7

Orbit Structure of Reductive Monoids

Let M be a reductive monoid with unit group G. We assume that M has a zero element $0 \in M$. The general case is not interestingly different (see Proposition 8 of [121]). From Theorem 4.2, M is regular, so that

$$M = GE(M) = E(M)G\ .$$

But we can do much better than this. Indeed, from Theorem 4.5,

$$M = \bigsqcup_{e \in \Lambda} GeG$$

where $\Lambda = \{e \in E(\overline{T}) \mid Be \subseteq eB\}$.

In this chapter we want to explain how M is "stuck together" using G, Λ and $P(e) = \{g \in G \mid ge = ege\}$. Since $P(e)$ is a parabolic subgroup containing B, the reader should take note of the key objective here: to obtain control of the structure of M in terms of something easily described in terms of the Coxeter-Dynkin complex of G, and the set of standard parabolic subgroups of G.

Our second objective here is to identify and record a large number of examples where we can determine Λ and $\Lambda \longrightarrow \mathcal{P}$, $e \rightsquigarrow P(e)$, explicitly. Notice that there is a canonical identification $\mathcal{P} = 2^S$, where S is the set of simple reflections. See Theorem 2.46. Thus we usually write the type map as $\lambda : \Lambda \longrightarrow 2^S$

7.1 The System of Idempotents and the Type Map

In this section we describe the orbit structure of a reductive monoid, assuming there is only one minimal, nonzero orbit. The results of this section are taken from [95].

Let M be reductive with unit group G, Borel subgroup $B \subseteq G$ and maximal torus $T \subseteq B$. $W = N_G(T)/T$. From Definition 4.6 we obtain the type map

$$\lambda : \Lambda \longrightarrow 2^S .$$

Recall that $\lambda(e) = \{s \in S \mid se = es\}$, where $S \subseteq W$ is the set of simple reflections of W relative to B, and Λ is the cross section lattice of M relative to B and T.

Notice that $\lambda(e)$ determines $P(e) = \{g \in G \mid ge = ege\}$ since $P(e)$ is generated by B and $\lambda(e)$. Note also that $P(e)$ and $P^-(e) = \{g \in G \mid eg = ege\}$ are opposite parabolic subgroups.

Definition 7.1. *Let*

$$E(\lambda) = \left\{ (J, P, Q) \,\middle|\, \begin{array}{l} J \in G\backslash M/G, \quad P \textit{ and } Q \textit{ are opposite parabolics,} \\ P = gP(e)g^{-1} \textit{ for some } g \in G, \textit{ where } J \cap \Lambda = \{e\} \end{array} \right\} .$$

A quasi-ordering *on a set E is a relation $\leq$ on E that is transitive and reflexive.*

Theorem 7.2. *Both $E(M)$ and $E(\lambda)$ have canonically defined quasi-orderings $\leq_\ell$ and $\leq_r$. Define*

$$\psi : E(M) \longrightarrow E(\lambda)$$

by $\psi(e) = (GeG,\ P(e),\ P^-(e))$. Then ψ is an isomorphism of biordered sets. $\leq_r$ and $\leq_\ell$ are defined as follows.

On $E(M)$ define

$$\begin{array}{ll} e \leq_r f & \textit{if } fe = e. \\ e \leq_\ell f & \textit{if } ef = e. \\ e \leq f & \textit{if } ef = fe = e. \end{array}$$

On $E(\lambda)$ define

$(J_1, P, Q)\mathcal{R}(J_2, P', Q')$ if $J_1 = J_2$ and $P = P'$.
$(J_1, P, Q)\mathcal{L}(J_2, P', Q')$ if $J_1 = J_2$ and $Q = Q'$.
$(J_1, P, Q) \leq (J_2, P', Q')$ if $J_1 \leq J_2$ and there exist opposite Borel subgroups B and B^-, such that $B \subseteq P \cap P'$ and $B^- \subseteq Q \cap Q'$.

Then define on $E(\lambda)$

$$\leq_r = \mathcal{R} \circ \leq$$

and

$$\leq_\ell = \mathcal{L} \circ \leq .$$

Proof. Assume that $GeG = GfG$. Then one checks, as in Lemma 3.4 of [95], that

$$\circledast \begin{cases} eM = fM & \text{if and only if } P(e) = P(f) \quad \text{and} \\ Me = Mf & \text{if and only if } P^-(e) = P^-(f) . \end{cases}$$

Next we check that ψ is bijective. Let $(J, P, Q) \in E(\lambda)$. Then $P = P(e)$ for some $e \in E(J)$. Further, by the results of [83], P is opposite to $P^-(e)$.

So by standard results there exists $g \in P$ such that $g^{-1}P^-(e)g = Q$. Thus, $(J, P, Q) = \psi(g^{-1}eg)$. Conversely, if $\psi(e) = \psi(f)$, then by $\circledast$ above that $eM = fM$ and $Me = Mf$. But then $e = f$, by an elementary semigroup calculation.

For the remainder of the proof we refer the reader to Theorem 3.5 of [95]. Notice that, in view of $\circledast$ above, all that remains here is to check that $e \leq f$ if and only if $\psi(e) \leq \psi(f)$.

The moral of the story is that, once we know λ, we obtain E automatically. If we also know G, then we can reconstruct M up to a kind of "central extension" abstractly. It can be seen direcly that $E(M)$ is a **biordered set** in the sense of Nambooripad [64].

7.2 The Cross Section Lattice and the Weyl Chamber

Let M be a reductive algebraic monoid. For the results of this section, it is not necessary to impose any other restrictions. We need to show how the cross-section lattice Λ can be described in terms relating $X(\overline{T})$ and the set of dominant weights

$$X(T)_+ = \{\chi \in X(T) \mid \Delta_\alpha(\chi) \geq 0 \quad \text{for all } \alpha \in \Delta\}$$

where $\Delta_\alpha : X(T) \longrightarrow \mathbb{Z}$ is defined by the equation

$$\chi - s_\alpha(\chi) = \Delta_\alpha(\chi)\alpha.$$

Let $e \in E(\overline{T})$. Then $\overline{eT}$ is also a D-monoid with unit group eT. So let $X(\overline{eT})$ denote the monoid of characters of $\overline{eT}$. Consider

$$\mu_e = \left\{ \chi \in X(e\overline{T}) \subseteq X(\overline{T}) \,\middle|\, \begin{matrix} \chi \neq 0 \text{ and} \\ \chi_{e\overline{T}\setminus eT} = 0 \end{matrix} \right\}$$

where $X(e\overline{T}) \subseteq X(\overline{T})$ via the map $\overline{T} \to e\overline{T}$, $z \to ez$. One can easily check that

$$X(\overline{T})\backslash\{0\} = \bigsqcup_{e \in E(\overline{T})} \mu_e \ .$$

Let Δ be the set of simple roots of G relative to B and T. For $\alpha \in \Delta$ let U_α be the one dimensional, unipotent subgroup of B, normalized by T with weight α.

Lemma 7.3. *Let $\alpha \in \Delta$ and $e \in E(\overline{T})$. The following are equivalent:*

a) $U_\alpha e = eU_\alpha e$;
b) *either* $U_\alpha e = eU_\alpha$, *or else* $U_\alpha f = f$ *for all* $f \in E_1(\overline{eT})$.

Proof. In case $eU_\alpha \neq e$ and $U_\alpha e \neq e$, one obtains that $U_\alpha e = eU_\alpha e$. Otherwise, if $U_\alpha f = f$ for all $f \in E_1(e\overline{T})$ yet $U_\alpha e \neq eU_\alpha e$ (i.e. $U_\alpha e \neq eU_\alpha$ and $U_\alpha e \neq e$), then $eU_\alpha = e \neq U_\alpha e$ is the only other possibility. Hence $\sigma_\alpha e \sigma_\alpha \neq e$ and thus, $\sigma_\alpha f \sigma_\alpha \neq f$ for some $f \in E_1(e\overline{T})$. But then $fU_\alpha = feU_\alpha = fe = f$. Hence $U_\alpha f \neq f$, since $f\sigma_\alpha \neq \sigma_\alpha f$. Contradiction.

Lemma 7.4. *a) The following are equivalent:*
i) $U_\alpha e = eU_\alpha$ *(equivalently,* $s_\alpha e = es_\alpha$*);*
ii) $\Delta_\alpha(\chi) = 0$ *for some* $\chi \in \mu_e$*.*
b) The following are equivalent (assuming $U_\alpha e \neq eU_\alpha$*).*
i) $U_\alpha f = f$ *for all* $f \in E_1(e\overline{T})$*;*
ii) $\Delta_\alpha(\chi) > 0$ *for some* $\chi \in \mu_e$*.*

Proof. For a), first note that, for $\chi \in \mu_e$, $\Delta_\alpha(\chi) = 0$ iff $s_\alpha(\chi) = \chi$. But $s_\alpha(\mu_e) = \mu_{e'}$ where $e' = s_\alpha e s_\alpha$. Hence $s_\alpha(\mu_e) \cap \mu_e \neq \phi$ iff $s_\alpha e s_\alpha = e$. Accordingly, if $\chi \in \mu_e$ and $s_\alpha(\chi) = \chi$, then $\chi \in s_\alpha(\mu_e) \cap \mu_e$. Conversely, if $\chi_1 \in s_\alpha(\mu_e) \cap \mu_e \neq \phi$, then $\chi_1 s_\alpha(\chi_1) \in \mu_e$ and $\Delta_\alpha(\chi) = 0$.

For b), assume first that $U_\alpha f = f$ for all $f \in E_1(\overline{eT})$. Now $\overline{fT} \cong K$ as algebraic varieties. Hence there is a unique character $\chi_f \in X(\overline{T})$ such that $\mathcal{O}(\overline{fT}) = K[\chi_f]$. But from Lemma 3.6 of [109], $\Delta_\alpha(\chi_f) \geq 0$. Now

$$K[\overline{eT}] = K[\chi \mid \chi^n \in \langle \chi_{f_1}, \ldots, \chi_{f_2} \rangle \text{ for some } n > 0] \qquad (*)$$

where $\{f_i\}_{i=1}^s = E_1(e\overline{T})$. Now if $\Delta_\alpha(\chi_f) = 0$ for all $f \in E_1(e\overline{T})$, then $s_\alpha(\chi_f) = \chi_f$ for all $f \in E_1(e\overline{T})$. Hence by $(*)$, $s_\alpha(K[e\overline{T}]) = K[e\overline{T}]$ and so $s_\alpha e = es_\alpha$. Thus $U_\alpha e = eU_\alpha$, a contradiction. Thus $\Delta_\alpha(\chi_f) > 0$ for some $f \in E_1(e\overline{T})$. Hence $\Delta_\alpha(\chi) > 0$ for all $\chi \in \mu_e$. Conversely, suppose that $\Delta_\alpha(\chi_f) > 0$ for some $f \in E_1(\overline{eT})$. Consider

$$\chi = \chi_f^N \chi_{f_2} \cdot \ldots \cdot \chi_{f_s} \in \mu_e$$

where $N > 0$ and $E_1(e\overline{T}) = \{f, f_2, \ldots, f_s\}$. Then $\Delta_\alpha(\chi) < 0$ if $N >> 0$. This is a contradiction.

Theorem 7.5. *The following are equivalent for* $e \in E(\overline{T})\backslash\{0\}$*.*

a) $e \in \Lambda\backslash\{0\}$*;*
b) there exists $\chi \in \mu_e$ *such that* $\Delta_\alpha(\chi) \geq 0$ *for all* $\alpha \in \Delta$*.*

Proof. Now $e \in \Lambda' := \Lambda\backslash\{0\}$ if and only if for all $\alpha \in \Delta$ either $U_\alpha e = eU_\alpha$ or else $U_\alpha e \neq eU_\alpha$ and $U_\alpha f = f$ for all $f \in E_1(\overline{eT})$. By Lemma 7.4 this is equivalent to:

For each $\alpha \in \Delta$, either

$$\Delta_\alpha(\chi) = 0 \text{ for some } \chi \in \mu_e$$

or else

$$\Delta_\alpha(\chi) > 0 \text{ for all } \chi \in \mu_e\ .$$

Thus, $e \in \Lambda\backslash\{0\}$ if and only if for all $\alpha \in \Delta$ either

i) $\Delta_\alpha(\chi) = 0$ for some $\chi \in \mu_e$, or else
ii) $\Delta_\alpha(\chi) > 0$ for all $\chi \in \mu_e$.

Hence b) implies a).

Conversely, if $e \in \Lambda\backslash\{0\}$ then $\Delta = \Delta_1 \sqcup \Delta_2$, where

$$\Delta_1 = \{\alpha \in \Delta \mid s_\alpha e = e s_\alpha\}$$

and

$$\Delta_2 = \{\alpha \in \Delta \mid s_\alpha e \neq e s_\alpha\}.$$

Let $\chi_0 \in \mu_e$ and define

$$\chi = \prod_{w \in W_{\Delta_1}} w(\chi_0) \in \mu_e \ .$$

Then $\Delta_\alpha(\chi) = 0$ for all $\alpha \in \Delta$. But $\Delta_\alpha(\chi) > 0$ for all $\alpha \in \Delta_2$.

Theorem 7.5 has a very appealing geometric interpretation.

