.*

Proof. We identify $\mathfrak{g}$ and $\mathfrak{g}^\vee$ via the Killing form. Let $\alpha \in \Delta$ be any root, and $e_\alpha \in \mathfrak{g}$ the corresponding root vector. Since $[\mathfrak{g}, e_\alpha]^\perp = \mathfrak{g}_{e_\alpha}$, we have

$$\left(\overline{\mathbb{P}(\mathrm{Ad}(G) \cdot e_\alpha)}\right)^* = \overline{\mathbb{P}(\mathrm{Ad}(G) \cdot \mathfrak{g}_{e_\alpha})}.$$

Let

$$\mathfrak{t}_\alpha = \{x \in \mathfrak{t} \,|\, \alpha(x) = 0\}, \quad \hat{\mathfrak{t}}_\alpha = \mathfrak{t}_\alpha + e_\alpha.$$

Then, clearly, $\hat{\mathfrak{t}}_\alpha \in \mathfrak{g}_{e_\alpha}$. Moreover, if α is long then $D_l|_{\hat{\mathfrak{t}}_\alpha} = D_l|_{\mathfrak{t}_\alpha} = 0$. If α is short then $D_s|_{\hat{\mathfrak{t}}_\alpha} = D_s|_{\mathfrak{t}_\alpha} = 0$. Therefore, in order to prove the theorem, it suffices to check that

$$\dim \mathrm{Ad}(G) \cdot \hat{\mathfrak{t}}_\alpha = n - 1. \tag{8.12}$$

Clearly, for generic $x \in \hat{\mathfrak{t}}_\alpha$ we have

$$\dim \mathrm{Ad}(G) \cdot \hat{\mathfrak{t}}_\alpha = \dim G + \dim \hat{\mathfrak{t}}_\alpha - \dim \mathrm{Tran}(x, \hat{\mathfrak{t}}_\alpha), \tag{8.13}$$

where

$$\mathrm{Tran}(x, \hat{\mathfrak{t}}_\alpha) = \{g \in G \,|\, \mathrm{Ad}(g)x \in \hat{\mathfrak{t}}_\alpha\}.$$

Let $x = y + e_\alpha \in \hat{\mathfrak{t}}_\alpha$, where $y \in \mathfrak{t}_\alpha$ is such that $\beta(y) \neq 0$ for any $\beta \in \Delta \setminus \{\pm\alpha\}$. Suppose that $\mathrm{Ad}(g)x \in \hat{\mathfrak{t}}_\alpha$. Since $[e_\alpha, \mathfrak{t}_\alpha] = 0$, it easily follows that $\mathrm{Ad}(g)y \in \mathfrak{t}_\alpha$ and $\mathrm{Ad}(g)e_\alpha = e_\alpha$. Under our assumptions on y this means that

$$\mathrm{Tran}(x, \hat{\mathfrak{t}}_\alpha) = (N_G(\mathfrak{t}_\alpha))_{e_\alpha},$$

where $N_G(\mathfrak{t}_\alpha)$ is the normalizer of t_α in G. Therefore, in order to prove (8.12) using (8.13), it suffices to check that

$$\dim(\mathfrak{g}_{\mathfrak{t}_\alpha})_{e_\alpha} = r, \tag{8.14}$$

where $\mathfrak{g}_{\mathfrak{t}_\alpha}$ is the centralizer of t_α in $\mathfrak{g}$. Now $\mathfrak{g}_{\mathfrak{t}_\alpha}$ is a Levi subalgebra equal to $\mathfrak{t} + \mathbb{C}e_\alpha + \mathbb{C}e_{-\alpha}$. Therefore $(\mathfrak{g}_{\mathfrak{t}_\alpha})_{e_\alpha} = \mathfrak{t}_\alpha + \mathbb{C}e_\alpha$ and (8.14) follows. $\square$

Remark 8.26 In A-D-E cases one can prove by inspection the following interesting formula:

$$2 \dim \mathfrak{g} = \operatorname{rank} \mathfrak{g} (\dim \mathcal{O} + 4).$$

8.6 Homogeneous Vector Bundles

8.6.1 Zeros of Generic Global Sections

Both resultants and discriminants are related to the following general construction. Suppose that X is a smooth projective variety and E is a vector bundle on X generated by global sections. Let $Z(s)$ denote the scheme of zeros of any global section $s \in H^0(X, E)$. One might expect that for generic s the scheme $Z(s)$ is a smooth variety of codimension $\dim E$. Then we can define the degeneration variety $D \subset H^0(X, E)$ parametrizing all global sections s such that $Z(s)$ is not smooth of expected codimension. For example, if E is a very ample line bundle then D is a cone over the dual variety. If E is a very ample vector bundle and $\dim E = \dim X + 1$ then D is the resultant variety. If E is a very ample vector bundle and $\dim E = \dim X$ then the corresponding homogeneous polynomial can be called Bézoutian. Indeed, if $X = \mathbb{P}^n$ and $E = \mathcal{O}(d_1) \oplus \ldots \oplus \mathcal{O}(d_n)$ then D parametrizes sets of homogeneous forms of degrees $d_1, \ldots, d_n$ such that the Bézout theorem is not applicable.

In general, in order to make the theory consistent, it is necessary to impose very strong conditions like ampleness on the vector bundle E. These conditions are not always satisfied even in the case of homogeneous vector bundles on flag varieties. Even the question whether or not a generic zero scheme is non-empty can be quite difficult.

Assume that G is a connected reductive group, T is a fixed maximal torus, B is a fixed Borel subgroup, $T \subset B \subset G$, B_- is the opposite Borel subgroup, P is a parabolic subgroup containing B_-, $\mathbf{X}(T)$ is the lattice of characters of T, and $\lambda \in \mathbf{X}(T)$ is the dominant weight. Consider the homogeneous vector bundle $\mathcal{L}_\lambda = G \times_P U_\lambda$ over G/P, where U_λ is the irreducible P-module with highest weight λ. By the Borel–Weyl–Bott Theorem 8.1, $V_\lambda = H^0(G/P, \mathcal{L}_\lambda)$ is an irreducible G-module with highest weight λ.

Theorem 8.27 *Let $s \in V_\lambda$ be a generic global section. Then*

(i) *If $\dim U_\lambda > \dim G/P$, the scheme of zeros Z_s is empty.*
(ii) *If $\dim U_\lambda \leq \dim G/P$, either Z_s is empty or s intersects the zero section of $\mathcal{L}_\lambda$ transversally and Z_s is a smooth unmixed subvariety of expected codimension $\dim U_\lambda$.*
(iii) *If $\dim U_\lambda = \dim G/P$, the geometric number of points in Z_s is equal to the top Chern class of $\mathcal{L}_\lambda$.*

Proof. (i) is obvious and follows by an easy dimension count. (ii) and (iii) follow from the fact that $\mathcal{L}_\lambda$ is generated by global sections. Let us recall this argument here. Let $\dim U_\lambda \leq \dim G/P$, and assume that every global section has a zero. We have to prove that a generic global section s intersects the zero section of $\mathcal{L}_\lambda$ transversally. This will imply, in particular, that Z_s is a smooth unmixed subvariety of codimension $\dim U_\lambda$. For simplicity we suppress the index λ. Consider the incidence variety

$$Z \subset G/P \times V, \quad Z = \{(x,s) \,|\, x \in (Z_s)_{red}\}.$$

Since G/P is homogeneous and Z is invariant, it follows that Z is obtained by spreading the fiber $Z_e = \{s \in V \,|\, s(eP) = 0\}$ by the group G. Since U is irreducible, we have $\dim Z_e = \dim V - \dim U$. Hence Z is a smooth irreducible subvariety of dimension $\dim V + \dim G/P - \dim U$.

