1

Introduction

1.1 Signals and Signal Processing

Wavelet analysis had its origins in the mid-eighties. From the very beginning it was driven by application needs: The desire to analyze seismic signals more sensitively than with Fourier techniques led to the first appearance of the continuous wavelet transform formula. In parallel it turned out that the new technique could be applied successfully to certain problems in theoretical physics as well as in pure mathematics. For one of the earliest collections of research and survey papers documenting the state of the art the reader is referred to [6].

It soon turned out that wavelet analysis successfully could be applied to many types of signal processing problems: In *signal analysis* the detection of discontinuities or irregularities was tackled with wavelets. The analysis of medical signals like electrocardiograms of the heart is one of the first reported examples of discontinuity detection (see [6]). For more applications, like the analysis of sensor signals in robotics, cf. sect. 2.3.1.

Signal compression is another impressive example of wavelet applications. JPEG 2000, the present version of the international standard on still image compression is based on wavelet techniques (see, e.g., [36]).

Wavelet applications both in signal analysis and signal compression shall be described in more detail in later sections. This chapter serves as a brief introduction to the main features of the wavelet transform by comparing wavelet transform with Fourier transform, the standard tool of signal analysis. For that purpose we shall work out the common aspects of wavelet and Fourier transforms and point out the main differences. For understanding the following section, the knowledge of the Fourier transform is not a necessary prerequisite. On the other hand, of course, it would be useful, if the reader already had some experience with applications of the Fourier transform. Basic facts about the Fourier transform are collected in the appendix, sections 5.2 and 5.3.

Mathematical symbols, used throughout this book, are explained upon their first appearance. They are collected in sect. 5.1 of the appendix.

1.2 Local Analysis

In this section we will deal with signals which may be represented by a function $f(t)$ depending on time t. We shall assume that t is a continuously varying parameter; thus $f(t)$ is called a "continuous-time signal".

We shall try to transform $f(t)$ into a representation, which encorporates the desired information about the signal as compactly as possible. The Fourier transform (cf. sections 1.2.2 and 1.2.3) supplies information about the contribution of certain frequencies to the signal, the wavelet transform (cf. sect. 1.2.4) indicates whether details of a certain size are present in a signal and quantifies their respective contribution. Both transforms are called "local" if they not only globally measure frequencies and detail sizes, respectively, but also indicate *where* they are located in the signal $f(t)$.

There are many applications for the kind of signal information described above – we explicitly mention signal classification and data compression. These applications are described in more detail later, in the subsections below we indicate *how* frequencies and detail sizes may be measured. Furthermore, we will work out the aspects which are common to both transforms and illuminate the respective differences. The transform results shall be visualized and we will give an example which serves as an illustration for the above-mentioned compactness of the respective signal representations.

The purpose of this chapter is to introduce the ideas underlying Fourier and wavelet transforms, respectively. For more - in particular for mathematical - details the reader is referred to the following chapters.

1.2.1 Transforms

All transforms of the signal $f(t)$ described in this section share a common computation principle: The signal is multiplied with a certain "analysis function" and integrated about the time domain. In a symbolic notation the prescription for performing a transform reads

$$f(t) \overset{transform}{\longrightarrow} \int_{-\infty}^{+\infty} f(u)\overline{g(u)}\,du \tag{1.1}$$

The "analysis function" $g(u)$ characterizes the chosen transform. In general it may be a complex function, the overline denotes the complex conjugate entity. $g(u)$ in a certain way depends on the parameters, i.e. frequencies or detail sizes, to be measured (see below). Thus, by the computation principle given above the transformed entity will depend on these parameters. In other words: the transformed entity again will be a function. These functions we shall denote with “transform” or “transformed signal”.

Another common aspect of all transforms discussed in this section is invertibility: From the transformed signal the original signal $f(t)$ may be reconstructed. This is essential for understanding the comparison experiment carried out in sect. 1.2.6.

1.2.2 Fourier Transform

The parameter relevant for the Fourier transform is the circular frequency ω, the analysis function reads $g_\omega(u) = e^{ju\omega}$. Thus the transformed signal is a function depending on ω and it is denoted with $\hat{f}(\omega)$:

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(u)\overline{g_\omega(u)}\, du \tag{1.2}$$

Figure 1.1 illustrates the above computation recipe by plotting both curves required for computing $\hat{f}(\pi)$. The signal is shown as a solid curve, the real part of the analysis function $g_\pi(u) = e^{ju\pi}$ is dashed. Obviously, it is an harmonic oscillation with circular frequency $\omega = \pi$.

