Introduction

Jean-Dominique Deuschel[1] and Andreas Greven[2]

[1] Fakultät II - Mathematik und Naturwissenschaften, Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, D-10623 Berlin
Tel. (0049 30) 314-25193, Fax: (0049 30) 314-21695
deuschel@math.tu-berlin.de

[2] Mathematisches Institut, Bismarckstr. 1 $\frac{1}{2}$, D-91054 Erlangen,
Tel. (0049 9131) 85-22454, Fax (0049 9131) 85-26214

This volume collects original work and reviews on work in the field of probability which took place within the framework of the DFG-Schwerpunkt: Interacting stochastic systems of high complexity. This research network started in May 1997 and was funded till May 2003. In this network between 20–30 (depending on the 2-year periods of grant renewal) groups from probability, statistical physics and mathematical statistics within Germany were active. An extensive international collaboration was an essential part of this network and here particularly intense contacts were built up between the DFG-Schwerpunkt and EURANDOM, a European institute for research in stochastics, which was founded in 1997. This partnership reached from joint workshops and colloquia to collaborations on specific projects.

The key scientific idea which was behind the research network was to explore and develop the connections between research in infinite dimensional stochastic analysis, statistical physics (Gibbs measures, random media), spatial population models from mathematical biology, complex models of financial markets or of stochastic models with interacting components in other sciences as for example in climatology. At the time the proposal was written such connections were about to emerge and become part of the proposals, other connections developed as the work in the network proceeded.

An example of these newly emerged connections is the appearance of Neveu's branching process in the theory of spin glasses as key model in the area of random media. In this particular area the connections planned in the proposal were more in the line of the application of renormalization methods and ideas common in the statistical physics community in the field of stochastic population models, a topic which also flourished during the time of the network. Other examples are the applications of the theory of random dynamical systems, part of the original proposal, which shifted from the development of the theoretical framework and questions of evolutionary biology to treating questions of the stability of ships.

The topics covered by the research in the "Schwerpunkt" were grouped in the following general themes:

– Interacting particle systems in statistical physics,
– random media and homogenization,
– branching models in population genetics,
– stochastic methods for the analysis of financial derivatives,
– consistency and efficiency of Monte Carlo algorithm,
– stochastic analysis: Dirichlet forms, partial stochastic differential equations and theory of large deviations.

This original plan for and the original structure of the network underwent rapidly changes as the scientific work proceeded. The theory of random Schrödinger operators became part of the random media section of the network, the theory of spin glasses led to a closer connection between random media and interacting particle systems in statistical physics, the theory of Dirichlet forms and large deviation merged with both these fields, besides financial markets and Monte Carlo algorithms new applications came into sight like sequence alignment or modelling of the climate using stochastic analysis and random dynamical systems. All these developments benefitted from the regular yearly meetings of the network, the workshops and summer schools.

In line with this development the "Schwerpunkt" participated in various joint efforts with other mathematical areas and with other disciplines in the sciences. Highlights were a symposium on entropy jointly organized by three mathematical "Schwerpunkte" together with the Max-Planck-Institute for the Physics of Complex Systems (Dresden) and workshops on mathematical biology jointly with the Max-Planck-Institute for Evolutionary Anthropology and the Max-Planck Institute for Mathematics in the Sciences, both in Leipzig. Over the whole period efforts were made to co-sponsor a series of workshops on climatology. In addition to that a quite successful series of workshops on statistical physics and on stochastic analysis (both jointly with EURANDOM) were organized supporting the collaboration between these institutions, within the network and with the international community of probabilists.

The contribution in the volume will clearly show the extensive international collaboration taking place in order to achieve the scientific progress made as the project was running. Through these activities it was possible to provide for a substantial number of young scientists the possibility to participate in workshops and joint projects with internationally leading scientists. This way the goal was reached to create in the field of stochastics in Germany a group of young researchers which are internationally competitive.

We describe next in more detail what happens in the different chapters of this volume and also give some references to work in the network which is not described in this volume.

The first chapter treats stochastic models from statistical physics and the closely related topic of processes in random media. The contributions to this field concentrate on two main topics. The first topic evolves around the Gibbs measure while the second had its focus on random Schroedinger operators, the parabolic Anderson model and spectra of random matrices. This chapter illustrates how modern stochastic methods can be very successfully applied in a very broad range of models of statistical physics and random media. The applications range from quantum electrodynamics, to quantum field theory and from magneto-hydrodynamics, to quantum anharmonic crystals and random Schrödinger operators.