Let $\pi : Z \to V$ be the restriction to Z of the projection of $G/P \times V$ on the second summand. By assumption, π is a surjection. By Sard's lemma for algebraic varieties (see [Mum1]), the differential $d\pi_{(x,s)}$ is surjective for a generic point $s \in V$ and any point (x,s) in $\pi^{-1}s$. We claim that s has the transversal intersection with the zero section. Indeed, Z can be regarded as a subbundle of the trivial bundle $G/P \times V$. Then $\mathcal{L} = (G/P \times V)/Z$ and the zero section of $\mathcal{L}$ is identified with $Z \bmod Z$. The section s is identified with $(G/P \times \{s\}) \bmod Z$. Hence, it is sufficient to prove that $G/P \times \{s\}$ is transversal to Z, which is equivalent to the following claim: $d\pi_{(x,s)}$ is surjective for all $(x,s) \in Z$. Now statement (iii) of the theorem follows from the standard intersection theory (see [Fu1]). □

Example 8.28 It should be noted that Theorem 8.27 cannot be strengthened to the point where the non-emptiness and the irreducibility of the scheme of zeros in Theorem 8.27 could be established apriori, as the following example shows. Consider the vector bundle $S^2\mathcal{S}^\vee$ on $\mathrm{Gr}(k,2n)$, where $\mathcal{S}$ is the tautological bundle. The dimension of a fiber does not exceed the dimension of the Grassmannian if $k \leq \dfrac{4n-1}{3}$, but a generic section (that is, a non-degenerate quadratic form in $\mathbb{C}^{2n}$) has a zero (that is, a k-dimensional isotropic subspace) only if $k \leq n$. For $k = n$ the scheme of zeros is a reducible variety of dimension $\dfrac{n(n-1)}{2}$ with two irreducible components (spinor varieties) that correspond to two families of maximal isotropic subspaces on an even-dimensional quadric.

Consider the P-submodule

$$M_\lambda = \{s \in H^0(G/P, \mathcal{L}_\lambda) \,|\, s(P) = 0\}$$

in V_λ. It is easy to see that M_λ can be characterized as the unique maximal proper P-submodule of V_λ. Clearly $V_\lambda/M_\lambda \simeq U_\lambda$ as P-modules. Consider the map

$$\Psi : G \times M_\lambda \to V_\lambda, \quad \Psi(g,v) = gv.$$

Then generic global sections of $\mathcal{L}$ have zeros iff Ψ is dominant iff the differential of Ψ at a generic point is surjective. We write $\mathfrak{g}$ and $\mathfrak{p}$ for Lie algebras of G and P, respectively. The natural "orbital" map $\mathfrak{g} \times V_\lambda \to V_\lambda$ defines the P-equivariant map $\psi : \mathfrak{g}/\mathfrak{p} \times M_\lambda \to U_\lambda$. Then Ψ is dominant iff $\psi(\cdot, x)$ is surjective at a generic point $x \in M_\lambda$. Let $Z_\lambda = \psi^{-1}(0)_{red} \subset \mathfrak{g}/\mathfrak{p} \times M_\lambda$ be the incidence variety. Then the following proposition follows from an easy dimension count:

Proposition 8.29 *Suppose that*

$$\dim Z_\lambda = \dim \mathfrak{g}/\mathfrak{p} + \dim M_\lambda - \dim U_\lambda.$$

Then generic global sections of $\mathcal{L}_\lambda$ have zeros.

Denote by π the restriction on Z_λ of the projection of $\mathfrak{g}/\mathfrak{p} \times M_\lambda$ to the first factor. The variety Z_λ could be rather complicated. Say, it is not irreducible in general. We shall use the fact that fibers $\pi^{-1}(x)$ are linear spaces. So consider the function $l_\lambda(x) = \dim \pi^{-1}(x)$, $x \in \mathfrak{g}/\mathfrak{p}$.

Conjecture 8.30 *There exists an algebraic stratification $\mathfrak{g}/\mathfrak{p} = \bigsqcup_{i=1}^{r} X_i$ such that for any λ the function $l_\lambda(x)$ is constant along each X_i:*

$$l_\lambda(x) = l_\lambda^i, \quad x \in X_i.$$

If such stratification exists then

$$\dim Z_\lambda = \max_i (\dim X_i + l_\lambda^i). \tag{8.15}$$

If P acts on $\mathfrak{g}/\mathfrak{p}$ with a finite number of orbits then we can take these orbits as X_i (since ψ is P-equivariant). Unfortunately, this class of parabolic subgroups is very small. The dual P-module $(\mathfrak{g}/\mathfrak{p})^\vee$ is isomorphic to the representation of P on the unipotent radical $\mathfrak{p}_n$ of $\mathfrak{p}$, and hence by the Pyasetskii Theorem 2.4 the action $P : \mathfrak{g}/\mathfrak{p}$ has a finite number of orbits iff the action $P : \mathfrak{p}_n$ satisfies this property. Such actions were studied in [PR], and the complete classification for classical groups was obtained in [HRo]. But even in these cases the problem of the complete description of the orbital decomposition seems very messy. In the next section we shall introduce another class of parabolic subgroups that satisfy the Conjecture 8.30.

The most wonderful class of parabolic subgroups is the class of parabolic subgroups P with abelian unipotent radical (aura) $\mathfrak{p}_n$; see Section 2.3.2. Let P^- be an opposite parabolic subgroup, let $L = P \cap P^-$ be a Levi subgroup with Lie algebra $\mathfrak{l}$, and let $\mathfrak{p}_n^-$ be a unipotent radical of $\mathfrak{p}^-$. Then $\mathfrak{p}_n$ is abelian iff the decomposition $\mathfrak{g} = \mathfrak{p}_n^- \oplus \mathfrak{l} \oplus \mathfrak{p}_n$ is a $\mathbb{Z}$-grading: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Notice that $\mathfrak{g}/\mathfrak{p}$ is isomorphic to $\mathfrak{p}_n^-$ as a L-module. It is well-known that the action of L on $\mathfrak{p}_n^-$ has a finite number of orbits in this case. Take the orbital decomposition $\mathfrak{p}_n^- = \bigcup_{i=1}^{r} Lf_i$, for whose detailed description see Theorem 2.10. Then (8.15) takes the form

$$\dim Z_\lambda = \max_i (\dim Lf_i + l_\lambda(f_i)). \tag{8.16}$$

It remains to calculate $l_\lambda(f_i)$. If $f_i = 0$ then $l_\lambda(f_i) = \dim M_\lambda$. In the opposite case we can include f_i in a $\mathfrak{sl}_2$-triple $\langle f_i, h_i, e_i \rangle$. Evidently we can assume that $h_i \in \mathfrak{l}$, $e_i \in \mathfrak{p}_n$. These h_i were also written down explicitly in Theorem 2.10. We shall use the following easy lemma from the $\mathfrak{sl}_2$-theory:

Lemma 8.31 *Let V be a finite-dimensional $\mathfrak{sl}_2$-module, with $\mathfrak{sl}_2 = \langle f, h, e\rangle$. Let $\mathfrak{b} = \langle h, e\rangle$. Suppose that $M \subset V$ is a $\mathfrak{b}$-submodule such that $eV \subset M$. Then*

$$\dim \operatorname{Coker}(M \subset V \xrightarrow{f} V \to V/M) = \dim(V/M)^h,$$

where $(V/M)^h = \{v \in V/M \mid hv = 0\}$.

Applying this Lemma to the representation of $\mathfrak{sl}_2$ in V_λ we get

$$l_\lambda(f_i) = \dim \operatorname{Ker} \psi(f_i, \cdot) = \dim M_\lambda - \dim U_\lambda + \dim \operatorname{Coker} \psi(f_i, \cdot) = \dim M_\lambda - \dim U_\lambda + \dim U_\lambda^{h_i}.$$

Notice that this formula remains valid for $f_i = 0$ if we assume that $h_i = 0$ in this case. Joining this formula with (8.16) and applying Lemma 8.31 we get the following result.

Theorem 8.32 ([T3]) *Suppose that for any i*

$$\dim U_\lambda^{h_i} \leq \operatorname{codim}_{\mathfrak{g}/\mathfrak{p}} Lf_i.$$

Then generic global sections of the bundle $\mathcal{L}_\lambda$ have zeros.