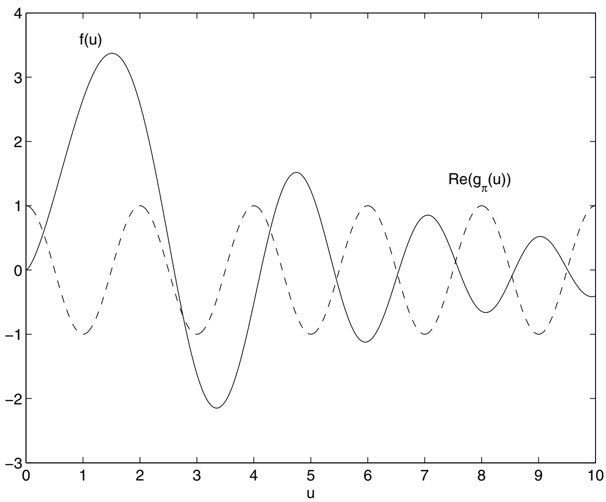

Fig. 1.1. Fourier transform: signal and analysis function

Why does $\hat{f}(\pi)$ measure the appearance of $\omega = \pi$ in the signal? The qualitative argument is as follows: If in some time interval the signal oscillates with circular frequency $\omega = \pi$, the signal and the analysis function have a constant mutual phase shift in this interval and therefore provide a nonzero contribution to $\hat{f}(\pi)$.

Yet there is no possibility to *localize* the appearance of the circular frequency: If (the absolute value of) $\hat{f}(\pi)$ is "large", we only know that the signal contains the circular frequency π, but we do not know *where* it appears, since the analysis function extends over the whole real axis. The only label parameterizing the analysis function is circular frequency.

1.2.3 Short Time Fourier Transform (STFT)

This transform sometimes also is called "Windowed Fourier Transform" (WFT). The STFT looks for the appearance of the circular frequency ω *at a certain time* t. The corresponding analysis function reads: $g_{(\omega,t)}(u) = e^{ju\omega}w(u-t)$. Here $w(u)$ is a "window function", usually centered about the origin (for an example see below). In the expression $w(u-t)$ this window is shifted to the desired time t.

Now the transformed signal depends on ω *and* t! Since it also will depend on the shape of the window function, it is denoted with $\hat{f}_w(\omega,t)$:

$$\hat{f}_w(\omega,t) = \int_{-\infty}^{+\infty} f(u)\overline{g_{(\omega,t)}(u)}\,du \tag{1.3}$$

For a box window w of width 2, centered symmetrically about 0, $\omega = \pi$ and $t = 8$, the computation principle is illustrated in Fig. 1.2. Again the dashed curve shows the real part of the analysis function $g_{(\pi,8)}(u)$; it is obviously now localized at $t = 8$, since $w(u-8)$ denotes the box window, shifted by 8 units to the right.

In general, the analysis function will be localized at the respective "analysis time" t . Therefore the transform provides not just global information about the appearance of a certain circular frequency, but in addition the time of this appearance.

The procedure described so far has a disadvantage: If in the above example one is interested in small details of the signal around $t = 8$, the corresponding frequency of the analysis function must be increased. As an example Fig. 1.2 is redrawn for $\omega = 6\pi$ in Fig. 1.3.

Obviously the window width is constant and non-adaptive: If one is interested in very tiny signal details (high frequencies) in only a small neighborhood of $t = 8$, eventually signal parts, which actually are "not of interest", also will be analyzed. Zooming into small details - analogously to a microscope - is not supported.

1.2.4 Wavelet Transform

The wavelet transform has such a zooming property. In contrast to the Fourier transform, the wavelet transform does not look for circular frequencies but