One of the central themes is *Gibbs distributions* and their properties. In the formalism of thermodynamics, Gibbs distributions describe the equilibrium states with respect to some given interaction. Some of the challenging questions deal with existence and uniqueness of the corresponding states. In the case of non uniqueness we have a phase transition, this corresponds to a strong interaction or the low temperature regime. In this regime the analysis is very delicate and various technique have been developed to study the properties in this regime and to establish the phase transition.

One technique relies on long range interactions of the mean field type, so called Kac models characterized by a parameter describing the range of interaction, see the paper of Bovier and Külske. For lattice dimensions greater than three, one can show the existence of ferromagnetic order Gibbs states for a range of temperatures that is uniform in the Kac parameter. The proof is based on a a coarse-graining procedure or renormalization argument, once the long-range random field model has been reduced to an effective short range contour model.This analysis sheds light into more realistic models with local interactions.

Another interesting application of Gibbs measures deals with quantum anharmonic crystals. The idea is to represent the crystal with the help of Euclidean Gibbs measures on loop lattice, and is developed in the work of Albeverio and Roeckner. The main result here is the existence and uniqueness of the Gibbs measure for sufficiently small values of the mass. The technique is based on an integration by parts formula, which gives an alternative description of Gibbs measures. In particular the Dobrushin uniqueness criterion, exponential decay of correlations are derived.

Gibbs measures on the space of continuous trajectories relative to Brownian motions as a model of quantum electrodynamic is the objective of Betz, Lörinczi and Spohn. Here the existence and path properties of such measures are investigated. In particular the existence of phase transition and a functional central limit theorem are proved using cluster expansion techniques.

Another model of quantum field is modelled by a Markov jump process with minimal jump rates, the so-called Bell process in the paper of Georgii and Tumulka. Here global existence is obtained for manifolds with boundaries.

The second main theme of the first section deals with *random media* which appears in the context of spatially inhomogeneous systems. One of the standard method used in studying spatially disordered system is what is called homogenisation. This means that under rescaling the system with random coefficients converges to a PDE with deterministic coefficients obtained by "properly averaging" random coefficients. However in many important situations, random systems cannot be adequately described by a deterministic approach. This is for instance the case for localization effects for the electron transport in disordered media given by the Schrödinger operators with random potentials. They are used to model quantum aspects of disorder electronic systems like unordered alloys, amorphous solids or liquids. In this range of questions belongs also the analysis of spectral properties of random matrices, a field in rapid development.

The main question in this setting concerns the phenomena of *localization* and *delocalization*. The localized phase corresponds to a pure point spectrum with only bounds states prohibiting transport of current, whereas the delocalized phase with absolutely continuous spectrum exhibits scattering states with electron transport. While periodic potentials have absolutely continuous spectrum, disordered systems have a tendency to localize states, the so-called Anderson localization.

Although situations with localized states are well known, so far delocalization has not been settled for genuine, i.e. translation invariant, random models. The object the work of Böcker, Kirsch and Stollmann is to investigate the spectrum and the density of states in the context of nonstationary random potentials with a delocalized phase. In particular different models with sparse random potentials and random surface models are presented.

While the previous paper deals with a discrete model, the paper of Leschke, Müller and Warzel is concerned with the spectrum of continuous models for amorphous solids. In this setting, translation invariance is assumed, and the paper analyzes the density of states and corresponding Lifshitz tail in localized regime for both generic Gaussian and Poissonian random potentials.

The paper of Gärtner and König studies the parabolic Anderson model, that is with the Cauchy problem for the heat equation with random potentials on the lattice. Unlike the two previous papers which dealt only with the spectrum, this paper focuses on the intermittency of the solutions. Intermittency takes place in the localized regime, when one expects that the solution of the system develops pronounced spatial structures on islands located far from each other. The aim of the paper is to give a mathematical description of this phenomena, which differs depending on the quenched setting with given "frozen" random media or the annealed setting, averaged over the media. In particular the paper illustrates the four different universal classes of possible asymptotics. Both time independent and time dependent random potentials such as the one generated by a Poisson field are considered.