8.6.2 Isotropic Subspaces of Forms

All ingredients of the formula in Theorem 8.32 can be easily computed in many particular cases. We will apply Theorem 8.32 for the proof of the following

Theorem 8.33 ([T3]) *Let $w \in S^d V^\vee$, resp. $w \in \Lambda^d V^\vee$, be a generic symmetric or skew-symmetric form. Then V contains a k-dimensional isotropic subspace of w if and only if*

$$n \geq \frac{\binom{d+k-1}{d}}{k} + k, \ \textit{resp. } n \geq \frac{\binom{k}{d}}{k} + k, \tag{8.17}$$

with the following exceptions: V contains a k-dimensional isotropic subspace if and only if

- *$n \geq 2k$ for $w \in S^2 V^\vee$ or $w \in \Lambda^2 V^\vee$.*
- *$k \leq n-2$ for $w \in \Lambda^{n-2} V^\vee$, n is even.*
- *$k \leq 4$ for $w \in \Lambda^3 V^\vee$, $n = 7$.*

Remark 8.34 The variety of 4-dimensional isotropic subspaces of a generic skew-symmetric 3-form in $\mathbb{C}^7$ is a smooth 8-dimensional Fano variety. Moreover, this variety is a compactification of the unique symmetric space of the simple algebraic group G_2.

Proof. We choose a basis $\{e_1, \ldots, e_n\}$ in V and identify GL_n with the group of non-singular matrices. Let T, B, B^- be the subgroups of diagonal, upper- and lower-triangular matrices. We fix an integer k. Consider the parabolic subgroup

$$P = \begin{pmatrix} A & 0 \\ * & B \end{pmatrix},$$

where B is a $k \times k$ matrix. Then G/P is the Grassmannian $\mathrm{Gr}(k, V)$. Consider the vector bundle $\mathcal{L} = S^d\mathcal{S}^\vee$ (resp. $\mathcal{L} = \Lambda^d\mathcal{S}^\vee$, but only in the case $k \geq d$) on G/P, where $\mathcal{S}$ is the tautological bundle. Then $\mathcal{L} = \mathcal{L}_\lambda$, where λ is the highest weight of the GL_n-module $S^dV^\vee$ (resp. $\Lambda^dV^\vee$). Let $w \in S^dV^\vee$ (resp. $w \in \Lambda^dV^\vee$), and let s_w be the corresponding global section. It is easy to see that $(Z_{s_w})_{red}$ coincides with the variety of k-dimensional isotropic subspaces of w. Notice that the inequalities (8.17) are equivalent to the condition $\dim U_\lambda \leq \dim G/P$. In the sequel we suppose that these inequalities hold.

We take

$$f_i = E_{1,n} + E_{2,n-1} + \ldots + E_{i,n+1-i}, \quad i = 0, \ldots, r = \min(k, n-k).$$

The unipotent radical $\mathfrak{p}_n^-$ can be identified with matrices of the shape $k \times (n-k)$, and then the orbit Lf_i is identified with the variety of matrices of rank i. Therefore

$$\mathrm{codim}_{\mathfrak{g}/\mathfrak{p}}\, Lf_i = (k-i)(n-k-i).$$

We have

$$h_i = (E_{n,n} - E_{1,1}) + (E_{n-1,n-1} - E_{2,2}) + \ldots + (E_{n+1-i,n+1-i} - E_{i,i}),$$

and hence

$$\dim U_\lambda^{h_i} = \binom{d+k-i-1}{d} \text{ if } V_\lambda = S^dV^\vee;$$
$$\dim U_\lambda^{h_i} = \binom{k-i}{d} \text{ if } V_\lambda = \Lambda^dV^\vee.$$

Applying Theorem 8.32 we get

Proposition 8.35

(i) *Suppose that for any $i = 0, \ldots, r$ we have*

$$\binom{d+k-i-1}{d} \leq (k-i)(n-k-i).$$

Then a generic form $w \in S^dV^\vee$ has a k-dimensional isotropic subspace.

(ii) *Suppose that for any $i = 0, \ldots, r$ we have*

$$\binom{k-i}{d} \leq (k-i)(n-k-i).$$

Then a generic form $w \in \Lambda^dV^\vee$ has a k-dimensional isotropic subspace.

It remains to clarify when the conditions of Proposition 8.35 follow from formulas (8.17).

A. Symmetric Case.

Clearly if $i = k$ then the conditions of Proposition 8.35 are satisfied. Therefore it suffices to find out when the inequality

$$n \geq \frac{\binom{d+k-i-1}{d}}{k-i} + k + i$$

follows from the inequality

$$n \geq \frac{\binom{d+k-1}{d}}{k} + k$$

as $i = 1, \ldots, \min(k-1, n-k)$. If $d = 1$ then Theorem 8.33 is obvious. If $d = 2$ it reduces to a well-known result about isotropic subspaces of quadratic form. So assume that $d \geq 3$. We use the following lemma that can be easily verified by induction:

Lemma 8.36 *Let $d \geq 3$ and $\alpha \geq 2$. Then* $\frac{\binom{d+\alpha-1}{d}}{\alpha} \geq \frac{\binom{d+\alpha-2}{d}}{\alpha-1} + 1$.

It follows from Lemma 8.36 that

$$n \geq \frac{\binom{d+k-1}{d}}{k} + k \geq \frac{\binom{d+k-2}{d}}{k-1} + k + 1 \geq$$
$$\frac{\binom{d+k-3}{d}}{k-2} + k + 2 \geq \ldots \geq \frac{\binom{d+k-1-(k-1)}{d}}{k-(k-1)} + k + k - 1.$$

□

B. Skew-symmetric Case.

Clearly if $i > k - d$ then the conditions of Proposition 8.35 are satisfied. Therefore it suffices to check when the inequality

$$n \geq \frac{\binom{k-i}{d}}{k-i} + k + i$$

follows from the inequality

$$n \geq \frac{\binom{k}{d}}{k} + k$$

as $i = 1, \ldots, \min(k-d, n-k)$. If $d = 1$ then Theorem 8.33 is obvious, if $d = 2$ it reduces to a well-known result about isotropic subspaces of a 2-form. So assume that $d \geq 3$. We use a following Lemma that can be easily verified by induction:

Lemma 8.37 *Let $3 \leq d \leq \alpha - 2$. Then* $\frac{\binom{\alpha}{d}}{\alpha} \geq \frac{\binom{\alpha-1}{d}}{\alpha-1} + 1$.

It follows from Lemma 8.37 that

$$n \geq \frac{\binom{k}{d}}{k} + k \geq \frac{\binom{k-1}{d}}{k-1} + k + 1 \geq$$
$$\frac{\binom{k-2}{d}}{k-2} + k + 2 \geq \ldots \geq \frac{\binom{k-(k-3-d)}{d}}{k-(k-3-d)} + k + (k-3-d).$$

It remains to consider 3 cases: $i = k-d-2$, $i = k-d-1$, $i = k-d$ as $i \leq n-k$.

Let $i = k-d-2$, $n \geq 2k-d-2$. We want to deduce the inequality $n \geq \frac{\binom{d+2}{d}}{d+2} + 2k - d - 2 = 2k - \frac{d-1}{2} - 2$ from $n \geq \frac{\binom{k}{d}}{k} + k$. It suffices to check $\binom{k}{d} \geq k(k - \frac{d-1}{2} - 2)$. Since $d \geq 3$ and $k-d = i+2 \geq 3$ we have $\binom{k}{d} \geq \binom{k}{3} \geq k(k-3) \geq k(k - \frac{d-1}{2} - 2)$.

Let $i = k-d-1$, $n \geq 2k-d-1$. We want to deduce the inequality $n \geq \frac{\binom{d+1}{d}}{d+1} + 2k - d - 1 = 2k - d$ from $n \geq \frac{\binom{d+1}{d}}{d+1} + 2k - d - 1 = 2k - d$. It suffices to check $\binom{k}{d} \geq k(k-d)$. Since $d \geq 3$ and $k-d = i+1 \geq 2$ we have $\binom{k}{d} \geq \binom{k}{3} \geq k(k-3) \geq k(k-d)$ as $k-d \geq 3$. If $k-d = 2$ then $\binom{k}{d} = \frac{(d+2)(d+1)}{2} \geq 2(d+2) \geq k(k-d)$.

Let $i = k-d$, $n \geq 2k-d$. We need to deduce the inequality $n \geq \frac{\binom{d}{d}}{d} + 2k - d = 2k - d + \frac{1}{d}$ from $n \geq \frac{\binom{k}{d}}{k} + k$. All exceptions are defined by the following system of equalities and inequalities:

$$n = 2k - d, \quad \binom{k}{d} \leq k(k-d), \quad k - d \geq 1.$$

There exist only two possibilities: either $k = d+1$, $n = d+2$ or $d = 3$, $k = 5$, $n = 7$.