One can identify $E(\overline{T})$ with the face lattice $\mathcal{F}$ of the rational polyhedral cone $X(\overline{T}) \otimes \mathbb{Q}^+ \subseteq X(T) \otimes \mathbb{Q}$. Furthermore, $X(\overline{T}) \otimes \mathbb{Q}^+$ is W-invariant. We can think of $\mu_e \otimes \mathbb{Q}^+$ as the topological interior of $X(\overline{eT}) \otimes \mathbb{Q}^+ \in \mathcal{F}$. Theorem 7.5 says that

$$\Lambda = \left\{ e \in E(\overline{T}) \,\middle|\, \begin{array}{l} \text{the interior of } X(e\overline{T}) \otimes \mathbb{Q}^+ \text{ meets} \\ X(\overline{T})_+ \otimes \mathbb{Q}^+ \end{array} \right\} .$$

Clearly, $|Cl_W(e) \cap \Lambda| = 1$ for all $e \in E(\overline{T})$.

Recall that a reductive monoid M is *semisimple* if the center of G is one-dimensional and M has a zero element. In this case the zero element of M is in the closure $\overline{Z}$ of Z the one-dimensional connected center of M. As Z is contained in any maximal torus T of G, we have in particular that $\overline{Z} \subseteq \overline{T}$. Thus we obtain the induced (dual) map on the corresponding character monoids:

$$\gamma : X(\overline{T}) \to X(\overline{Z}) \cong \mathbb{N}.$$

This γ determines, on the associated rational polyhedral cones, a homomorphism

$$\zeta : X(\overline{T}) \otimes \mathbb{Q}^+ \to X(\overline{Z}) \otimes \mathbb{Q}^+ \cong \mathbb{Q}^+,$$

by setting $\zeta = \gamma \otimes 1$. For M semisimple we make the following definition.

Definition 7.6. *Let*

$$\mathcal{P} = \zeta^{-1}(1).$$

$\mathcal{P}$ *is the* polytope of M.

From the above results, $\mathcal{P}$ is W-invariant, and the face lattice $\mathcal{F}$ of $\mathcal{P}$ is canonically identified with $E(\overline{T})$. Furthermore, we can identify Λ as a subset of $\mathcal{F}$ using Theorem 7.5.

Example 7.7. Let $M = M_n(K)$, the semisimple monoid of $n \times n$ matrices over K. In this case $Z = \{\alpha I_n \mid \alpha \in K^* \}$, where I_n is the identity $n \times n$ matrix. If T is the D-group of invertible diagonal matrices then $\overline{T}$ is the set of diagonal matrices and $\zeta : X(\overline{T}) \otimes \mathbb{Q}^+ \to X(\overline{Z}) \otimes \mathbb{Q}^+$ is easily identified with the map

$$\rho : (\mathbb{Q}^+)^n \to \mathbb{Q}^+$$

defined by $\rho(s_1, ..., s_n) = \sum_i s_i$. The polytope here is

$$\mathcal{P} = \{(s_1, ..., s_n) \in (\mathbb{Q}^+)^n \mid \sum_i s_i = 1 \}.$$

The face lattice of $\mathcal{P}$ is easily identified with $E(\overline{T})\backslash\{0\}$. Notice that characters are written additively in this setup.

7.3 $\mathcal{J}$-irreducible Monoids

We start with a simple lemma to focus our discussion.

Lemma 7.8. *Let M be a reductive monoid with zero $0 \in M$. Let $\Lambda \subseteq E(\overline{T})$ be a cross section lattice. The following are equivalent.*

a) $\Lambda\backslash\{0\}$ has a unique minimal element e_0 (so that $e_0 f = e_0$ for all $f \in \Lambda\backslash 0$);
b) there exists a rational representation $\rho : M \longrightarrow End(V)$ such that
  i) V is irreducible over M.
  ii) ρ is a finite morphism.

Proof. Assume that $\rho : M \longrightarrow End(V)$ is as in b). Suppose that $e_1, e_2 \in \Lambda\backslash 0$ are minimal elements yet $e_1 \neq e_2$. Then $e_1 M e_2 = 0$. But $\rho(Me_1)V$ and $\rho(Me_2)V$ are M-submodules of V. Thus, $\rho(Me_1)V = \rho(Me_2)V = V$. But then $\rho(e_1)V = \rho(e_1)\rho(Me_2)V = \rho(e_1Me_2)V = 0$, so that $\rho(e_1) = 0$. But this is impossible since ρ is a finite morphism.

Now assume that $\Lambda\backslash\{0\}$ has a unique minimal element e. Let $\rho : M \longrightarrow End(W)$ be a finite morphism of algebraic monoids [82]. If we replace W by $\overline{W} = \bigoplus_{i=1}^{n} W_i/W_{i-1}$, where $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n = W$ is a composition series of W, then $gr(\rho) : M \longrightarrow End(\overline{W})$ is also a finite morphism since, by regularity of M, $gr(\rho)^{-1}(0) = \{0\}$. So assume that $W = \bigoplus_{i=1}^{n} V_i$ where each V_i is irreducible. Now $\rho(e) \neq 0$, since ρ is finite. Say $\rho(e)(V_1) \neq 0$. Thus we let $\rho_1 = \rho|V_1$. Then $\rho : M \longrightarrow End(V_1)$ is the desired irreducible representation.

Definition 7.9. *Let M be as in Lemma 7.8. We say that M is* $\mathcal{J}$-irreducible.

The major purpose of this chapter is to determine the cross section lattices and the type maps of $\mathcal{J}$-irreducible reductive moniods. But first, let us notice what determines the cross section lattice Λ and the type map $\lambda : \Lambda \to 2^S$.

Given M as in Lemma 7.8, we observe several discrete invariants.

i) the *type* of the representation $\rho : M \longrightarrow End(V)$
ii) $\lambda(e_0) = J_0 = \{s \in S \mid se_0 = e_0 s\} \subseteq S$
iii) $\{g \in G \mid ge_0 = ege_0\} = P(e_0) < G$.

To define the *type* of ρ, let $B \subseteq G$ be a Borel subgroup and let $L \subseteq V$ be the line such that $\rho(B)L = L$. Then the *type of* ρ is the parabolic subgroup

$$P = \{g \in G \mid \rho(g)L = L\} \ .$$

These invariants all amount to the same thing. Indeed, $P = P(e_0) = \bigsqcup_{w \in W_{J_0}} BwB$ and $L = e_0(V)$. So our mission here is as follows.

> Determine the type map $\lambda : \Lambda \to 2^S$ in terms of $\lambda(e_0) = J_0 \subseteq S$ where $e_0 \in \Lambda \backslash \{0\}$ is the minimal element.

For the remainder of this section we assume that M is a $\mathcal{J}$-irreducible monoid.

Lemma 7.10. *Let $e, f \in E(M)$ be nonzero idempotents. Then $P(e) = P(f)$ if and only if $e\mathcal{R}f$.*

Proof. If $e\mathcal{R}f$, then $f = eg$ for some $g \in G$. Hence $fe = e$. If $x \in \mathcal{R}(e)$, then $xe = exe$ and so $xf = xeg = exeg = fexeg = fexf$. Hence $f(xf) = f(fexf) = fexf = xf$. Thus, $P(e) \subseteq P(f)$. By symmetry, $P(f) \subseteq P(e)$.

Conversely, assume that $P(e) = P(f)$. Assume that $e \in \overline{T} \subseteq \overline{P(e)}$. Now there exists $g \in P(f)$ such that $f' = gfg^{-1} \in \overline{T}$. Then $f\mathcal{R}f'$ and $P(f) = P(f')$. So without loss of generality, $f = f'$. Now let $h \in E_1(\overline{T})$ be such that $he = eh = h$. Then there exists a cross section lattice Λ such that $e, h \in \Lambda$. Then

$$\begin{aligned} B &= \{g \in G \mid gh = hgh \text{ for all } h \in \Lambda\} \\ &\subseteq P(e) \end{aligned}$$

since $e \in \Lambda$. But $B \subseteq P(e) = P(f)$. Hence by definition,

$$f \in \{h \in E(\overline{T}) \mid gh = hgh \text{ for all } g \in B\} = \Lambda \ .$$

Since $h \in \Lambda$ is the unique, nonzero, minimal element of Λ, we have $fh = h$. But this is true for any $h \in E_1(\overline{eT}) = \{h_1, \ldots, h_s\}$, the set of minimal, nonzero idempotents of $\overline{eT}$. Thus $f = e$ since $e = h_1 \vee \cdots \vee h_s$. Similarly, $f = fe$. So $e = f$.

Lemma 7.10 is the crux of the matter. Indeed, let

$$\lambda : \Lambda\backslash\{0\} \longrightarrow 2^S ,$$

where $\lambda(e) = \{s \in S \mid se = es\}$, be the type map. Then $P(e) = P_{\lambda(e)}$. Hence by Lemma 7.10, λ is injective. Thus it remains to find $\lambda(\Lambda\backslash\{0\}) \subseteq 2^S$ and to recover the $\mathcal{J}$-ordering on Λ from this image.

Definition 7.11. *For $e \in \Lambda$ define*

$$\lambda^*(e) = \{s \in S \mid se = es \neq e\}$$
$$\lambda_*(e) = \{s \in S \mid se = es = e\}.$$

It is easy to check that $e \geq f$ implies that both $\lambda^*(f) \subseteq \lambda^*(e)$ and $\lambda_*(e) \subseteq \lambda_*(f)$. In particular, $\lambda_*(e) \subseteq \lambda_*(e_0) = J_0$. But we can do much more here. It turns out that we can characterize $\{I \in 2^S \mid I = \lambda^*(e)$ for some $e \in \Lambda\backslash 0\}$ and that $\lambda^*(e)$ determines $\lambda_*(e) \subseteq J_0$.

First recall the graph structure on S:

$$s \text{ and } t \text{ are joined by an edge if } st \neq ts.$$

Therefore we can talk about the **connected components** of any subset of S. The following theorem is the main result of [95].

Theorem 7.12. *a) The following are equivalent for $I \subseteq S$.*
i) $I = \lambda^*(e)$ for some $e \in \Lambda\backslash\{0\}$.
ii) No connected component of I lies entirely in J_0.
Furthermore, if $e \geq f$ then $\lambda^*(e) \supseteq \lambda^*(f)$
b) For any $e \in \Lambda\backslash\{0\}$, $\lambda_*(e) = \{s \in J_0\backslash\lambda^*(e) \mid st = ts$ for all $t \in \lambda^*(e)\}$.

Proof. For b) we refer the reader to the straightforward calculation of Lemma 4.10 of [95]. For a), first notice that $e \mapsto \lambda^*(e)$ is an injection $\Lambda\backslash\{0\} \longrightarrow 2^S$ since λ is injective and it is determined by λ^*. Furthermore, if $e \geq f$ then $eMe \supseteq fMf$, while $\lambda^*(e)$ is canonically identified with the simple reflections of eMe (and similarly for f). Hence $\lambda^*(e) \supseteq \lambda^*(f)$.

To see why i) and ii) are equivalent, we start with e_0 and notice that $\lambda^*(e_0) = \phi$; and then work our way "up". The key step is Theorem 4.13 of [95].

For $e \in \Lambda\backslash\{0\}$ there is a canonical bijection between $\{f \in \Lambda\backslash\{0\} \mid f$ covers $e\}$ and

$$\{s \in S \mid se \neq es\}. \tag{$*$}$$

Hence f corresponds to the unique s with $\lambda^*(f) = \lambda^*(e) \cup \{s\}$.

To find f given s, consider W_I where $I = \lambda^*(e) \cup \{s\}$. Then it is easily checked that there is a minimal element $e' \in \Lambda^{W_I} = \{e \in \Lambda \mid we = ew$ for all $w \in W_I\}$ such that $e'e = ee' = e$ and $e' \neq e$. This gives us "a foot in the doorway" since $\lambda^*(e') \supseteq \mu(e) \cup \{s\}$. We can now find f by induction on $\dim M$ using the $\mathcal{J}$-irreducible monoid $e'Me'$.

Now, given $I \subseteq S$ as in a) ii), let

$$\begin{aligned}
K_1 &= S \backslash J_0 \\
K_2 &= K_2 \cup \{s \in I \backslash K_1 \mid st \neq ts_1, \text{ some } t \in K_1\} \\
&\vdots \\
K_i &= K_{i-1} \cup \{s \in I \backslash K_{i-1} \mid st \neq ts, \text{ some } t \in K_{i-1}\} \\
&\vdots
\end{aligned}$$

By definition $I = K_s$ for some $s > 0$. But from $(*)$ applied repeatedly, there exists $e_i \in \Lambda \backslash \{0\}$ such that $\lambda^*(e_i) = K_i$.

Remark 7.13. a) Theorem 7.12 provides an algorithm for calculating Λ and $\lambda : \Lambda \longrightarrow 2^S$ for any $\mathcal{J}$-irreducible monoid M in terms of $J_0 = \lambda(e_0) = \lambda_*(e_0)$. In each case, $S \backslash J_0$ corresponds to the set of fundamental dominant weights involved in the associated irreducible representation of M.

b) One defines a reductive monoid M, with zero, to be $\mathcal{J}_i$-irreducible if $|\Lambda_j| = 1$ for all $j \leq i$. The reader can check that
   i) M is $\mathcal{J}_2$-irreducible if and only if $J_0 = S \backslash \{s\}$ for some $s \in S$,
   ii) M is $\mathcal{J}_3$-irreducible if and only if $J_0 = S \backslash \{s\}$ where s corresponds to an end node on the Dynkin diagram of G. See Figure 7.1 below.

c) One can use Theorem 7.12 to characterize other classes of $\mathcal{J}$-irreducible monoids.
   i) We say that a semisimple monoid M is **$\mathcal{J}$-simple** if each $\mathcal{H}$-class of M has at most one simple component. It turns out that M is $\mathcal{J}$-simple if and only if S is connected and M is either $\mathcal{J}_2$-irreducible or $S \backslash J_0 = \{s, t\}$ where $st \neq ts$. See Figure 7.2 below and Exercise 3 of 7.7.1.
   ii) $\Lambda(M)$ is a distributive lattice if and only if $S \backslash J_0$ is connected.