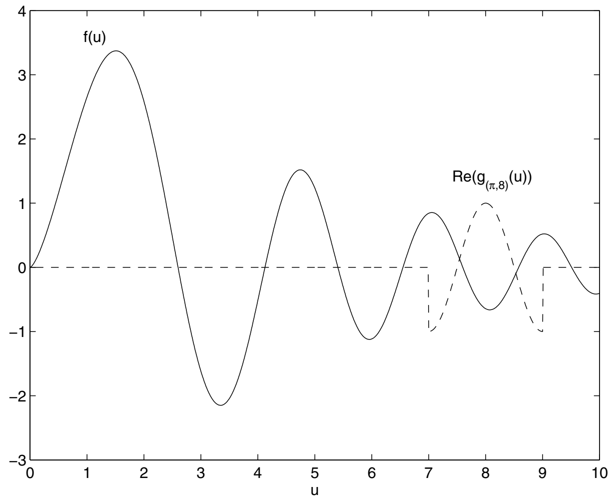

Fig. 1.2. STFT: signal and analysis function for $\omega = \pi$

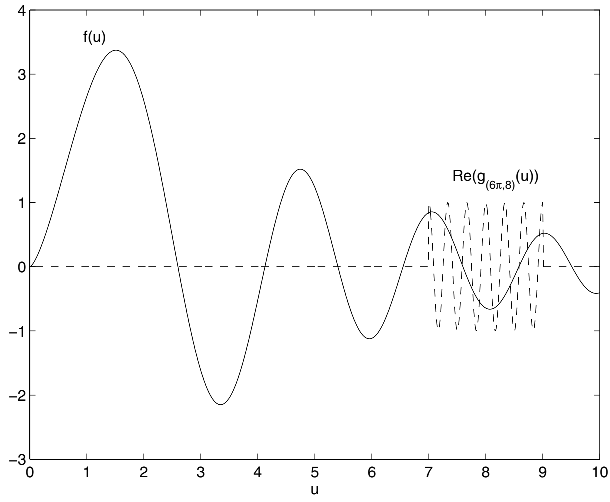

Fig. 1.3. STFT: signal and analysis function for $\omega = 6\pi$

rather for detail sizes a at a certain time t. Instead of detail sizes we also will speak of "scale factors", both notions will be used equivalently. As mentioned already, high frequencies correspond to small details and vice versa, thus, when comparing wavelet with Fourier transforms we have to take into account that frequencies and detail sizes are inversely proportional to each other: There exists a constant β such that

$$a = \frac{\beta}{\omega}. \tag{1.4}$$

We shall now briefly indicate, how the wavelet transform is computed.

Consider a (real or complex) analysis function g, oscillating around the u-axis (mathematically: $\int\limits_{-\infty}^{+\infty} g(u)\, du = 0$) and decreasing rapidly for $u \to \pm\infty$. Such a function is called a "wavelet". In eq. 1.4, relating scale factors with frequencies, the constant β depends on g.

Starting from g consider the following family of functions: $g_{(a,t)}(u) = \frac{1}{\sqrt{a}} g\left(\frac{u-t}{a}\right)$. The members of this family are generated from g by shifting the function to t followed by shrinking ($a < 1$) or dilating ($a > 1$) the width of the function. The wavelet transform now reads:

$$L_g f(a,t) = \int\limits_{-\infty}^{+\infty} f(u)\overline{g_{(a,t)}(u)}\, du \tag{1.5}$$

For the "Haar-wavelet"

$$g(u) = \begin{cases} 1 & 0 \le u < \frac{1}{2} \\ -1 & \frac{1}{2} \le u < 1 \\ 0 & \text{else} \end{cases}$$

the computation of $L_g f(a,t)$ with $a = \frac{1}{2}$ and $t = 8$ is illustrated in Fig. 1.4.

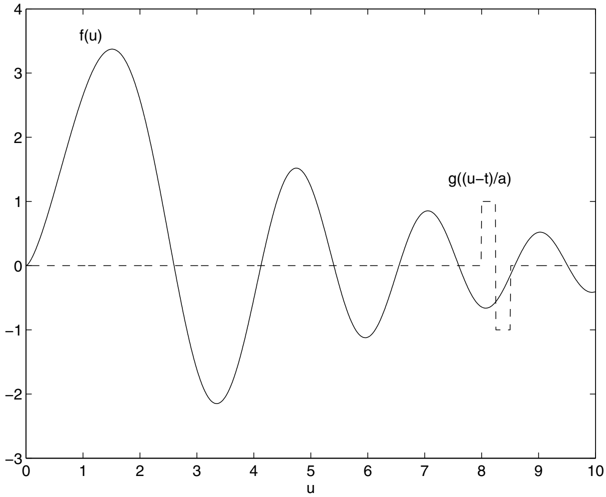

Fig. 1.4. Wavelet transform: signal and analysis function for $a = \frac{1}{2}$

The reader may note that the Haar-wavelet, originally situated in the interval $[0, 1)$ now has been shifted to the right by 8 units and its width has shrunk by the factor $\frac{1}{2}$, corresponding to the chosen values of t and a. For $a = \frac{1}{4}$ and $t = 8$ we obtain Fig. 1.5.