In the first case we need to clarify whether a generic form $w \in \Lambda^{n-2}V^\vee$ has a $(n-1)$-dimensional isotropic subspace or not. This holds iff a generic 2-form in $V^\vee$ has a non-zero kernel. It is well-known that this is true only for odd n.

To complete the proof it remains to find out whether a generic form $w \in \Lambda^3(\mathbb{C}^7)^\vee$ has a 5-dimensional isotropic subspace. In suitable coordinates this isotropic subspace will coincide with the linear span $\langle e_1, \ldots, e_5 \rangle$. Therefore $w = \sum_{1 \leq i < j < k \leq 7} \alpha_{ijk} x_i \wedge x_j \wedge x_k$, where $\alpha_{ijk} = 0$ as $k \leq 5$. Consider the one-parameter subgroup

$$H(t) = \mathrm{diag}(t^2, t^2, t^2, t^2, t^2, t^{-5}, t^{-5})$$

in SL_7. Clearly, $\lim_{t \to 0} H(t)w = 0$. Therefore w belongs to the null-cone of the action $\mathrm{SL}_7 : \Lambda^3(\mathbb{C}^7)^\vee$. It follows that any non-constant homogeneous invariant of this action vanishes at w. But it is known that this action has non-constant invariants (see [PV]), for example, the discriminant is well-defined in this case. Therefore generic forms do not admit 5-dimensional isotropic subspaces. $\square$

Remark 8.38 A 3-form in the 7-dimensional vector space has a 5-dimensional isotropic subspace if and only if its discriminant is equal to zero.

8.6.3 Moore–Penrose Inverse and Applications

The nice notion of a generalized inverse of an arbitrary matrix (possibly singular or even non-square) has been discovered independently by Moore [Mo] and Penrose [Pe]. The following definition belongs to Penrose (Moore's definition is different but equivalent):

Definition 8.39 A matrix A^+ is called an *a MP-inverse* of a matrix A if

$$AA^+A = A, \quad A^+AA^+ = A^+,$$

and AA^+, A^+A are Hermitian matrices.

It is quite surprising, but the MP-inverse always exists and is unique. Since the definition is symmetric with respect to A and A^+ it follows that $(A^+)^+ = A$. If A is a non-singular square matrix then A^+ coincides with an ordinary inverse matrix A^{-1}. The theory of MP-inverses and their numerous modifications becomes now a separate subfield of Linear Algebra [CM] with various applications. Here we show that this notion quite naturally arises in the theory of shortly graded simple Lie algebras, and we give applications. To explain this connection let us first give another definition of the MP-inverse.

Let $A \in \mathrm{Mat}_{n,m}(\mathbb{C})$. Then it is easy to see that a matrix $A^+ \in \mathrm{Mat}_{m,n}(\mathbb{C})$ is an MP-inverse of A if and only if there exist Hermitian matrices

$$B_1 \in \mathrm{Mat}_{n,n}(\mathbb{C}) \text{ and } B_2 \in \mathrm{Mat}_{m,m}(\mathbb{C})$$

such that the following matrices form an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_{n+m}(\mathbb{C})$:

$$E = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ A^+ & 0 \end{pmatrix}.$$

By an $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ in a Lie algebra $\mathfrak{g}$ we mean a collection of (possibly zero) vectors such that

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

In other words, an $\mathfrak{sl}_2$-triple is a homomorphic image of canonical generators of $\mathfrak{sl}_2$ with respect to some homomorphism of Lie algebras $\mathfrak{sl}_2 \to \mathfrak{g}$.

This definition admits an immediate generalization. Suppose that $\mathfrak{g}$ is a simple complex Lie algebra, and G is a corresponding simple simply-connected Lie group. Suppose further that P is a parabolic subgroup of G with abelian unipotent radical (with *aura*). Then $\mathfrak{g}$ admits a short grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with only three nonzero parts. Here $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie algebra of P and $\exp \mathfrak{g}_1$ is the abelian unipotent radical of P. Let $\mathfrak{k}_0$ be a compact real form of $\mathfrak{g}_0$. In this section we shall permanently consider compact real forms of reductive subalgebras of simple Lie algebras. These subalgebras will always be Lie algebras of algebraic reductive subgroups of a corresponding simple complex algebraic group. Their compact real forms will always be understood as Lie algebras of compact real forms of corresponding algebraic groups. For example, a Lie algebra of an algebraic torus has a unique compact real form.

Suppose now that $e \in \mathfrak{g}_1$. It is well-known that there exists a *homogeneous* $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$.

Definition 8.40 An element $f \in \mathfrak{g}_{-1}$ is called *an MP-inverse* of $e \in \mathfrak{g}_1$ if there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ with $h \in i\mathfrak{k}_0$.

MP-inverses of elements $f \in \mathfrak{g}_{-1}$ are defined in the same way. It is clear that if f is an MP-inverse of e then e is an MP-inverse of f.

Example 8.41 Suppose that $G = \mathrm{SL}_{n+m}$ and $P \subset G$ is a maximal parabolic subgroup of block triangular matrices of the form

$$\begin{pmatrix} B_1 & A \\ 0 & B_2 \end{pmatrix}, \quad \text{where} \quad B_1 \in \mathrm{Mat}_{n,n},\ A \in \mathrm{Mat}_{n,m},\ B_2 \in \mathrm{Mat}_{m,m}\,.$$

The graded components of the correspondent grading consist of matrices of the following form:

$$\mathfrak{g}_{-1} = \begin{pmatrix} 0 & 0 \\ A' & 0 \end{pmatrix}, \quad \mathfrak{g}_0 = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad \mathfrak{g}_1 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

where $A' \in \mathrm{Mat}_{m,n}$, $B_1 \in \mathrm{Mat}_{n,n}$, $B_2 \in \mathrm{Mat}_{m,m}$, and $A \in \mathrm{Mat}_{n,m}$. One can take $\mathfrak{k}_0$ to be the real Lie algebra of block diagonal skew-Hermitian matrices with zero trace. Then $i\mathfrak{k}_0$ is a vector space of block diagonal Hermitian matrices with zero trace. Therefore in this case we return to a previous definition of a Moore–Penrose inverse.

Theorem 8.42 ([T4]) *For any $e \in \mathfrak{g}_1$ there exists a unique Moore–Penrose inverse $f \in \mathfrak{g}_{-1}$.*

The proof is similar to the proof of Theorem 8.44 below.

It obviously follows that for any non-zero $f \in \mathfrak{g}_{-1}$ there exists a unique MP-inverse $e \in \mathfrak{g}_1$. So taking an MP-inverse is a well-defined involutive operation. In general, it is not equivariant with respect to a Levi subgroup $L \subset P$ with Lie algebra $\mathfrak{g}_0$, but only with respect to its maximal compact subgroup $K_0 \subset L$.

Let us give an intrinsic description of the Moore-Penrose inverse in all cases arising from short gradings of classical simple Lie algebras. Exceptional cases may be found in [T4].

Linear Maps.

This is, of course, the classical Moore–Penrose inverse. Let us recall its intrinsic description. Suppose that $\mathbb{C}^n$ and $\mathbb{C}^m$ are vector spaces equipped with standard Hermitian scalar products. The Moore-Penrose inverse of a linear map $F : \mathbb{C}^n \to \mathbb{C}^m$ is a linear map $F^+ : \mathbb{C}^m \to \mathbb{C}^n$ defined as follows. Let $\operatorname{Ker} F \subset \mathbb{C}^n$ and $\operatorname{Im} F \subset \mathbb{C}^m$ be a kernel and an image of F. Let $\operatorname{Ker}^\perp F \subset \mathbb{C}^n$ and $\operatorname{Im}^\perp F \subset \mathbb{C}^m$ be their orthogonal complements with respect to the Hermitian scalar products. Then F defines via restriction a bijective linear map $\tilde{F} : \operatorname{Ker}^\perp F \to \operatorname{Im} F$. Then $F^+ : \mathbb{C}^m \to \mathbb{C}^n$ is the unique linear map such that $F^+|_{\operatorname{Im}^\perp F} = 0$ and $F^+|_{\operatorname{Im} F} = \tilde{F}^{-1}$. This MP-inverse corresponds to short gradings of $\mathfrak{sl}_{n+m}$.