7.4 Explicit Calculations of the Type Map

In this section we illustrate Theorem 7.12 by using it to calculate the type maps of several interesting classes of $\mathcal{J}$-irreducible monoids. In our first example we calculate the type maps associated with the adjoint representation.

7.4.1 The Type Map for the Adjoint Representations

In this subsection we illustrate Theorem 7.12 by calculating the type maps for the monoids associated with the adjoint representations of simple groups. Here $M = \overline{K^* Ad(G)} \subseteq End(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G. Also, by b) of Theorem 7.12, it suffices to calculate

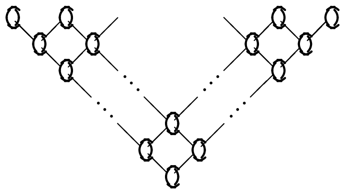
$$\{\lambda^*(e) \mid e \in \Lambda\backslash\{0\}\}$$

in each case. We also include the Hasse diagram of Λ in each case, along with an illustration of the corresponding extended Dynkin diagram. The reader can use the extended Dynkin diagram to "see" how Theorem 7.12 is used to calculate Λ.

a) **Type** A_ℓ:

$$\begin{aligned} &S = \{s_1, \ldots, s_\ell\} \\ &s_i s_j \neq s_j s_i \quad \text{if } |i-j| = 1 \\ &J_0 = \{s_2, \ldots, s_{\ell-1}\} \end{aligned}$$

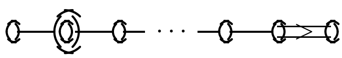
$$\lambda^*(\Lambda\backslash\{0\}) = \{S\backslash\{s_i, s_{i+1}, \ldots, s_j\} \mid 1 \leq i \leq j \leq \ell\} \cup \{S\} .$$

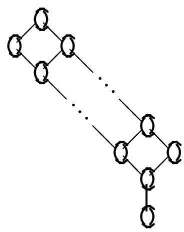


b) **Type** B_ℓ:

$$\begin{aligned} &S = \{s_1, \ldots, s_e\} \\ &s_i s_j \neq s_j s_i \quad \text{if } |i-j| = 1 \\ &J_0 = \{s_1, s_3, \ldots, s_\ell\} \end{aligned}$$

$$\begin{aligned} \lambda^*(\Lambda\backslash 0) = \{&\phi; \{s_2\}; \{s_1, s_2\}, \{s_2, s_3\}; \ldots \\ &\ldots; \{s_1, s_2, \ldots, s_i\}, \{s_2, s_3, \ldots, s_{i+1}\}; \ldots \\ &\ldots; \{s_1, \ldots, s_{\ell-1}\}, \{s_2, \ldots, s_\ell\}; \{s_1, \ldots, s_\ell\}\}. \end{aligned}$$

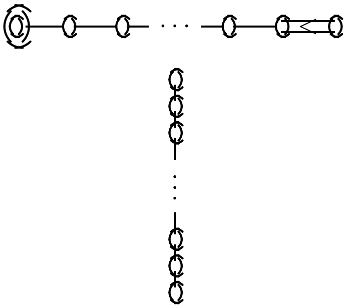




c) **Type C_ℓ**:

$$S = \{s_1, \ldots, s_\ell\}$$
$$s_i s_j \neq s_j s_i \quad \text{if } |i-j| = 1$$
$$J_0 = \{s_2, s_3, s_4, \ldots, s_\ell\}$$

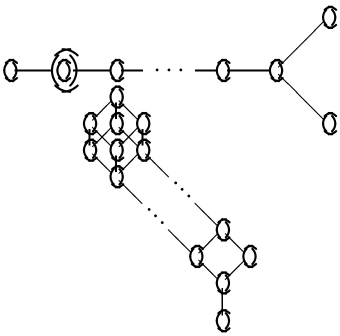
$$\lambda^*(\Lambda \backslash 0) = \{\phi; \{s_1\}; \{s_1, s_2\}; \ldots; \{s_1, \ldots, s_i\}; \ldots, \{s_1, \ldots, s_\ell\}\}\ .$$



d) **Type D_ℓ**:

$$S = \{s_1, \ldots, s_{\ell-2}, s_{\ell-1}, s_\ell\}$$
$$s_i s_j \neq s_j s_i \quad \text{if } |i-j| = 1 \text{ and } i, j \leq \ell - 1 \text{ or } \{i, j\} = \{\ell-2, \ell\}\ .$$
$$J_0 = \{s_1, s_3, s_4, \ldots, s_\ell\}$$

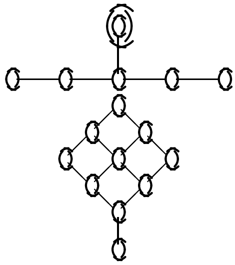
$$\begin{aligned}\lambda^*(\Lambda \backslash 0) = \{&\phi; \{s_2\}; \{s_1, s_2\}, \{s_2, s_3\}; \{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}; \ldots;\\ &\{s_1, s_2, \ldots, s_{\ell-3}\}, \{s_2, s_3, \ldots, s_{\ell-2}\};\\ &\{s_1, s_2, \ldots, s_{\ell-2}\}, \{s_2, s_3, \ldots, s_{\ell-2}, s_\ell\}, \{s_2, s_3, \ldots, s_{\ell-1}\};\\ &\{s_1, s_2, \ldots, s_{\ell-2}, s_\ell\}, \{s_1, s_2, \ldots, s_{\ell-1}\}, \{s_2, s_3, \ldots, s_\ell\};\\ &\{s_1, s_2, \ldots, s_\ell\}\}.\end{aligned}$$



e$_6$) **Type E_6**:

$$S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$$
$$s_is_j \neq s_js_i \quad \text{if } \{i,j\} \in \{\{1,2\},\{2,3\},\{3,4\},\{3,5\},\{5,6\}\}$$
$$J_0 = \{s_1, s_2, s_3, s_5, s_6\}$$

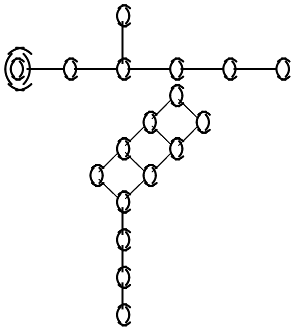
$$\begin{aligned}\lambda^*(\Lambda\backslash 0) = \{&\phi; \{s_4\}; \{s_3, s_4\}; \{s_2, s_3, s_4\}, \{s_3, s_4, s_5\};\\ &\{s_1, s_2, s_3, s_4\}, \{s_2, s_3, s_4, s_5\}, \{s_3, s_4, s_5, s_6\};\\ &\{s_1, s_2, s_3, s_4, s_5\}, \{s_2, s_3, s_4, s_5, s_6\}; \{s_1, s_2, s_3, s_4, s_5, s_6\}\}.\end{aligned}$$



e$_7$) **Type** E_7:

$$S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$$
$$s_is_j \neq s_js_i \quad \text{if } \{i,j\} \in \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{4,6\},\{6,7\}\}$$
$$J_0 = \{s_2, s_3, s_4, s_5, s_6, s_7\}$$

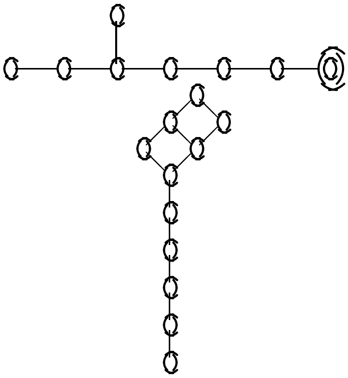
$$\begin{aligned}\lambda^*(\Lambda\backslash 0) = \{&\phi; \{s_1\}; \{s_1, s_2\}, \{s_1, s_2, s_3\}; \{s_1, s_2, s_3, s_4\}, \{s_1, s_2, s_3, s_5\};\\ &\{s_1, s_2, s_3, s_4, s_5\}, \{s_1, s_2, s_3, s_5, s_6\};, \{s_1, s_2, s_3, s_4, s_5, s_6\},\\ &\{s_1, s_2, s_3, s_5, s_6, s_7\}; \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}\}.\end{aligned}$$



e$_8$) **Type** E_8:

$$S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$$
$$s_is_j \neq s_js_i \quad \text{if } \{i,j\} \in \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{5,7\},\{7,8\}\}$$
$$J_0 = \{s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$$

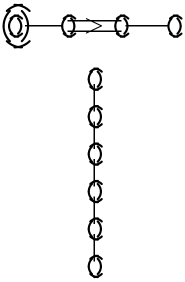
$$\lambda^*(\Lambda\backslash 0) = \{\phi; \{s_1\}; \{s_1, s_2\}; \{s_1, s_2, s_3\}; \{s_1, s_2, s_3\}; \{s_1, s_2, s_3, s_4\}, \{s_1, s_2, s_3, s_4, s_5\}; \{s_1, s_2, s_3, s_4, s_5, s_6\}, \{s_1, s_2, s_3, s_4, s_5, s_7\}; \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}, \{s_1, s_2, s_3, s_4, s_5, s_7, s_8\}; \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}\}.$$



f) **Type** F_4:

$$S = \{s_1, s_2, s_3, s_4\}$$
$$s_i s_j \neq s_j s_i \quad \text{if } |i - j| = 1$$
$$J_0 = \{s_2, s_3, s_4\}$$

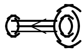
$$\lambda^*(\Lambda\backslash 0) = \{\phi; \{s_1\}; \{s_1, s_2\}; \{s_1, s_2, s_3\}; \{s_1, s_2, s_3, s_4\}\} .$$



g) **Type** G_2:

$$S = \{s_1, s_2\}$$
$$s_1 s_2 \neq s_2 s_1$$
$$J_0 = \{s_1\}$$

$$\lambda^*(\Lambda\backslash 0) = \{\phi; \{s_1\}; \{s_1, s_2\}\} .$$



In each of these examples one may also interpret $\Lambda\backslash\{0\}$ as the lattice of centers of unipotent radicals of standard parabolic subgroups.

7.4.2 Further Examples of the Type Map

In this subsection we illustrate Theorem 7.12 with two classes of pictorial diagrams, Figure 7.1 and Figure 7.2.

In Figure 7.1 we calculate (λ, Λ) for all $\mathcal{J}$-irreducible monoids with $J_0 = S\backslash\{s\}$. These cross section lattices correspond to $\mathcal{J}$-irreducible monoids that arise from dominant weights μ of the form $\mu = a\omega$, where ω is a fundamental dominant weight.

In Figure 7.2 we calculate (λ, Λ) for all $\mathcal{J}$-irreducible monoids with $J_0 = S\backslash\{s, t\}$ and $st \neq ts$. These cross section lattices correspond to $\mathcal{J}$-irreducible monoids that arise from dominant weights μ of the form $\mu = a\omega_1 + b\omega_2$ where ω_1 and ω_2 are adjacent fundamental dominant weights. See Exercise 3 of 7.7.1 for another characterization of this class of $\mathcal{J}$-irreducible monoids.

The structure of $\mathcal{J}$-irreducible monoids has a peculiar, but interesting relationship with irreducible representations. The following result is originally due to S. Smith [126]. It becomes useful in the development of Putcha's abstract theory of monoids of Lie type. See Chapter 10.

Proposition 7.14. *Let $\rho : G \longrightarrow G\ell(V)$ be an irreducible representation and let $P < G$ be parabolic with $U = R_u(P)$ such that $B \subseteq P$. Let $M = \overline{K^*\rho(G)}$ be the associated $\mathcal{J}$-irreducible monoid, with Λ, T and B as usual. Then*

a) V^U is an irreducible $P/R_u(P)$-module
b) $e \mapsto V^{R_u(P(e))}$ is a 1-1 correspondence between $\Lambda\backslash\{0\}$ and $\{V^{R_u(P)} \mid P \supseteq B\}$.

Proof. Assume first that $P = P(e)$ for some $e \in \Lambda\backslash\{0\}$. Let $W = V^U$. Since V is irreducible, we have that $W^{B_u/U} = V^{B_u}$ is one-dimensional. Hence W is an indecomposable P/U-module. Now $e \in \overline{C_G(e)}$ and so $e : W \rightarrow W$ is a $C_G(e)$-module homomorphism. Hence

$$W = e(W) \oplus ker(e)$$

as $C_G(e)$-modules. But, as already mentioned, W is indecomposable. Hence $ker(e) = 0$ and $W = e(W) = e(V)$. But $e(V)$ is irreducible over eMe by an easy calculation as in Proposition 5.1 of [95]. Thus, $e(V) = V^U$ is irreducible over P.