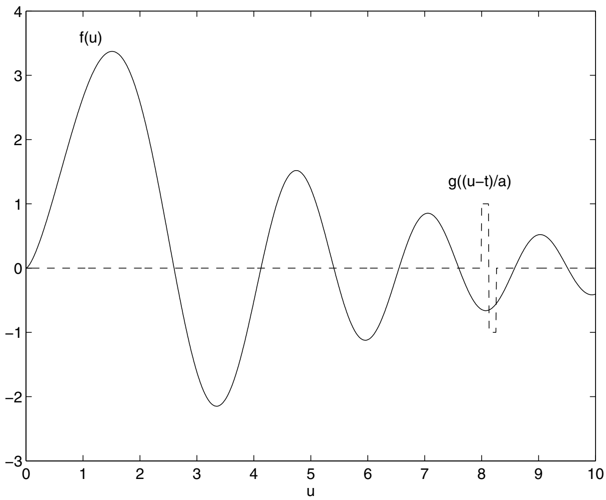

Fig. 1.5. Wavelet transform: signal and analysis function for $a = \frac{1}{4}$

Compare Figs. 1.4 and 1.5 with Figs. 1.2 and 1.3 and note that the wavelet transform shows the desired zooming property in contrast to the STFT: When searching for smaller and smaller details (higher and higher frequencies) with the wavelet transform, the corresponding analysis function is oscillating faster *and* is contracted.

1.2.5 Visualization

Both the STFT $\hat{f}_w(\omega, t)$ and the wavelet transform $L_g f(a, t)$ are functions depending on two variables. A suitable visualization of these functions is of essential importance in signal analysis. A wide-spread graphical representation of two-dimensional functions is the use of contour lines. In signal analysis one usually prefers the visualization of the absolute values of the respective transforms by gray values. High values are coded with bright, low values with dark gray values.

Figure 1.6 shows such a visualization for the STFT (above) and the wavelet transform (below). As a signal in both cases the “chirp” $f(t) = \sin(t^2)$ has been used.

The chirp is an harmonic oscillation $\sin(\omega t)$, whose circular frequency increases with t: $\omega = t$. The linear increase of frequency is clearly visible with the STFT (see the upper part of Fig. 1.6).

Since (cf. eq. 1.4) detail size a and frequency ω are inversely proportional with respect to each other, for the wavelet transform one would expect a

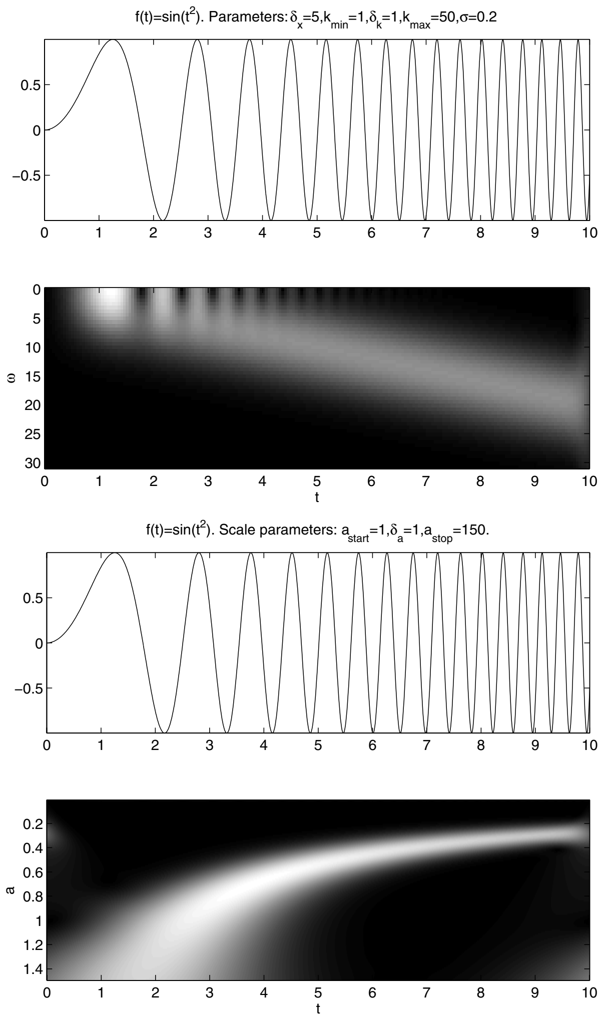

Fig. 1.6. Above: STFT of the chirp-signal $f(t) = \sin(t^2)$. Below: Wavelet transform of the chirp-signal $f(t) = \sin(t^2)$.

behavior corresponding to a hyperbola (i.e. proportional to $\frac{1}{t}$). The lower part of Fig. 1.6 shows exactly this behavior.