The last paper of this section deals with the topic of random matrices. In this area quite some development took place while the research network was active and the group around F. Götze started working in this field. In this contribution Götze and Merkl review results on the limit distributions and the rate of convergence for the spectra of certain ensembles of random matrices. The ensembles include the Wigner anf general unitary ensemble (GUE), the Laguerre ensembles and the circular unitary ensemble (CUE). Connections to quantuum field theory and distribution of the zeros of the zeta function is also illustrated.

A central aim of the research in the network was besides the application of stochastic analysis to further develop the foundations of stochastic analysis. Some of this work developed in close connection with certain applications as for example the work of Alberverio and Röckner found in the chapter on statistical physics models some of the work however focussed on structural aspects, which is collected in section 3, which we discuss next, except the contribution concerned with the concept of random dynamical systems, this will be discussed in context of the last chapter.

The paper of Sturm investigates the space of probability measures over a Euclidean or Riemannian space. One of the objective introduce nonlinear diffusion equations describing the flow of the distribution of interacting stochastic particles. These diffusions can be characterized as the gradient flow of an appropriate free energy functional. Moreover the barycenter map is shown to be contractive in metric spaces with nonpositive curvarture. This allows to generalize the classical law of large number numbers to this abstract setting. Also contraction properties for the heat semigroup on a Riemanian manifold with its relations to bounds on the Ricci curvature are discussed.

In the work of Deuschel, a wide class the qualitative properties of interactive diffusion processes in time and space are studied. The aim of the paper is to present a technique, the random walk representation which allow to express the space-time covariances in terms of the Green function of a random walk in random environment. In particular existence of invariant measures, convergence rate to equilibrium and various aging phenomena can be derived for various models which in the language of mathematical physics are described as massless, see also the references to effective interface models in this paper.

The work by Klenke and Mörters is concerned with the intersection local time of two independent Brownian paths in $\mathbb{R}^d$ for $d = 2$ and 3. This intersection set has been studied for typical points by its Hausdorf dimension and Hausdorf gauge function. The work here is concerned with the question whether there are thick or thin points in S with local dimension lower or higher than typically expected. The results disprove some assertions made by the multi-fractal formalism from statistical physics.

Besides the application of stochastic analysis in models from statistical physics a central goal of the network was to apply stochastic methods in models from mathematical biology. The main focus was on evolutionary questions in a broad sense. The chapter on population models exhibits very well the spectrum of tasks mathematicians face modelling and analysing the evolution of populations. This spectrum reaches from the effort to model a biological phenomenon appropriately in mathematical terms accessible to statistical methods, the development of the statistical theory up to the final task the construction of a transparent mathematical theory for the qualitative behavior of stochastic population models.

The building block of population models are stochastic systems in which the reproduction and the migration of individuals in geographic space can be described. A paradigmatic process is the so-called branching random walk and its relatives super random walk and super Brownian motion which arise by considering continuum limits in the mass, i.e. replace individuals by smaller and smaller masses with at the same time increasing number of individuals, and rapid reproduction, respectively continuum limits in space replacing for example the lattice Z^d as geographic space by $\mathbb{R}^d$.

The research on the qualitative properties of such models has been focussed on a number of directions. First try to understand not only the size and distribution of the population at given times, but try to understand the complete history of such individuals and their ancestors. This means besides location and type of individuals record their whole *genealogy* and describe qualitative properties of this object. The second direction has been to find the classes of models which lead when considered on large space and time scales to the same qualitative behavior. The key word here is *universality.* This is a theme which connects the theory of these biologically motivated systems with the corresponding attempts in statistical physics and in particular systems in inhomogeneous media. The third direction of great importance in applications is to incorporate in population models mechanisms representing mutation, recombination and selection which are responsible for many effects observed in nature. This leads to many new and difficult problems and in particular requires a close connection to the methods of infinite dimensional analyses in order to be able to construct the appropriate mathematical models and to develop methods to tackle these highly nonlinear systems.