Symmetric and Skew-symmetric Bilinear Forms.

Let $V = \mathbb{C}^n$ be a vector space equipped with a standard Hermitian scalar product. The Moore-Penrose inverse of a symmetric (resp. skew-symmetric) bilinear form ω on V is a symmetric (resp. skew-symmetric) bilinear form ω^+ on $V^\vee$ defined as follows. Let $\operatorname{Ker}\,\omega \subset V$ be the kernel of ω. Then ω induces a non-degenerate bilinear form $\tilde{\omega}$ on $V/\operatorname{Ker}\,\omega$. Let $\operatorname{Ann}(\operatorname{Ker}\,\omega) \subset V^\vee$ be an annihilator of $\operatorname{Ker}\,\omega$. Then $\operatorname{Ann}(\operatorname{Ker}\,\omega)$ is canonically isomorphic to the dual of $V/\operatorname{Ker}\,\omega$. Therefore the form $\tilde{\omega}^{-1}$ on $\operatorname{Ann}(\operatorname{Ker}\,\omega)$ is well-defined. The form ω^+ is defined as the unique form whose its restriction on $\operatorname{Ann}(\operatorname{Ker}\,\omega)$ coincides with $\tilde{\omega}^{-1}$ and whose kernel is $\operatorname{Ann}(\operatorname{Ker}\,\omega)^\perp$, the orthogonal complement with respect to a standard Hermitian scalar product on $V^\vee$. This MP-inverse corresponds to the short grading of $\mathfrak{sp}_{2n+2}$ (resp. $\mathfrak{so}_{2n+2}$).

Vectors in a Vector Space With the Scalar Product.

Suppose that $V = \mathbb{C}^n$ is a vector space with the standard bilinear scalar product $(\cdot,\cdot)$. The Moore-Penrose inverse $v^\vee$ of a vector $v \in V$ is again a vector in V defined as follows:

$$v^\vee = \begin{cases} \dfrac{2v}{(v,v)}, & \text{if } (v,v) \neq 0 \\ \dfrac{\overline{v}}{(\overline{v},v)}, & \text{if } (v,v) = 0,\ v \neq 0 \\ 0, & \text{if } v = 0. \end{cases}$$

This MP-inverse corresponds to the short grading of $\mathfrak{so}_{n+2}$.

It is quite natural to ask whether it is possible to extend the notion of the Moore–Penrose inverse from parabolic subgroups with aura to arbitrary parabolic subgroups. It is also interesting to consider the "non-graded" situation. Let us start with this situation. Suppose that G is a simple connected simply-connected Lie group with Lie algebra $\mathfrak{g}$. We fix a compact real form $\mathfrak{k} \subset \mathfrak{g}$.

Definition 8.43 A nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ is called *a Moore–Penrose orbit* if for any $e \in \mathcal{O}$ there exists an $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in i\mathfrak{k}$.

It turns out that it is quite easy to find all Moore-Penrose orbits. Recall that the height $\mathrm{ht}(\mathcal{O})$ of a nilpotent orbit $\mathcal{O} = \mathrm{Ad}(G)e$ is equal to the maximal integer k such that $\mathrm{ad}(e)^k \neq 0$. Clearly $\mathrm{ht}(\mathcal{O}) \geq 2$.

Theorem 8.44 *$\mathcal{O}$ is a Moore–Penrose orbit if and only if* $\mathrm{ht}(\mathcal{O}) = 2$. *In this case for any $e \in \mathcal{O}$ there exists a unique $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in i\mathfrak{k}$.*

Proof. Let $x \to \overline{x}$ denotes a complex conjugation in $\mathfrak{g}$ with respect to the compact form $\mathfrak{k}$. Therefore $x = \overline{x}$ iff $x \in \mathfrak{k}$ and $x = -\overline{x}$ iff $x \in i\mathfrak{k}$. Let $B(x, y) = \mathrm{Tr}\ \mathrm{ad}(x)\,\mathrm{ad}(y)$ be the Killing form of $\mathfrak{g}$. Finally, let $H(x, y) = -B(x, \overline{y})$ be a positive-definite Hermitian form on $\mathfrak{g}$.

Lemma 8.45 *We fix a nilpotent element $e \in \mathfrak{g}$. Suppose that $\langle e, h, f\rangle$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ such that $h \in i\mathfrak{k}$. Then for any other $\mathfrak{sl}_2$-triple $\langle e, h', f'\rangle$ we have $H(h, h) < H(h', h')$. In particular, if there exists an $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ with $h \in i\mathfrak{k}$ then the $\mathfrak{sl}_2$-triple with this property is unique.*

Proof. Recall that if $\langle e, h, f\rangle$ is an $\mathfrak{sl}_2$-triple then h is called *a characteristic* of e. Consider the subset $\mathcal{H} \subset \mathfrak{g}$ consisting of all possible characteristics of e. It is well-known that $\mathcal{H}$ is an affine subspace in $\mathfrak{g}$ such that the corresponding linear subspace is precisely the unipotent radical $\mathfrak{z}_{\mathfrak{g}}^u(e)$ of the centralizer $\mathfrak{z}_{\mathfrak{g}}(e)$ in $\mathfrak{g}$ of the element e. Since $H(h', h')$ is a strongly convex function on $\mathcal{H}$, there exists a unique element $h_0 \in \mathcal{H}$ such that $H(h_0, h_0) < H(h', h')$ for any $h' \in \mathcal{H}$, $h' \neq h_0$. We need to show that $h_0 = h$. It is clear that an element $h_0 \in \mathcal{H}$ minimizes $H(h, h)$ on $\mathcal{H}$ iff $H(h_0, \mathfrak{z}_{\mathfrak{g}}^u(e)) = 0$ iff $B(\overline{h}_0, \mathfrak{z}_{\mathfrak{g}}^u(e)) = 0$. If $h \in \mathcal{H} \cap i\mathfrak{k}_0$ then $\overline{h} = -h$ and we have

$$B(\overline{h}, \mathfrak{z}_{\mathfrak{g}}^u(e)) = -B(h, \mathfrak{z}_{\mathfrak{g}}^u(e)) = -B([e, f], \mathfrak{z}_{\mathfrak{g}}^u(e)) = B(f, [e, \mathfrak{z}_{\mathfrak{g}}^u(e)]) = 0.$$

Therefore $h = h_0$. □

Suppose that $\langle e, h, f\rangle$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Consider the grading $\mathfrak{g} = \bigoplus_k \mathfrak{g}_k$ such that $x \in \mathfrak{g}_k$ iff $[h, x] = kx$. Let $\mathfrak{n}_+ = \bigoplus_{k>0} \mathfrak{g}_k$ and $\mathfrak{n}_- = \bigoplus_{k<0} \mathfrak{g}_k$. It is well known that $\mathfrak{z}_{\mathfrak{g}}^u(e) \subset \mathfrak{n}_+$.