Since the STFT depends on t and ω, the gray value coding of the STFT has been performed on the t-ω-plane. This plane is also called “phase plane”, the corresponding gray value coding “phase space representation” of the STFT. Analogously the t-a-plane is called “scale plane” and the corresponding gray value coding of the wavelet transform “scale space representation” of the wavelet transform.

1.2.6 Fourier vs. Wavelet Transform - A Comparison Experiment

In a certain sense, the zooming property of the wavelet transform ensures that characteristic features of the analyzed signal on a certain scale are well represented by the transform values corresponding to this scale factor, i.e. not distributed among other scale factors. Moreover, these transform values will be localized at the respective signal parts, where the above-mentioned features are present. These concentration properties - both with respect to scale and time - may be formulated mathematically in a more rigorous way; the purpose of this section is, to give a plausibility argument for the above statement by performing a comparison experiment with the Fourier transform.

The signal displayed in Fig. 1.7 is a section from the beginning of an audio signal. Roughly in the middle, the sudden start of sound clearly can be recognized.

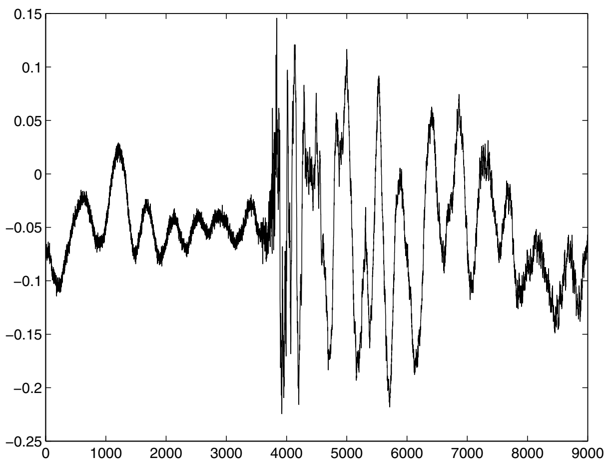

Fig. 1.7. "Attack-signal"

Such signals ("attack-signals") shall *locally*, i.e. in a small neighborhood of the point, where the sound starts, contain *high frequencies*, equivalently, there will be drastic changes on a *small scale*. In such a situation the zooming property of the wavelet transform should be advantageous when compared with the Fourier transform. To confirm this conjecture, the following experiment has been carried out:

1. Compute the Fourier transform of the signal, keep those 4% of the values of the transformed signal, having the largest absolute values. Put the remaining transform values equal to zero and reconstruct the signal from this modified transform (remember that, as stated in sect. 1.2.1, all transforms discussed here are invertible).

2. Perform the same procedure with the wavelet transform instead of the Fourier transform.

The results of the experiment are shown in Fig. 1.8. The dashed curve shows the result of the Fourier-reconstruction, the wavelet-reconstruction is displayed by +-symbols, the solid line represents the original signal. When comparing with Fig. 1.7, observe that the curves show an enlarged section of the signal from Fig. 1.7 in a neighborhood of the point where the sound starts.

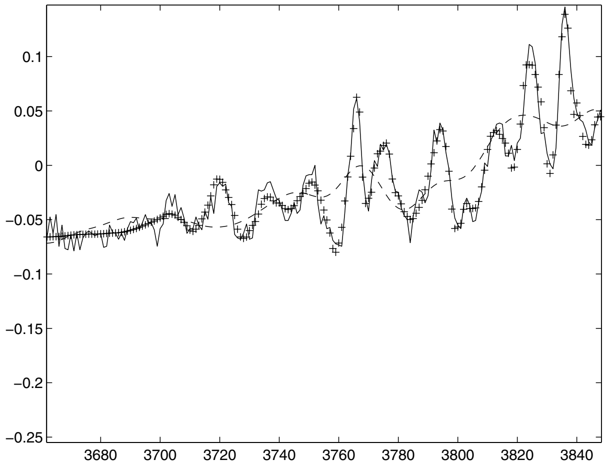

Fig. 1.8. "Attack-signal": reconstructions

Obviously, when using the Fourier transform, the suppression of 96% of the transform signal values - namely those transform signal values having lower absolute values than the retained ones - leads to a global smoothing (low pass filtering) and therefore the local peaks during the attack phase are not reproduced any more. In contrast, applying the same suppression procedure to the wavelet transform does *not* disturb the reproduction of these peaks. This is a clear indication for the above-mentioned concentration properties of the wavelet transform values.