The analysis of population models evolves now on three levels. On the mathematical level where properties of these models are derived on the basis of rigorous mathematical reasoning, on the level of a statistical theory which allows the choice of parameters in those models on the basis of data and finally the level on which the probabilist tries to adapt models he can handle both mathematically and computationally to the situations and the questions raised by the biologist. The latter process of balancing mathematical tractability with the search for realistic models is a very delicate task due to the complexity of the biological situation with often overwhelming amounts of data versus the mathematical restriction as to which nonlinear effects can still be rigorously analysed.

The level of mathematical theory is here represented by two articles, the contribution of Birkner, Geiger and Kersting and the contribution of Greven.

The first one shows how insight on branching populations can be gained by focussing on the whole genealogical tree and how new results can be obtained this way even on questions one can raise on the basis of population size alone. More examples of such results can be found in the references in this chapter. The contribution by Greven deals with a mathematically rigorous approach to the problem of understanding the universality classes of neutral population models and desribes the progress which has been made in passing from single type to multitype population models. On the level of modelling the contribution by Fleiss ner, Metzler, von Haeseler and Wakolbinger describes the progress which has been made in understanding the procedure of sequence alignment based on stochastic models.

Besides this work in population models documented here research on population models in random environment was pursued by Greven, Klenke and Wakolbinger in a series of papers ([GKW99, GKW01, GKW02]) and by Fleischmann and his international collaborators ([DFMPX]). The main focus of these papers is the rigorous construction of catalytic branching processes and the determination of its longtime behavior in its dependence on the dimension of the geographic space in which the population lives. This work gained substantially by ideas developing in the circle working on random media problems motivated by statistical physics. Here the fundaments have been laid to proceed to models where medium and process interact as for example for the process called mutually catalytic branching ([CDG]).

In addition to the themes mentioned the research network was active in a broad field of applications reaching from climatology, stability of ships, stochastics in financial markets to the study of algorithms. The last chapter of this book tries to give an impression of this work.

A concept with a broad range of applications is the concept of a random dynamical system. This concept has been extensively studied by the group around Arnold in Bremen and Imkeller and Scheutzow in Berlin. In this volume this work is represented on a more theoretical level by contribution of Chueshov, Scheutzow and Schmalfuß, by Hermann, Imkeller and Pavlyukevich and a contribution on the applied side by Arnold, Chueshov and Ochs. The first contribution focusses on continuity properties of the inertial manifold in stochastic retarded equations in the regime of small delay times and in the regime of small noise. Another topic relevant in the sciences and falling in the framework of random dynamical systems is what is called stochastic resonance of a random dynamical system. Here the article by Herrmann, Imkeller and Pavlyukevich contributes by analysing a one-dimensional diffusion driven by a Brownian motion with variance ε and a double well potential providing the drift. Here the theory of large deviation and couplings with Markov chains allowing for metastable states play a central role to study the qualitative behavior of this system.

The contribution of Arnold, Chueshov and Ochs uses the concept and the theory of random dynamical systems to analyse in a case study a nonlinear model under the point of view of stochastic stability. The concrete case chosen here is the motion of a ship. The model arises by describing the sea as a stationary random field and formulating the equations of motions from which then a simpler nonlinear random differential equation is extracted for the roll motion. This random dynamical system is then analysed.

A key question in financial markets is how to model and deal with the possibility of a crash. In the contribution of Korn and Menkens the concept of the worst-case portfolio optimization is described and analysed under the point of view of the expected utility. The analysis is focussed on model assumptions relevant for an insurer.

The contribution of Neininger and Rüschendorf on algorithms of divide and conquer type has a quite different flavour. In the center of this work is analysis of additive recursive sequences for which the contraction method is exploited, which is a useful tool for the analysis of stochastic algorithms. The technique allows to derive quite general limit theorems.

Further work not covered here concerns the application of stochastic analysis in climatology, we refer the reader to the proceedings book of Imkeller and von Storch ([IvS01]). The stochastics of financial markets was investigated by the group around Föllmer using stochastic analysis. For example interacting Markov chains, a theme, which is present in most of the other topics of the Schwerpunkt, are applied to the theory of the stochastics of financial markets by Föllmer and Horst in ([FH01]).

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http://www.springer.com/978-3-540-23033-5

Interacting Stochastic Systems
Deuschel, J.-D.; Greven, A. (Eds.)
2005, XII, 450 p., Hardcover
ISBN: 978-3-540-23033-5