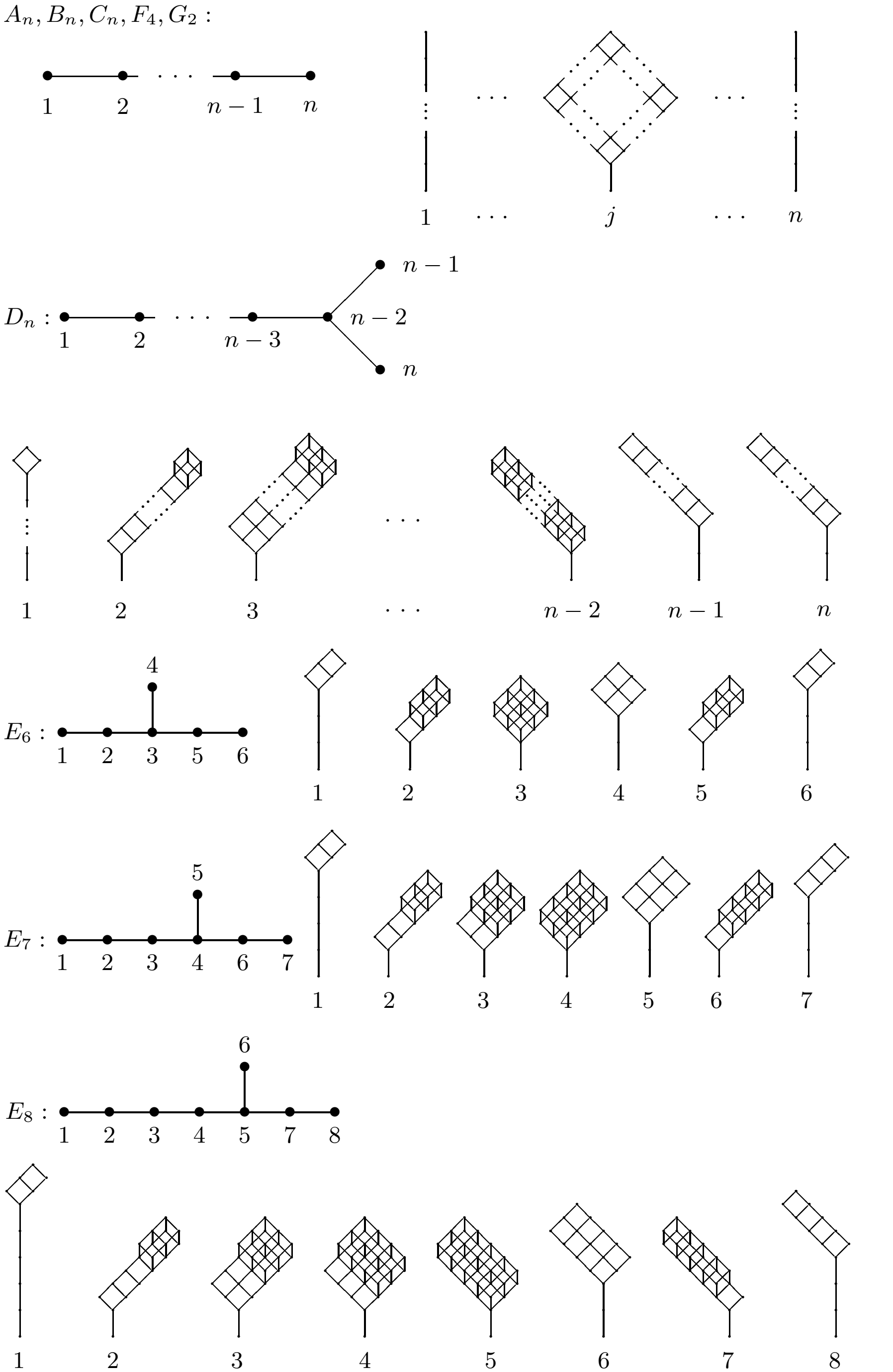


Fig. 7.1. $\Lambda\backslash\{0\}$ for $\mathcal{J}_2$-irreducible monoids. This is the case where $J_0 = S\backslash\{s\}$. Each lattice is labeled by $S\backslash J_0$.

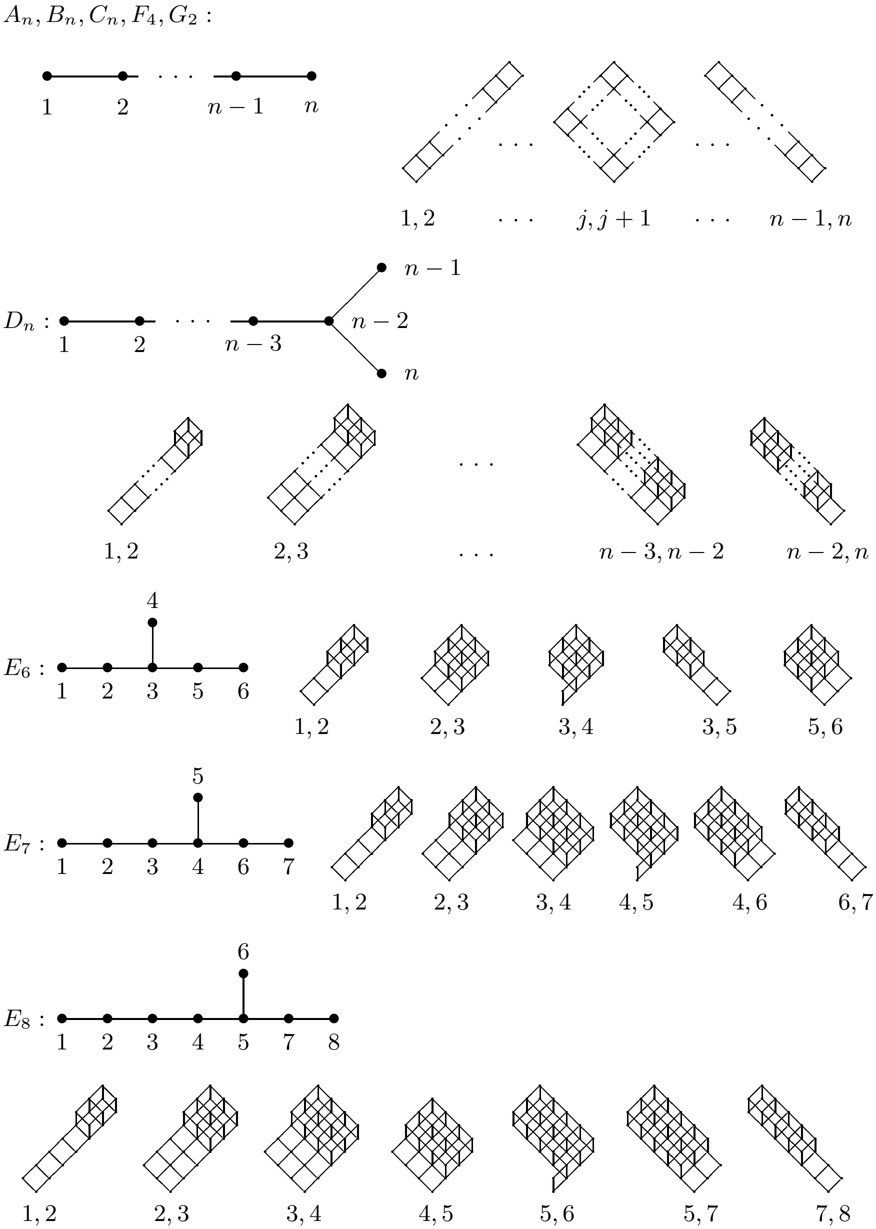


Fig. 7.2. $\Lambda \backslash \{0\}$ for $\mathcal{J}$-irreducible monoids with $J_0 = S \backslash \{s, t\}$ and $st \neq ts$. Again each lattice is labeled by $S \backslash J_0$.

Now assume that $P < G$ is any standard parabolic subgroup. We can then find $e \in \Lambda\backslash\{0\}$ such that $eC_G(e) = eC_P(e)$ and $P \subseteq P(e)$. Indeed, let $P = P_I$, and write $I = \lambda^* \sqcup \lambda_*$ where $\lambda_* \subseteq J_0$ consists of all connected components of I lying in J_0. Then $e \in \Lambda\backslash\{0\}$ is the unique idempotent with $\lambda^*(e) = \lambda^*$. Hence

$$e(V) = V^{R_u(P(e))} \supseteq V^{R_u(P)} .$$

But $R_u(P)e = \{e\}$ and so $e(V) \subseteq V^{R_u(P)}$ also. This completes the proof.

It turns out that any subspace of V, of the form V^U (where U is the unipotent radical of some standard parabolic subgroup $Q = P_I$), is already of the form $V^U = e(V)$ where $e \in \Lambda\backslash\{0\}$. In fact, $e \in \Lambda\backslash\{0\}$ is the unique minimal element of $\{e \in \Lambda\backslash\{0\} | se = es \text{ for all } s \in I\}$. For more detail, see Corollary 5.4 of [95].

7.5 2-reducible Reductive Monoids

In this section we study the orbit structure of semisimple algebraic monoids with exactly two nonzero minimal orbits. These results were first obtained in a joint paper with Putcha [98].

The case of one minimal orbit was diccussed in the previous section. The present situation is more complicated, but our results are still very precise and revealing. We associate with each 2-reducible monoid M, certain invariants (I_+, I_-) and (Δ_+, Δ_-). These invariants are not entirely independent, but should be regarded as the minimal information needed to determine the much sought after type map of M. We end the discussion with two carefully chosen examples. The first one illustrates how the Cartan matrix is used in calculating (Δ_+, Δ_-) from (I_+, I_-) and the polytope of M.

Vinberg obtained a similar description of the $G \times G$-orbits of his universal, flat deformation monoid $Env(G_0)$ of the semisimple group G_0. See § 6.3 for a summary of these results.

A reductive monoid M is **2-reducible** if $M\backslash\{0\}$ has exactly two minimal $G \times G$-orbits. Given a 2-reducible monoid M, we obtain certain invariants (I_+, I_-) and (Δ_+, Δ_-). From these, we calculate the cross section lattice Λ, and the type map of M. But (I_-, I_+) and (Δ_+, Δ_-) are not entirely independent; and it appears that the final answer depends on the "shape" of the inverse of the Cartan matrix; and not just the shape of the Dynkin diagram.

7.5.1 Reductive Monoids and Type Maps

Let M be reductive with unit group G, and let Λ be the cross-section lattice of M, relative to T and B. Then

$$\Lambda = \{e \in E(\overline{T}) \mid Be = eBe\}.$$

We let

$$\Lambda' = \Lambda \backslash \{0\},$$

so that by Theorem 4.5 c)

$$M = \bigsqcup_{e \in \Lambda} GeG.$$

Since Λ is a lattice, it has two operations, the **meet** $\wedge$, and the **join** $\vee$. The meet of $e, f \in \Lambda$ is just there product ef in M. The join of e and f is the smallest idempotent $h \in \Lambda$ with $he = e$ and $hf = f$.

In this section, we regard the type map

$$\lambda : \Lambda \to 2^{\Delta}$$

as taking values in the set of subsets of Δ, the set of simple roots. This slight change of notation should not cause difficulties.

Lemma 7.15. *Let $W(e) = W_{\lambda(e)} = \{w \in W \mid we = ew\}$, the associated parabolic subgroup of W. Let $W_*(e) = \{w \in W \mid we = ew = e\}$, and $W^*(e) = \{w \in W \mid we = ew \neq e\}$. Then*

a) $\lambda_*(e) = \bigcap_{f \leq e} \lambda(f)$ and $W_*(e) = \bigcap_{f \leq e} W(f)$;
b) $\lambda^*(e) = \bigcap_{f \geq e} \lambda(f)$ and $W^*(e) = \bigcap_{f \geq e} W(f)$.

It follows from Lemma 7.15 that

i) for $e \in \Lambda$, $\lambda(e) = \lambda^*(e) \sqcup \lambda_*(e)$;
ii) for $e, f \in \Lambda$, $\lambda(e) \cap \lambda(f) \subseteq \lambda(e \vee f) \cap \lambda(e \wedge f)$;
iii) for $e \in \Lambda$, $W(e) = W^*(e) \times W_*(e)$;
iv) if $e \geq f$ then $\lambda_*(e) \subseteq \lambda_*(f)$ and $\lambda^*(f) \subseteq \lambda^*(e)$. Furthermore, λ^* restricted to eMe is the λ^* of eMe, and λ_* restricted to M_e is the λ_* of M_e.

Here, eMe is the reductive monoid with unit group $eC_G(e)$, and $M_e = \overline{G_e}$, where $G_e = \{g \in G \mid ge = eg = e\}^0$; M_e is also a reductive monoid.

Definition 7.16. *Let M, Λ and λ be as above. Let $\Lambda_1 \subseteq \Lambda$ be the subset of nonzero minimal elements.*

a) The core *C of Λ is*

$$C = \{e \in \Lambda \mid e = e_1 \vee \cdots \vee e_k, \ for\ some\ e_i \in \Lambda_1\}.$$

b) Define $\theta : \Lambda' \to C$ by

$$\theta(e) = \vee\{f \in \Lambda_1 \mid f \leq e\}$$

so that, if $e_1 \leq e_2$, then $\theta(e_1) \leq \theta(e_2)$.
c) Write $\Lambda' = \bigsqcup_{h \in C} \Lambda_h$, where $\Lambda_h = \theta^{-1}(h)$

Recall that a reductive monoid M with zero is *semisimple* if $dim(Z(G)) = 1$. For any semisimple monoid M, there is a special relationship between Λ' and $S \subseteq X(T)$. If $\alpha \in \Delta$ then by Theorem 10.20 of [82] there exists $e_\alpha \in \Lambda'$ such that $P(e_\alpha) = P_{\Delta\setminus\{\alpha\}}$. Morover, e_α is unique. See Lemma 7.22 below.

Definition 7.17. *Let $C \subseteq \Lambda'$ be the core.*

a) Define $\pi : \Delta \to C$ by $\pi(\alpha) = \theta(e_\alpha)$.
b) Write $\Delta = \sqcup_{h \in C} \Delta_h$, where $\Delta_h = \pi^{-1}(h)$.

