Preface

This volume is a collection of chapters reflecting the status of much of the current research in K-theory. As editors, our goal has been to provide an entry and an overview to K-theory in many of its guises. Thus, each chapter provides its author an opportunity to summarize, reflect upon, and simplify a given topic which has typically been presented only in research articles. We have grouped these chapters into five parts, and within each part the chapters are arranged alphabetically.

Informally, K-theory is a tool for probing the structure of a mathematical object such as a ring or a topological space in terms of suitably parameterized vector spaces. Thus, in some sense, K-theory can be viewed as a form of higher order linear algebra that has incorporated sophisticated techniques from algebraic geometry and algebraic topology in its formulation. As can be seen from the various branches of mathematics discussed in the succeeding chapters, K-theory gives intrinsic invariants which are useful in the study of algebraic and geometric questions. In low degrees, there are explicit algebraic definitions of K-groups, beginning with the Grothendieck group of vector bundles as K_0, continuing with H. Bass's definition of K_1 motivated in part by questions in geometric topology, and including J. Milnor's definition of K_2 arising from considerations in algebraic number theory<. On the other hand, even when working in a purely algebraic context, one requires techniques from homotopy theory to construct the higher K-groups K_i and to achieve computations. The resulting interplay of algebra, functional analysis, geometry, and topology in K-theory provides a fascinating glimpse of the unity of mathematics.

K-theory has its origins in A. Grothendieck's formulation and proof of his celebrated Riemann-Roch Theorem [5] in the mid-1950's. While K-theory now plays a significant role in many diverse branches of mathematics, Grothendieck's original focus on the interplay of algebraic vector bundles and algebraic cycles on algebraic varieties is much reflected in current research, as can be seen in the chapters of Part II. The applicability of the Grothendieck construction to algebraic topology was quickly perceived by M. Atiyah and F. Hirzeburch [1], who developed topological K-theory into the first and most important example of a "generalized cohomology theory". Also in the 1960's, work of H. Bass and others resulted in the formulation and systematic investigation of constructions in geometric topology (e.g., that of the Whitehead group and the Swan finiteness obstruction) involving the K-theory of non-commutative rings such as the group ring of the fundamental group of a manifold. Others soon saw the relevance of K-theoretic techniques to number theory, for example in the solution by H. Bass, J. Milnor, and J.-P. Serre [2] of the congruence subgroup problem and the conjectures of S. Lichtenbaum [6] concerning the values of zeta functions.

In the early 1970's, D. Quillen [8] provided the now accepted definition of higher algebraic K-theory and established remarkable properties of "Quillen's K-groups", thereby advancing the formalism of the algebraic side of K-theory and enabling various computations. An important application of Quillen's theory is the identification by A. Merkurjev and A. Suslin [7] of $K_2 \otimes Z/n$ of a field with n-torsion in the Brauer group. Others soon recognized that many of Quillen's techniques could be applied to rings with additional structure, leading to the study of operator algebras and to L-theory in geometric topology. Conjectures by S. Bloch [4] and A. Beilinson [3] concerning algebraic K-theory and arithmetical algebraic geometry were also formulated during the 1970's; these conjectures prepared the way for many current developments.

We now briefly mention the subject matter of the individual chapters, which typically present mathematics developed in the past twenty years.

Part I consists of five chapters, beginning with Gunnar Carlsson's exposition of the formalism of infinite loop spaces and their role in K-theory. This is followed by the chapter by Daniel Grayson which discusses the many efforts, recently fully successful, to construct a spectral sequence converging to K-theory analogous to the very useful Atiyah-Hirzebruch spectral sequence for topological K-theory. Max Karoubi's chapter is dedicated to the exposition of Bott periodicity in various forms of K-theory: topological K-theory of spaces and Banach algebras, algebraic and Hermitian K-theory of discrete rings. The chapters by Lars Hesselholt and Charles Weibel present two of the most successful computations of algebraic K-groups, namely that of truncated polynomial algebras over regular noetherian rings over a field and of rings of integers in local and global fields. These computations are far from elementary and have required the development of many new techniques, some of which are explained in these (and other) chapters.

Some of the important recent developments in arithmetic and algebraic geometry and their relationship to K-theory are explored in Part II. In addition to a discussion of much recent progress, the reader will find in these chapters considerable discussion of conjectures and their consequences. The chapter by Thomas Geisser gives an exposition of Bloch's higher Chow groups, then discusses algebraic K-theory, étale K-theory, and topological cyclic homology. Henri Gillet explains how algebraic K-theory provides a useful tool in the study of intersection theory of cycles on algebraic varieties. Various constructions of regulator maps are presented in the chapter by Alexander Goncharov in order to investigate special values of L-functions of algebraic varieties. Bruno Kahn discusses the interplay of algebraic K-theory, arithmetic algebraic geometry, motives and motivic cohomology, describing fundamental conjectures as well as some progress on these conjectures. Marc Levine's chapter consists of an overview of mixed motives, including various constructions and their conjectural role in providing a fundamental understanding of many geometric questions.

Part III is a collection of three articles dedicated to constructions relating algebraic K-theory (including the K-theory of quadratic spaces) to "geometric topology" (i.e., the study of manifolds). In the first chapter, Paul Balmer gives a modern and general survey of Witt groups constructed in a fashion analogous to the construction of algebraic K-groups. Jonathan Rosenberg's chapter surveys a great range of topics in geometric topology, reviewing recent as well as classical applications of K-theory to geometry. Bruce William' chapter emphasizes the role of the K-theory of quadratic forms in the study of moduli spaces of manifolds.

In Part IV are grouped three chapters whose focus is on the (topological) K-theory of C^*-algebras and other topological algebras which arise in the study of differential geometry. Joachim Cuntz presents in his chapter an investigation of the K-theory, K-homology and bivariant K-theory of topological algebras and their relationship with cyclic homology theories via Chern character transformations. In their long survey, Wolfgang Lueck and Holger Reich discuss the significant progress made towards the complete solution of important conjectures which would identify the K-theory or L-theory of group rings and C^*-algebras with appropriate equivariant homology groups. In the chapter by Jonathan Rosenberg, the relationship between operator algebras and K-theory is motivated, investigated, and explained through applications.

The fifth and final part presents other forms and approaches to K-theory not found in earlier chapters. Eric Friedlander and Mark Walker survey recent work on semi-topological K-theory that interpolates between algebraic K-theory of varieties and topological K-theory of associated analytic spaces. Alexander Merkurjev develops the K-theory of G-vector bundles over an algebraic variety equipped with an action of a group G and presents some applications of this theory. Stephen Mitchell's chapter demonstrates how algebraic K-theory provides an important link between techniques in algebraic number theory and sophisticated constructions in homotopy theory. The final chapter by Amnon Neeman provides a historical overview and through investigation of the challenge of recovering K-theory from the structure of a triangulated category.

Finally, two Bourbaki articles (by Eric Friedlander and Bruno Kahn) are reprinted in the appendix. The first summarizes some of the important work of A. Suslin and V. Voevodsky on motivic cohomology, whereas the second outlines the celebrated theorem of Voevodsky establishing the validity of a conjecture by J. Milnor relating $K(-) \otimes Z/2$, Galois cohomology, and quadratic forms.

Some readers will be disappointed to find no chapter dedicated specifically to low-degree (i.e., classical) algebraic K-groups and insufficient discussion of the role of algebraic K-theory to algebraic number. We fully acknowledge the many limitations of this handbook, but hope that readers will appreciate the expository effort and skills of the authors.

April, 2005

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