Lemma 8.46 *Suppose that $\mathfrak{z}_{\mathfrak{g}}^u(e) = \mathfrak{n}_+$. Then $\mathcal{O} = \mathrm{Ad}(G)e$ is a Moore–Penrose orbit.*

Proof. We need to prove that for any element $e' \in \mathcal{O}$ there exists an $\mathfrak{sl}_2$-triple $\langle e', h', f'\rangle$ such that $h' \in i\mathfrak{k}$, where $\mathfrak{k}$ is a *fixed* compact real form of $\mathfrak{g}$. Clearly it is sufficient to prove that for an *arbitrary* compact real form $\mathfrak{k}$ there exists an $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ with $h \in i\mathfrak{k}$. According to the proof of Lemma 8.45 we should choose h to be a unique characteristic such that $B(\overline{h}, \mathfrak{z}_{\mathfrak{g}}^u(e)) = 0$, where $x \to \overline{x}$ denotes a complex conjugation in $\mathfrak{g}$ with respect to the compact form $\mathfrak{k}$. It remains to prove that $h \in i\mathfrak{k}$. Since B is a non-degenerate ad-invariant scalar product on $\mathfrak{g}$ and $\mathfrak{z}^u_{\mathfrak{g}}(e) = \mathfrak{n}_+$, it follows that $\overline{h} \in \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n}_+$. Let $\mathfrak{l}$ be some "standard" compact real form of $\mathfrak{g}$ such that $h \in i\mathfrak{l}$ and $\tilde{\mathfrak{n}}_\pm = \mathfrak{n}_\mp$, where $x \to \tilde{x}$ denotes a complex conjugation in $\mathfrak{g}$ with respect to the compact form $\mathfrak{l}$. Let $P \subset G$ be a parabolic subgroup of G with the Lie algebra $\mathfrak{p}$, and let $H \subset P$ be its Levi subgroup with the Lie algebra $\mathfrak{g}_0$. Since all compact real forms of a semisimple complex Lie algebra are conjugated by elements of any fixed Borel subgroup it follows that there exists $g \in P$ such that $\mathrm{Ad}(g)\mathfrak{k} = \mathfrak{l}$. Therefore

$$\widetilde{\mathrm{Ad}(g)}h = \mathrm{Ad}(g)\overline{h} \subset \mathrm{Ad}(g)(\mathfrak{p}) = \mathfrak{p}.$$

We can express g as a product uz, where $u \in \exp(\mathfrak{n}_+)$ and $\mathrm{Ad}(z)h = h$. Then $\widetilde{\mathrm{Ad}(g)}h = \widetilde{\mathrm{Ad}(u)}h$. If u is not the identity element of G then $\mathrm{Ad}(u)h = h + \xi$, where $\xi \in \mathfrak{n}_+$ and $\xi \neq 0$. Therefore $\widetilde{\mathrm{Ad}(u)}h = -h + \tilde{\xi}$. But $\tilde{\xi} \in \mathfrak{n}_-$ and hence $\widetilde{\mathrm{Ad}(u)}h \notin \mathfrak{p}$: a contradiction. Therefore u is trivial and since $\mathrm{Ad}(z)h = h$ we finally get

$$\overline{h} = \tilde{h} = -h.$$

□

Now we shall try to reverse this argument.

Lemma 8.47 *Suppose that $\mathcal{O} = \mathrm{Ad}(G)e$ is a Moore–Penrose orbit. Then $\mathfrak{z}^u_{\mathfrak{g}}(e) = \mathfrak{n}_+$.*

Proof. We choose a standard compact real form $\mathfrak{l}$ as in the proof of Lemma 8.46. Clearly, $\mathfrak{z}^u_{\mathfrak{g}}(e)$ is a graded subalgebra of $\mathfrak{n}_+ = \bigoplus_{k>0} \mathfrak{g}_k$. Suppose, on the contrary, that $\mathfrak{z}^u_{\mathfrak{g}}(e) \neq \mathfrak{n}_+$. Let $\xi \in \mathfrak{g}_p$, $p > 0$, be a homogeneous element that does not belong to $\mathfrak{z}^u_{\mathfrak{g}}(e)$. Let $u = \exp(\xi)$ and $e' = \mathrm{Ad}(u)e$. We claim that all characteristics of e' do not belong to $i\mathfrak{l}$. Indeed, all characteristics of e' have the form $\mathrm{Ad}(u)h + \mathrm{Ad}(u)x$, where $x \in \mathfrak{z}^u_{\mathfrak{g}}(e)$. Suppose that for some x we have $\mathrm{Ad}(u)h + \mathrm{Ad}(u)x \in i\mathfrak{l}$. Since $h \in i\mathfrak{l}$, $\tilde{\mathfrak{n}}_\pm = \mathfrak{n}_\mp$, and $\mathrm{Ad}(u)(h+x) - h \in \mathfrak{n}_+$ it follows that $\mathrm{Ad}(u)(h+x) = h$. In $\mathfrak{n}_+$ modulo $\bigoplus_{k>p} \mathfrak{g}_k$ we obtain the equation $[\xi, h] + x = 0$, but $[h, \xi] = p\xi$ and therefore $\xi \in \mathfrak{z}^u_{\mathfrak{g}}(e)$. This is the required contradiction. □

Now we can finish the proof of Theorem 8.44. Combining previous lemmas we see that $\mathcal{O}$ is a Moore–Penrose orbit if and only if $\mathfrak{z}^u_{\mathfrak{g}}(e) = \mathfrak{n}_+$. It follows from the $\mathfrak{sl}_2$-theory that $\dim \mathfrak{z}^u_{\mathfrak{g}}(e) = \dim \mathfrak{g}_1 + \dim \mathfrak{g}_2$. Therefore $\mathfrak{z}^u_{\mathfrak{g}}(e) = \mathfrak{n}_+$ if and only if $\mathfrak{g}_p = 0$ for $p > 2$. Clearly, this is precisely equivalent to $\mathrm{ht}(\mathcal{O}) = 2$. In this case for any $e \in \mathcal{O}$ there exists a unique $\mathfrak{sl}_2$-triple $\langle e, h, f \rangle$ such that $h \in i\mathfrak{k}$ by Lemma 8.45. □

Now let us turn to the graded situation. Suppose that $\mathfrak{g}$ is a $\mathbb{Z}$-graded simple Lie algebra with $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$. Let $P \subset G$ be a parabolic subgroup with Lie algebra $\mathfrak{p} = \bigoplus_{k\geq 0} \mathfrak{g}_k$. Let $L \subset P$ be a Levi subgroup with Lie algebra $\mathfrak{g}_0$. We choose a compact real form $\mathfrak{k}_0$ of $\mathfrak{g}_0$. Suppose now that $e \in \mathfrak{g}_k$. It is well-known that there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f \rangle$ with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-k}$.

Definition 8.48 Take any $k > 0$ and any L-orbit $\mathcal{O} \subset \mathfrak{g}_k$. Then $\mathcal{O}$ is called *a Moore-Penrose orbit* if for any $e \in \mathcal{O}$ there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in i\mathfrak{k}_0$. In this case f is called an MP-inverse of e. A grading is called *a Moore–Penrose grading in degree* $k > 0$ if all L-orbits in $\mathfrak{g}_k$ are Moore-Penrose. A grading is called *a Moore–Penrose grading* if it is a Moore–Penrose grading in any positive degree. A parabolic subgroup $P \subset G$ is called *a Moore–Penrose parabolic subgroup* if there exists a Moore–Penrose grading $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$ such that $\mathfrak{p} = \bigoplus_{k\geq 0} \mathfrak{g}_k$ is a Lie algebra of P.

One should be careful comparing graded and non-graded situations: if $\mathcal{O} \subset \mathfrak{g}_k$ is a Moore–Penrose L-orbit then $\mathrm{Ad}(G)\mathcal{O} \subset \mathfrak{g}$ is not necessarily a Moore–Penrose G-orbit. Let us give a criterion for an L-orbit to be Moore–Penrose. Suppose that $\mathcal{O} = \mathrm{Ad}(L)e \subset \mathfrak{g}_k$. Take any homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$. Then h defines a grading $\mathfrak{g}_0 = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_0^n$, such that $\mathrm{ad}(h)|_{\mathfrak{g}_0^n} = n \cdot \mathrm{Id}$.

Theorem 8.49 ([T4]) *$\mathcal{O}$ is a Moore–Penrose orbit if and only if* $\mathrm{ad}(e)g_0^n = 0$ *for any $n > 0$. In this case for any $e' \in \mathcal{O}$ there exists a unique homogeneous $\mathfrak{sl}_2$-triple $\langle e', h', f'\rangle$ such that $h' \in i\mathfrak{k}_0$.*

The proof is similar to the proof of Theorem 8.44.

Example 8.50 Suppose that G is a simple group of type G_2. We fix a root decomposition. There are two simple roots α_1 and α_2 such that α_1 is short and α_2 is long. There are 3 proper parabolic subgroups: the Borel subgroup B and two maximal parabolic subgroups P_1 and P_2 such that a root vector of α_i belongs to a Levi subgroup of P_i. Then the following is an easy application of Theorem 8.49. B is a Moore–Penrose parabolic subgroup (actually Borel subgroups in all simple groups are Moore–Penrose parabolic subgroups with respect to any grading). P_1 is not Moore–Penrose, but it is a Moore–Penrose parabolic subgroup in degree 2 (with respect to the natural grading of height 2). P_2 is a Moore–Penrose parabolic subgroup.