Proposition 7.18. *a) If $e \in \Lambda_h$ then*

$$\lambda_*(e) = \{\alpha \in \lambda_*(h) \mid s_\alpha s_\beta = s_\beta s_\alpha \textit{ for all } \beta \in \lambda^*(e)\}.$$

b) If $e \in \Lambda_h$ and $f \in \Lambda_k$ then

$$e \leq f \textit{ if and only if } h \leq k \textit{ and } \lambda^*(e) \subseteq \lambda^*(f).$$

Proof. Consider a). Since $e \geq h$, $\lambda_*(e) \subseteq \lambda_*(h)$. Let $\alpha \in \lambda^*$. Then since $W(e) = W^*(e) \times W_*(e)$, $s_\alpha s_\beta = s_\beta s_\alpha$ for all $\beta \in \lambda^*(e)$.

So it remains to prove the reverse inclusion. Now $E(e\overline{T})$ is the face lattice of a polytope (see Section 4). Therefore e is the join of the nonzero minimal idempotents of $E(e\overline{T})$. Hence

$$e = \vee\{xe'x^{-1} \mid e' \in \Lambda_1,\ e \geq e',\ x \in W^*(e)\}.$$

Let $\alpha \in \lambda_*(h)$ be such that $s_\alpha s_\beta = s_\beta s_\alpha$ for all $\beta \in \lambda^*(e)$. Then $s_\alpha x = x s_\alpha$ for all $x \in W^*(e)$. Let $e' \in \Lambda_1$ be such that $e \geq e'$. Since $e \in \Lambda_h, h \geq e'$. Let $x \in W^*(e)$. However, $\alpha \in \lambda^*(h)$. Hence

$$s_\alpha x e' x^{-1} = x s_\alpha e' x^{-1} = x e' x^{-1} = x e' s_\alpha x^{-1} = x e' x^{-1} s_\alpha.$$

By the above join formula for e, and Proposition 7.5 of [82], $s_\alpha x = x s_\alpha$. Thus, $s_\alpha \in W(e)$. Now s_α commutes with all the nonzero minimal idempotents in $E(e\overline{T})$, and thus, es_α has the same property. Thus, es_α commutes with all idempotents of of $e\overline{T}$. Since $eW(e)$ acts faithfully on $E(e\overline{T})$, it follows from Chapter 10 of [82] that $es_\alpha = e$. Hence $s_\alpha \in W_*(e)$ and $\alpha \in \lambda_*(e)$.

For b), let $h \leq k$ and $\lambda^*(e) \subseteq \lambda^*(f)$. Let $e' \in \Lambda_1$ be such that $e \geq e'$. Then $e' \leq h \leq k \leq f$. Let $x \in W^*(e) \subseteq W^*(f)$. Then,

$$fxe'x^{-1} = xfe'x^{-1} = xe'x^{-1}.$$

Hence $xe'x^{-1} \leq f$. Therefore by the above join formula for e, $e \leq f$. The converse is clear.

7.5.2 The Type Map of a 2-reducible Monoid

Let M be a 2-reducible, semisimple monoid. Our terminology is well chosen because of the following proposition.

Proposition 7.19. *Let M be a semisimple monoid. The following are equivalent.*

a) M is 2-reducible;
b) i) there is a rational representation $\rho : M \to End(V \oplus W)$ so that ρ is finite as a morphism, and V and W are irreducible M-summands;
ii) M is not 3-irreducible.

Proof. If M is 2-reducible, let $\Lambda_1 = \{e, f\}$. There exist irreducible representations $\rho_1 : M \to End(V)$ and $\rho_2 : M \to End(W)$ such that $\rho_1(e) \neq 0$ and $\rho_2(f) \neq 0$. It is easy to check that $\rho = \rho_1 \oplus \rho_2$ does the job. Conversely, if the conditions of b) are satisfied, let $\Lambda_1 = \{e_1, \ldots, e_r\}$, where $r \geq 2$. We can assume that $e_1(V) \neq 0$. But then $e_1(V)$ generates V as an M-module, and so $e_2(V) = 0$ since $e_1 G e_2 = 0$. Thus $e_2(W) \neq 0$. But now for any $i > 2$, $e_i(V \oplus W) = 0$. Thus $r = 2$.

In this section, we determine Λ and $\lambda : \Lambda \to 2^{\Delta}$ in terms of certain invariants (I_+, I_-) and (Δ_+, Δ_-).

Write

$$\Lambda_1 = \{e_+, e_-\}.$$

Then

$$C = \{e_+, e_-, e_0\},$$

where $e_0 = e_+ \vee e_-$. Let

$$I_+ = \lambda_*(e_+),\ I_- = \lambda_*(e_-) \text{ and } I_0 = \lambda_*(e_0).$$

Then

$$I_0 = I_+ \cap I_-.$$

By 7.16 c),

$$\Lambda' = \Lambda_+ \sqcup \Lambda_- \sqcup \Lambda_0$$

and by 7.17 b)

$$\Delta = \Delta_+ \sqcup \Delta_- \sqcup \Delta_0$$

where $\Delta_+ = \pi^{-1}(e_+), \Delta_- = \pi^{-1}(e_-)$, and $\Delta_0 = \pi^{-1}(e_0)$. Hence

i) $\alpha \in \Delta_+$ if $e_\alpha \geq e_+$ and $e_\alpha \not\geq e_-$;
ii) $\alpha \in \Delta_-$ if $e_\alpha \geq e_-$ and $e_\alpha \not\geq e_+$;
iii) $\alpha \in \Delta_0$ if $e_\alpha \geq e_+$ and $e_\alpha \geq e_-$.

See the paragraph preceding Definition 7.17 the definition of e_α. By Proposition 7.18, our problem is reduced to determining $\lambda^*(\Lambda_+), \lambda^*(\Lambda_-)$ and $\lambda^*(\Lambda_0)$.

Remark 7.20. If M is 2-reducible but not semisimple, then $dim(Z(G)) = 2$. One can then show that, in this situation,

a) $\lambda^*(\Lambda_+) = \{X \subseteq \Delta |$ no component of X is in $I_+\}$
b) $\lambda^*(\Lambda_-) = \{X \subseteq \Delta |$ no component of X is in $I_-\}$
c) $\lambda^*(\Lambda_0) = \{X \subseteq \Delta |$ no component of X is in $I_0\}$.

In this case, M is a special case of the *multilined closure* with $n = 2$. Here n is the number of minimal $G \times G$ orbits of $M \backslash \{0\}$. The multilined closure is an appealing situation where the lattice of orbits and the type map can be written down directly in terms of the types of the minimal orbits. This has been described in generality in Chapter 6. See also Remark 7.21 below for the case $n = 1$. In any case, the semisimple case is more complicated. It is also more interesting.

Remark 7.21. We shall freely use the results from § 7.3 about $\mathcal{J}$-irreducible monoids in the proof of Theorem 7.23 below. Notice that Proposition 7.18 includes Theorem 7.12 as a special case.

We now return to the 2-reducible case.

Lemma 7.22. *$\Delta_+ \neq \phi$ and $\Delta_- \neq \phi$.*

Proof. Choose a maximal $e \in \Lambda_+$. Then e is covered by some $f \in \Lambda_0$. Furthermore, f is unique, since if e is also covered by $h \in \Lambda_0$, and $f \neq h$, then $e = fh \geq e_0$, a contradiction. Thus, both fMf and M_f are $\mathcal{J}$-irreducible, and hence semisimple. Hence $\lambda(e) = \Delta \backslash \{\alpha\}$ for some $\alpha \in \Delta$. This $e \in \Lambda$ is actually unique with $\lambda(e) = \Delta \backslash \{\alpha\}$. (The connnected center Z of $C_G(e)$ is two dimensional. So $\overline{Z}$ has exactly four idempotents $\{e, f, 0, 1\}$. $P(f)$ is the opposite parabolic of $P(e)$. But then $B \not\subseteq P(f)$, so that $f \notin \Lambda$.) In any case, $\alpha \in \Delta_+$. Similarly, $\Delta_- \neq \phi$.

As we already mentioned, we want to determine $\lambda : \Lambda \to 2^{\Delta}$ in terms of I_+, I_-, Δ_+ and Δ_-. By Proposition 2.5, it suffices to determine the sets $\lambda^*(\Lambda_+), \lambda^*(\Lambda_-)$ and $\lambda^*(\Lambda_0)$. Let

$\mathcal{A}_+ = \{X \subseteq \Delta \mid$ no component of X is contained in $I_+,\ \Delta_+ \not\subseteq X\}$
$\mathcal{A}_- = \{X \subseteq \Delta \mid$ no component of X is contained in $I_-,\ \Delta_- \not\subseteq X\}$
$\mathcal{A}_0 = \{X \subseteq \Delta \mid$ no component of X is contained in I_0, and either $\Delta_+ \not\subseteq X$ and $\Delta_- \not\subseteq X$ or else $\Delta_+ \cup \Delta_- \subseteq X\}$.

Theorem 7.23.
a) $\lambda^*(\Lambda_+) = \mathcal{A}_+$;
b) $\lambda^*(\Lambda_-) = \mathcal{A}_-$;
c) $\lambda^*(\Lambda_0) = \mathcal{A}_0$.
In all cases, λ^* is injective.

Proof. Suppose first that $X \in \mathcal{A}_+$. Then $\alpha \notin X$ for some $\alpha \notin \Delta_+$. By Theorem 10.20 of [82], there exists $e \in \Lambda_+$ such that $\lambda(e) = \Delta\backslash\{\alpha\}$. Hence $X \subseteq \lambda(e)$. By Proposition 7.18 a), $X \subseteq \lambda^*(e)$. Now eMe is a $\mathcal{J}$-irreducible monoid of type $I_+ \cap \lambda^*(e)$. Since no component of X is contained in I_+, there exists $f < e$ such that $\lambda^*(f) = X$. Clearly, $f \in \Lambda_+$.

Conversely, let $f \in \Lambda_+$. Let $e \in \Lambda_+$ be maximal such that $f \leq e$. By the proof of Lemma 7.22, $|\lambda(e)| = |\Delta| - 1$. Hence $\lambda(e) = \Delta\backslash\{\alpha\}$ for some $\alpha \in \Delta_+$. Also, $\lambda_*(f) \subseteq \lambda_*(e) \subseteq \Delta\backslash\{\alpha\}$ by iv) following Definition 7.16. Hence $\lambda^*(f) \in \mathcal{A}_+$, by Proposition 7.18 a).

Similarly, $\lambda^*(\Lambda_-) = \mathcal{A}_-$.

To prove c), we proceed by induction on $dim(M)$. Let $f \in \Lambda_0$. Then $f \leq e$ for some maximal $e \neq 1$. So eMe is a 2-reducible monoid. First, suppose that eMe is not semisimple. Then by Remark 7.20, $\lambda(e) = \lambda^*(e) = \Delta\backslash\{\alpha_1, \alpha_2\}$. By Proposition 7.18, $\alpha_1, \alpha_2 \notin \Delta_0$. Suppose that $\alpha_1, \alpha_2 \in S_+$. Then there exist $e_1, e_2 \in \Lambda_+$ such that $\lambda(e_1) = \Delta\backslash\{\alpha_1\}$ and $\lambda(e_2) = \Delta\backslash\{\alpha_2\}$. By Remark 7.20, there exists $h \in \Lambda_+$ such that e covers h and $\lambda(h) \subseteq \Delta\backslash\{\alpha_1, \alpha_2\}$. By Proposition 7.18 b), $h < e_1$ and $f < e_2$. But $\{1, e_2, h\}$ is a maximal chain in $E(\overline{T}_h)$. So $dim(T_h) = 2$, while $\{1, e_1, e_2, e, h\} \subseteq E(\overline{T}_h)$. This is a contradiction since $|E(\overline{T}_h)| = 4$ for such D-monoids. Similarly, $\alpha_1, \alpha_2 \in \Delta_-$ leads to a contradiction. So assume that $\alpha_1 \in \Delta_+$ and $\alpha_2 \in \Delta_-$. Then by Proposition 7.18 b), $\lambda^*(f) \subseteq \lambda^*(e) = \Delta\backslash\{\alpha_1, \alpha_2\}$. Hence $\lambda^*(f) \in \mathcal{A}_0$.

Next assume that eMe is semisimple. Then $\lambda(e) = \lambda^*(e) = \Delta\backslash\{\beta\}$ for some $\beta \in \Delta_0$. Correspondingly, in eMe, let

$$\Delta\backslash\{\beta\} = \Delta'_+ \sqcup \Delta'_0 \sqcup \Delta'_-.$$

Let λ_1 denote λ in eMe. We claim that $\Delta_+ = \Delta'_+$. Let $\alpha \in \Delta_+$. Since eMe is a semisimple monoid, there exists $e_1 < e$ such that $\lambda_1(e_1) = \Delta\backslash\{\alpha, \beta\}$. If $\lambda(e_1) = \Delta\backslash\{\alpha\}$, then $e_1 \in \Lambda_+$, and hence $\alpha \in \Delta'_+$. So assume that $\lambda(e_1) = \Delta\backslash\{\alpha, \beta\}$. Now $\lambda(e_2) = \Delta\backslash\{\alpha\}$ for some $e_2 \in \Lambda_+$. However,

$$\beta \notin \lambda^*(e_2) \implies \lambda^*(e_2) \subseteq \Delta\backslash\{\beta\} \implies e_2 \leq e.$$

But

$$e_2 \leq e \implies \lambda_1(e_2) = \Delta\backslash\{\alpha, \beta\} \implies e_1 = e_2 \implies \alpha \in \Delta'_+$$

Therefore let $\beta \in \lambda^*(e_2)$. Since $e_2 \in \Lambda_+$, e_2Me_2 is $\mathcal{J}$-irreducible, and hence semisimple. Let λ_2 denote λ for e_2Me_2. There exists $e_3 < e_2$ such that $\lambda_2(e_3) = \lambda_2(e_2)\backslash\{\beta\}$. hence

$$\Delta\backslash\{\alpha, \beta\} = \lambda_*(e_2) \cup (\lambda^*(e_2)\backslash\{\beta\}) \subseteq \lambda_2(e_3) \cup \lambda_*(e_2) \subseteq \lambda(e_3).$$

If $\lambda(e_3) = \Delta\backslash\{\beta\}$, then $e_3 = e \in \Lambda_0$, a contradiction. Hence $\lambda(e_3) = \Delta\backslash\{\alpha, \beta\}$. By Proposition 7.18 b), $e_3 < e$ and so $\alpha \in \Delta'_+$. Thus, $\Delta_+ \subseteq \Delta'_+$. Similarly, $\Delta_- \subseteq \Delta'_-$.