As a final remark we should indicate that the original signal was not represented as a continuous-time signal, but it was discretely sampled. For such signals there exist variants of the continuous formulae 1.2 and 1.5, respectively, with which the above experiment has been carried out. These "discrete transforms" shall be described in later sections.

Moreover, as with any wavelet transform application, the result of the above experiment depends on the chosen wavelet g. The results displayed in Fig. 1.8 have been obtained using the db4-wavelet described in a later section.

1.3 A Roadmap for the Book

The topics sketched in this introduction will be described in more detail in chap. 2. Questions like "what is an optimal window function for the Short Time Fourier Transform?" or "how may visualizations as displayed in Fig. 1.6 be accomplished?" will be treated there. Moreover, we will indicate how the original signal may be reconstructed from the respective transforms.
This is of crucial importance for applications like signal compression: There, transforms of the original signal usually are computed in a first step. Subsequently these transforms are modified in such a way that the required storage space for the transformed signal is reduced considerably. Finally, in the "decompression" step the signals are reconstructed from the modified transforms.
In chap. 2 also a brief survey is given of typical signal analysis applications of the Short Time Fourier Transform and the wavelet transform, respectively. Two industrial case studies are described in more detail. Section 2.4 contains some exercises. Partly these exercises will be of "paper-and-pen-type", but they also will consist in writing computer programs.

As a programming platform we use the tool MATLAB, which is widespread and a de-facto-standard in the engineering community. Of course it is possible to use the book just as a reference guide to wavelet techniques without performing any programming. For readers interested in using the software described in this book and in developing their own programs, however, the use of MATLAB will be a prerequisite. Readers having already some familiarity with programming languages and looking for a compactly written introduction to MATLAB are referred to [12]. This book provides a very nice and efficient description of MATLAB's main features. It is written in German; if this turns out to be an obstacle, reference [30] is highly recommended.
The MATLAB software which is discussed in this book and has been written by the author may be downloaded from

`www.springeronline.com/de/3-540-23433-0`

Most of it requires only the basis version of MATLAB. Some programs, however, make use of the MATLAB Wavelet Toolbox, a collection of wavelet-related signal processing algorithms. Its use is described in the user's guide [24], which is not just a software handbook. It is moreover a beautifully written introduction to wavelet techniques. Since in the author's opinion MATLAB together with the wavelet toolbox will turn out to be a standard software platform for wavelet-related signal processing, the present book also will provide a short introduction to the most important components of the MATLAB Wavelet Toolbox.

In chapters 1 and 2, respectively, signals are considered to be *continuous-time signals*, even though computerized versions of the algorithms described there necessarily involve a discretization. A different perspective is given in chap. 3: Here from the very beginning signals are *discrete sequences of numbers* and the wavelet transform described there is designed for such sequences.

The corresponding notions, wavelet constructions, signal transform and reconstruction formulae are given in this chapter. Most important for practical applications is the existence of fast algorithms both for transformation and reconstruction. They are also described in this chapter together with MATLAB implementations. Thereafter again applications and case studies are presented. In this context we shall also comment on real-time properties of fast wavelet algorithms. Section 3.7 provides exercises.

Chapter 4 is devoted to a more detailed description of some additional applications of the wavelet transform. First we shall focus on the most popular application, namely data compression. Subsequently, an application related to retrieving images from database management systems is described. Again at the end of this chapter some exercises are presented.

As far as mathematics is concerned, we assume some familiarity of the reader with basic calculus like integral calculus and complex numbers. We shall give no rigorous proofs of mathematical statements, rather we want to provide some intuitive insight into the essence of these statements and their practical meaning. More mathematical details related to background and applications of wavelets are collected in the appendix. It is intended to support the reader, if he or she feels that some additional information would be helpful to understand the main body of the text. Moreover, the appendix contains solutions to selected problems from the exercises and provides a list of frequently used symbols and notations.

http://www.springer.com/978-3-540-23433-3

Wavelets and Signal Processing
An Application-Based Introduction
Stark, H.-G.
2005, X, 150 p. 67 illus., Hardcover
ISBN: 978-3-540-23433-3