Example 8.51 Suppose that $G = \mathrm{SL}_n$. We fix positive integers $d_1, \ldots, d_k$ such that $n = d_1 + \ldots + d_k$. We consider the parabolic subgroup $P(d_1, \ldots, d_k) \subset \mathrm{SL}_n$ that consists of all upper-triangular block matrices with sizes of blocks equal to $d_1, \ldots, d_k$. We take a standard grading. Then $\mathfrak{g}_1$ is identified with the linear space of all tuples of linear maps $\{f_1, \ldots, f_k\}$,

$$\mathbb{C}^{d_1} \xleftarrow{f_1} \mathbb{C}^{d_2} \xleftarrow{f_2} \ldots \xleftarrow{f_{k-1}} \mathbb{C}^{d_k},$$

$\mathfrak{g}_{-1}$ is identified with the linear space of all tuples of linear maps $\{g_1, \ldots, g_k\}$,

$$\mathbb{C}^{d_1} \xrightarrow{g_1} \mathbb{C}^{d_2} \xrightarrow{g_2} \ldots \xrightarrow{g_{k-1}} \mathbb{C}^{d_k},$$

and Levi subgroup $L(d_1, \ldots, d_k)$ is just the group

$$(A_1, \ldots, A_k) \in \mathrm{GL}_{d_1} \times \ldots \times \mathrm{GL}_{d_k} \quad \text{such that} \quad \det(A_1) \cdot \ldots \cdot \det(A_k) = 1.$$

$L(d_1, \ldots, d_k)$ acts on these spaces of linear maps in an obvious way. The most important among L-orbits are varieties of complexes. To define them, let us fix in addition non-negative integers $m_1, \ldots, m_{k-1}$ such that $m_{i-1} + m_i \leq d_i$ (we set $m_0 = m_k = 0$), and consider the subvariety of all tuples $\{f_1 \ldots, f_{k-1}\}$ as above such that $\operatorname{rk} f_i = m_i$ and $f_{i-1} \circ f_i = 0$ for any i. These tuples form a single L-orbit $\mathcal{O}$ called a variety of complexes. For any tuple $\{f_1, \ldots, f_{k-1}\} \in \mathcal{O}$ consider the tuple $\{f_1^+, \ldots, f_{k-1}^+\} \in \mathfrak{g}_{-1}$, where f_i^+ is a classical "matrix" Moore–Penrose inverse of f_i. It is easy to see that this new tuple is again a complex and, moreover, this complex is a Moore–Penrose inverse (in our latest meaning of this word) of an original complex. In particular, orbits of complexes are Moore–Penrose orbits.

At first glance only few parabolic subgroups are Moore–Penrose. But this is scarcely true. For example, we have the following theorem:

Theorem 8.52 ([T4]) *Any parabolic subgroup in SL_n is Moore–Penrose.*

To explain our interest in Moore–Penrose parabolic subgroups let us recall Conjecture 8.30 from the previous section. Suppose once again that G is a simple connected simply-connected Lie group, P is its parabolic subgroup, and $\mathfrak{p} \subset \mathfrak{g}$ are their Lie algebras. We take any irreducible G-module V. There exists a unique proper P-submodule M_V of V. We have the inclusion $i : M_V \to V$, the projection $\pi : V \to V/M_V$ and the map $R_V : \mathfrak{g} \to \operatorname{End}(V)$ defining the representation. Therefore we have a linear map $\tilde{R}_V : \mathfrak{g} \to \operatorname{Hom}(M_V, V/M_V)$, namely $\tilde{R}_V(x) = \pi \circ R_V(x) \circ i$. Clearly $\mathfrak{p} \subset \operatorname{Ker} \tilde{R}_V$. Finally, we have a linear map $\Psi_V : \mathfrak{g}/\mathfrak{p} \to \operatorname{Hom}(M_V, V/M_V)$. The Conjecture 8.30 states that there exists an algebraic stratification $\mathfrak{g}/\mathfrak{p} = \bigsqcup\limits_{i=1}^{n} X_i$ such that for any V the function $\operatorname{rk} \Psi_V$ is constant along each X_i.

The following theorem shows the connection of this problem with the Moore–Penrose inverse.

Theorem 8.53 ([T4]) *Suppose that a grading $\mathfrak{g} = \bigoplus\limits_{k\in\mathbb{Z}} \mathfrak{g}_k$ is a Moore–Penrose grading in all positive degrees except at most one. Then the Conjecture 8.30 is true for the corresponding parabolic subgroup P.*

Proof. Suppose that G is a connected reductive group with a Lie algebra $\mathfrak{g}$. For any elements $x_1, \ldots, x_r \in \mathfrak{g}$ let $\langle x_1, \ldots, x_r \rangle_{alg}$ denote the minimal algebraic Lie subalgebra of $\mathfrak{g}$ that contains $x_1, \ldots, x_r$ (algebraic subalgebras are the Lie algebras of algebraic subgroups). By a theorem of Richardson [Ri1] $\langle x_1, \ldots, x_r \rangle_{alg}$ is reductive if and only if an orbit of the r-tuple $(x_1, \ldots, x_r)$ in $\mathfrak{g}^r$ is closed with respect to the diagonal action of G. Suppose now that $h_1, \ldots, h_r$ are semi-simple elements of $\mathfrak{g}$. Consider the closed subvariety $\hat{\mathcal{O}} = (\operatorname{Ad}(G)h_1, \ldots, \operatorname{Ad}(G)h_r) \subset \mathfrak{g}^r$. For any closed G-orbit $\mathcal{O} \subset \hat{\mathcal{O}}$ let us denote by $G(\mathcal{O})$ the conjugacy class of the reductive subalgebra $\langle x_1, \ldots, x_r \rangle_{alg}$ for $(x_1, \ldots, x_r) \in \mathcal{O}$.

Lemma 8.54 *There are only finitely many conjugacy classes $G(\mathcal{O})$.*

Proof. We shall use induction on $\dim \mathfrak{g}$. Suppose that the claim of the Lemma is true for all reductive groups H with $\dim H < \dim G$. Let $\mathfrak{z} \subset \mathfrak{g}$ be the center of $\mathfrak{g}$, and $\mathfrak{g}' \subset \mathfrak{g}$ be its commutant. Consider two canonical homomorphisms

$$\mathfrak{g} \xrightarrow{\pi} \mathfrak{g}' \quad \text{and} \quad \mathfrak{g} \xrightarrow{\pi'} \mathfrak{z}.$$

We take any closed G-orbit $\mathcal{O} \subset \hat{\mathcal{O}}$. Let $(x_1, \dots, x_r) \in \mathcal{O}$, $y_i = \pi(x_i)$ for $i = 1, \dots, r$. Then $\langle y_1, \dots, y_r \rangle_{alg} = \pi(\langle x_1, \dots, x_r \rangle_{alg})$ and is therefore reductive. Let us consider two cases.

Suppose first that $\langle y_1, \dots, y_r \rangle_{alg} = \mathfrak{g}'$. Then $\mathfrak{g}'$ is a commutor subgroup of $\langle x_1, \dots, x_r \rangle_{alg}$ and, therefore, $\langle x_1, \dots, x_r \rangle_{alg} = \langle \pi'(h_1), \dots, \pi'(h_r) \rangle \oplus \mathfrak{g}'$. In this case we get one conjugacy class.