Suppose that $\alpha \in \Delta'_+$, $\alpha \notin \Delta_+$. Then $\alpha \notin \Delta_-$ since $\Delta_- \subseteq \Delta'_-$. Hence $\alpha \in \Delta_0$. There exists $e_1 \in \Lambda_+$ with $e_1 < e$ such that $\lambda_1(e_1) = \Delta\backslash\{\alpha, \beta\}$. Since

$\alpha \notin \Delta_+$, $\lambda(e_1) \neq \Delta\backslash\{\alpha\}$. Hence $\lambda(e_1) = \Delta\backslash\{\alpha, \beta\}$. Now $\lambda(e_2) = \Delta\backslash\{\alpha\}$ for some $e_2 \in \Lambda_0$, since $\alpha \in \Delta_0$. By Proposition 7.18 b), $e_1 < e_2$. Hence $e_1 < ee_2$. By ii) of § 7.5.1

$$\Delta\backslash\{\alpha, \beta\} = (\Delta\backslash\{\alpha\}) \cap (\Delta\backslash\{\beta\}) \subseteq \lambda(ee_2).$$

By Proposition 7.18 b), $\lambda(ee_2) \neq \Delta\backslash\{\alpha\}$ or $\Delta\backslash\{\beta\}$. Hence $\lambda(ee_2) = \Delta\backslash\{\alpha, \beta\}$. Hence $\lambda_1(e_1) = \lambda_1(ee_2) = \Delta\backslash\{\alpha, \beta\}$. Since eMe is semisimple, $e_1 = ee_2 \in \Lambda_0$, a contradiction. Hence $\Delta'_+ \subseteq \Delta_+$ and so $\Delta'_+ = \Delta_+$. Similarly, $\Delta'_- = \Delta_-$. By the induction hypothesis, $\lambda^*(f) \in \mathcal{A}_0$. Thus, $\lambda^*(\Lambda_0) \subseteq \mathcal{A}_0$.

Conversely, let $X \in \mathcal{A}_0$. Suppose first that $\Delta_+ \cup \Delta_- \subseteq X, X \neq \Delta$. Then $X \subseteq \Delta\backslash\{\beta\}$ for some $\beta \in \Delta_0$. There exists $f \in \Lambda_0$ such that $\lambda(f) = \Delta\backslash\{\beta\}$. If fMf is semisimple, then $\Delta'_+ = \Delta_+$ and $\Delta'_- = \Delta_-$ as above; and by the induction hypothesis, $\lambda^*(f') = X$ for some $f' \in \Lambda_0, f' \leq f$. If fMf is not semisimple, then the same is true by Remark 7.21.

Suppose next that $\Delta_+ \not\subseteq X$ and $\Delta_- \not\subseteq X$. Let $\alpha \in \Delta_+, \beta \in \Delta_-$ be such that $X \subseteq \Delta\backslash\{\alpha, \beta\}$. We first show that there exists $f \in \Lambda_0$ such that $\lambda(f) = \Delta\backslash\{\alpha, \beta\}$. Now there exists $e \in \Lambda_+$ such that $\lambda(e) = \Delta\backslash\{\alpha\}$. Then M_e and eMe are both semisimple. Suppose that $\beta \in \lambda_*(e)$. Then there exists $f > e$ such that $\lambda_1(f) = \lambda^*(e)\backslash\{\beta\}$, where λ_1 is λ for M_e. So in M (using iv) following Lemma 7.15),

$$\Delta\backslash\{\alpha, \beta\} = (\lambda_*(e)\backslash\{\beta\}) \cup \lambda^*(e) \subseteq \lambda(f).$$

Since $f > e \geq e_+$, $f \notin \Lambda_-$. Hence, $\lambda(f) \neq \Delta\backslash\{\beta\}$ and so $\lambda(f) = \Delta\backslash\{\alpha, \beta\}$. So e is central in fMf, and thus fMf is not $\mathcal{J}$-irreducible. Hence, $f \notin \Lambda_+$. Thus $f \in \Lambda_0$.

Assume next that $\beta \in \lambda^*(e)$. Then there exists $e_1 < e$ such that $\lambda_1(e_1) = \lambda_*(e)\backslash\{\beta\}$, where λ_1 is λ for eMe. So by iv) just following Lemma 7.15,

$$\Delta\backslash\{\alpha, \beta\} = (\lambda^*(e)\backslash\{\beta\}) \cup \lambda_*(e) \subseteq \lambda(e_1).$$

Since $e_1 < e, e_1 \notin \Lambda_-$. Hence $\lambda(e_1) \neq \Delta\backslash\{\beta\}$ and so $\lambda(e_1) = \Delta\backslash\{\alpha, \beta\}$. Hence e is central in M_{e_1}. Thus M_{e_1} has at least four central idempotents. So let f be a central idempotent of M_{e_1} such that $f \notin \{1, e, e_1\}$. Then $\Delta\backslash\{\alpha, \beta\} \subseteq \lambda(f)$ by iv) again. Since $f > e_1$, $f \notin \Lambda_-$ and so $\lambda(f) \neq \Delta\backslash\{\beta\}$. Since $f \neq e$, $\lambda(f) \neq \Delta\backslash\{\alpha\}$. Thus $\lambda(f) = \Delta\backslash\{\alpha, \beta\}$. If $f \in \Lambda_+$, then fMf is $\mathcal{J}$-irreducible, and e_1 is a central idempotent: a contradiction. Hence $f \in \Lambda_0$.

There exists $f \in \Lambda_0$ such that $\lambda(f) = \Delta\backslash\{\alpha, \beta\}$. Hence either M_f is not semisimple, or fMf is not semisimple. Suppose that M_f is not semisimple. There exists $f' > f$, $f \neq 1$, such that f' is central in M_f. By iv) just after Lemma 7.15, $\Delta\backslash\{\alpha, \beta\} \subseteq \lambda(f')$. Since $f' \in \Lambda_0$, $\alpha \in \Delta_+$, $\beta \in \Delta_-$, $\lambda(f') \neq \Delta\backslash\{\alpha\}$ and $\lambda(f') \neq \Delta\backslash\{\beta\}$. Hence $\lambda(f') = \Delta\backslash\{\alpha, \beta\}$. Thus by iv) again, and Proposition 7.18 b), $f = f'$: a contradiction. Consequently M_f is semisimple. But then fMf is not semisimple. Since $X \subseteq \Delta\backslash\{\alpha, \beta\}$, $X \subseteq \lambda_*(f)$. By Remark 7.21, $\lambda^*(f') = X$ for some $f' \in \Lambda_0$, $f' \leq f$. Thus $\mathcal{A} \subseteq \lambda^*(\Lambda_0)$. This concludes the proof.

Corollary 7.24. *The partial order on Λ is determined as follows. Let $e, f \in \Lambda$. Then the following are equivalent:*

a) $e \leq f$*;*
b) *i)* $\lambda^*(e) \subseteq \lambda^*(f)$*, and*
ii) $e, f \in \Lambda_+$*;* $e, f \in \Lambda_-$*;* $e, f \in \Lambda_0$*;* $e \in \Lambda_+, f \in \Lambda_0$*; or* $e \in \Lambda_-, f \in \Lambda_0$*.*

Proof. This is straightforward using Proposition 7.18 and Theorem 7.23.

7.5.3 Calculating the Type Map Geometrically

In the previous section we found the exact description of the type map

$$\lambda : \Lambda \to 2^{\Delta}$$

of a 2-reducible monoid by first identifying the necessary combinatorial invariants (I_+, I_-) and $(\Delta_+, \Delta_-, \Delta_0)$. In this section we determine some geometric refinements of that situation by calculating the decomposition

$$\Delta = \Delta_+ \sqcup \Delta_- \sqcup \Delta_0$$

in terms of the coordinates of $\Lambda_1 = \{e_+, e_-\}$, thought of as vertices of the polytope $\mathcal{P}$ of M. The problem here is to determine which decompositions of Δ are possible for a 2-reducible monoid M of type (I_+, I_-). This is no longer a purely combinatorial problem.

Let M be a 2-reducible, semisimple monoid, and let T, $\overline{T}$, Λ, etc. have the usual meanings. As above, let $\mathcal{P}$ be the polytope of M. By Theorem 7.5, we have a canonical bijection

$$\iota : \Lambda_1 \to \{x, y\}.$$

We write $\iota(e_+) = x$ and $\iota(e_-) = y$ where $\{x, y\}$ is the set of vertices of $\mathcal{P}$ that are contained in $X(\overline{T})^+ \otimes \mathbb{Q}^+$. Let $Bd(\mathcal{P})$ be the boundry of $\mathcal{P}$. For $\alpha \in \Delta$ let

i) $H_\alpha = Span_{\mathbb{Q}}(\Delta \backslash \{\alpha\})$
ii) $H_\alpha^+ = Cone_{\mathbb{Q}^+}(\Delta \backslash \{\alpha\})$.

For $\alpha \in \Delta$, let $\omega_\alpha \in X(\overline{T})^+ \otimes \mathbb{Q}^+$ be the fundamental dominant weight that is orthogonal to H_α.

Lemma 7.25. *For any $\alpha \in \Delta$ there is a unique $z_\alpha \in \mathbb{Q}^+\omega_\alpha$ such that*

$$(z_\alpha + H_\alpha) \cap \mathcal{P} = (z_\alpha + H_\alpha) \cap Bd(\mathcal{P}) \neq \phi.$$

Furthermore,

i) $z_\alpha \in Bd(\mathcal{P})$
ii) $(z_\alpha + H_\alpha) \cap \mathcal{P}$ *is the face F of $\mathcal{P}$ corresponding to e_α.*

Proof. Let $e = e_\alpha$ be the unique idempotent such that $\lambda(e_\alpha) = \Delta\backslash\{\alpha\}$ (see Lemma 7.22). Let $F \in \mathcal{F}$ be the face of $\mathcal{P}$ corresponding to $e \in \Lambda$. Then

$$\mathbb{Q}^+\omega_\alpha \subseteq \mu_e,$$

and thus $\mathbb{Q}^+\omega_\alpha \cap F = \{z_\alpha\}$ (since F is a subset of $z_\alpha + H_\alpha$, it must be orthogonal to $\mathbb{Q}\omega_\alpha$). Clearly, $F \subseteq Bd(\mathcal{P})$.

Let $I = \Delta\backslash\{\alpha\}$. Then F is W_I-invariant. Thus $F - z_\alpha$ is also W_I-invariant. But $\mathbb{Q}\omega_\alpha \cap (F - z_\alpha) = \{0\}$, and so $(F - z_\alpha)^{W_I} = \{0\}$. Thus $F - z_\alpha \subseteq H_\alpha$. Hence $F \subseteq H_\alpha + z_\alpha$.

The author would like to thank Hugh Thomas for the proof of the following Lemma.

Lemma 7.26. *The following are equivalent:*

a) $x \in z_\alpha + H_\alpha$
b) $x \in z_\alpha + H_\alpha^+$
c) $e_+ \leq e_\alpha$.

The corresponding result holds with x *replaced by* y *and* e_+ *replaced by* e_-.

Proof. For $\alpha \in \Delta$, let $C_1 = Cone(\{\omega_\alpha\})$ and $C_2 = Cone((\Delta\backslash\{\alpha\}) \cup \{\omega_\alpha\})$. We claim that $C_1 \subseteq C_2$. It suffices to show that $\omega_\beta \in C_2$ for any $\beta \in \Delta\backslash\{\alpha\}$. Now

$$X(T) \otimes \mathbb{Q} = H_\alpha \oplus \mathbb{Q}\omega_\alpha,$$

an orthogonal decomposition. So let

$$\omega_\beta = x + c\omega_\alpha.$$

It suffice to show that

i) $c \geq 0$, and
ii) $x \in Cone(\Delta\backslash\{\alpha\})$.

To get i), we use the inner product. Since $\omega_\beta = x + c\omega_\alpha$, we obtain

$$< \omega_\beta, \omega_\alpha >=< x, \omega_\alpha > +c < \omega_\alpha, \omega_\alpha > .$$

But $< x, \omega_\alpha >= 0$, so that

$$c =< \omega_\beta, \omega_\alpha > / < \omega_\alpha, \omega_\alpha >,$$

and it is well known that this is non-negative.

To get ii), first notice that

$$< \beta, x >=< \beta, \omega_\beta - c\omega_\alpha >=< \beta, \omega_\beta >= 1.$$

But if $\gamma \neq \beta, \alpha$, we obtain

$$< \gamma, x >=< \gamma, \omega_\beta - c\omega_\alpha >= 0.$$

So x is the dual of β in the root system $(H_\alpha, \Delta \backslash \{\alpha\})$. But it is well known that, for any root system, the cone generated by the fundamental weights is contained in the cone generated by the positive roots, since the inverse of the Cartan matrix has positive entries. This proves the claim.