Suppose next that $\langle y_1, \dots, y_r \rangle_{alg} \neq \mathfrak{g}'$. Then $\langle y_1, \dots, y_r \rangle_{alg}$ is contained in some maximal reductive Lie subalgebra of $\mathfrak{g}'$. It is well-known (and not difficult to prove) that in a semisimple Lie algebra there are only finitely many conjugacy classes of maximal reductive subalgebras. Let $\mathfrak{h}'$ be one of them, $\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{h}' \subset \mathfrak{g}$. Let H be a corresponding reductive subgroup of G. It is sufficient to prove that, for any closed G-orbit $\mathcal{O}$ of $\hat{\mathcal{O}}$ that meets $\mathfrak{h}^r$, there are only finitely many possibilities for $G(\mathcal{O})$. It easily follows from Richardson's Lemma [Ri] that for any i the intersection $\mathrm{Ad}(G)h_i \cap \mathfrak{h}$ is a union of finitely many closed H-orbits, say $\mathrm{Ad}(H)h_i^1, \dots, \mathrm{Ad}(H)h_i^{s_i}$. It remains to prove that if, for some r-tuple $(x_1, \dots, x_r) \in \mathrm{Ad}(H)h_1^{k_1} \times \dots \mathrm{Ad}(H)h_r^{k_r}$, the corresponding subalgebra $\langle x_1, \dots, x_r \rangle_{alg}$ is reductive then there are only finitely many possibilities for its conjugacy class. But this is precisely the claim of the lemma for the group H, which is true by the induction hypothesis. □

Suppose that $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$.

Lemma 8.55 *If an r-tuple $(x_1, \dots, x_r)$ belongs to $(i\mathfrak{k})^r$, then its G-orbit is closed in $\mathfrak{g}^r$.*

Proof. Indeed, let B be a non-degenerate ad-invariant scalar product on $\mathfrak{g}$, which is negative-definite on $\mathfrak{k}$. Let $H(x) = -B(\overline{x}, x)$ be a positive-definite $\mathfrak{k}$-invariant Hermitian quadratic form on $\mathfrak{g}$, where the complex conjugation is taken with respect to $\mathfrak{k}$. Let H^r be a corresponding Hermitian quadratic form on $\mathfrak{g}^r$. More precisely, $H^r(x_1, \dots, x_r) = H(x_1) + \dots + H(x_r)$. By a Kempf–Ness criterion [PV], in order to prove that the G-orbit of $(x_1, \dots, x_r)$ is closed, it is sufficient to prove that the real function $H^r(\cdot)$ has a critical point on this orbit. Let us show that $(x_1, \dots, x_r)$ is this critical point. Indeed, for any $g \in \mathfrak{g}$

$$-B(\overline{x}_1, [g, x_1]) - \dots - B(\overline{x}_r, [g, x_r]) = B(x_1, [g, x_1]) + \dots + B(x_r, [g, x_r]) = 0.$$

□

Now let G be a simple simply-connected Lie group, and let $\mathfrak{g}$ be its Lie algebra with a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$. Let r be a maximal integer such that

$\mathfrak{g}_r \neq 0$. We denote the non-positive part of the grading $\underset{k\le 0}{\oplus}\mathfrak{g}_k$ by $\mathfrak{p}$. Let $P \subset G$ be a parabolic subgroup with the Lie algebra $\mathfrak{p}$. We shall identify $\mathfrak{g}/\mathfrak{p}$ with $\underset{k>0}{\oplus}\mathfrak{g}_k$. Let $L \subset G$ be a connected reductive subgroup with Lie algebra $\mathfrak{g}_0$. Let V be an irreducible G-module. If we choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}_0$ then the grading of $\mathfrak{g}$ originates from some $\mathbb{Z}$-grading on $\mathfrak{t}^\vee$. Therefore there exists a $\mathbb{Z}$-grading $V = \underset{k\in\mathbb{Z}}{\oplus} V_k$ such that $\mathfrak{g}_i V_j \subset V_{i+j}$. Let R be a maximal integer such that $V_R \neq 0$. It is easy to see that $M_V = \underset{k<R}{\oplus} V_k$ (notice that V_R is an irreducible L-module).

Now we can finish the proof of Theorem 8.53. It is sufficient to prove that there exists a finite set of points $\{x_1, \ldots, x_N\} \subset \mathfrak{g}/\mathfrak{p}$ such that for any $x \in \mathfrak{g}/\mathfrak{p}$ and for any V we have $\mathrm{rk}\,\Psi_V(x) = \mathrm{rk}\,\Psi_V(x_i)$ for some i. Recall that L has finitely many orbits on each $\mathfrak{g}_k$; see Theorem 2.6. We pick some L-orbit $\mathcal{O}_i$ in each $\mathfrak{g}_i$. Then it is sufficient to find a finite set of points as above only for points $x \in \mathfrak{g}/\mathfrak{p}$ of a form $x = x_1 + \ldots + x_r$, where $x_i \in \mathcal{O}_i$. For any orbit $\mathcal{O}_i$ let $\mathcal{H}_i$ denote the set of all possible homogeneous characteristics of all elements from $\mathcal{O}_i$. Clearly $\mathcal{H}_i$ is a closed $\mathrm{Ad}(L)$-orbit. Let $\hat{\mathcal{O}} = \mathcal{H}_1 \times \ldots \times \mathcal{H}_r \subset \mathfrak{g}_0^r$. Then by Lemma 8.54 the set of conjugacy classes of subgroups $G(\mathcal{O})$ for closed L-orbits $\mathcal{O}$ in $\hat{\mathcal{O}}$ is finite. Let us show that for any r-tuple $(x_1, \ldots, x_r) \in \mathcal{O}_1 \times \ldots \times \mathcal{O}_r$ there exists an r-tuple $(h_1, \ldots, h_r) \in \hat{\mathcal{O}}$ such that h_i is a homogeneous characteristic of x_i and an L-orbit $\mathrm{Ad}(L)(h_1, \ldots, h_r)$ is closed. Indeed, after simultaneous conjugation of elements x_i by some element $g \in L$ we may suppose that any x_i has a homogeneous characteristic $h_i \in i\mathfrak{k}_0$ (in all degrees except at most one, no conjugation is needed because of Moore–Penrose property; for one degree this is obvious). Then by Lemma 8.55 an orbit $\mathrm{Ad}(L)(h_1, \ldots, h_r)$ is closed. Since all functions $\mathrm{rk}\,\Psi_V$ are L-invariant, we may restrict ourselves to the points $x = \sum_i x_i \in \mathfrak{g}/\mathfrak{p}$ such that $x_i \in \mathcal{O}_i$, any x_i has a homogeneous characteristic $h_i \in i\mathfrak{k}_0$, and a conjugacy class of $\langle h_1, \ldots, h_r\rangle_{alg}$ is fixed. We claim that any function $\mathrm{rk}\,\Psi_V$ is constant along the set of these points. Moreover, we shall prove that

$$\mathrm{rk}\,\Psi_V(x) = \dim V_R - \dim V_R^{\langle h_1,\ldots,h_r\rangle_{alg}}. \tag{8.18}$$

Indeed,

$$\mathrm{rk}\,\Psi_V(x) = \dim \sum_i \mathrm{Im}\left(\mathrm{ad}(x_i)|_{V_{R-i}}\right).$$

Clearly V_R is $\mathrm{ad}(h_i)$-invariant and is killed by $\mathrm{ad}(e_i)$, therefore from the $\mathfrak{sl}_2$-theory we find that $V_R = \underset{k\ge 0}{\oplus} V_R^k$, where $\mathrm{ad}(h_i)|_{V_R^k} = k \cdot \mathrm{Id}$. Moreover,

$$\mathrm{Im}\left(\mathrm{ad}(x_i)|_{V_{R-i}}\right) = \underset{k>0}{\oplus} V_R^k.$$

Let H be a contravariant Hermitian form on V_R with respect to the compact form $\mathfrak{k}_0$ of $\mathfrak{g}_0$. Since $h_i \in i\mathfrak{k}_0$ and H is a contravariant form we find that $\underset{k>0}{\oplus} V_R^k = (V_R^0)^\perp$. Therefore

$$\sum_i \operatorname{Im}\left(\operatorname{ad}(x_i)|_{V_{R-i}}\right) = \left(\cap_i V_R^{h_i}\right)^{\perp} = \left(V_R^{\langle h_1,\ldots,h_r\rangle_{alg}}\right)^{\perp}.$$

The formula (8.18) follows. □

For example, combining Theorem 8.53 and Theorem 8.52 we get the following corollary:

Corollary 8.56 *Conjecture 8.30 is true for any parabolic subgroup in* SL_n.

Though the conditions of Theorem 8.53 are not always satisfied, it seems that one can prove Conjecture 8.30 for any simple group using the same ideas.

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