Now let $\mathcal{C} = Cone(\{\omega_\alpha | \alpha \in \Delta\})$. We claim now that

$$(z_\alpha + H_\alpha) \cap \mathcal{C}) = (z_\alpha + H_\alpha^+) \cap \mathcal{C}.$$

From our first claim,

$$\mathcal{C} = \bigcup_{r \geq 0} (rz_\alpha + H_\alpha^+) \cap \mathcal{C} = \mathcal{C} = \bigcup_{r \geq 0} (rz_\alpha + H_\alpha) \cap \mathcal{C}.$$

But $(rz_\alpha + H_\alpha) \cap (sz_\alpha + H_\alpha) = \phi$ if $r \neq s$. Hence

$$(z_\alpha + H_\alpha) \cap \mathcal{C}) \subseteq (z_\alpha + H_\alpha^+) \cap \mathcal{C},$$

and this establishes the second claim.

Now assume that $x \in z_\alpha + H_\alpha$. Then since $x \in \mathcal{P} \cap \mathcal{C}$, we get from the claim that $x \in z_\alpha + H_\alpha^+$. So clearly, a) and b) are equivalent. Also a) and c) are equivalent since, from Lemma 7.25, $(z_\alpha + H_\alpha) \cap \mathcal{P}$ is the face of $\mathcal{P}$ corresponding to $e_\alpha \in \Lambda$; while $x \in \mathcal{P}$ is the vertex of $\mathcal{P}$ corresponding to e_+. This completes the proof.

Corollary 7.27. *For each $\alpha \in \Delta$, either $x \in z_\alpha + H_\alpha$, or else $y \in z_\alpha + H_\alpha$.*

Proof. $\{e_\alpha, e_+, e_-\} \subseteq \Lambda'$, *while* $\Lambda_1 = \{e_+, e_-\}$. *Thus* $e_\alpha \geq e_+$ *or else* $e_\alpha \geq e_-$.

Theorem 7.28. *Write $x - y = \sum_{\alpha \in \Delta} r_\alpha \alpha$, where $r_\alpha \in \mathbb{Q}$.*

a) The following are equivalent:
 i) $r_\alpha > 0$
 ii) $e_\alpha \in \Lambda_+$
 iii) $x \in z_\alpha + H_\alpha^+$, $y \notin z_\alpha + H_\alpha^+$.

b) The following are equivalent:
 i) $r_\alpha < 0$
 ii) $e_\alpha \in \Lambda_-$
 iii) $y \in z_\alpha + H_\alpha^+$, $x \notin z_\alpha + H_\alpha^+$.

c) The following are equivalent:
 i) $r_\alpha = 0$
 ii) $e_\alpha \in \Lambda_0$
 iii) $x \in z_\alpha + H_\alpha^+$, $y \in z_\alpha + H_\alpha^+$.

Proof. In each case, it suffices to show that i) and ii) are equivalent since, by Lemma 7.26, ii) and iii) are equivalent. By Corollary 7.27, exactly one of a) iii), b) iii) or c) iii) occurs.

In case a), $x \in z_\alpha + H_\alpha^+$ and $y \notin z_\alpha + H_\alpha^+$. Then

$$x = z_\alpha + \sum_{\beta \neq \alpha} a_\beta \beta$$

and

$$y = z_\alpha + \sum_{\beta \in \Delta} b_\beta \beta.$$

But $b_\alpha < 0$, since y lies in the bounded part of $\mathcal{C} \backslash (z_\alpha + H_\alpha)$, and thus "below" the hyperplane $z_\alpha + H_\alpha$. Hence

$$x - y = \sum_{\beta \neq \alpha} a_\beta \beta - \sum_{\beta \in \Delta} b_\beta \beta = \sum_{\beta \neq \alpha} (a_\beta - b_\beta)\beta \ - \ b_\alpha \alpha.$$

Hence $r_\alpha = -b_\alpha > 0$ here. Case b) is similar to case a).

In case c) we can write

$$x = z_\alpha + \sum_{\beta \neq \alpha} a_\beta \beta$$

and

$$x = z_\alpha + \sum_{\beta \neq \alpha} b_\beta \beta.$$

Hence

$$x - y = \sum_{\beta \neq \alpha} (a_\beta - b_\beta)\beta$$

and thus, $r_\alpha = 0$ in this case.

7.5.4 Monoids with $I_+ = \Delta \backslash \{\alpha\}$ and $I_- = \Delta \backslash \{\beta\}$

In this section we exhibit some explicit calculations of the type maps of 2-reducible monoids. We restrict our attention to certain monoids with group $G = Gl_{n+1}(K)$. The general problem here is to determine all possible $(+, -, 0)$-decompositions of Δ that can actually occur for the given I_+ and I_-. We do not yet have a general solution to this intriguing problem. However, our calculations indicate that it has something to do with linear programming problems involving the inverse of the Cartan matrix.

So let $G = Gl_{n+1}(K)$, and let us consider 2-reducible, semisimple monoids M with unit group G. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots of G, and $\{\omega_1, \ldots, \omega_n\}$ the set of fumdamental, dominant weights. Then it is well known that, for $i = 0, \ldots, n-1$,

$$(n+1)\omega_{i+1} = (n-i)\alpha_1 + 2(n-i)\alpha_2 + \cdots + (i+1)(n-i)\alpha_{i+1} + \cdots + (i+1)\alpha_n.$$

For convenience, we let

$$x_{i+1} = (n+1)\omega_{i+1}.$$

Let M be a 2-reducible, semisimple monoid with unit group G and assume that $I_+ = \Delta\backslash\{\alpha_1\}$, $I_- = \Delta\backslash\{\alpha_{i+1}\}$. The polytope $\mathcal{P}$ of M is the convex hull of the W-orbit of $\{x, y\} \subseteq X(T_0) \otimes \mathbb{Q}^+$. Hence x is a rational multiple of x_1, and y is a rational multiple of x_{i+1}. Without loss of generality, $x = x_1$ and $y = rx_{i+1}$ for some $r > 0$. By the results of Theorem 7.28, we need to calculate

$$x - y = \sum_{i=1}^{n} r_i\alpha_i.$$

But that is elementary, and we obtain

i) $r_j = n - j + 1 - j(r(n-i))$ if $j \leq i$
ii) $r_j = (1-(i+1)r)(n-j+1)$ if $j > i$.

By Corollary 7.27, we must have

i) $n - r(n-i) > 0$, and
ii) $(1-(i+1)r) < 0$.

Hence

$$1/(i+1) \ < \ r \ < \ n/(n-i).$$

For certain special values of r, r_j can be zero. These values are

$$r = (n-j)/(j+1)(n-i).$$

In any case, it is an elementary calculation. We summarize our results as follows.

Theorem 7.29. *Let M be a 2-reducible, semisimple monoid with unit group $Gl_{n+1}(K)$, and assume that $I_+ = \Delta\backslash\{\alpha_1\}$, $I_- = \Delta\backslash\{\alpha_{i+1}\}$. Write $x = x_1$, $y = rx_{i+1}$ as above. Then*

a) $1/(i+1) < r < l/(l-i)$;
b) if $1 \leq j \leq i-1$ *and* $r = (n-j)/(j+1)(n-i)$ *then*
$\Delta_+ = \{\alpha_1, \ldots, \alpha_j\}$
$\Delta_- = \{\alpha_{j+2}, \ldots, \alpha_n\}$;
c) if $0 \leq j \leq i-1$ *and* $(n-j-1)/(j+2)(n-i) < r < (n-j)/(j-i)(n-i)$ *then*
$\Delta_+ = \{\alpha_1, \ldots, \alpha_{j+1}\}$
$\Delta_- = \{\alpha_{j+2}, \ldots, \alpha_n\}$.

It is now possible to calculate Λ and λ in each case using Theorem 7.23. The details are left to the reader.

7.5.5 Monoids with $I_+ = \phi$ and $I_- = \phi$

It is easy to characterize the pairs (I_+, I_-) that can actually occur as $(\lambda_*(e_+), \lambda_*(e_-))$ for some 2-reducible semisimple monoid M with $\Lambda_1 = \{e_+, e_-\}$. Indeed, let $A, B \subseteq \Delta$ be any two proper subsets. Then $(A, B) = (I_+, I_-)$ for some semisimple, 2-reducible monoid M if and only if either

i) $A \neq B$, or else
ii) $A = B$ and $|\Delta \backslash A| \geq 2$.

In particular, $I_+ = I_- = \phi$ is possible; in fact generic. Notice that this is equivalent to $\{x, y\}$ being a subset of $\mathcal{C}^0$, the interior of $\mathcal{C}$.

Theorem 7.30. *The following are equivalent:*

a) there exists a 2-reducible, semisimple monoid M with $I_+ = I_- = \phi$ and $(\Delta_+, \Delta_-) = (U, V)$;
b) $U \neq \phi, V \neq \phi$ and $U \cap V = \phi$.

Proof. Obviously, a) implies b). So assume that $U, V \subseteq \Delta$ satisfy b). Define

$$\delta = \sum_{\alpha \in U} \alpha - \sum_{\beta \in V} \beta.$$

It is then easy to find $x, y \in \mathcal{C}^0$ so that $x - y = \delta$. Then apply Theorem 7.28.

7.5.6 $(\mathcal{J},\sigma)$-irreducible Monoids Revisited

In this section we use the results of Theorems 7.23 and 7.28 to study the orbit structure of certain reductive monoids M with involution $\sigma : M \to M$.

Definition 7.31. *Let M be an reductive monoid with zero, and suppose that $\sigma : M \to M$ is a bijective morphism of algebraic monoids. We say that (M, σ) is* $(\mathcal{J},\sigma)$-irreducible *if the map induced by σ is transitive on the set of minimal $G \times G$-orbits of $M \backslash \{0\}$.*

$(\mathcal{J},\sigma)$-irreducible monoids were studied systematically by Z. Li and the other authors of [51, 52, 53]. In all cases, except those that contain D_4 as a component, σ^2 induces the identity morphism on the set of $G \times G$-orbits of M. In such cases, M is a 2-reducible monoid precisely when $M \backslash \{0\}$ has exactly two minimal $G \times G$-orbits and σ exchanges these orbits. In this section, we discuss several examples where M is 2-reducible and semisimple, and σ is actually an automorphism of M of order two. The purpose of Theorems 7.23 and 7.28 is to identify the minimal information (i.e. Δ_+ and Δ_-) needed to get the type map of M.

Example 7.32. Let M be a 2-reducible, semisimple monoid with unit group $Gl_6(K)$. Assume that there is an automorphism $\sigma : M \to M$ such that $\sigma^2 = id$ and $\sigma | Gl_6(K)$ is transpose-inverse.

Let $F = \{\lambda_1, \ldots, \lambda_5\}$ be the set of fundamental dominant weights of $Sl_6(K)$. Then σ induces the following involution σ^* on F:

$$\sigma^*(\lambda_i) = \lambda_{6-i}.$$

From Table 2 on page 295 of [69] we obtain

$$\lambda_1 - \lambda_5 = \frac{1}{6}(4\alpha_1 + 2\alpha_2 - 2\alpha_4 - 4\alpha_5),$$

and

$$\lambda_2 - \lambda_4 = \frac{1}{6}(2\alpha_1 + 4\alpha_2 - 4\alpha_4 - 2\alpha_5).$$

Now any 2-reducible, semisimple monoid M has a representaion $\rho : M \to End(V \oplus W)$, as in Proposition 7.19. If V is the irreducible M-module with highest weight $\lambda \in X(T)_+$, then W is the irreducible M-module with highest weight $\sigma^*(\lambda) \neq \lambda$. Write

$$\lambda = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 + a_5\lambda_5$$

where $a_i \geq 0$, and either $a_1 \neq a_5$ or else $a_2 \neq a_4$ (so that $\sigma^*(\lambda) \neq \lambda$). In any case,

$$\begin{aligned} \lambda - \sigma^*(\lambda) &= \frac{1}{6}([4(a_1 - a_5) + 2(a_2 - a_4)]\alpha_1 + [2(a_1 - a_5) + 4(a_2 - a_4)]\alpha_2) \\ &\quad - \frac{1}{6}([2(a_1 - a_5) + 4(a_2 - a_4)]\alpha_4 + [4(a_1 - a_5) + 2(a_2 - a_4)]\alpha_5). \end{aligned}$$

Now

$$\begin{aligned} I_+ &= \{\alpha_i \mid a_i \neq 0\} \\ I_- &= \{\alpha_i \mid a_{6-i} \neq 0\}. \end{aligned}$$

Notice that in all cases $\Delta_- = \{\alpha_{6-i} \mid \alpha_i \in \Delta_+\}$, while $\alpha_3 \notin \Delta_+ \sqcup \Delta_-$. So it suffices to calculate the possibilities for Δ_+ in terms of λ.

1. $\Delta_+ = \{\alpha_1, \alpha_2\}$ if $2(a_1 - a_5) + (a_2 - a_4) > 0$ and $(a_1 - a_5) + 2(a_2 - a_4) > 0$.
2. $\Delta_+ = \{\alpha_1, \alpha_4\}$ if $2(a_1 - a_5) + (a_2 - a_4) > 0$ and $(a_1 - a_5) + 2(a_2 - a_4) < 0$.
3. $\Delta_+ = \{\alpha_1\}$ if $2(a_1 - a_5) + (a_2 - a_4) > 0$ and $(a_1 - a_5) + 2(a_2 - a_4) = 0$.
4. $\Delta_+ = \{\alpha_2\}$ if $2(a_1 - a_5) + (a_2 - a_4) = 0$ and $(a_1 - a_5) + 2(a_2 - a_4) > 0$.

All other feasible data are obtained by reversing the rôles of λ and $\sigma^*(\lambda)$. But we obtain no new monoids. The potential cases with $\Delta_+ = \{\alpha_1, \alpha_5\}$ or $\{\alpha_2, \alpha_4\}$ are not possible. Also, any situation where $|\Delta_+| \geq 3$ is not possible.

We see from Theorems 7.23 and 7.28 that the type map of M is now determined in each case.

Example 7.33. Let M be a 2-reducible, semisimple monoid with unit group $K^*SO_{2n}(K) \subseteq Gl_{2n}(K)$. Assume that there is an automorphism $\sigma : M \to M$ such that $\sigma^2 = id$ and $\sigma|SO_{2n}(K)$ is transpose-inverse.

Let $F = \{\lambda_1, \ldots, \lambda_{n-2}, \lambda_{n-1}, \lambda_n\}$ be the set of fundamental dominant weights of $SO_{2n}(K)$. Then σ induces the following involution σ^* on F:

$$\sigma^*(\lambda_i) = \lambda_i \text{ if } i \leq n-2,$$

and

$$\sigma^*(\lambda_{n-1}) = \lambda_n,\ \sigma^*(\lambda_n) = \lambda_{n-1}.$$

From Table 2 on page 296 of [69] we obtain

$$\lambda_m - \lambda_{n-1} = \frac{1}{2}(-\alpha_{n-1} + \alpha_n).$$

As in the previous example, any 2-reducible, semisimple monoid M has a representaion $\rho : M \to End(V \oplus W)$, according to Proposition 7.19. If V is the irreducible M-module with highest weight $\lambda \in X(\overline{T})_+$, then W is the irreducible M-module with highest weight $\sigma^*(\lambda) \neq \lambda$. Write

$$\lambda = a_1\lambda_1 + a_2\lambda_2 + \cdots + a_{n-2}\lambda_{n-2} + a_{n-1}\lambda_{n-1} + a_n\lambda_n$$

where $a_n \neq a_{n-1}$, (so that $\sigma^*(\lambda) \neq \lambda$). Then

$$\lambda - \sigma^*(\lambda) = \frac{a_n - a_{n-1}}{2}(-\alpha_{n-1} + \alpha_n).$$

Now

$$I_+ = \{\alpha_i \mid a_i \neq 0\}$$
$$I_- = \{\alpha_i \mid \overline{a_i} \neq 0\}.$$

where $\overline{a_n} = a_{n-1}$, $\overline{a_{n-1}} = a_n$, and $\overline{a_i} = a_i$ if $i < n-1$. Notice again that, in all cases, $\Delta_- = \{\overline{\alpha} \mid \alpha \in \Delta_+\}$, and so we only need to consider the possibilities for Δ_+ in terms of λ. There are just two cases:

1. $\Delta_+ = \{\alpha_{n-1}\}$ if $a_{n-1} \geq a_n$;
2. $\Delta_+ = \{\alpha_n\}$ if $a_n \geq a_{n-1}$.

Again we see from Theorems 7.23 and 7.28 how the type map of M is completely determined in each case. The details are left to the reader. Notice that the two cases yield the same monoid M, since σ^* exchanges α_{n-1} and α_n.

7.6 Type Maps in General

It is certain that the combinatorial classification of type maps of all semisimple monoids M is a "dead end" problem. Indeed, it appears to include the combinatorial classification of all rational polytopes as a proper subproblem. But there are still some interesting questions here. It is clear that the type map is the combinatorial glue that makes the monoid structure possible. But it may also be (as it is for the case of two 0-minimal $\mathcal{J}$-classes in Theorem 7.28) an important combinatorial manifestation of the classification data of reductive monoids.

In this section, we speculate on the likelihood that the set of isomorphism classes of reductive monoids may have the structure of a union of rational polyhedral cones, similar to the data one obtains from a non affine torus embedding. Each face appears to represent the set of isomorphism classes of monoids with the same (fixed) type map. The order relation between these faces should represent a particular combinatorial degeneracy of that type map. This speculation leads us to a number of interesting results about the geometric underpinnings of type maps.

Let G be a semisimple algebraic group with maximal torus T. Let $X(T)$ be the set of characters of T and let $\Delta \subseteq X(T)$ be the set of simple roots. As usual, let

$$\mathcal{C} = \{x \in X(T) \otimes \mathbb{Q} \mid \; < \alpha, x > \geq 0 \text{ for all } \alpha \in \Delta \}$$

be the Weyl chamber of $E = X(T) \otimes \mathbb{Q}$ associated with Δ.

Definition 7.34. *If $x_1, \ldots, x_n \in \mathcal{C}$, we say that $\{x_1, \ldots, x_n\}$ is* stable *if, for each $i \neq j$,*

$$x_i - x_j = \sum_{\alpha \in \Delta} r_\alpha \alpha$$

has the property that $r_\alpha < 0$ for some $\alpha \in \Delta$, and $r_\beta > 0$ for some $\beta \in \Delta$.

Conjecture 7.35. The following are equivalent for $\{x_1, \ldots, x_n\} \subseteq \mathcal{C}$:

a) $\{x_1, \ldots, x_n\}$ is stable,
b) Each x_i is an extreme point of the convex hull of $\{w(x_i) \mid w \in W, \; i = 1, \ldots, n\}$.

Question 7.36. Write $\Lambda_1 = \{x_1, \ldots, x_n\} \subseteq \mathcal{C}$. Define, for $\alpha \in \Delta$,

$$r_\alpha : \Lambda_1 \times \Lambda_1 \to \mathbb{Q}$$

by the rule

$$x - y = \sum_{\alpha \in \Delta} r_\alpha(x, y)\alpha.$$

Then

a) $r_\alpha(x,y) + r_\alpha(y,z) = r_\alpha(x,z)$;
b) $r_\alpha(x,y) = -r_\alpha(y,x)$;
c) if $x \neq y$ then there exists $\alpha, \beta \in \Delta$ such that $r_\alpha(x,y) > 0$ and $r_\beta(x,y) < 0$.

Does any collection $\{r_\alpha : \Lambda_1 \times \Lambda_1 \to \mathbb{Q}\}$ satisfyting a), b) and c) come from a subset $\Lambda_1 \subseteq \mathcal{C}$? If not, is there an interpretation?

Given $\Lambda_1, \Lambda_1' \subseteq \mathcal{C}$ as in Definition 7.34 we say that Λ_1 and Λ_1' have the same **shape** if there is a bijection $\rho : \Lambda_1 \to \Lambda_1'$ such that

a) $W_x = W_{\rho(x)}$ for each $x \in \Lambda_1$
b) $r_\alpha(x,y) = 0$ if and only if $r_\alpha(\rho(x), \rho(y)) = 0$
c) $r_\alpha(x,y) > 0$ if and only if $r_\alpha(\rho(x), \rho(y)) > 0$.

We do not claim here that, if Λ_1 and Λ_1' have the same shape, then they come from monoids with the same type map. This does not seem to be true, although we do not yet have any revealing examples.

Proposition 7.37. *Assuming the above conjecture is true, the bijection $\rho : \Lambda_1 \to \Lambda_1'$ is unique if it exists.*

Proof. Suppose that there there are two, say $\rho : \Lambda_1 \to \Lambda_1'$ and $\sigma : \Lambda_1 \to \Lambda_1'$. Then let $\psi = \rho^{-1} \circ \sigma : \Lambda_1 \to \Lambda_1$. Notice that ψ satisfies a), b) and c) above. By Conjecture 7.35, there exists $\alpha \in \Delta$ such that

$$r_\alpha(x, \psi(x) > 0.$$

Thus,

$$r_\alpha(\psi(x), \psi^2(x)) > 0,$$
$$\vdots$$
$$r_\alpha(\psi^{n-1}(x), \psi^n(x)).$$

Also, where we assume that $\psi^n(x) = x$. In any case,

$$\sum_{i=1}^{n} r_\alpha(\psi^{i-1}(x), \psi^i(x)) > 0.$$

However,

$$\begin{aligned} 0 &= (x - \psi(x)) + (\psi(x) - \psi^2(x)) + \cdots + (\psi^{n-1}(x) - \psi^n(x)) \\ &= \sum_{\alpha \in \Delta} r_\alpha(x, \psi(x))\alpha + \cdots + \sum_{\alpha \in \Delta} r_\alpha(\psi^{n-1}(x), \psi^n(x))\alpha \\ &= \sum_{\alpha \in \Delta} (\sum_{i=1}^{n} r_\alpha(\psi^{i-1}(x), \psi^i(x)))\alpha. \end{aligned}$$

Thus, $\sum_{i=1}^{n} r_\alpha(\psi^{i-1}(x), \psi^i(x)) = 0$, since $\Delta \subseteq E$ is a $\mathbb{Q}$-basis. This contradiction finishes the proof.

We conclude that, if Λ_1 and Λ_1' have the same shape, then we can add them as follows.
Let $\rho : \Lambda_1 \to \Lambda_1'$ be the unique bijection that evidences Λ_1 and Λ_1' of the same shape. Then define the **sum** of Λ_1 and Λ_1' as

$$\Lambda_1'' = \{x + \rho(x) \mid x \in \Lambda_1\}.$$

Proposition 7.38. *Λ_1'' has the same shape as Λ_1.*

Proof. Define $\psi : \Lambda_1 \to \Lambda_1''$ by $\psi(x) = x + \rho(x)$. Now for $x \in \Lambda_1$,

$$x, \rho(x) \in (E^{W_x} \cap \mathcal{C})^0,$$

which is closed under addition. Hence $x + \rho(x) \in (E^{W_x} \cap \mathcal{C})^0$ as well, and thus $W_x = W_{x+\rho(x)}$, since

$$(E^{W_x} \cap \mathcal{C})^0 = \{y \in \mathcal{C} \mid W_y = W_x\}.$$

To finish the proof, notice that

$$x + \rho(x) - (y + \rho(y)) = \sum_{\alpha \in \Delta} (r_\alpha(x, y) + r_\alpha(\rho(x), \rho(y))\alpha.$$

Hence $r_\alpha(x, y) = 0$ implies that $r_\alpha(\rho(x), \rho(y)) = 0$, which implies that $r_\alpha(x + \rho(x), y + \rho(y)) = 0$. Similarly for $>$ and $<$. Thus Λ_1 and Λ_1'' have the same shape.

One can pose the dual problem using Λ^1 and λ_*. One can determine the type map in terms of colors (λ_*) and divisors (λ^1) (see § 5.3.3). One might then be able to define the addition of polytopes and cross section lattices (in that setup) in terms of the associated valuations comming from Λ^1.

7.7 Exercises

7.7.1 The Cross Section Lattice

1. Let M be reductive, and let $\Lambda \subseteq E(\overline{T})$ be a cross section lattice. Prove that the number of maximal chains in $E(\overline{T})$ is equal to the number of maximal chains of Λ times the order of the Weyl group.
2. One defines a reductive monoid M, with zero, to be $\mathcal{J}_i$-*irreducible* if $|\Lambda_j| = 1$ for all $j \leq i$. Prove that
   i) M is $\mathcal{J}_2$-irreducible if and only if $J_0 = S \backslash \{s\}$ for some $s \in S$
   ii) M is $\mathcal{J}_3$-irreducible if and only if $J_0 = S \backslash \{s\}$ where s corresponds to an end node on the Dynkin diagram of G.
3. One can use Theorem 7.12 also to characterize other classes of $\mathcal{J}$-irreducible monoids.

i) M is $\mathcal{J}$-simple if and only if S is connected and M is either $\mathcal{J}_2$-irreducible or $S\backslash J_0 = \{s,t\}$ where $st \neq ts$. Here, we say a $\mathcal{J}$-irreducible monoid is $\mathcal{J}$-*simple* if $\lambda^*(e)$ is a connected subset of the Dynkin diagram for each $e \in \Lambda$.
ii) $\Lambda(M)$ is a distributive lattice if and only if $S\backslash J_0$ is connected.

4. Let M be reductive, and let Λ be a cross section lattice of M. Prove that there is a one-to-one correspondence between the set of two-sided ideals of M and the set of poset ideals of Λ.

7.7.2 Idempotents

1. Let $\psi : M \to N$ be a finite dominant morphism of irreducible algebraic monoids.
   a) Prove that $\mathcal{U}(\psi) : \mathcal{U}(M) \to \mathcal{U}(N)$ is bijective.
   b) Prove that $E(\psi) : E(M) \to E(N)$ is bijective.
2. Let M be irreducible with unit group G and maximal torus T. Let B be a Borel subgroup containing T. Let α be a positive root, and consider $U_\alpha \subseteq B, e \in E(\overline{T})$.
   a) Prove the following are equivalent:
      i) $eU_\alpha = U_\alpha e$,
      ii) $s_\alpha e = es_\alpha$.
   b) Prove the following are equivalent:
      i) $eU_\alpha = U_\alpha e \neq \{e\}$,
      ii) $s_\alpha e = es_\alpha \neq e$.
3. Let M be reductive with $e \leq f \leq g$. Assume that $e, f, g \in E(\overline{T})$. As usual, let $S = \{s_\alpha \in W \mid \alpha \in \Delta\}$, and identify S with the set of nodes on the Dynkin diagram. Prove that each connected component of $\lambda^*(f)$ is contained in either $\lambda^*(e)$ or $\lambda^*(g)$.

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